The Étale Topology

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1 Smooth morphisms and the étale site

1.1 A fact about étale maps

We defined the notion of étale morphism last time, as an algebraic analogue of the inverse function theorem in analytic geometry. To support that analogy, it is at least psychologically helpful to be aware of the following result:

Theorem 1.1.1. *For a map $f\colon X \to Y$ of locally finite type $\mathbf{C}$-schemes, the associated map of complex-analytic spaces $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a local isomorphism if and only if f is étale.*

We present this as motivation for our definition of étale morphism; we won't spend time justifying this. For the implication $\implies$, one uses that the natural map

$$\widehat{O}_{X,x} \to \widehat{\mathcal{O}}_{X^{\mathrm{an}},x}$$

is an isomorphism (and both local rings are Noetherian before completion) and that one can check étaleness for maps between locally finite type schemes over a field k by using k-finite artin local points (thereby enabling one to check the infinitesimal criterion by computations with completions of local rings).

The converse implication is essentially the "inverse function theorem", which in the absence of smoothness for X and Y rests on the local structure theorem for analytic maps near points isolated in their fiber. (The beautiful book *Coherent Analytic Sheaves* by Grauert and Remmert develops the theory of complex-analytic spaces from scratch, and this local structure theorem is the Proposition in §1.2 of Chapter 3 of that book.)

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1.2 Smooth maps

Definition 1.2.1. A map of schemes $f\colon X \to S$ is *smooth* if it is Zariski-locally of the form

$$\begin{array}{ccc} \operatorname{Spec} B & \subset & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \subset & S \end{array}$$

where

$$B = (A[T_1, \ldots, T_n]/(f_1, \ldots, f_r))_h$$

with $(\frac{\partial f_i}{\partial T_j})$ has rank r at all points of Spec B.

As before, we have several equivalent formulations (with the proof of equivalences discussed very nicely in Chapter 2 of the book *Néron Models*):

Theorem 1.2.2. *For $f\colon X \to S$, the following are equivalent to smoothness:*

1. *f is locally finitely presented and for $R \to R_0$ any square-zero thickening (i.e., the kernel I has $I^2 = 0$) the map $X(R) \twoheadrightarrow X(R_0)$ is surjective; in other words, for any diagram as below there exists a lift:*

$$\begin{array}{ccc} \operatorname{Spec} R_0 & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} R & \longrightarrow & S \end{array}$$

2. *X has a Zariski cover $\{X_i\}$ admitting étale morphisms $X_i \to \mathbf{A}_S^{n_i}$ over S.*

3. *f is locally finitely presented, flat, and each geometric fiber $X_{\overline{s}} := X_s \otimes_{k(s)} \overline{k(s)}$ is regular. (This regularity can equivalently be checked for* any *algebraically closed $K/k(s)$.)*

4. *f is locally finitely presented, flat, and $\Omega^1_{X/S}$ is locally free of rank $\dim_x X_{f(x)}$ near each $x \in X$.*

It is relatively easy to go from our definition of smoothness to the first condition: one can express the existence of the lift as a linear algebra problem with values in the ideal I of R_0 in R. Once in the realm in linear algebra, the existence is equivalent to checking after localizing at any prime of R, and over local rings the existence is guaranteed by the condition on the Jacobian.

The hardest part of the Theorem is (1) $\Longrightarrow$ (2). The idea is to (locally) embed X into an affine space over S and use the lifting condition to deduce the pointwise rank condition on the defining equations.

1.3 The étale site

Definition 1.3.1. The *étale site* $S_{\text{ét}}$ of a scheme S is the category of étale maps $U \to S$ with S-maps as morphisms, and a *cover* of $U \to S$ is a collection of maps

$$\begin{array}{ccc} U_i & \xrightarrow{h_i} & U \\ & \searrow \qquad \swarrow & \\ & S & \end{array}$$

such that $\bigcup_i h_i(U_i) = U$.

