

# Purity of Real Sheaves

Zev Rosengarten

## 1 Introduction

Throughout these notes, let  $\kappa$  be a finite field (of order  $q$ , say),  $X_0$  a normal geometrically irreducible finite type  $\kappa$ -scheme,  $X := X_0 \otimes_{\kappa} \bar{\kappa}$ . Let  $\mathcal{G}_0$  be a smooth Weil  $\bar{\mathbb{Q}}_l$ -sheaf on  $X_0$ , and let  $\mathcal{G}$  be the pullback of  $\mathcal{F}_0$  to  $X$ . Given a geometric point  $\bar{x}$  of  $X_0$ , recall that the geometric Frobenius  $F_x$  at  $x$  acts on the stalk  $\mathcal{G}_{0,\bar{x}}$  of  $\mathcal{G}_0$ , which is a finite-dimensional  $\bar{\mathbb{Q}}_l$ -vector space. Suppose that we are given an embedding  $\tau : \bar{\mathbb{Q}}_l \hookrightarrow \mathbf{C}$ . If for all geometric points  $\bar{x}$  of  $X$  and all eigenvalues  $\alpha$  of  $F_x$ , we have

$$|\tau(\alpha)|^2 = N(x)^\beta$$

then we say that  $\mathcal{F}_0$  is  $\tau$ -pure of weight  $\beta$ .

The goal of this lecture is to show that sheaves that are  $\tau$ -real (meaning that all of the characteristic polynomials of Frobenius at all closed points have real coefficients after applying  $\tau$ ) are also  $\tau$ -pure (of some weight). The assumption of  $\tau$ -realness may seem excessively strong, but actually, one can produce from any sheaf  $\mathcal{G}_0$  whatsoever a ( $\tau$ -)real one, by considering  $\mathcal{G}_0 \otimes \overline{\mathcal{G}_0}$ . In this manner, the purity of general sheaves will be reduced to the real case.

While there are many details in the proof of purity of real sheaves, the basic idea of the proof is quite simple. First, one uses the fact that purity holds for all one-dimensional sheaves. This was treated in an earlier lecture, and is a consequence of the fact that any character of the Weil group of  $X_0$  restricts to a finite-order character on the geometric fundamental group (cf. [KW, Th. 3.1 and Cor.]). Now one tries to get at what one hopes are the actual weights of a sheaf  $\mathcal{G}_0$  by studying a poor man's version, the so-called determinant weights.

**Definition 1.1.** *Let  $\mathcal{F}_0$  be an irreducible component of rank  $r$  of the smooth Weil sheaf  $\mathcal{G}_0$  on  $X_0$ , and let  $\beta$  be the ( $\tau$ )-weight of the one-dimensional sheaf  $\Lambda^r \mathcal{F}_0$ . Then the number  $\beta/r$  is called the determinant weight of  $\mathcal{G}_0$  with respect to  $\mathcal{F}_0$  and  $\tau$ .*

The first step is to study the arithmetic monodromy group, defined approximately as the Zariski closure of the image of the Weil group in  $GL(V)$ , where  $V$  is the representation of the Weil group associated to the sheaf  $\mathcal{G}_0$  (this is not quite correct; the arithmetic monodromy group is not actually a finite type group scheme, only a locally finite type one). The key

point is that, via the degree map to the Weil group of  $\bar{\kappa}/\kappa$ , there is a central element of this group of positive degree. By Schur's Lemma, this element acts as a scalar on every irreducible constituent of  $\mathcal{G}_0$ . Since the weight of a one-dimensional sheaf may be measured by looking at the eigenvalues of any element of positive degree, we may use this central element, whose action is given simply by a scalar (at least on irreducible components) to understand how determinant weights behave under various operations on sheaves.

