Constructible Sheaves, Stalks, and Cohomology

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1 Constructible Sheaves

We would like to understand cohomology with coefficients in some constant abelian group, like $\mathbf{Z}/l^n\mathbf{Z}$. But in order to get étale cohomology off the ground, one has to carry out various dévissage arguments, and so we are quickly forced to pass to a larger class of sheaves that behave well under the basic operations; e.g. higher pushforwards, extension by 0, etc. This is the class of so-called constructible sheaves, which are built from constant sheaves.

Given a scheme S and a set Σ, one can construct the constant étale sheaf $\underline{\Sigma}_S$, which is the sheafification of the presheaf Σ^{pre} defined by $\Gamma(U, \Sigma^{\text{pre}}) = \Sigma$. Equivalently, $\underline{\Sigma}_S(U) :=$ {locally constant functions $U \to \Sigma$}. Note that if U has finitely many connected components $\{U_i\}_{i=1}^n$, then $\Sigma(U) = \prod_{i=1}^n \Sigma$. We will often abuse notation and simply write Σ or Σ_S to denote the constant sheaf on S associated to Σ. Note that constant sheaves are representable by étale S-schemes: Σ_S is represented by $S \times \Sigma := \amalg_{\sigma\in\Sigma} S_\sigma$, since for any étale S-scheme U, $\mathrm{Hom}_S(U, S \times \Sigma) = \{\text{partitions } U = \amalg_{\sigma\in\Sigma} U_\sigma \mid U_\sigma \subset U \text{ is open}\} =$ {locally constant functions U $\to \Sigma$}.

A locally constant sheaf $\mathscr{F} \in \text{Ét}(S)$ is a sheaf that is constant locally for the étale topology. That is, there is an étale cover $\{U_i \to S\}$ such that each $\mathscr{F}|_{U_i}$ is constant. If the associated set for each U_i is finite, then we say that $\mathscr{F}$ is locally constant constructible (lcc).

Example 1.1. For any positive integer n, consider the sheaf μ_n on $\mathrm{Spec}(\mathbf{Q})_{\text{ét}}$, defined by $\Gamma(U, \mu_n) = \{f \in \Gamma(U, \mathcal{O}_U) \mid f^n = 1\}$. For $n > 2$, μ_n is lcc, but not constant. Indeed, the pullback of μ_n to the étale $\mathbf{Q}$-scheme $\mathrm{Spec}(\mathbf{Q}(\zeta_n))$ is constant (here ζ_n denotes a primitive nth root of unity), non-canonically isomorphic to $\mathbf{Z}/n\mathbf{Z}$. But $\mu_n(\mathrm{Spec}(\mathbf{Q})) = \{\pm 1\}$, not $\mathbf{Z}/n\mathbf{Z}$.

Similarly, for any $n > 2$ the étale sheaf μ_n on $\mathrm{Spec}(\mathbf{Z}[1/n])$ is lcc, but not constant. It becomes constant after pullback to the étale $\mathbf{Z}[1/n]$-scheme $\mathrm{Spec}(\mathbf{Z}[1/n][\zeta_n]) \simeq \mathrm{Spec}(\mathbf{Z}[1/n][T]/(\Phi_n))$, where $\Phi_n \in \mathbf{Z}[T]$ is the nth cyclotomic polynomial.

An important fact about lcc sheaves is that they correspond to finite étale covers:

Theorem 1.2. *The functor $X \mapsto \underline{X}$ is an equivalence between the category of finite étale S-schemes and the category of lcc sheaves on S.*

Proof. (Sketch) The functor is fully faithful by Yoneda's Lemma. To check that it lands in the expected target we may work Zariski-locally on S so that the rank of each X_s is equal to an integer $n \geq 1$. The diagonal $X \to X \times_S X$ is étale and a closed immersion, hence an open immersion onto a clopen subset, so $X \times_S X = X \amalg Y$, where Y is of degree $n-1$ over X and $X \times_S X$ is an X-scheme via pr_1. By induction (applied over the base X!) this splits étale locally on X, so the induction is finished. We may therefore assume that $X = S \times \Sigma$ for some (finite) set Σ. But then X represents the constant sheaf $\underline{\Sigma}$.

For the converse, we will not give all of the details (see Thm. 1.1.7.2 of Brian's notes), but the idea is to work Zariski-locally on S and then pass to a quasi-compact étale cover of S over which the lcc sheaf $\mathscr{F}$ is actually constant, hence representable by a finite disjoint union of copies of the base scheme. One then uses descent theory to obtain a scheme over S representing $\mathscr{F}$, and this scheme is affine over S by design; it is finite étale over S because this holds after an étale surjective base change. $\square$

Example 1.3. Let K be a field, $\mathscr{F}$ an lcc sheaf on $\mathrm{Spec}(K)$. Then $\mathscr{F}$ corresponds to some finite discrete $\Gamma := \mathrm{Gal}(K_s/K)$-set M. To describe M, by breaking M up into its Γ-orbits we may assume that Γ acts transitively on M. Let $H \subset \Gamma$ be the subgroup of elements that act trivially on M. Then M corresponds to the finite separable extension $L := K_s^H$ of K.

Here is a very useful corollary (of particular interest for number-theoretic applications; think of R as the ring of integers of a local or global field).

Corollary 1.4. *Let R be a Dedekind domain with fraction field K, $i : \mathrm{Spec}(K) \hookrightarrow \mathrm{Spec}(R)$ the canonical inclusion. Then the functor $\mathscr{F} \mapsto i^*\mathscr{F}$ is an equivalence between the category of lcc sheaves on $\mathrm{Spec}(R)$ and the category of lcc sheaves on $\mathrm{Spec}(K)$ corresponding to discrete $\mathrm{Gal}(K_s/K)$-sets that are unramified at all closed points of $\mathrm{Spec}(R)$.*

Proof. We need to show that if X_K is finite étale over K, then there is a finite étale R-scheme X such that $X \times_R K = X_K$ precisely when $X(K_s)$ is unramified at all closed points of $\mathrm{Spec}(R)$, and we need to show that X is functorial in X_K. For this, we may assume that $X_K = \mathrm{Spec}(L)$ for some finite separable extension L/K. Since R is normal, X must be as well, and it must also be integral over $\mathrm{Spec}(R)$, since it is finite over it. Hence if it exists, X must equal $\mathrm{Spec}(\widetilde{R})$, where $\widetilde{R}$ is the integral closure of R in L. We therefore take this as our definition of X. This is étale over R precisely when the extension L/K is unramified over R; i.e., the inertia groups in $\mathrm{Gal}(K_s/K)$ at all maximal ideals of R act trivially on L when embedded into K_s over K, which is the same as saying that they act trivially on $\mathrm{Hom}_K(L, K_s) = X_K(K_s)$. $\square$

Definition 1.5. Let X be a noetherian topological space. A *stratification* of X is a finite partition $X = \amalg_i S_i$ of X into locally closed subsets S_i such that $\overline{S_i}$ is a union of S_j's for each i. If S is a noetherian scheme, then a sheaf $\mathscr{F}$ on $S_{\text{ét}}$ is *constructible* if there is a stratification $\{S_i\}$ of S such that each $\mathscr{F}|_{S_i}$ is lcc.

The notion of constructibility is independent of the chosen scheme structure on the S_i since the étale topos of any scheme is equivalent to that of a closed subscheme with the same underlying spaces. The notion of a stratification comes up naturally in proofs that use noetherian induction as follows. One chooses a dense open subset $U_0 \subset S$ where some property holds, and then similarly chooses some dense open subset U_1 of the closed complement $Z_0 = S - U_0$ where the property holds, etc. This process terminates after finitely many steps because S is noetherian. In this way one obtains a stratification of S such that one may obtain the desired property piece by piece. One thereby often reduces proofs of the preservation of constructibility under various operations to showing that the desired result holds generically, and then applies noetherian induction.