Remark 1.3.2. It is an exercise using the infinitesimal criterion to show that each h_i is automatically étale in this situation, and so in particular every $h_i(U_i)$ is Zariski-open in U.

Definition 1.3.3. A *sheaf of sets* $\mathcal{F}$ on $S_{\text{ét}}$ is a contravariant functor $\mathcal{F}\colon S_{\text{ét}} \to \mathbf{Set}$ such that for any cover $(U_i \to U)$,

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_S U_j)$$

is an equalizer diagram. (A presheaf is simply the functor, without the equalizer condition.)

For any $h : V \to U$ in $S_{\text{ét}}$ and $t \in \mathcal{F}(U)$, we often denote $\mathcal{F}(h)(t) \in \mathcal{F}(V)$ as $t|_V$, suppressing h from the notation. This keeps things uncluttered, but to be rigorous one must always rememeber the depending on the map h since (unlike in classical sheaf theory!) there can be *many maps* between a given pair of objects in $S_{\text{ét}}$ (and even many *automorphisms* $h : U \simeq U$ of a single object U, in whichh case the notation $t|_U$ as shorthand for $\mathcal{F}(h)(t)$ could lead to confusion!).

Remark 1.3.4. There are set-theoretic issues, but for *sheaves* they can be resolved using the notion of "$\mathcal{B}$-sheaves". Any cover of any $W \in S_{\text{ét}}$ can be refined by a cover whose members belong to the *set* $\mathcal{B}_S$ of étale morphisms of the form $U \to V \subset S$ where V is open affine in S and U is affine étale over V.

Notation. We denote by $\text{Ét}(S)$ (or $S^{\sim}$) the category of sheaves of sets on $S_{\text{ét}}$. Note that a map $\mathcal{F} \to \mathcal{G}$ between such sheaves is determined by $\mathcal{F}(U) \to \mathcal{G}(U)$ for $U \in \mathcal{B}_S$, so there are no set-theoretic issues in considering $\text{Hom}_{S_{\text{ét}}}(\mathcal{F}, \mathcal{G})$ as a *set* (in contrast with the collection of natural transformations between presheaves, which one never needs to do). We denote by $\text{Ab}(S) \subset \text{Ét}(S)$ the subcategory of abelian group objects (equivalently, sheaves of abelian groups) with additive morphisms.

Remark 1.3.5. The sheaf axioms imply $\mathcal{F}(\emptyset) = \{*\}$ (consider the cover of $\emptyset$ given by an empty collection, for example, or just take this as part of the definition of a sheaf if such logic makes you uneasy), and for $U := \coprod U_i$ we have $U_i \times_U U_j = \emptyset$ for all $i \neq j$, so the equalizer condition gives that the natural map

$$\mathcal{F}(\coprod U_i) \to \prod \mathcal{F}(U_i)$$

is bijective.

2 Example: fields

2.1 The basic construction

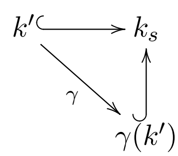
Let $S = \operatorname{Spec} k$ for k a field, and $\Gamma = \operatorname{Gal}(k_s/k)$ for a choice of separable closure k_s/k. For any affine $\operatorname{Spec}(A)$ over S we define $\mathcal{F}(A) := \mathcal{F}(\operatorname{Spec}(A))$ for a sheaf $\mathcal{F}$ on $S_{\text{ét}}$; this is covariant in A since sheaves and Spec are contravariant. We may then form the natural transformation

$$\text{Ét}(S) \to \{\text{discrete } \Gamma\text{-sets}\}$$

via

$$\mathcal{F} \rightsquigarrow \varinjlim_{\substack{k' \subset k_s \\ k'/k \text{ finite}}} \mathcal{F}(k'),$$

where the transition maps in the direct limit are injective (as for any $k' \subset k'' \subset k_s$ the map $\operatorname{Spec}(k'') \to \operatorname{Spec}(k')$ in $S_{\text{ét}}$ is a cover), and this latter set has a natural Γ-action induced by the maps $\mathcal{F}(\gamma) : \mathcal{F}(k') \to \mathcal{F}(\gamma(k))$ via the commutative diagram



It is easy to check that this Γ-set has open stabilizers, so it is a discrete Γ-set, as required.