Next, we turn to the study of real sheaves. In order to prove the desired purity, we use a slicing argument to reduce to the case when  $X$  is an affine curve. For simplicity, let us assume here that  $\mathcal{G}_0$  is irreducible (we will treat the general case in the notes). Poincaré duality first allows us to relate the eigenvalues of Frobenius on the representation  $V$  associated to  $\mathcal{G}_0$  to its eigenvalues on  $H_c^2(X, \mathcal{G})$ . This will give an upper bound on the eigenvalues of Frobenius

On the other hand, since  $\mathcal{G}_0$  is  $(\tau)$ -real, the characteristic polynomials of Frobenius on the stalks of its even tensor powers  $\mathcal{G}_0^{\otimes 2k}$  have positive coefficients. Using the Lefschetz trace formula

$$\prod_{x \in |X_0|} \tau \det(1 - F_x t^{d(x)}, \mathcal{G}_{0\bar{x}}^{\otimes 2k})^{-1} = \frac{\tau \det(1 - Ft, H_c^1(X, \mathcal{G}^{\otimes 2k}))}{\tau \det(1 - Ft, H_c^2(X, \mathcal{G}^{\otimes 2k}))}$$

(the product on the left is over the closed points of  $X_0$ ; there is no  $H^0$  term because  $H_c^0(X, \mathcal{G}) = 0$  due to the affineness of  $X$ ) and the positivity of the coefficients of all of the terms on the left side, we are able to deduce an upper bound for the zeroes of each term on the left; that is, a lower bound for the eigenvalues of Frobenius at each closed point  $x$ . Applying a similar argument to the dual sheaf of  $\mathcal{G}_0$  yields the same lower bound. Combining the upper and lower bounds therefore yields that the eigenvalues of Frobenius at all points of  $X_0$  have the same weights. That is,  $\mathcal{G}_0$  is  $\tau$ -pure.

## 2 Semisimplicity of Geometric Monodromy

Let  $(V, \rho)$  be the  $\bar{\mathbf{Q}}_l$ -representation corresponding to  $\mathcal{G}_0$ . Define the geometric monodromy group  $G_{\text{geom}}$  to be the Zariski closure of  $\rho(\pi_1(X, \bar{x})) \subset GL(V)$ . This is a smooth linear algebraic group over  $\bar{\mathbf{Q}}_l$ . Every element of  $\rho(W(X_0, \bar{x}))$  normalizes  $G_{\text{geom}}$ , so choosing an arbitrary element  $\sigma \in W(X_0, \bar{x})$  of degree 1, we get a section  $W(\bar{\kappa}/\kappa) \rightarrow W(X_0, \bar{x})$ , which yields an action of  $W(\bar{\kappa}/\kappa)$  on  $G_{\text{geom}}$ . We may therefore define the arithmetic monodromy group  $G$  by

$$G_{\text{geom}} := W(\bar{\kappa}/\kappa) \times G_{\text{geom}}$$

This is a locally finite type group scheme over  $\bar{\mathbf{Q}}_l$ . Here is the main result of this section.

**Proposition 2.1.** *Let  $\mathcal{G}_0$  be a geometrically semisimple smooth sheaf.*

(i)  $G_{\text{geom}}$  is a semisimple algebraic group.

(ii) Let  $Z$  denote the center of  $G(\overline{\mathbf{Q}}_l)$ . Then the map  $Z \rightarrow \mathbf{W}(\overline{\kappa}/\kappa)$  has finite kernel and cokernel. In particular,  $Z$  contains an element of positive degree.

*Proof.* (i) We can replace  $X_0$  by a connected étale covering and thereby assume that  $G_{\text{geom}}$  is connected. First we check that  $G_{\text{geom}}$  is reductive. This follows from general principles. Indeed, let  $U$  be the unipotent radical of  $G$ . Let  $W \subset V$  be a nonzero irreducible  $G$ -subspace. Then  $W^U \neq 0$  by the Lie-Kolchin Theorem. Since  $U$  is *normal* in  $G$ ,  $W^U$  is  $G$ -invariant, hence  $W^U = W$ . Since  $V$  is a semisimple  $G$ -representation, it is generated by its irreducible subrepresentations, hence  $V^U = V$ . Since  $\rho$  is a faithful representation of  $G_{\text{geom}}$  (by definition), it follows that  $U = 1$ , so  $G$  is reductive.