We remark that constructibility is an étale-local condition. To see this, one applies noetherian induction (and the openness of étale maps) as above to allow us to replace S with a neighborhood of the generic point of one of its irreducible components. We may therefore assume that S is irreducible, and that $S' \to S$ is a finite étale cover such that $\mathscr{F}|_{S'}$ is constructible. Shrinking S further (again using the openness of étale maps), we may assume that $\mathscr{F}|_{S'}$ is lcc. But then $\mathscr{F}$ is lcc, since lcc-ness is étale-local by definition.

As another illustration of noetherian induction, we will show that if $X \to S$ is quasi-compact and étale, then the sheaf $\underline{X}$ represented by X is constructible. By Noetherian induction, we may shrink S around its generic points, and after doing this, X becomes finite étale over S, hence $\underline{X}$ is lcc, so we're done.

Constructibility is also preserved by many of our favorite functors; e.g. pullback, image under a map of sheaves, and finite direct limits. To see this, we may assume by noetherian induction that the input sheaves are lcc, and even constant, due to the étale-local nature of constructibility. But then in the case of pullback, the output is the sheaf represented by the same set, for images it is the image of the corresponding map of sets, and for finite direct limits it is represented by the corresponding limit of sets.

Constructibility is also preserved by extension by $\emptyset$ for a quasi-compact étale map. Indeed, suppose that $j : U \to S$ is quasi-compact étale, and let $\mathscr{F}$ be a constructible sheaf on U. As usual, noetherian induction (together with the compatibility of $j_!$ with restriction) allows us to shrink S and thereby assume that j is finite étale and $\mathscr{F}$ is lcc. We may also work étale-locally on S and thereby assume that j is split: $U = S \times \Sigma$ for some finite set Σ. Let $\mathscr{F}_\sigma$ denote the restriction of $\mathscr{F}$ to $S \times \{\sigma\}$. Then $\mathrm{Hom}_S(j_!\mathscr{F}, \mathscr{G}) = \mathrm{Hom}_U(\mathscr{F}, j^*\mathscr{G}) = \prod_{\sigma\in\Sigma} \mathrm{Hom}_S(\mathscr{F}_\sigma, \mathscr{G}) = \mathrm{Hom}_S(\amalg \mathscr{F}_\sigma, \mathscr{G})$, so $j_!\mathscr{F} = \amalg \mathscr{F}_\sigma$, which is constructible since it is a finite direct limit of constructible sheaves. The same argument applies to extension by 0 for constructible abelian sheaves.

Example 1.6. Here is an example of a constructible sheaf that is not lcc. Let R be a dvr with fraction field K, $S := \mathrm{Spec}(R)$, $U := \mathrm{Spec}(K)$, and let s be the closed point of X. Let $j : U \hookrightarrow S$ and $i : s \hookrightarrow S$ be the obvious maps. Then the sheaf $\mathscr{F} := j_!(\mathbf{Z}/2\mathbf{Z})$ is constructible by the previous paragraph, but I claim that it is not lcc. Indeed, if it were, then there would be some étale morphism $X \to S$ with nonempty special fiber such that $\mathscr{F}|_X$ is constant. Since $i^*\mathscr{F} = 0$, it would follows that $\mathscr{F}|_X$ is the 0 sheaf. But since étale maps are open, X has nonempty generic fiber, so since $j^*\mathscr{F} = \mathbf{Z}/2\mathbf{Z}$ is nonzero, we must have $\mathscr{F}|_X \neq 0$. So $\mathscr{F}$ is not lcc, as claimed.

The following result reduces the proofs of many statements to the case of constructible sheaves.

Theorem 1.7. *Let S be a noetherian scheme.*

(i) An object in $\acute{\mathrm{E}}\mathrm{t}(S)$ is noetherian (meaning its subobjects satisfy the ascending chain condition) if and only if it is constructible. Consequently, subsheaves of constructible sheaves are constructible. The same holds among torsion abelian sheaves.

(ii) Every $\mathscr{F} \in \acute{\mathrm{E}}\mathrm{t}(S)$ is the filtered direct limit of its constructible subsheaves.

(iii) If every section of $\mathscr{F} \in \mathrm{Ab}(S)$ is locally killed by a nonzero integer, then $\mathscr{F}$ is the direct limit of its constructible abelian subsheaves.

Proof. The idea of the proof (in the sheaf of sets case) is to show that for an arbitrary étale sheaf of sets $\mathscr{F}$, there is a surjection $\amalg_{i\in I}\underline{X_i} \to \mathscr{F}$ where the X_i are separated quasi-compact étale S-schemes, and $\underline{X_i}$ denotes the sheaf represented by X_i. There are then two key points: (i) constructible sheaves are precisely those for which one may take I to be finite; and (ii) subsheaves of sheaves representable by quasi-compact separated étale S-schemes are also so represented. These two claims easily imply what we want. See [FK, Ch. 1, §4, pp. 42-44] for details. □

2 Henselian rings

An essential role in étale cohomology is played by strictly Henselian local rings. These rings play the role in the étale topology played by local rings for the Zariski topology. If R is a local ring, then the only neighborhood of the closed point is all of $\mathrm{Spec}(R)$. Similarly, if R is a strictly Henselian local ring, then it will turn out that any étale map $U \to S := \mathrm{Spec}(R)$ with nonempty fiber over the closed point of S (which will itself be a geometric point by definition), may be refined by the trivial cover $S \to S$. That is, there are no nontrivial étale neighborhoods of the closed point. The formation of the "strict Henselization" R^{sh} of a local ring R will be an "algebraic" substitute for completion in many situations that is well-behaved without noetherian hypotheses (whereas completion works nicely only in the noetherian case).

Let us establish some notation. Throughout this section, R denotes a local ring, $\mathfrak{m}$ its maximal ideal, and $k := R/\mathfrak{m}$ its residue field. Let $S = \mathrm{Spec}(R)$ denote the corresponding local scheme, and s the closed point of S.

Before coming to the definitions, let us motivate them. For a scheme X and a point $x \in X$, what should we expect a "local ring" for the étale topology around x to look like? Since étale maps are open, such a local ring should factor through the local ring of x. So we can concentrate on the case of a local scheme $S = \mathrm{Spec}(R)$, with $x = s$ the closed point.

By the characterization of étale morphisms to spectra of fields, in case $R = k$ is a field, we should expect that the local ring is just a separable closure k_s of k. Correspondingly, the residue field of the "étale local ring" R^{sh} should be a separable closure k_s of k. What other properties should R^{sh} satisfy? Given a monic polynomial $P \in R^{\mathrm{sh}}[T]$ such that the reduction $\overline{P} \in k_s[T]$ is separable (and therefore has a root over k_s), the ring $A := R[T]/(P)$ is an étale R^{sh}-algebra, and should therefore have a section (since R^{sh} is supposed to be a "local ring for the étale topology"). That is, P should have a root in R. So we want R^{sh} to have separably closed residue field and to satisfy Hensel's Lemma (i.e., be "Henselian," a term that we shall soon define).

Moreover, as was discussed in Brian's lecture, every étale morphism is essentially given locally by adjoining a variable, quotienting out by a monic polynomial, and then passing to a localization near a residually simple root. A ring satisfying these two properties of being Henselian with separably closed residue field will becalled a *strictly Henselian* local ring.

Before becoming more friendly with strictly Henselian local rings, we should first introduce Henselian local rings. These are the rings that satisfy Hensel's Lemma:

Definition 2.1. A local ring R is *Henselian* if for any monic $P \in R[T]$ and any simple root $\alpha \in k$ of $\overline{P} \in k[T]$, α lifts to a root of P in R.

Example 2.2. Any complete local noetherian ring is Henselian, by Hensel's Lemma.

Here are some equivalent formulations of the Henselian property.