In the other direction, to a discrete Γ-set M we associate a sheaf $\mathcal{F}_M$ on $S_{\text{ét}}$ defined by the (necessary) condition

$$\mathcal{F}_M\left(\coprod_{i\in I} \operatorname{Spec} k_i\right) = \prod_{i \in I} \mathcal{F}_M(k_i)$$

with $\mathcal{F}_M(k')$ for a *field* k' defined by the recipe

$$\mathcal{F}_M(k') = \left\{ (m_j) \in \prod_{\substack{j \colon k' \hookrightarrow k_s \\ \text{over } k}} M \mid \gamma(m_j) = m_{\gamma \circ j} \text{ for all } \gamma \in \Gamma \right\}.$$

(The idea of this mouthful of a definition is that k'/k has no *preferred* embedding into k_s, so we consider all embeddings at once and make equivariant choices.)

See pp. 9–12 in the main notes for why $\mathcal{F}_M$ is a sheaf and that these two constructions are naturally inverse to each other. The *key situation* in the verification that $\mathcal{F}_M$ is a sheaf is

$$V = \mathrm{Spec}\ k'' \to \mathrm{Spec}\ k' = U$$

with k''/k' a finite Galois extension with Galois group G. We want to verify the equalizer property for the diagram

$$\mathcal{F}_M(k') \to \mathcal{F}_M(k'') \rightrightarrows \mathcal{F}_M(k'' \otimes_{k'} k'').$$

Fix an embedding $k'' \hookrightarrow k_s$ over k, corresponding to an open subgroup $\Gamma'' \subset \Gamma$, and let $\Gamma' \subset \Gamma$ be the open subgroup corresponding to $k' \subset k_s$, so Γ'' is open normal in Γ' with $\Gamma'/\Gamma'' = G$. Projecting the equivariant tuples in M to factors corresponding to these compatible embeddings $k' \hookrightarrow k'' \hookrightarrow k_s$ identifies $\mathcal{F}_M(k'')$ with $M^{\Gamma''}$ and identifies $\mathcal{F}_M(k')$ with $M^{\Gamma'}$ (check!) in such a way that $\mathcal{F}_M(k') \to \mathcal{F}_M(k'')$ goes over to the inclusion $M^{\Gamma'} \hookrightarrow M^{\Gamma''}$.

2.2 Some calculations

The essential observation to prove that $\mathcal{F}_M$ is a sheaf is that

$$\mathcal{F}_M(k'' \otimes_{k'} k'') = \prod_{g \in G} \mathcal{F}_M(k'')$$

using the identification

$$k'' \otimes_{k'} k'' = \prod_G k''$$

defined by

$$a' \otimes b' \mapsto \prod_g (a' g(b'))_g.$$

Via this identification, we see that the two maps

$$M^{\Gamma''} = \mathcal{F}_M(k'') \rightrightarrows \mathcal{F}_M(k'' \otimes_{k'} k'') = \prod_{g \in G} \mathcal{F}_M(k'')$$

are respectively the diagonal and $s \mapsto (g \cdot s)_g$ with $g \in G = \Gamma'/\Gamma''$. The equalizer diagram then just expresses the obvious fact that the elements of $M^{\Gamma''}$ having the same image in $\prod_{g \in G} M^{\Gamma''}$ under these two maps are the elements of $(M^{\Gamma''})^G = M^{\Gamma'} = \mathcal{F}_M(k')$, so the sheaf axiom is affirmed in this key situation.