Now we turn to semisimplicity, where there is actual content. Let  $T$  be the maximal central torus of  $G_{\text{geom}}$ . We want to show that  $T = 0$ . Choose  $\sigma \in \mathbf{W}(X_0, \overline{\kappa})$  of degree 1. Since  $T$  is a characteristic subgroup of  $G_{\text{geom}}$ , conjugation by  $\rho(\sigma)$  preserves it. There are only finitely many outer automorphisms of the reductive  $G_{\text{geom}}$  preserving  $T$ . (This follows from the structure theory of reductive and semisimple groups, which imply that such automorphisms are determined by the induced automorphism of the Dynkin diagram of the semisimple derived group of  $G_{\text{geom}}$ .) Therefore, replacing  $\sigma$  with some power  $\sigma^n$ , the induced conjugation action on  $G_{\text{geom}}$  is inner. We may replace  $\kappa$  with an extension of degree  $n$ , hence renaming  $\sigma^n$  as  $\sigma$ , we may continue to assume that  $\sigma$  has degree 1, and that

$$\zeta \cdot h \cdot \zeta^{-1} = g^{-1} \cdot h \cdot g$$

for some  $g \in G_{\text{geom}}(\overline{\mathbf{Q}}_l)$ , and functorially for all  $h \in G_{\text{geom}}$ , where  $\zeta := \rho(\sigma)$  has degree 1. Then  $g\zeta$  lies in the center of  $G(\overline{\mathbf{Q}}_l)$ .

This degree 1 element therefore defines a splitting  $G(\overline{\mathbf{Q}}_1) \simeq G_{\text{geom}}(\overline{\mathbf{Q}}_1) \times \mathbf{Z}$ . We use this splitting to define a map  $\mathbf{W}(X_0, \overline{\kappa}) \rightarrow G_{\text{geom}}(\overline{\mathbf{Q}}_l)$  as the composition of  $\rho$  and projection onto the first factor. If  $G_{\text{geom}}$  is not semisimple, then it admits a nontrivial character  $G_{\text{geom}} \rightarrow \mathbf{G}_m$ , which then extends to a character  $\mathbf{W}(X_0, \overline{\kappa}) \rightarrow \overline{\mathbf{Q}}_l^\times$  whose restriction to  $\pi_1(X, \overline{\kappa})$  has Zariski dense image inside  $\mathbf{G}_m$ . But this is impossible, by [KW, Th. 3.1].

(ii) The finiteness of the kernel follows immediately from the semisimplicity of  $G_{\text{geom}}$ , which implies that  $G_{\text{geom}}(\overline{\mathbf{Q}}_l)$  has finite center. The finiteness of the cokernel is equivalent to the assertion that  $Z$  contains an element of positive degree. The whole argument here involves dealing with the potential disconnectedness of  $G_{\text{geom}}$ . (If  $G_{\text{geom}}$  is connected, then the element  $g\zeta$  constructed in part (i) is central of positive degree.) An essentially equivalent argument to the one above allows us to construct an element  $z \in G(\overline{\mathbf{Q}}_l)$  of degree 1 such that  $\zeta := z^m$  commutes with  $G_{\text{geom}}^0(\overline{\mathbf{Q}}_l)$  for some  $m > 0$ , and (replacing  $m$  with a multiple of itself) such that conjugation by  $\zeta$  acts trivially on the finite group  $G_{\text{geom}}/G_{\text{geom}}^0$ .

Then for any  $g \in G(\overline{\mathbf{Q}}_l)$ , the map  $\phi_g : \mathbf{Z} \rightarrow G_{\text{geom}}^0(\overline{\mathbf{Q}}_l)$  defined by

$$\phi_g(n) := g\zeta^n g^{-1} \zeta^{-n}$$

defines a cocycle of  $\mathbf{Z}$  valued in  $G_{\text{geom}}^0(\overline{\mathbf{Q}_l})$ . That is,  $\phi_g(n+m) = \phi_g(n)\zeta^n\phi_g(m)\zeta^{-n}$ . Since  $\zeta$  acts trivially on  $G_{\text{geom}}^0$ , this map therefore defines a homomorphism  $\phi_g : \mathbf{Z} \rightarrow G_{\text{geom}}^0(\overline{\mathbf{Q}_l})$ . One easily checks that