Theorem 2.3. *Let R be a local ring. The following are equivalent:*

- *(i) R is Henselian.*
- *(ii) Every étale map $X \to S = \mathrm{Spec}(R)$ is a local isomorphism at any point $x \in X$ above the closed point of s such that $k(s) = k(x)$.*
- *(iii) Every finite R-algebra is a product of local rings.*
- *(iv) For any quasi-finite morphism $X \to S$ and any $x \in X$ lying above s, there is a neighborhood U of x such that $U \to S$ is finite.*

Proof. See [BLR, §2.3] (which rests on some deep input from EGA, especially Zariski's Main Theorem). □

Now we can introduce strictly Henselian local rings.

Definition 2.4. A *strictly Henselian* local ring is a Henselian local ring with separably closed residue field.

As we mentioned earlier, strictly Henselian local rings are for the étale topology what local rings are for the Zariski topology. As a first step toward fully realizing this analogy, we have the following result.

Theorem 2.5. *Let R be a strictly Henselian local ring. Any étale neighborhood of the closed point s of $S := \mathrm{Spec}(R)$ may be refined by the trivial cover $S \to S$.*

Proof. This is immediate from Theorem 2.3(ii), together with the fact that any étale cover of s is a disjoint union of copies of s. $\square$

In order to conclude that strictly Henselian local rings are the true analogs of local rings, we need to be able to construct, for any local ring R, a strictly Henselian local ring R^{sh} that is a directed limit of essentially étale local R-algebras.

First, define an *essentially étale* R-algebra to be a localization of an étale R-algebra at a multiplicative set. Fix a separable closure $i : R/\mathfrak{m} \hookrightarrow k_s$. Consider the category of pairs (R', i') with R' an essentially étale local R-algebra and $i' : R'/\mathfrak{m}' \hookrightarrow k_s$ an embedding lying over i. The morphisms are the obvious thing, namely local R-homomorphisms respecting the embeddings of the residue fields into k_s. Since local schemes are connected, by the elementary rigidity for pointed étale maps (see Lemma 3.1, which does not rely on the present considerations) and denominator-chasing (because we are using local rings that are essentially étale over R, rather than actually being étale), any two objects in the category have at most one morphism between them. By passing to local rings over fiber products, we can also check that this is a filtered category. Thus, such pairs (R', i') form a directed system in an unambiguous manner. We may then define the strict Henselization R_i^{sh} of (R, i) to be the direct limit of the pairs (R', i'). The ring R^{sh} should be thought of as the local ring of the geometric point i of $S := \mathrm{Spec}(R)$ for the étale topology.

Lemma 2.6. *The ring R_i^{sh} is a strictly Henselian local ring.*

Proof. Note that it is clearly local, being a direct limit of local rings with local transition maps. Next we check that the residue field k is separably closed. Observe that by limit considerations, the only essentially étale local R_i^{sh}-algebra is R_i^{sh} itself. Let $\overline{P} \in k[T]$ be a monic separable polynomial. Lift it to some monic $P \in R_i^{\mathrm{sh}}[T]$. Then the ring $B = (R_i^{\mathrm{sh}}[T]/(P))[1/P']$ is an étale R_i^{sh}-algebra with nonempty special fiber, so localizing at a point of the special fiber yields an essentially étale local R_i^{sh}-algebra A, which therefore has a section, hence $\overline{P}$ (as well as the lift P) has a root, so $k = k_s$ is separably closed. By factoring $\overline{P}$ into linear factors, and applying a similar argument (localizing B in a more controlled manner), we see that R_i^{sh} is also Henselian, hence strictly Henselian. $\square$

The map $\phi : R \to R_i^{\mathrm{sh}}$ is universal among local maps from R into strictly Henselian local rings whose (separably closed) residue field is equipped with a specified embedding of k_s. More precisely, given any local homomorphism $f : R \to R'$ with R' strictly Henselian, and an inclusion $j : k_s \hookrightarrow R'/\mathfrak{m}'$ such that the map $R/\mathfrak{m} \to R'/\mathfrak{m}'$ induced by ϕ equals $j \circ i$, there is a unique map $g : R_i^{\mathrm{sh}} \to R'$ lifting j such that $f = g \circ \phi$. The idea of the proof is to consider a local essentially étale R-algebra A, and note that $A \otimes_R R'$ is essentially étale over R', hence has a section, which is unique once we impose the compatiblities of the maps on residue fields. Taking a direct limit over all such A then does the job. In particular, a strictly Henselian local ring is its own strict Henselization.

Let us actually formalize the claim that the strict Henselization is the correct notion of local ring for the étale topology.

Theorem 2.7. *Let X be a scheme, $x \in X$, $\overline{x}$ a geometric point above x such that $k(\overline{x})$ is a separable closure of $k(x)$. Let $\mathcal{O}_{X,x}^{\mathrm{sh}}$ denote the strict Henselization of the local ring $\mathcal{O}_{X,x}$ of x with respect to the inclusion $k(x) \hookrightarrow k(\overline{x})$, and let $i : \mathrm{Spec}(\mathcal{O}_{X,x}^{\mathrm{sh}}) \to X$ denote the natural map. Let $\mathscr{F}$ be an étale sheaf on X. Then we have an equality $\mathscr{F}_{\overline{x}} = (i^*\mathscr{F})(\mathcal{O}_{X,x}^{\mathrm{sh}})$.*

Proof. By construction, $\mathcal{O}_{X,x}^{\mathrm{sh}}$ *factors uniquely* through all *pointed* étale neighborhoods of $\overline{x}$. Moreover, sheaf-pullback to $\mathrm{Spec}(\mathcal{O}_{X,x}^{\mathrm{sh}})$ requires *no sheafification* for the formation of global sections of the pullback because every étale cover admits a section! Therefore, essentially by construction

$$(i^*\mathscr{F})(\mathcal{O}_{X,x}^{\mathrm{sh}}) = \varinjlim_{(U,\overline{u})\to(X,\overline{x})} \mathscr{F}(U)$$

(we say "essentially" because we have Zariski-localized étale neighborhoods of $(X, \overline{x})$ to become pointed local-étale neighborhoods $(U, \overline{u})$, a harmless maneuver that turns the non-directed limit defining presheaf-pullback into an actual direct limit), and by definition the right side is $\mathscr{F}_{\overline{x}}$. $\square$

In an analogous manner to how we defined the strict Henselization of R, we may also define the *Henselization* R^{h} of R: this is the direct limit of the filtered category of essentially étale local R-algebras R' such that $R/\mathfrak{m} = R'/\mathfrak{m}'$. That is, we take the (essentially) étale local R-algebras having the same residue field. One can show that this ring is Henselian and has the obvious universal property with respect to local maps from R to Henselian local rings.

The rings $R^{\mathrm{h}}, R_i^{\mathrm{sh}}$ are local and faithfully flat over R. The locality is clear. To see faithful flatness, we recall that for local homomorphisms, flat is the same as faithfully flat, so we only have to check flatness, and this is easily deduced from the fact that these rings are obtained as filtered direct limits of flat R-algebras. Much less obvious is the fact that if R is noetherian, then so are $R^{\mathrm{h}}, R_i^{\mathrm{sh}}$ (use the criterion in [EGA, 0_{III}, 10.3.1.3]).

Let us work out an example of interest in number theory to illustrate these concepts.

Example 2.8. Let R be normal with fraction field K and maximal ideal $\mathfrak{m}$, and let K_s be a separable closure of K with Galois group $G := \mathrm{Gal}(K_s/K)$. We wish to describe the Henselization and strict Henselization of R. Since normality is inherited by étale maps, we are led to look at local rings of integral closures of R in finite extensions of K inside K_s.