Example 2.2.1. For any S-scheme Z, $\mathrm{Hom}_S(-, Z) = \underline{Z}$ on $S_{\text{ét}}$ *is* a sheaf due to the combination of the obvious fact that representable functors are Zariski sheaves and the much less evident fact (discussed in §8.1/1 of *Néron Models*, building on Grothendieck's descent theory in §6.1–§6.2 of that book) that such functors on schemes are sheaves relative to faithfully flat quasi-compact morphisms. (Here we are using crucially the openness of étale morphisms, to ensure that any étale cover of a scheme admits, Zariski-locally on the base, a refinement consisting of a single quasi-compact étale surjection, hence an fpqc morphisms.)

Beware that you cannot necessarily recover Z from $\underline{Z}$ when Z is not étale over S! You might think that this should be possible by Yoneda's Lemma, but recall that in that in the proof of Yoneda the object needs to be in the category. For instance if $S = \mathrm{Spec}\,(k)$ for a separably closed field k then for any k-scheme Z we have $\underline{Z}(\coprod_i \mathrm{Spec}\ k) = \prod_i Z(k)$, so the k-scheme Z and the huge "discrete" (even étale!) k-scheme $\coprod_{Z(k)} \mathrm{Spec}\ k$ give the *same* sheaf on $S_{\text{ét}}$.

Proposition 2.2.2 (Corollary 1.1.4.6 of the notes)**.** *The functor $X \rightsquigarrow \underline{X}$ is an equivalence from the étale site of* $\mathrm{Spec}\,(k)$ *onto the category of sheaves of sets on the étale site.*

The proof amounts to the fact that étale k-schemes are explicitly given by $\coprod_{i \in I} \mathrm{Spec}\ k_i$. Spectra of fields are the *only* class of (interesting) schemes for which one has such a concrete description of all étale objects, especially that étale maps between finite locally on the source (which fails badly even for spectra of discrete valuation rings, over which there is no result like the one over fields).

3 Étale sheaves

For any map of schemes $f\colon X \to Y$, we have

$$f_*\colon \text{Ét}(X) \to \text{Ét}(Y)$$

by defining $f_*\mathcal{F}$ via the "expected" formula

$$f_*\mathcal{F}(U \to Y) = \mathcal{F}(U \times_Y X \to X).$$

To construct f^*, as a left adjoint to f_*, we usually need a sheafification. But there is one *easy case*: if $f\colon X \hookrightarrow Y$ is an étale map (akin to an open embedding in classical sheaf theory), then

$$f^*\mathcal{F} := \mathcal{F}|_{X_{\text{ét}}(\subset Y_{\text{ét}})}$$

is easily seen to work.

Question. What about $j_!$ for étale $j\colon U \to S$? (This should be "extension by $\emptyset$" for Ét, and "extension by 0" for Ab.) In classical sheaf theory even this construction involves a sheafification step, and the same will happen for the étale topology.

So we next take up the issue of sheaficiation, which introduces many new subtlties because there can be many maps between a pair of objects in $S_{\text{ét}}$ (and even many automorphisms of a single object), which has no counterpart for classical sheaf theory.

3.1 Sheafification

For a contravariant functor $\mathcal{F}\colon S_{\text{ét}} \to \mathbf{Set}$, define the direct limit

$$\mathcal{F}_0(U \to S) := \varinjlim_{\mathfrak{U}=(U_i \to U)} \mathrm{H}^0(\mathfrak{U}, \mathcal{F})$$

where for a given cover $\mathfrak{U}$ of U we define

$$\mathrm{H}^0(\mathfrak{U}, \mathcal{F}) = \ker\left(\prod_k \mathcal{F}(U_k) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times U_j)\right).$$