$$\begin{aligned}\phi_{(gg'g^{-1})g} &= \phi_g \\ \phi_{g'g}(n) &= g'\phi_g(n)(g')^{-1}\end{aligned}$$

for all  $g \in G_{\text{geom}}(\overline{\mathbf{Q}_l})$ ,  $g' \in G_{\text{geom}}^0(\overline{\mathbf{Q}_l})$ . So we deduce (since  $G_{\text{geom}}^0$  is normal in  $G_{\text{geom}}$ ) that  $\phi_g(n) = g'\phi_g(n)(g')^{-1}$ . Hence  $\phi_g$  takes values in the center of the semisimple group  $G_{\text{geom}}^0(\overline{\mathbf{Q}_l})$ . This center is finite, say of order  $n$ , so  $\phi_g(n) = 0$  for all  $g \in G_{\text{geom}}(\overline{\mathbf{Q}_l})$ . That is,  $\zeta^n$  commutes with  $G_{\text{geom}}(\overline{\mathbf{Q}_l})$ . Since  $G(\overline{\mathbf{Q}_l})$  is generated by  $G_{\text{geom}}$  and by  $z$ , we deduce that  $\zeta \in Z$ , and this gives us our central element of nonzero degree.  $\square$

Schur's Lemma says that the element  $\zeta \in Z$  of positive degree constructed above acts by scalars on each irreducible component of  $\mathcal{G}_0$ , so the fact that weights of rank one sheaves may be computed by looking at the eigenvalues of *any* element of positive degree (and of course dividing the resulting "weight" by the degree of this element in order to determine the weight of Frobenius) thanks to [KW, Th. 3.1], the scalar by which the element  $\zeta \in Z$  acts on each component may be used to determine the determinant weights of  $\mathcal{G}_0$ . This immediately proves the first two assertions of the following corollary.

**Corollary 2.2.** *Suppose given smooth sheaves  $\mathcal{F}_0, \mathcal{G}_0$  on the normal geometrically irreducible  $\kappa$  scheme  $X_0$  of finite type.*

- (i) *If  $\alpha_1, \dots, \alpha_n$  are the determinant weights of  $\mathcal{F}_0$  with respect to  $\tau$ , and  $\beta_1, \dots, \beta_m$  are those of  $\mathcal{G}_0$ , then the  $\alpha_i + \beta_j$  are those of  $\mathcal{F}_0 \otimes \mathcal{G}_0$ .*
- (ii) *For  $\gamma \in \mathbf{R}$ , let  $r(\gamma)$  denote the sum of the ranks of all irreducible constituents of  $\mathcal{F}_0$  which have determinant weight  $\gamma$  with respect to  $\tau$ . Then the determinant weights of  $\Lambda^r \mathcal{F}_0$  are the numbers  $\sum_{\gamma \in \mathbf{R}} n(\gamma)\gamma$  with  $\sum_{\gamma \in \mathbf{R}} n(\gamma) = r$ ,  $0 \leq n(\gamma) \leq r(\gamma)$ ,  $n(\gamma) \in \mathbf{Z}$ .*

Finally, we mention the following result.

**Proposition 2.3.** *Let  $f_0 : X'_0 \rightarrow X_0$  be a  $\kappa$ -morphism with dense image between normal geometrically irreducible  $\kappa$ -schemes of finite type, and let  $\mathcal{G}_0$  be a smooth sheaf on  $X_0$ . Then  $\mathcal{G}_0$  and  $f_0^*\mathcal{G}_0$  have the same determinant weights (with respect to any  $\tau$ ).*