Fix a prime ideal $\mathfrak{m}_s$ over $\mathfrak{m}$ on the integral closure R_s of R in K_s. The decomposition group of $\mathfrak{m}_s$ is defined by

$$D := \{\sigma \in G \mid \sigma(\mathfrak{m}_s) = \mathfrak{m}_s\}.$$

The Henselization is the localization of the ring R_s^D at the prime $\mathfrak{m}_s \cap R_s^D$. The strict Henselization of R (more precisely, with respect to the inclusion $R/\mathfrak{m} \hookrightarrow R_s/\mathfrak{m}_s$) is the analogous construction using the inertia group

$$I := \{\sigma \in G \mid \sigma(x) \equiv x \bmod \mathfrak{m}_s\}.$$

For full proofs of these assertions, see [BLR, §2.3, Prop. 11]. Note that there is *no completion* involved in these descriptions.

One last property of Henselian local rings that we should mention is that their étale sites are determined by that of their residue fields. More precisely, we have the following result.

Theorem 2.9. *Let R be a Henselian local ring with residue field k. Then the functor $X \rightsquigarrow X \otimes_R k$ is an equivalence between the category of finite étale R-schemes and the category of finite étale k-schemes.*

Consequently, if we let $i : \mathrm{Spec}(k) \to \mathrm{Spec}(R)$ denote the obvious map, then the functors i^*, i_* are inverse equivalences between the category of sheaves of sets on the category $\mathrm{Spec}(R)_{\text{fét}}$ of finite étale R-schemes and on $\mathrm{Spec}(k)_{\text{ét}}$, and likewise for abelian sheaves.

Before we proceed with the proof, we need the following lemma.

Lemma 2.10. *Let R be a Henselian local ring, R' a finite local R-algebra. Then R' is also Henselian.*

Proof. We use Theorem 2.3(iii). Any finite R'-algebra is also a finite R-algebra, hence a product of local rings. □

Proof of Theorem 2.9. The second assertion follows from the first, since the connected finite étale R-algebras R' are local and functorially determined by their residue field (and i_* preserves global section). Hence, we focus on the first assertion.

We begin by proving that the functor is fully faithful. Let $S = \mathrm{Spec}(R)$, and let X, Y be finite étale S-schemes. We want to show that the natural map $\mathrm{Hom}_S(X, Y) \to \mathrm{Hom}_k(X_k, Y_k)$ is a bijection. To prove injectivity, we note that the map $Y \xrightarrow{\Delta} Y \times_S Y$ is an open immersion onto a closed subset. Hence $Y \times_S Y = Y \amalg Z$, where Y denotes the diagonal. If we have a pair of maps $f, g : X \rightrightarrows Y$ that agree on the special fiber, then the map $f \times g : X \to Y \times_S Y$, has special fiber contained in the diagonal, hence is entirely contained in the diagonal (since the only clopen subschemes of the R-finite X that contains X_k is X itself, due to the locality of connected finite R-schemes), so $f = g$.

For surjectivity, we may suppose that X, Y are connected, hence they are Henselian local rings, by Theorem 2.3(iii) and Lemma 2.10. Suppose that we have a k-morphism $\overline{\phi} : X_k \to Y_k$. We want to lift it to an R-morphism. The map $\overline{\phi}$ corresponds to a section $\overline{\phi}'$ over the special fiber of the finite étale map $X \times_S Y \to X$. Note that X_k is the (reduced) closed point of X, since $X \to S$ is *finite étale* and X is connected (so local). Since X is the spectrum of a Henselian local ring, by Theorem 2.3(ii), the section $\overline{\phi}'$ lifts to a section $\phi' : X \to X \times_S Y$, which corresponds to an S-morphism $\phi : X \to Y$ lifting $\overline{\phi}$.

Now we prove the essential surjectivity of the functor. Given a finite étale k-scheme Y, we need to construct a finite étale R-scheme X such that $X_k \simeq Y$. We may assume that $Y = \mathrm{Spec}(L)$ for some finite separable extension L/k. By the primitive element theorem, we have $L = k[T]/(\overline{P})$ for some separable polynomial $\overline{P} \in k[T]$. Lift $\overline{P}$ to a monic $P \in R[T]$. Then $Z := \mathrm{Spec}((R[T]/(P))[1/P'])$ is an étale quasi-finite S-scheme with special fiber $\mathrm{Spec}(L)$. By Theorem 2.3(iv), there is an open subscheme $X \subset Z$ that is finite étale over S with special fiber $\mathrm{Spec}(L)$. □

3 The étale fundamental group

For a reasonable path-connected topological space X and any $x \in X$, the fundamental group $\pi_1(X, x)$ classifies the covering spaces of X. In this section we will define an analogous notion of fundamental group for the étale topology, emphasizing the analogy with topology throughout. Because we are restricted to algebraic maps, covering spaces of infinite degree, like $\mathbf{R} \to S^1$, $x \mapsto e^{2\pi i x}$, don't really make sense in this context. The étale fundamental group will therefore classify *finite* étale covers of a (connected) scheme. Since any finite étale cover of a connected scheme has *constant* fiber-degree (why?), any separation of such a cover has its two non-empty parts each with clopen and hence full image in the base and of strictly smaller fiber-degree. Hence, we can "separate" the cover at most finitely many times, so its connected components are clopen and thus nothing weird happens.

As in topology, the étale fundamental group will be defined with reference to a base point (and will be independent of this choice up to inner automorphism). As we have seen, the analog of a point in the étale topology is a geometric point. We will refer to a scheme together with a choice of geometric point as a pointed scheme. For a covering map $Y \to X$ of reasonable topological spaces, a reasonable connected space Z, and a map $f : Z \to X$, a lift $g : Z \to Y$ of f is determined by the image of any given point $z \in Z$. Similarly, a lift through reasonable étale maps is determined by the image of any geometric point:

Lemma 3.1 (rigidity lemma)**.** *If $f, g : S' \rightrightarrows S''$ are S-morphisms to a separated étale S-scheme S'', and $(S', \overline{s}')$ is a pointed connected scheme such that $f(\overline{s}') = g(\overline{s}')$, then $f = g$.*

Proof. The diagonal map $S'' \to S'' \times_S S''$ is a closed immersion, but it is also étale, hence an open immersion onto a clopen subset. Thus, $S'' \times_S S'' = S'' \amalg Y$, where S'' is the diagonal. Since S' is connected and the image of $f \times g : S' \to S'' \times_S S''$ intersects the diagonal by hypothesis, it must factor through the diagonal. That is, $f = g$. □

Let $(S, \overline{s})$ be a pointed connected scheme. Given a connected finite étale cover $S' \to S$ of degree n, we have $\#\operatorname{Aut}(S'/S) \leq n$, since the rigidity lemma implies that an automorphism is determined by the image of any point above $\overline{s}$, and there are n such points (the fiber $S'_{\overline{s}}$ is a disjoint union of copies of $\overline{s}$).

If $S' = \operatorname{Spec}(L), S = \operatorname{Spec}(K)$ are spectra of fields, this expresses the classical fact that $\#\operatorname{Gal}(L/K) \leq \deg(L/K)$. Equality holds precisely when L/K is Galois. By analogy, we say that a connected finite étale cover S'/S is *Galois* if we have equality above; i.e., $\#\operatorname{Aut}(S'/S) = \#\deg(S'/S)$. As for fields, we have an analogue of Galois closure: for any connected finite étale cover $S' \to S$, there is a finite étale map $S'' \to S'$ with connected S'' such that S''/S is Galois. For the proof, see [SGA1, Exp. V, §2-§4]. If S'/S is Galois, then we define the *Galois group* of S'/S by the formula $\operatorname{Gal}(S'/S) := \operatorname{Aut}(S'/S)^{\mathrm{op}}$. (To see why we use the opposite group, think of the case when S', S are spectra of fields. Then we need to take the opposite group to recover the usual Galois group due to the contravariance of the functor $\operatorname{Spec}(\cdot)$.)