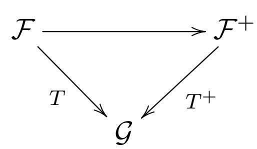
For a refinement $\mathfrak{U}' = (U'_i \to S)$ of $\mathfrak{U}$ (i.e., a cover admitting a *choice* of S-maps $\tau_\alpha : U'_\alpha \to U_{i(\alpha)}$ for each α – note that we have to choose both the index $i(\alpha)$ *and* the map, in contrast with classical sheaf theory), the induced map

$$\rho_\tau : \mathrm{H}^0(\mathfrak{U}, \mathcal{F}) \to \mathrm{H}^0(\mathfrak{U}', \mathcal{F})$$

is easily seen (check!) to be independent of the choices τ, so the limit in the definition of $\mathcal{F}_0$ is over a *directed set* (require each U_i to come from $\mathcal{B}_S$ to avoid set-theoretic issues).

One checks by arguments similar to the classical context that $\mathcal{F}_0$ is a separated presheaf, and even a sheaf if $\mathcal{F}$ is separated. This leads us to:

Definition 3.1.1. For a presheaf $\mathcal{F}$, we define the *sheafification* to be $\mathcal{F}^+ = (\mathcal{F}_0)_0$. It is "easy" to check that for any sheaf $\mathcal{G}$ and natural transformation $T\colon \mathcal{F} \to \mathcal{G}$, there is a unique natural transformation $T^+\colon \mathcal{F}^+ \to \mathcal{G}$ such that the diagram



commutes.

Remark 3.1.2. Note that $\mathcal{F} \hookrightarrow \mathcal{G} \implies \mathcal{F}_0 \hookrightarrow \mathcal{G}_0$ from the construction (check!), so $\mathcal{F}^+ \hookrightarrow \mathcal{G}^+$. In other words, sheafification carries subpresheaves to subsheaves; this is extremely important and used all the time.

Key points.

1. For all $s \in \mathcal{F}^+(U)$, there exists a cover $(U_i \to U)$ such that

$$s|_{U_i} \in \mathrm{Im}\,(\mathcal{F}(U_i) \to \mathcal{F}^+(U_i)).$$

2. For $s, t \in \mathcal{F}(U)$ with the same image in $\mathcal{F}^+(U)$, there exists a cover $(U_i \to U)$ such that
$$s|_{U_i} = t|_{U_i} \in \mathcal{F}(U_i) \text{ for all } i.$$

Example 3.1.3. For $\varphi\colon \mathcal{F} \to \mathcal{G}$, we can build $\mathrm{Im}\,(\varphi) \subset \mathcal{G}$ by sheafifying

$$U \rightsquigarrow \mathrm{Im}\,(\mathcal{F}(U) \to \mathcal{G}(U)) \subset \mathcal{G}(U).$$

Exercise 3.1.4. Check that $\mathcal{F} \to \mathcal{G}$ is an epimorphism as sheaves of sets (or as sheaves of abelian groups) if and only if $\mathrm{Im}\,(\varphi) = \mathcal{G}$ (i.e. "locally surjective").

Example 3.1.5. We can make $\varinjlim \mathcal{F}_\alpha$ with the expected universal property by sheafifying the "direct limit" presheaf.

3.2 Kummer and Artin-Schreier sequences

The construction of images and cokernels via sheafification yields as in classical sheaf theory that $\mathrm{Ab}(S)$ is an abelian category. Two fundamental short exact sequences for getting computations off the ground later are:

Example 3.2.1. For $n \in \mathbf{Z}^+$ a unit on S, we have a *Kummer sequence*

$$1 \to \mu_n \to \mathbf{G}_m \xrightarrow{t^n} \mathbf{G}_m \to 1.$$

The right-exactness is the only non-trivial part, and amounts to the fact that for any $\mathbf{Z}[1/n]$-algebra A and unit $u \in A^\times$, it becomes an nth power in locally finitely presented and flat A-scheme $\mathrm{Spec}\,(A[T]/(T^n - u))$ that is is a (finite) *étale* cover (as we may check on fibers, noting that $T^n - c$ is a separable polynomial over any field k and $c \in k^\times$ when $\mathrm{char}(k) \nmid n$).