*Proof.* Let  $\pi : G' \rightarrow G$  be the induced map of arithmetic monodromy groups. By our earlier discussion, it suffices to show that there is an element  $z' \in Z'$  (with  $Z'$  the  $(\overline{\mathbf{Q}_l}$ -points of the) center of the arithmetic monodromy group of  $X'_0$ ) that has positive degree and such that  $\pi(z') \in Z$ . Choose  $z' \in Z'$  and  $z \in Z$  of positive degree. By replacing each of  $z', z$  with a suitable positive power, we may assume that they have the same degree and lie in  $G_{\text{geom}}^0(\overline{\mathbf{Q}_l})$ . Then  $\pi(z')z^{-1} \in G_{\text{geom}}^0(\overline{\mathbf{Q}_l})$ . But since the image of  $\pi$  has finite index in  $G$ ,  $(G')_{\text{geom}}^0 = G_{\text{geom}}^0$ . It follows that  $\pi(z')z^{-1}$  lies in the center of the semisimple group  $G_{\text{geom}}^0$ , which is finite. Hence, replacing  $z', z$  with  $z'^n, z^n$  for some  $n > 0$ , we have  $\pi(z') = z \in Z$ , as desired.  $\square$

### 3 Real Sheaves

In order to prove the purity of real sheaves in general, we will first treat the case of affine curves, for which there are few interesting cohomology groups, and so the situation is quite simple. So (maintaining our usual notational conventions) let  $X_0$  be an affine normal geometrically connected curve over  $\kappa$ , and let  $\mathcal{G}_0$  be a smooth Weil  $\overline{\mathbf{Q}}_l$ -sheaf on  $X_0$ . Fix a geometric point  $\bar{x}$  lying over the closed point  $x$  of  $X_0$ . Let  $\pi := \pi_1(X, \mathcal{G})$  denote the geometric fundamental group, and let  $V$  be the representation of the Weil group  $W(X_0, \bar{x})$  corresponding to  $\mathcal{G}_0$ . Since  $H^0(X, \mathcal{G}) = V^\pi$ , Poincaré duality implies that  $H_c^2(X, \mathcal{G}) = V_\pi(-1)$ , where  $V_\pi$  is as usual the space of coinvariants of  $V$  (that is, the largest quotient of  $V$  on which  $\pi$  acts trivially). The Weil group  $W(\bar{\kappa}/\kappa)$  acts on  $V_\pi$ , and for every eigenvalue  $\alpha$  of the geometric Frobenius  $F : V_\pi(-1) \rightarrow V_\pi(-1)$ ,  $\alpha q^{-1}$  is an eigenvalue of  $F$  on  $V_\pi$ , hence  $(\alpha q^{-1})^{d(x)}$  is an eigenvalue of  $F_x$  acting on  $\mathcal{G}_{0\bar{x}}$ , where  $d(x) := [\kappa(x) : \kappa]$  is the degree of the closed point  $x$ . Now  $W(\bar{\kappa}/\kappa)$  acts on  $V_\pi$ , so  $V_\pi$  is a quotient sheaf of  $\mathcal{G}_0$ , hence we deduce that its determinant weights appear among those of  $\mathcal{G}_0$ . We therefore have the following lemma.

**Lemma 3.1.** *Let  $\alpha$  be an eigenvalue of  $F : H_c^2(X, \mathcal{G}) \rightarrow H_c^2(X, \mathcal{G})$ . Then  $\log(|\tau(\alpha q^{-1})|^2)/\log(q)$  is a determinant weight of  $\mathcal{G}_0$ .*

Now we turn to the study of real sheaves.

**Definition 3.2.** *The Weil sheaf  $\mathcal{G}_0$  on the finite type  $\kappa$ -scheme  $X_0$  is said to be  $\tau$ -real if for all closed points  $x$  of  $X_0$  and geometric points  $\bar{x}$  over  $x$ , the characteristic polynomial  $\tau \det(1 - F_x t, \mathcal{G}_{0\bar{x}})$  of geometric Frobenius has real coefficients.*

Suppose now that  $\mathcal{G}_0$  is  $\tau$ -real. Then the logarithmic derivative of the power series  $\tau \det(1 - F_x t, \mathcal{G}_{0\bar{x}}^{\otimes k})^{-1}$  is

$$f(t) := \sum_{n=1}^{\infty} \tau(\mathrm{Tr}(F_x^n)^k) t^{n-1}$$

so  $\tau \det(1 - F_x t, \mathcal{G}_{0\bar{x}}^{\otimes k})^{-1} = e^{\int f(t) dt}$ . In particular,  $\mathcal{G}_{0\bar{x}}$  is still  $\tau$ -real, and if  $k$  is even, then the inverse of its characteristic polynomial has positive coefficients. We will now use this positivity in order to deduce  $\tau$ -purity.