If S'/S is finite Galois with Galois group G, then G acts on S' on the right, yielding a map $S' \times G \to S' \times_S S'$ given by $(s', g) \mapsto (s', s'g)$. Checking on geometric fibers, we see that the map (which is finite étale) has all fibers of degree 1, so it is an isomorphism. Thus, $S' \times_S S'$ has $|G|$ connected components. Conversely, if $S' \times_S S'$ has $n := \deg(S'/S)$ connected components, then S'/S is Galois. Indeed, by degree considerations each component is isomorphic to S' via the projections $p_i : S' \times_S S' \to S'$. Then each component C_i yields an S-automorphism of S' via $S' \underset{\sim}{\xrightarrow{p_1^{-1}}} C_i \underset{\sim}{\xrightarrow{p_2}} S'$, with C_i giving the graph of that automorphism, so distinct C_i's yield distinct S-automorphisms. Thus, the inequality $\#\operatorname{Aut}(S'/S) \leq n$ is an equality.

Suppose that we are given two pointed connected finite Galois covers $(S', \overline{s}'), (S'', \overline{s}'')$ of the connected pointed scheme (S, s). By the rigidity lemma, there is at most one S-morphism $\pi : (S'', \overline{s}'') \to (S', \overline{s}')$. If such a map exists, then for any $f'' \in \operatorname{Aut}(S''/S)$, there exists a unique $f' \in \operatorname{Aut}(S'/S)$ making the following diagram commute:

$$\begin{tikzcd} S'' \arrow[r, "f''"] \arrow[d, "\pi"] & S'' \arrow[d, "\pi"] \\ S' \arrow[r, "f'"] & S' \end{tikzcd}$$

Indeed, f', if it exists, is unique due to the rigidity lemma. To see that it exists, note that Galois property for S'/S implies that $\operatorname{Aut}(S'/S)$ act transitively on the geometric points of S' lying above $\overline{s}$. Thus, there exists $f' \in \operatorname{Aut}(S'/S)$ such that $f'(\pi(\overline{s}'')) = \pi(f''(\overline{s}''))$. The rigidity lemma again then implies that $f' \circ \pi = \pi \circ f''$. This yields a group homomorphism

$$\operatorname{Aut}(S''/S) \to \operatorname{Aut}(S'/S) \tag{3.1}$$

via $f'' \mapsto f'$, and this map is surjective because $\operatorname{Aut}(S''/S)$ acts *transitively* on the set of geometric points of $S''_{\overline{s}}$ (by rigidity and counting).

We may now define the étale fundamental group $\pi_1(S,\overline{s})$ to be the profinite group

$$\pi_1(S,\overline{s}) := \varprojlim_{(S',\overline{s}')} \mathrm{Aut}(S'/S)^{\mathrm{op}} = \varprojlim_{(S',\overline{s}')} \mathrm{Gal}(S'/S)$$

where the inverse limit is over all connected pointed finite Galois covers $(S',\overline{s}') \to (S,\overline{s})$. Due to the surjectivity of (3.1), the map $\pi_1(S,\overline{s}) \to \mathrm{Gal}(S'/S)$ is surjective for each finite Galois S'/S.

We would of course like to do better and obtain covariant functoriality of π_1. Let $f : (T,\overline{t}) \to (S,\overline{s})$ be a map of connected pointed schemes. Let $(S',\overline{s}') \to (S,\overline{s})$ be finite Galois (with our usual convention that $k(\overline{s}') = k(\overline{s})$). Then $(S' \times_S T, \overline{t}' := \overline{s}' \otimes_{\overline{s}} \overline{t}) \to (T,\overline{t})$ is finite étale over T; this cover of T is typically disconnected, but it is a right G-torsor since S' is a right G-torsor over S. Thus, if T' is the connected component containing $\overline{t}'$ and $H \subset G$ is the stabilizer of T' then $T' \to T$ is a right H-torsor (why?), so T' is Galois over T with $H = \mathrm{Gal}(T'/T)$. In this way we obtain a continuous map

$$\pi_1(T,\overline{t}) \twoheadrightarrow \mathrm{Gal}(T'/T) = H \hookrightarrow G = \mathrm{Gal}(S'/S).$$

These fit together to yield a map $\pi_1(T,\overline{t}) \to \pi_1(S,\overline{s})$.

Example 3.2. Let k be a field, $i : k \hookrightarrow k_s$ a separable closure of k. The connected pointed finite Galois covers of $\mathrm{Spec}(k)$ are in bijection with the finite Galois extensions k'/k with $k' \subset k_s$. We therefore have $\pi_1(\mathrm{Spec}(k), i) = \mathrm{Gal}(k_s/k)$. That is, étale fundamental groups recover Galois groups in the case of fields. As we will see later, étale fundamental groups are only independent of the base point up to inner automorphism. Correspondingly, the absolute Galois group of k is only canonical up to inner automorphism. This is the sense in which classifying the abelian extensions of a field k is "intrinsic" to k whereas trying to classify the nonabelian extensions involves keeping track of auxiliary isomorphisms.

There is a simple criterion for surjectivity of a map of π_1's:

Theorem 3.3. *Let $f : (X,\overline{x}) \to (Y,\overline{y})$ be a map of connected pointed schemes. The induced map $\pi_1(X,\overline{x}) \to \pi_1(Y,\overline{y})$ is surjective if and only if for every connected finite Galois covering $Y' \to Y$, $X \times_Y Y'$ is also connected.*

Proof. The idea is that if $X \times_Y Y'$ were disconnected, then letting X' be the connected component lying above $\overline{x} \otimes_{\overline{y}} \overline{y}'$, we would obtain

$$\#\,\mathrm{Gal}(X'/X) = \#\,\deg(X'/X) < \deg(X \times_Y Y'/X) = \deg(Y'/Y),$$

so the map $\mathrm{Gal}(X'/X) \to \mathrm{Gal}(Y'/Y)$ can't be surjective. Conversely, if the map is surjective then X' must be all of $X \times_Y Y'$; i.e., this latter scheme is connected. For more details, see Thm. 1.2.2.1 of Brian's notes. □

If X is a reasonable path-connected topological space, then $\pi_1(X,x)$ classifies the covering spaces of X. Similarly, the étale fundamental group classifies the finite étale coverings of a connected scheme.

Theorem 3.4. *Let $(S,\overline{s})$ be a connected scheme. The functor $S' \mapsto S'(\overline{s})$ is an equivalence of categories between the finite étale covers $S' \to S$ and the finite discrete left $\pi_1(S,\overline{s})$-sets.*

Connected covers correspond to sets with transitive π_1-action. Further, this equivalence is functorial in the sense that if $f:(T,\overline{t}) \to (S,\overline{s})$ is a map of connected pointed schemes then the equality $T \times_S S'(\overline{t}) = S'(\overline{s})$ respects π_1-actions, where $\pi_1(T,\overline{t})$ acts on $S'(\overline{s})$ via $\pi_1(f)$.

Proof. We may assume that S' is connected. Choose S''/S Galois such that we have a factorization $S'' \xrightarrow{\pi} S' \to S$. Given $\overline{s}' \in S'(\overline{s})$, choose $\overline{s}'' \in S''(\overline{s}')$. Then we define the π_1-action by $g \cdot \overline{s}' := \pi \circ g(\overline{s}'')$ for $g \in \mathrm{Gal}(S''/S)$. Then by rigidity, π and $\pi \circ g$ agree precisely when $g \cdot \overline{s}' = \overline{s}'$. This yields a bijection

$$\mathrm{Aut}(S''/S)^{\mathrm{op}} / \mathrm{Aut}(S''/S')^{\mathrm{op}} \simeq S'(\overline{s}')$$

One checks that this is functorial in $(S'',\overline{s}'')$, that every discrete finite left π_1-set with transitive action arises in this manner, and that this construction is functorial. □

Thanks to our classification of lcc sheaves as those represented by finite étale S-schemes, this immediately implies the following useful result.