Example 3.2.2. Let κ be a finite field of size q and let S be a κ-scheme. We have the associated *Artin-Schreier sequence*

$$0 \to \underline{\kappa} \to \mathbf{G}_a \xrightarrow{t^a - t} \mathbf{G}_a \to 0$$

on $S_{\text{ét}}$ (with left term denoting the constant sheaf associated to the finite abelian group κ). Here, the first map is defined by the natural map $\kappa \to \Gamma(U, \mathcal{O}_U)$ for any étale S-scheme U (made into a κ-scheme via the structure map $U \to S$), and this diagram is obviously a complex with vanishing kernel on the left (even if S is empty!). Exactness on the right amounts to the fact that for any κ-algebra A each $a \in A$ admits a solution to $T^q - T = a$ over the locally finitely presented and flat A-scheme $\mathrm{Spec}\,(A[T]/(T^q - T - a))$ that is a (finite) étale cover.

Exactness in the middle requires a moment of thought since we allow A to be very far from a domain (so it may be non-reduced, etc.). It suffices to show that the only solutions to $t^q = t$ in a κ-algebra A are those $a \in A$ that *Zariski-locally* on $\mathrm{Spec}\,(A)$ arise from κ. By direct limit considerations we may restrict attention to local A, in which case we claim that the only solutions to $t^q = t$ in A are elements of κ. The image of a solution in the residue field of A is certainly an element of κ, so by subtracting that off in A we reduce to showing that the only solution to $t^q = t$ in $\mathfrak{m}_A$ is 0. If $a \in A$ is such a solution then the ideal (a) is equal to $(a^q) = (a)^{q-1}(a) \subset \mathfrak{m}_A(a)$, so $(a) = 0$ by Nakayama's Lemma (!).

To compute étale cohomology, the starting point will be to use descent theory to make étale sheaves associated to quasi-coherent sheaves and show that their étale cohomology coincides with their Zariski cohomology (so we get a first vanishign result on affine schemes). That will give us some leverage in characteristic p via the Artin–Schreier sequence, but to really get going we need to get a handle on the étale cohomology of $\mathbf{G}_m$, and that will require much more serious input (ultimately boils down to descent theory and hard input from Galois cohomology).

3.3 Lower shriek.

If $f\colon U \to S$ is étale and $\mathcal{F}$ is an étale sheaf on U, then define $f_!\mathcal{F}$ to be the sheafification of:

$$S' \mapsto \coprod_{\substack{S' \to U \\ \text{over } S}} \mathcal{F}(S' \xrightarrow{h} U)$$

One checks (do it!) that this admits a bifunctorial identification

$$\mathrm{Hom}_{S_{\text{ét}}}(f_!\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{U_{\text{ét}}}(\mathcal{F}, f^*\mathcal{G});$$

although we have not discussed sheaf-pullback for general scheme maps, we have seen that it is easy to make for étale maps, such as f here.

By the construction, one sees that $f_!$ preserves monicity. The functors $f_!$ for étale $f: U \to S$ will underlie the later proof (via Grothendieck's method) that $\mathrm{Ab}(S_{\text{ét}})$ has enough injectives, as we need to define sheaf cohomology. (Čech-theoretic methods will have a role to play in the foundations, but they are of very limited use in actual calculations, in total contrast to the situation with Zariski cohomology for quasi-coherent sheaves. So derived functor cohomology is really the only option for defining cohomology relative to the étale topology.)