**Lemma 3.3.** *Let  $\mathcal{G}_0$  be a smooth Weil sheaf on a geometrically connected smooth affine curve  $X_0$  over  $\kappa$ . If  $\mathcal{G}_0$  is  $\tau$ -real, then all of its irreducible components are  $\tau$ -pure. The  $\tau$ -weights are the determinant weights of the corresponding constituents.*

*Proof.* The second assertion follows immediately once we have the first. We therefore only have to show the  $\tau$ -purity of the constituents. We first use Lemma 3.1 in order to give an upper bound for the eigenvalue of Frobenius on  $H_c^2(X, \mathcal{G})$ . Let  $\beta$  denote the largest determinant weight of  $\mathcal{G}_0$ . Then  $2k\beta$  is the largest determinant weight of  $\mathcal{G}_0^{\otimes 2k}$  by Corollary

2.2(i). If  $t_0$  is a zero of  $\tau \det(1 - Ft, H_c^2(X, \mathcal{G}^{\otimes 2k}))$ , then  $t_0^{-1}$  is an eigenvalue of  $F$  acting on  $H_c^2(X, \mathcal{G}^{\otimes 2k})$ . By Lemma 3.1, therefore,  $\log(t_0^{-1}q^{-1})^2/\log(q)$  is a determinant weight of  $\mathcal{G}_0^{\otimes 2k}$ , hence  $|t_0^{-1}q^{-1}|^2 \leq q^{2k\beta}$  since  $2k\beta$  is the maximum determinant weight of  $\mathcal{G}_0^{\otimes 2k}$ . That is,

$$|t_0| \geq q^{-(2k\beta+2)/2}$$

On the other hand, the Lefschetz trace formula gives

$$\prod_{x \in |X_0|} \tau \det(1 - F_x t^{d(x)}, \mathcal{G}_{0\bar{x}}^{\otimes 2k})^{-1} = \frac{\tau \det(1 - Ft, H_c^1(X, \mathcal{G}^{\otimes 2k}))}{\tau \det(1 - Ft, H_c^2(X, \mathcal{G}^{\otimes 2k}))}$$

(here,  $|X_0|$  denotes the set of closed points of  $X_0$ ) since  $H_c^0(X, \mathcal{G}) = 0$  because  $X$  is affine. The product on the left therefore converges for all  $|t| < q^{-(2k\beta+2)/2}$ . Since all factors have nonnegative coefficients (This is where the  $\tau$ -realness gets used!) and constant term 1, each local  $L$ -factor converges on the same radius. Therefore, for each  $x \in |X_0|$ , the polynomial  $\tau \det(1 - F_x t^{d(x)}, \mathcal{G}_{0\bar{x}}^{\otimes 2k})$  is zero-free for  $|t| < q^{-(2k\beta+2)/2}$ . Let  $\alpha$  be an eigenvalue of  $F_x : \mathcal{G}_{0\bar{x}} \rightarrow \mathcal{G}_{0\bar{x}}$ . Then  $\alpha^{2k}$  is an eigenvalue of  $F_x$  acting on  $\mathcal{G}_{0\bar{x}}^{\otimes 2k}$ . The above inequality then implies that

$$|\tau(\alpha)|^2 \leq q^{d(x)(2k\beta+2)/2k} = N(x)^{\beta + \frac{1}{k}}$$

Letting  $k \rightarrow \infty$  then gives

$$\tau(\alpha)^2 \leq N(x)^\beta$$

That is, for every closed point  $x \in X_0$ , and every eigenvalue  $\alpha$  of  $F_x : \mathcal{G}_{0\bar{x}} \rightarrow \mathcal{G}_{0\bar{x}}$ , we have