Theorem 3.5. *Let S be a connected scheme, $\overline{s}$ a geometric point. The functor $\mathscr{F} \to \mathscr{F}_{\overline{s}}$ is an equivalence between the category of lcc sheaves of sets on $S_{\text{ét}}$ and the category of finite discrete left $\pi_1(S,\overline{s})$-sets, and likewise for finite discrete π_1-modules and lcc abelian sheaves.*

Note that the category of connected finite étale S-schemes makes no reference to the basepoint $\overline{s}$. As a consequence, we may deduce that $\pi_1(S,\overline{s})$ is independent of $\overline{s}$, canonically up to inner automorphism. Indeed, suppose that we have two profinite groups G, H and an equivalence F between the categories of finite left G-sets with transitive action and finite left H-sets with transitive action. This arises from an isomorphism $\phi: G \to H$ that is canonical up to innner automorphism on H, as follows. The open normal subgroups of G correspond to finite discrete left G-sets with transitive G-action such that the stabilizer of any (hence every) point is normal. Indeed, any such G-set X is isomorphic to G/A for some normal open subgroup $A \subset G$. But the bijection $G/A \to X$ is non-canonical. It may be modified by an inner automorphism of G by composing the orbit map $G \to X$ $(g \mapsto g \cdot x)$ with the action on X by some $g_0 \in G$. Thus the isomorphism ϕ is determined up to inner automorphism.

Now let us give a few examples of computations of fundamental groups.

Example 3.6. Let k be a separably closed field. The curve $\mathbf{P}^1_k$ is analogous to the Riemann sphere, so we expect it to be simply-connected. That is, we claim that it has no nontrivial connected finite étale covers. (We must assume that k is separably closed in order for this to hold; otherwise we have the constant étale covers arising by extending scalars to some finite separable extension of k.) Suppose that $f : X \to \mathbf{P}^1_k$ is finite étale and X is connected. We endow X with the structure of a k-scheme via the composition $X \to \mathbf{P}^1_k \to \mathrm{Spec}(k)$. Then X is a smooth proper curve over k, geometrically connected since $k = k_s$. We may therefore apply the Riemann–Hurwitz formula to conclude that $2g-2 = -2d$, where g is the genus of X and d is the degree of f. We immediately conclude that $d = 1$, so f is an isomorphism. Thus the only connected finite étale cover of $\mathbf{P}^1_k$ is the trivial one, so $\pi_1(\mathbf{P}^1, \overline{s}) = 0$.

Example 3.7. Let k be a separably closed field, and let E be an elliptic curve over k. Then E should be analogous to a genus one surface, so we might guess that its fundamental group should be $\mathbf{Z}^2$. Actually, since the étale fundamental group is defined as an inverse limit over finite Galois covers, a profinite completion enters the picture. A second guess might therefore be something like $\widehat{\mathbf{Z}}^2$. This is almost correct, but there is a bit of a wrinkle in characteristic $p > 0$, as we shall see.

One can show via Riemann–Hurwitz that any connected finite étale cover of E (equipped with a choice of rational point over 0, as we may do) is itself an elliptic curve mapping to E via an étale isogeny. The covering group in such a situation is given by translation against (rational) points in the (constant) kernel, so these covering groups are all *abelian*. By composing with the dual isogeny, we see that a cofinal system of connected finite étale covers of degree prime to p is given by the multiplication by N morphisms $[N] : E \to E$ for $p \nmid N$. The automorphisms of these covers are given by translation by an N-torsion point. Therefore, $\pi_1(E, \overline{s})[\text{non-}p] = \varprojlim_{p \nmid N} E(k)[N] \simeq \prod_{l \neq p} \mathbf{Z}_l^2$. So we get the expected answer away from the characteristic.

What about at p? Similar reasoning shows that the pro-p part of $\pi_1(E, \overline{s})$ is

$$\varprojlim_n E[p^n](k) = \begin{cases} 0 & \text{if } E \text{ is supersingular,} \\ \widehat{\mathbf{Z}}_p & \text{if } E \text{ is ordinary.} \end{cases}$$

A similar calculation (using the Lang–Serre theorem in place of Riemann–Hurwitz) shows that $\pi_1(A, \overline{s})[\text{non-}p] \simeq \prod_{l \neq p} \widehat{\mathbf{Z}}_l^{2g}$ for any g-dimensional abelian variety A over k, with the rank of the p-part depending on the amount of geometric p-torsion.

Example 3.8. Generalizing the previous two examples, let C be a smooth proper connected curve of genus g over the separably closed field k. Then we should expect that the maximal prime-to-p quotient $\pi_1(C, \overline{s})^{(p)} \simeq (\widehat{F_{2g}/w})^{(p)}$, where F_{2g} is the free group on $2g$ generators $x_i, y_i, i = 1, \ldots, g$, and $w = \prod_{i=1}^g [x_i, y_i]$. This is indeed the case, but lies quite deep and is one of the main theorems in [SGA1]. The structure of the pro-p part is very mysterious!

Example 3.9. Let R be a Dedekind domain with fraction field K. We have already seen (in the language of lcc sheaves rather than finite étale covers) that the connected finite étale covers of $\mathrm{Spec}(R)$ are all obtained as follows: for any finite separable extension L/K

unramified above every place of R, the integral closure R_L of R inside L is a finite étale R-algebra. If we fix a separable closure $\overline{s} : K \to K_s$ of K then $\pi_1(\mathrm{Spec}(R), \overline{s}) = \mathrm{Gal}(K^{\mathrm{nr}}/K)$, the Galois group of the maximal extension K^{nr} of K that is unramified above *every place* of R.

Example 3.10. As an application of Example 3.9, $\mathrm{Spec}(\mathbf{Z})$ is simply-connected. This is equivalent to showing that the only finite extension of $\mathbf{Q}$ that is unramified above every prime of $\mathbf{Z}$ is $\mathbf{Q}$ itself, and that is a well-known theorem of Hermite.

Example 3.11. Let K be a number field. By class field theory, the Galois group of the maximal *abelian* extension of K that is unramified above every finite place is canonically isomorphic to the ray class group I/P of K, where I denotes the group of fractional ideals of K, and P is the subgroup of principal ideals with a generator that is totally positive (i.e., positive under every embedding $K \hookrightarrow \mathbf{R}$). Thus, $\pi_1(\mathrm{Spec}(\mathcal{O}_K), \overline{s})^{\mathrm{ab}} \simeq I/P$. A similar calculation shows that for any finite set S of places of K containing the archimedean ones, we have $\pi_1(\mathcal{O}_{K,S}, \overline{s})^{\mathrm{ab}} \simeq I_S/P_S$, where I_S is the group of fractional ideals supported away from S, and P_S is the subgroup of principal ideals generated by an element $\alpha \equiv 1 \pmod{\prod_{\mathfrak{p} \in S} \mathfrak{p}}$, where for real places $\sigma \in S$, this is taken to mean that $\sigma(\alpha) > 0$.

Example 3.12. Let R be a Henselian local ring with residue field k. Fix a separable closure $\overline{s} : k \hookrightarrow k_s$. Then by Theorem 2.9, the canonical inclusion $(\mathrm{Spec}(k), \overline{s}) \hookrightarrow (\mathrm{Spec}(R), \overline{s})$ induces an isomorphism $\pi_1(\mathrm{Spec}(k), \overline{s}) \xrightarrow{\sim} \pi_1(\mathrm{Spec}(R), \overline{s})$. That is, $\pi_1(\mathrm{Spec}(R), \overline{s}) \simeq \mathrm{Gal}(k_s/k)$. So for example, $\pi_1(\mathrm{Spec}(\mathbf{Z}_p), \overline{s}) \simeq \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p) \simeq \widehat{\mathbf{Z}}$, recovering the well-known description of the Galois group of the maximal unramified extension of $\mathbf{Q}_p$.