3.4 Pullback

If $f\colon S' \to S$ is any map, and $\mathcal{F} \in \text{Ét}(S)$, we can form the presheaf

$$(f^{-1}\mathcal{F})(U' \xrightarrow{h'} S') = \varinjlim_{\substack{U' \xrightarrow{\phi} U \\ \downarrow \quad \downarrow \\ S' \to S}} (\mathcal{F}(U \xrightarrow{h} S), \phi)$$

We stress that this limit is *not* over a directed system but rather is just over a diagram whose terms are indexed by all such commutative squares, because the indexing tracks the map ϕ (which is an essential part of the data: the same set $\mathcal{F}(h : U \to S)$ may shown up *many* times, as we vary the possibilities for ϕ). This is a filtered limit (via fiber-product considerations), but it is not a directed one.

In §1.1.5 of the main notes on étale cohomology one finds the general formalism of forming limits over diagrams. For peace of mind with set theory issues, we could limit ourselves to $U' \to S'$ in $\mathcal{B}_{S'}$ and $U \to S$ in $\mathcal{B}_S$ (since we're going to sheafify in a moment).

One can adapt the argument from the classical case to establish a natural bijection

$$\mathrm{Hom}_{S'}(f^{-1}\mathcal{F}, \mathcal{F}') = \mathrm{Hom}_S(\mathcal{F}, f_*\mathcal{F}')$$

for any presheaf $\mathcal{F}'$ on $S'_{\text{ét}}$, where we can limit these to functors on $\mathcal{B}_S$ and $\mathcal{B}_{S'}$ to avoid set-theoretic issues (i.e., to ensure these "Hom-sets" are really sets).

We then define

$$f^*\mathcal{F} = (f^{-1}\mathcal{F})^+,$$

and the above "universal property" at the presheaf level then yields the desired adjointness at the sheaf level. The construction shows that f^* preserves monicity, and it is easy to check that f^* is also a left adjoint to f_* between categories of abelian sheaves too.

Right-exactness for f_* follows from it being a left adjoint, so overall f^* is *exact* (in the appropriate sense for sheaves of sets via equalizer and co-equalizer diagrams, and for abelian sheaves in the usual sense). This exactness for the left adjoint ensures that f_* carries injectives to injectives (between categories of abelian sheaves) by arguing as in the classical case, a fact which will later underlie the construction to Leray spectral sequences (as in the classical case).

Example 3.4.1. For Spec $k' \to$ Spec k corresponding to an *arbitrary* extension of fields any any *choice* of compatible embeddings into separable closures

$$\begin{array}{ccc} k' & \hookrightarrow & k'_s \\ | & & | \\ k & \hookrightarrow & k_s \end{array}$$

we have a continuous map $\Gamma' \to \Gamma$ between associated absolute Galois groups via restriction of automorphisms. Composing actions with that continuous map defines a functor from discrete Γ-sets to discrete Γ'-sets (!).

Exercise 3.4.2. Check that this functor between categories of discrete Galois-sets is identified with f^* via the earlier equivalence $M \rightsquigarrow \mathcal{F}_M$.

Example 3.4.3. If $X \to S$ is étale, then the fibered product over $f\colon S' \to S$

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \text{étale}} \\ S' & \xrightarrow[f]{} & S \end{array}$$

satisfies $f^*(\underline{X}) = \underline{X}'$. Indeed,

$$\mathrm{Hom}_{S'}(f^*(\underline{X}), \mathcal{F}') = \mathrm{Hom}_S(\underline{X}, f_*\mathcal{F}') = (f_*\mathcal{F}')(X)$$

and by Yoneda this in turn is $\mathcal{F}'(X') = \mathrm{Hom}_{S'}(\underline{X}', \mathcal{F}')$, so a further application of Yoneda provides the asserted identification.

We emphasize that this identification can and does generally fail if the original map $X \to S$ is not étale. For example, if $S = \mathrm{Spec}\,(k)$ for a separably closed field k and $X = \mathbf{G}_{a,k}$ then $\underline{X}$ is the constant sheaf associated to the set $\Sigma = \mathbf{G}_a(k) = k$, so $f^*(\underline{X})$ is the constant sheaf over S' associated to the same set k! In particular, this pullback is absolutely not the functor of points on $S'_{\text{ét}}$ represented by $\mathbf{G}_{a,S'}$ in general (e.g., if $S' = \mathrm{Spec}\,(k')$ for a non-trivial separably closed extension k'/k).