$$|\tau(\alpha)|^2 \leq N(x)^\beta \tag{3.1}$$

where  $\beta$  is the largest determinant weight of  $\mathcal{G}_0$  with respect to  $\tau$ . To illustrate how the proof may now be completed, let us first consider the special case in which  $\mathcal{G}_0$  is irreducible. Then  $\beta$  is its only determinant weight, and  $-\beta$  is the only determinant weight of the dual sheaf  $\mathcal{G}_0$ , which is also  $\tau$ -real. Applying the above inequality to this dual sheaf then yields  $|\tau(1/\alpha)|^2 \leq N(x)^{-\beta}$ , i.e.,  $|\tau(\alpha)|^2 \geq N(x)^\beta$ . Combining this with the reverse inequality (3.1) then yields  $|\tau(\alpha)|^2 = N(x)^\beta$  for every eigenvalue of the Frobenius  $F_x$  acting on  $\mathcal{G}_{0\bar{x}}$ ; that is,  $\mathcal{G}_0$  is  $\tau$ -pure of weight  $\beta$ .

In the general, not necessarily irreducible case, the argument is more complicated but proceeds along similar lines. It is tempting to try to replace  $\mathcal{G}_0$  with each of its irreducible components; the problem is that these need not be  $\tau$ -real, so we have to do something a bit different. We will do this by considering various exterior powers of  $\mathcal{G}_0$ . We first may assume that  $\mathcal{G}_0$  is semisimple, by replacing it with its semisimplification (which has no effect on the characteristic polynomials, hence no effect on  $\tau$ -realness). Suppose that the determinant weights of  $\mathcal{G}_0$  are  $\gamma_1 > \gamma_2 > \dots > \gamma_r$ . Let  $\mathcal{G}_0(i)$  denote the the direct sum of all irreducible constituents of  $\mathcal{G}_0$  of determinant weight  $\gamma_i$ , and let  $r(i)$  denote the rank of  $\mathcal{G}_0(i)$ . For any  $0 \leq n < r$ , let  $N = \sum_{i=1}^n r(i)$ . Then the  $\tau$ -real sheaf

$$\Lambda^{N+1} \mathcal{G}_0 = \mathcal{G}_0(n+1) \otimes_{i=1}^n \det(\mathcal{G}_0(i)) \oplus \dots$$

has largest determinant weight  $\gamma_{n+1} + \sum_{i=1}^n r(i)\gamma_i$  by Corollary 2.2. One particular eigenvalue of  $F_x : \Lambda^{N+1}\mathcal{G}_0 \rightarrow \Lambda^{N+1}\mathcal{G}_0$  is  $\alpha_i^{(n+1)} \prod_{j=1}^n \prod_{j=1}^{r(i)} \alpha_j^{(i)}$ , where  $\alpha_1^{(i)}, \dots, \alpha_{r(i)}^{(i)}$  are the eigenvalues (with multiplicity) of  $F_x : \mathcal{G}_0(i)_{\bar{x}} \rightarrow \mathcal{G}_0(i)_{\bar{x}}$ . Applying the inequality (3.1) above therefore shows that

$$|\tau(\alpha_i^{(n+1)} \prod_{i=1}^n \prod_{j=1}^{r(i)} \alpha_j^{(i)})|^2 \leq N(x)^{\gamma_{n+1} + \sum_{i=1}^n r(i)\gamma_i}$$

By the definition of determinant weights, we also have

$$|\tau(\prod \alpha_j)|^2 = N(x)^{r\gamma_i}$$

where the  $\alpha_j$  are the eigenvalues of  $F_x$  on a given irreducible component of  $\mathcal{G}_0(i)$  of rank  $r$ . Multiplying this over all such irreducible of all  $\mathcal{G}_0(i)$  ( $1 \leq i \leq n$ ) yields

$$|\tau(\prod_{i=1}^n \prod_{j=1}^{r(i)} \alpha_j^{(i)})|^2 = N(x)^{\sum_{i=1}^n r(i)\gamma_i}$$

hence

$$|\tau(\alpha_i^{(n)})|^2 \leq N(x)^{\gamma_n}$$

for all  $1 \leq n \leq r$ , and all  $1 \leq i \leq r(n)$ . As above, applying the same argument to the dual sheaf yields the opposite inequality, and thereby completes the proof.  $\square$

Now we sketch the proof of the general case (beyond the case of affine curves).