4 Classification of local constancy via specialization

If S is a scheme with two points $x, y \in S$ such that $x \in \overline{\{y\}}$, then we say that x is a *specialization* of y. This is equivalent to saying that the natural map $\mathrm{Spec}(k(y)) \to S$ factors as a composition $\mathrm{Spec}(k(y)) \to \mathrm{Spec}(\mathcal{O}_{S,x}) \to S$. For any Zariski sheaf $\mathscr{F}$ on S, we have a specialization map $\mathscr{F}_x \to \mathscr{F}_y$, defined as follows: for an element of $\mathscr{F}_x$ represented by a section $f \in \mathscr{F}(U)$ with U a neighborhood of x, U is also a neighborhood of y and so we may send this element to the class of f in $\mathscr{F}_y$.

There is a similar notion of specialization in étale cohomology. Suppose that we are given two geometric points $\overline{s}, \overline{\eta}$ of S. Let us make the (harmless) assumption that $k(\overline{s}), k(\overline{\eta})$ are separable closures of $k(s), k(\eta)$, as opposed to some much larger separably closed fields containing these residue fields. We say that $\overline{s}$ is a *specialization* of $\overline{\eta}$ if $\overline{\eta}$ factors through every étale neighborhood of $\overline{s}$. This is equivalent to saying that there is an S-morphism $j : \overline{\eta} \to \mathrm{Spec}(\mathcal{O}_{S,\overline{s}}^{\mathrm{sh}})$. It is also equivalent to saying that for the underlying physical points s, η, we have $s \in \overline{\{\eta\}}$. Just as in the Zariski case, given any étale sheaf $\mathscr{F}$ on S, we obtain a morphism of stalks $\mathscr{F}_{\overline{s}} \to \mathscr{F}_{\overline{\eta}}$ (via pullback along j), called the *specialization map*; this depends on j, but in practice we only need to use one j per pair of such geometric points.

The specialization map is clearly functorial in $\mathscr{F}$. Such maps provide a convenient criterion for local constancy:

Theorem 4.1. *Let S be a noetherian scheme, $\mathscr{F}$ a constructible sheaf on S. Then $\mathscr{F}$ is locally constant (hence lcc) if and only if all of the specialization maps $\mathscr{F}_{\overline{s}} \to \mathscr{F}_{\overline{\eta}}$ are bijective.*

Proof. First suppose that $\mathscr{F}$ is locally constant. The bijectivity of the specialization maps is clearly an étale-local condition, so we may assume that $\mathscr{F}$ is constant, say $\mathscr{F} = \underline{\Sigma}$, in which case the stalks are just Σ, and the specialization map is the identity, hence bijective.

Conversely, suppose that all of the specialization maps are bijective. Let $\overline{s}$ be a geometric point of S. We will show that $\mathscr{F}$ is constant in an étale neighborhood of $\overline{s}$. Let $\Sigma := \mathscr{F}_{\overline{s}}$. Since $\mathscr{F}$ is constructible, Σ is finite. In some étale neighborhood of $\mathscr{F}$, therefore, we have a map $\phi : \underline{\Sigma} \to \mathscr{F}$ inducing a bijection on $\overline{s}$-stalks. Since all of the specialization maps for $\underline{\Sigma}$ and $\mathscr{F}$ are bijective, $\phi_{\overline{s}'}$ is bijective for any geometric point $\overline{s}'$ which may be linked to $\overline{s}$ by finitely many specializations and generizations. By passing through generic points of irreducible components, this includes all points lying in an irreducible component containing $\overline{s}$. So ϕ is an isomorphism over an open neighborhood of $\overline{s}$ (as S is noetherian, so its collection of irreducible components is finite), so $\mathscr{F}$ is locally constant. □

5 Cohomology

In this section we will define the étale cohomology groups of an étale abelian sheaf on the scheme S. This proceeds exactly as for Zariski cohomology once we verify that the category $\mathrm{Ab}(S)$ of étale abelian sheaves on S has enough injectives.

Lemma 5.1. *The abelian category* $\mathrm{Ab}(S)$ *has enough injectives.*

Proof. We use a criterion due to Grothendieck for an abelian category to have enough injectives. This says that the abelian category has enough injectives if it has following three properties: (i) it admits arbitrary direct sums; (ii) for any monomorphism $B \hookrightarrow B'$ and any increasing filtered family $\{A_i\}$ of subobjects of B', we have

$$B \cap (\sum A_i) = \sum (B \cap A_i);$$

and (iii) it has a generating object. The first two axioms are easily verified for $\mathrm{Ab}(S)$. Let us explain the meaning of the third. A generating object in an abelian category $\mathscr{A}$ is an object A of $\mathscr{A}$ such that for any monomorphism $B \hookrightarrow B'$ in $\mathscr{A}$ that is not an isomorphism, the induced map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$ is not bijective. For a proof of this criterion, see [To, Thm. 1.10.1].

As a generating object in $\mathrm{Ab}(S)$, we may take the sheaf $\mathscr{F} := \oplus j_! \mathbf{Z}$, where the direct sum is over a cofinal set (there is such a set!) of étale morphisms $j : U \to S$. □

We may now define étale cohomology. (Below, Ab denotes the category of abelian groups.)

Definition 5.2. Let S be a scheme. The étale cohomology groups $\mathrm{H}^i_{\text{ét}}(S, \cdot)$ are the right derived functors of the left exact functor $\mathrm{Ab}(S) \to \mathrm{Ab}$ defined by $\mathscr{F} \mapsto \Gamma(S, \mathscr{F})$. If $f : X \to S$ is a map of schemes, then the functor $\mathrm{R}^\bullet f_*$ is the derived functor of the left-exact functor f_*.

Let us note some useful basic properties of étale cohomology.

Étale cohomology is functorial in the scheme. More precisely, given a morphism $f : X \to S$ of schemes, we have for any abelian sheaf $\mathscr{F}$ on S a natural map $\mathrm{H}^0(S, \mathscr{F}) \to \mathrm{H}^0(X, f^*\mathscr{F})$. Since f^* is exact, this map extends to a map of δ-functors $\mathrm{H}^i(S, \mathscr{F}) \to \mathrm{H}^i(X, f^*\mathscr{F})$, and it is transitive with respect to composition in f, because it is in degree 0.

Since f_* has exact left adjoint f^*, it carries injectives to injectives and so we obtain for any sheaf $\mathscr{F}$ on X a Leray spectral sequence

$$E_2^{p,q} = \mathrm{H}^i(S, \mathrm{R}^q f_* \mathscr{F}) \Longrightarrow \mathrm{H}^{p+q}(X, \mathscr{F})$$

Given a scheme S and an open immersion $j : U \to S$, the groups $\mathrm{H}^i_{\mathrm{Zar}}(U, j^* \cdot)$ are the derived functors of $\mathrm{H}^0(U, j^* \cdot)$. We would like the same thing to be true for étale cohomology where now j is an étale morphism rather than an open immersion. Since j^* is exact, we only need to show that it sends injectives to injectives. Since j^* has left adjoint $j_!$, it suffices to show that $j_!$ is exact. Being a left adjoint, $j_!$ is right-exact, and it preserves injectivity by construction, so it is exact. We may therefore unambiguously identify $\mathrm{H}^i(U, \mathscr{F}|_U)$ with $\mathrm{H}^i(U, \mathscr{F})$.

Finally, just as for Zariski cohomology, given a commutative diagram of morphisms of schemes

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

we obtain a base change morphism $g^* \circ \mathrm{R}\, f_* \to (\mathrm{R}\, f'_*) \circ g'^*$ between δ-functors from $\mathrm{Ab}(X)$ to $\mathrm{Ab}(S')$. We will later see that in favorable situations this map induces an isomorphism for suitable sheaves.