Example 3.4.4. For scheme maps $S'' \overset{g}{\to} S' \overset{f}{\to} S$ one can make an "explicit" isomorphism

$$\theta_{f,g}\colon g^* \circ f^* \to (f \circ g)^*$$

(use "concatenation of factorization squares" at the level of the presheaf operations f^{-1}, g^{-1}, and $(f \circ g)^{-1}$) that is adjoint to the isomorphism

$$f_* \circ g_* \overset{\sim}{\leftarrow} (f \circ g)_*$$

so $\theta_{f,g}$ *is* an *isomorphism*. One can then use this adjunction to deduce that $\theta_{f,g}$ is "associative" relative to composing with a third map $h : S''' \to S''$.

Remark 3.4.5 (1.1.6.4)**.** Consider $f\colon S' \to S$ that is a universal homeomorphism (which by EGA IV_4 18.12.11 is equivalent to being integral, surjective, and radicial). Key examples of interest are closed immersions defined by an ideal of locally nilpotent functions and any base change of a purely inseparable field extension. Then the functor

$$S_{\text{ét}} \to S'_{\text{ét}}$$

defined by

$$X \mapsto X \times_S S'$$

induces an equivalence of categories (see Remark 1.1.6.4 in the main notes for references on the proof of this remarkable fact). From this one can obtain equivalences of categories of sheaves of sets (and then of abelian groups) via pullback and pushforward: the natural transformations $f^*f_* \to \mathrm{Id}$ and $\mathrm{Id} \to f_*f^*$ are isomorphisms.

3.5 Stalks

Consider a map $\overline{s}$: Spec $k \to S$ with $k = k_s$. (This is the notion of a "geometric point" for the étale topology.) For $\mathcal{F} \in \text{Ét}(S)$, we define its *stalk* at $\overline{s}$ to be

$$\mathcal{F}_{\overline{s}} := (\overline{s}^*\mathcal{F})(k).$$

Example 3.5.1. The compatibility of sheaf-pullback with base change for functors represented by étale schemes over the base yields that for any $U \to S$ étale,

$$(\underline{U})_{\overline{s}}(k) = U_{\overline{s}}(k).$$

Since there is no sheafification necessary when making constructions on the étale site of a separably closed field (i.e., all sheaves are constant, determined by their global sections), from the construction of sheaf pullback we have

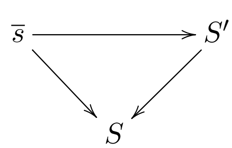
$$\mathcal{F}_{\overline{s}}(k) = \varinjlim_{\substack{\overline{s} \xrightarrow{h} U \\ \text{over } S}} (\mathcal{F}(U \to S), h)$$

where again the limit is over a diagram of all possible factorizations (and it is a filtered limit, but not directed).

In view of this formula and the openness of étale morphisms, one can check whether a map of sheaves is a monomorphism, epimorphism, or isomorphism at the level of the stalks. The key point is that we "spread out" from $\overline{s}$ to some étale map $U \to S$ hitting the image of $\overline{s}$, so if we vary through geometric points over all actual points of S then we obtain an étale cover of S, and that in turn can be used to test properties of maps of sheaves (noting that any geometric point of a scheme lifts through some member of any étale cover, since any étale non-empty scheme over a separably closed field is just a non-empty disjoint union of rational points). One also has natural isomorphisms

$$(f^*\mathcal{F})_{\overline{s}} \simeq \mathcal{F}_{\overline{s}}$$

for any $f : S' \to S$ and factorization



because of the compatibility of sheaf-pullback with composition (applied to the factorization diagram).