**Theorem 3.4.** *Let  $X$  be a finite  $\kappa$ -scheme,  $\mathcal{G}_0$  a  $\tau$ -real Weil sheaf on  $X_0$ .*

(i) *The sheaf  $\mathcal{G}_0$  is  $\tau$ -mixed.*

(ii) *If  $X_0$  is irreducible and normal and  $\mathcal{G}_0$  is smooth, then the irreducible constituents of  $\mathcal{G}_0$  are  $\tau$ -pure.*

*Proof.* Part (ii) follows from (i) and [KW, Th. I.2.8(3)]. For part (i), we may assume that  $X_0$  is reduced (since the étale sites of  $X_0$  and  $(X_0)_{\text{red}}$  agree). Let  $j_0 : U_0 \hookrightarrow X_0$  be an open embedding. Then the exact sequence

$$0 \longrightarrow (j_0)_! j_0^* \mathcal{G}_0 \longrightarrow \mathcal{G}_0 \longrightarrow (i_0)_* i_0^* \mathcal{G}_0 \longrightarrow 0$$

shows that it suffices to show that  $j_0^* \mathcal{G}_0$  and  $i_0^* \mathcal{G}_0$  are  $\tau$ -mixed. This allows us to proceed by Noetherian induction. So we only need to find a nonempty  $U_0 \subset X_0$  such that  $j_0^* \mathcal{G}_0$  is  $\tau$ -mixed. We may also replace  $\kappa$  with a finite extension field, since pulling back and then pushing forward  $\mathcal{G}_0$  by such an extension, the resulting sheaf contains  $\mathcal{G}_0$  as a subsheaf, so if the pullback is  $\tau$ -mixed, then so is  $\mathcal{G}_0$ .

We may therefore assume that  $X_0$  is a smooth irreducible affine  $\kappa$ -scheme of finite type and that  $\mathcal{G}_0$  is a smooth sheaf on  $X_0$ . Replacing  $\kappa$  with its algebraic closure inside the function field of  $X_0$ , we may also assume that  $X_0$  is geometrically irreducible. If  $\dim(X_0) = 1$ , then we are done by Lemma 3.3, so we may assume that  $\dim(X_0) > 1$ . We may also extend  $\kappa$  and thereby assume that all irreducible constituents of  $\mathcal{G}_0$  are absolutely irreducible.

Embed  $X_0$  into some projective space  $\mathbf{P}_0^N$ . Let  $\mathcal{F}_0$  be an irreducible constituent of  $\mathcal{G}_0$ . We want to show that there is some nonempty open  $U_0 \subset X_0$  such that  $\mathcal{F}_0|_{U_0}$  is  $\tau$ -pure. Consider the linear subspaces  $L$  of  $\mathbf{P}^N$  of codimension  $\dim(X_0) - 1$ . Those  $L$  for which  $L \cap X$  is a nonempty smooth irreducible curve such that  $\mathcal{F}|_C$  is irreducible form a dense open in the Grassmannian of all codimension  $\dim(X) - 1$  linear subspaces, by a suitable Bertini Theorem [KW, App. B, Th. 1]. For such an  $L$ , there is a finite extension  $\kappa'/\kappa$  and a curve  $C_0 \subset X_0 \otimes_{\kappa} \kappa'$  such that  $C_0 \otimes_{\kappa'} \bar{\kappa} = C$ . Then the pullback of  $\mathcal{F}_0$  to  $C_0$  is an irreducible constituent of the corresponding pullback of  $\mathcal{G}_0$  (which is still  $\tau$ -real), hence it is  $\tau$ -pure of weight equal to the determinant weight of  $\mathcal{F}_0$  by Lemma 3.3. Letting  $L$  vary over the good open subset of the Grassmannian mentioned above, we see that there is a nonempty open subscheme  $U \subset X$  such that all points  $x \in U$  lie in a good linear subspace. The open set  $U$  descends to an open  $U'$  over some finite extension  $\kappa'/\kappa$ , and we may then set  $U_0$  to be the intersection of the finitely many Galois conjugates of  $U'$ .  $\square$

## References

- [KW] R. Kiehl and R. Weissauer, *Weil Conjectures, Perverse Sheaves, and l'adic Fourier Transform*, Springer-Verlag (2001), Berlin.