6 Čech cohomology and torsors

Given an étale morphism $j : U \to S$ and an injective sheaf $\mathscr{I}$ on S, we have $\mathrm{H}^i(U, j^*\mathscr{I}) = 0$ for all $i > 0$, since $j^*\mathscr{I}$ is still injective. For any étale cover $\mathfrak{U}$ of S, therefore, and any sheaf $\mathscr{F}$ on S, we obtain a Čech to derived-functor spectral sequence

$$E_2^{p,q} = \check{\mathrm{H}}^p(\mathfrak{U}, \underline{\mathrm{H}}^q(\mathscr{F})) \Longrightarrow \mathrm{H}^{p+q}(S, \mathscr{F})$$

where $\underline{\mathrm{H}}^q(\mathscr{F})$ denotes the presheaf $U \mapsto \mathrm{H}^q(U, \mathscr{F})$. This sequence is compatible with refinement of $\mathfrak{U}$, hence taking a direct limit over a cofinal system of covers yields a spectral sequence

$$E_2^{p,q} = \check{\mathrm{H}}^p(S, \underline{\mathrm{H}}^q(\mathscr{F})) \Longrightarrow \mathrm{H}^{p+q}(S, \mathscr{F})$$

By computing in terms of an injective resolution, we see that for $q > 0$, any section of $\underline{\mathrm{H}}^q(\mathscr{F})$ vanishes étale-locally, so the separated presheaf associated to the sheaf $\underline{\mathrm{H}}^q(\mathscr{F})$ is 0. But this separated presheaf is $U \mapsto \check{\mathrm{H}}^0(U, \underline{\mathrm{H}}^q(\mathscr{F}))$. It follows that $E_2^{0,q} = \check{\mathrm{H}}^0(S, \underline{\mathrm{H}}^q(\mathscr{F}))$ vanishes for $q > 0$. In particular, we deduce that the map $\check{\mathrm{H}}^1(S, \mathscr{F}) = E_2^{1,0} \to E^1 = \mathrm{H}^1(S, \mathscr{F})$ is an isomorphism. We record this important (and much more generally valid) result.

Theorem 6.1. *Let $\mathscr{F} \in \mathrm{Ab}(S)$. The edge map $\check{\mathrm{H}}^1(S, \mathscr{F}) \to \mathrm{H}^1(S, \mathscr{F})$ is an isomorphism.*

This is great because by well-known explicit calculations, $\check{\mathrm{H}}^1(S, G)$ classifies étale left G-torsor sheaves over S, for any group object $G \in \text{Ét}(S)$. Let us briefly recall how this goes. Given a G-torsor sheaf E over S, there is an étale cover $\mathfrak{U} := \{U_i \to S\}$ such that we have isomorphisms $\phi_i : E|_{U_i} \simeq G_{U_i}$ as G-torsor sheaves over $(U_i)_{\text{ét}}$. Restricting ϕ_i, ϕ_j over $U_i \times_S U_j$, we obtain an automorphism $\phi_i \circ \phi_j^{-1}$ of the trivial G-torsor $G|_{U_i \times_S U_j}$. Any such automorphism is given by an element of $G(U_i \times_S U_j)$ (by looking at where $1 \in G(U_i \times_S U_j)$ gets sent). We therefore obtain an element of $\prod_{i,j} G(U_i \times_S U_j)$, and it clearly satisfies the cocycle condition, hence yields an element of $\check{\mathrm{H}}^1(\mathfrak{U}, G)$.

One can check that changing ϕ_i by an automorphism of $G|_{U_i}$ (that is, an element of $G(U_i)$) produces a cohomologous cocycle, hence this class is well-defined. The local nature of sheaves shows that this map yields a bijection

$$\{G\text{-torsor sheaves that split over each } U_i\} \xleftrightarrow{\sim} \check{\mathrm{H}}^1(\mathfrak{U}, G).$$

Further, this map is compatible with refinement of $\mathfrak{U}$, so we obtain a bijection

$$\{G\text{-torsor sheaves over } S\} \xleftrightarrow{\sim} \check{\mathrm{H}}^1(S_{\text{ét}}, G).$$

So the group $\mathrm{H}^1(S_{\text{ét}}, G)$ classifies G-torsor sheaves when G is an abelian sheaf. In case G is (represented by) a *finite étale* commutative S-group, all such sheaves are lcc (as we may check locally) and so are G-torsors in the scheme sense (for the étale topology). If G is finite étale over a connected S, we can relate this back to the fundamental group $\pi_1(S, \overline{s})$ as follows. The equivalence of Theorem 3.5 specializes to yield a bijection between étale G-torsors X over S and $G_{\overline{s}}$-torsors in the category of finite discrete left $\pi_1(S, \overline{s})$-sets, via $X \rightsquigarrow X_{\overline{s}}$. This yields a canonical isomorphism of groups $\mathrm{H}^1(S, G) \simeq \mathrm{H}^1(\pi_1(S, \overline{s}), G_{\overline{s}})$ for commutative finite étale S-groups G.

The finiteness assumption on G is essential in the above equivalence in terms of scheme-torsors rather than just sheaf-torsors. Indeed, we always have

$$\mathrm{H}^1(\pi_1(S, \overline{s}), \mathbf{Z}) = \mathrm{Hom}_{\text{cts}}(\pi_1(S, \overline{s}), \mathbf{Z}) = 0$$

since $\pi_1(S, \overline{s})$ is profinite. But one can construct examples of schemes with nontrivial étale $\mathbf{Z}$-torsors, so that $\mathrm{H}^1(S, \mathbf{Z}) \neq 0$. As an example, consider the nodal cubic C: $y^2w = x^3 - x^2w$. This is isomorphic to $\mathbf{G}_\mathrm{m}$ away from the node. If we take a bunch of copies of $\mathbf{P}^1$ indexed by $\mathbf{Z}$, glue each one at 0 to the next one at ∞, and call the resulting scheme C_∞, then we obtain a map $C_\infty \to C$ by using the isomorphism between C minus the node and $\mathbf{G}_\mathrm{m}$, and sending each 0 and ∞ to the node. This is clearly étale away from the copies of 0 and ∞, and it is étale even at these points because the node is "analytically isomorphic" to the transversal intersection of two lines (that is, the induced map on completed local rings is an isomorphism). The C-scheme C_∞ carries an obvious $\mathbf{Z}$-action over C, where 1 acts by sending each $\mathbf{P}^1$ to the next one, and this makes it into an étale $\mathbf{Z}$-torsor over C. This cover has no sections (why not?), and is therefore a nontrivial $\mathbf{Z}$-torsor.

If S is connected, noetherian, and normal then the preceding phenomenon *cannot happen*: any locally constant étale sheaf $\mathscr{F}$ on S is split by a finite étale cover, so the equivalence between $\mathrm{H}^1(S, G)$ and $\mathrm{H}^1(\pi_1(S, \overline{s}), G)$ holds for any abelian group G. To see this, first note that since S is connected, noetherian, and normal, it is irreducible. Let η be its generic point. Connected finite étale covers of S correspond to closed (or equivalently, open) subgroups $H \subset \pi_1(S, \overline{\eta})$ of finite index, by Theorem 3.4. In order to show that $\mathscr{F}$ is split by a finite étale cover of S, by the "locally constant" property it follows that $\mathscr{F}$ is split by any cover of S for which the generic fiber is constant. Thus, it suffices to find an open subgroup $H \subset \pi_1(S, \overline{\eta})$ of finite index that acts trivially on $\mathscr{F}_{\overline{\eta}}$. We first note that there is such an open subgroup of finite index inside of $\pi_1(\eta, \overline{\eta})$, since any étale cover of η may be refined by a finite étale one (because η is Spec of a field). It therefore suffices to show that the map $\pi_1(\eta, \overline{\eta}) \to \pi_1(S, \overline{\eta})$ is surjective.

By Theorem 3.3, in order to prove this surjectivity it is equivalent to show that S'_η is connected for any connected finite étale cover $S' \to S$. Since $S' \to S$ is flat, S'_η is the disjoint union of the generic points of S', so we want to show that S' is irreducible. But S' is connected and, being finite étale over S, it is noetherian and normal, so it is irreducible, as desired.

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