Zariski-Étale Comparison, Kummer and Artin-Schreier Sequences, Cohomology of Curves

David Sherman

October 26, 2016

1 Comparison of the Zariski and Étale Sites

For a scheme S, let S_{Zar} denote the site induced by the Zariski topology on S and $\text{Zar}(S)$ the corresponding category of sheaves of sets. Then we have a continuous map of Grothendieck topologies $S_{\text{ét}} \xrightarrow{\iota} S_{\text{Zar}}$, where $\iota^{-1}(U \subset S)$ is the inclusion of U into S considered as an étale map.

This may seem a bit abstract, but the point is the following: just as with the continuous map induced by a morphism of schemes, we can use ι to define pullback and pushforward functors between $\text{Ét}(S)$ and $\text{Zar}(S)$. Concretely,

$$(\iota_* \mathcal{F})(U) = \mathcal{F}(U)$$

for $\mathcal{F}$ an étale sheaf and U a Zariski-open set in S. This is restriction to the Zariski topology, and hence $\iota_* \mathcal{F}$ is a sheaf. On the other hand, as is typical when defining pullbacks, sheafification is necessary. So we first set

$$(\iota^* \mathcal{F})^{\text{pre}}(h : U \to S) = \varinjlim_{V \supseteq h(U)} \mathcal{F}(V)$$

for $\mathcal{F}$ a Zariski sheaf. Note that the direct limit on the right above is actually equal to $\mathcal{F}(h(U))$, since étale maps are open. Still, $(\iota^* \mathcal{F})^{\text{pre}}$ is typically not an étale sheaf. Thus, we define $\iota^* \mathcal{F}$ to be its étale sheafification.

As one should expect, (ι^*, ι_*) form an adjoint pair. Especially, this implies ι^* is right-exact. Also familiar is that the construction of ι^* implies that it commutes with finite limits and finite fiber products, so ι^* is exact. These properties also hold when we restrict to abelian sheaves.

Now, there's a natural map on global sections $\mathcal{F}(S) \to (\iota^* \mathcal{F})(S)$ (coming from the map $(\iota^* \mathcal{F})^{\text{pre}} \to \iota^* \mathcal{F}$). For $\mathcal{F}$ an abelian sheaf, exactness of ι^* guarantees this extends uniquely to a map of δ-functors (the source being a universal δ-functor):

$$\text{H}^\bullet(S, \mathcal{F}) \to \text{H}^\bullet_{\text{ét}}(S, \iota^* \mathcal{F}).$$

We call this the *Zariski-étale comparison morphism.* Note that we shouldn't expect this to be an isomorphism. For example, suppose S is an irreducible variety over $\mathbf{C}$ and $\mathcal{F}$ is a finite constant sheaf. Then, as promised in Lecture 1, étale cohomology with $\mathcal{F}$ coefficients will prove to be isomorphic to singular cohomology of $S(\mathbf{C})$. On the other hand, the higher Zariski cohomology vanishes! (for flasqueness reasons). But there is an improvement of our comparison morphism for $\mathcal{F}$ an $\mathscr{O}_S$-module, which we'll see is an isomorphism when $\mathcal{F}$ is quasi-coherent (and $\iota^*\mathcal{F}$ is replaced with an appropriate "sheaf of modules" pullback).

2 Quasi-coherent Zariski-étale comparison

Assume for this section that $\mathcal{F}$ is an $\mathscr{O}_S$-module. We'd like to define a variant of $\iota^*\mathcal{F}$ that does something useful with the $\mathscr{O}_S$-module structure. As with quasi-coherent pullback in usual sheaf theory, the variant involves tensoring. We thus need a replacement for the étale site of the structure sheaf $\mathscr{O}_S$. The group scheme $\mathbf{G}_a$ gives us a sheaf of rings on the étale site, which we denote by $\mathscr{O}_{S_{\text{ét}}}$:

$$\mathscr{O}_{S_{\text{ét}}}(U) := \mathbf{G}_{a,S}(U) = \mathscr{O}_U(U).$$

We have a natural map of sheaves of rings (on $\mathrm{Zar}(S)$) $\mathscr{O}_S \to \iota_*\mathscr{O}_{S_{\text{ét}}}$. The adjoint map $\iota^*\mathscr{O}_S \to \mathscr{O}_{S_{\text{ét}}}$ induces the map $\mathscr{O}_{S,s} \to \mathscr{O}_{S,\overline{s}}^{\text{sh}}$ on stalks at a geometric point $\overline{s}$ over $s \in S$. In particular, this is a flat extension of rings. We define the $\mathscr{O}_{S_{\text{ét}}}$-module

$$\mathcal{F}_{\text{ét}} := \mathscr{O}_{S_{\text{ét}}} \otimes_{\iota^*\mathscr{O}_S} \iota^*\mathcal{F},$$

and $\mathcal{F} \rightsquigarrow \mathcal{F}_{\text{ét}}$ is exact (by exactness of ι^* and the flatness noted above). In particular, we get an $\mathscr{O}_S$-module version of the Zariski-étale comparison morphism (and this version is also δ-functorial):

$$\mathrm{H}^\bullet(S,\mathcal{F}) \to \mathrm{H}^\bullet_{\text{ét}}(S,\iota^*\mathcal{F}) \to \mathrm{H}^\bullet_{\text{ét}}(S,\mathcal{F}_{\text{ét}}).$$

Now we can state the main theorem of this section:

Theorem 2.1. *If the $\mathscr{O}_S$-module $\mathcal{F}$ is quasi-coherent, the map $\Psi_S : \mathrm{H}^\bullet(S,\mathcal{F}) \to \mathrm{H}^\bullet_{\text{ét}}(S,\mathcal{F}_{\text{ét}})$ defined above is an isomorphism.*

Before proving the theorem, we need an alternate description of $\mathcal{F}_{\text{ét}}$:

Lemma 2.2. *Let $\mathcal{F}$ be a quasi-coherent $\mathscr{O}_S$-module. The étale presheaf $\widetilde{\mathcal{F}} : (U \xrightarrow{h} S) \rightsquigarrow \Gamma(U, h^*\mathcal{F})$ on $S_{\text{ét}}$ is naturally isomorphic to $\mathcal{F}_{\text{ét}}$ (where h^* denotes module pullback for the Zariski topology). In particular, $\widetilde{\mathcal{F}}$ is an étale sheaf.*

Proof. That $\widetilde{\mathcal{F}}$ is a sheaf follows from fpqc descent for quasi-coherent sheaves (since any étale cover of S can be refined over open affines on S to one which is quasi-compact over the base and hence is an fpqc cover).

By inspection $\widetilde{\mathscr{O}_S} = \mathscr{O}_{S_{\text{ét}}}$. For arbitrary $\mathcal{F}$, we have an evident natural morphism $\mathcal{F} \to \iota_* \widetilde{\mathcal{F}}$ which is linear over the morphism $\mathscr{O}_S \to \iota_* \mathscr{O}_{S_{\text{ét}}}$. By adjunction, we get a map $\iota^* \mathcal{F} \to \widetilde{\mathcal{F}}$ which is linear over $\iota^* \mathscr{O}_S \to \mathscr{O}_{S_{\text{ét}}}$. This extends by linearity to a map $\mathcal{F}_{\text{ét}} \to \widetilde{\mathcal{F}}$. To see this is an isomorphism, we shall check on stalks at geometric points.

Recall that in ordinary sheaf theory, the stalk of a tensor product of sheaves is the tensor product of the stalks. The same is true in the étale setting via the same proof. Thus, the stalks of $\mathcal{F}_{\text{ét}}$ and $\widetilde{\mathcal{F}}$ are each $\mathscr{O}^{\text{sh}}_{S,\bar{s}} \otimes_{\mathscr{O}_{S,s}} \mathcal{F}_s$ in such a way that our map induces the identity on stalks. □

An important example/application of the above lemma is that if $h : U \to S$ is étale then the restriction of $\mathcal{F}_{\text{ét}}$ to the Zariski topology on U is the quasi-coherent pullback $h^* \mathcal{F}$:

$$(\widetilde{\mathcal{F}}|_U)(U' \subset U) = \Gamma(U', h|_{U'}^* \mathcal{F}) = (h^* \mathcal{F})(U'),$$

since quasi-coherent pullback to a Zariski open set is the same as restriction. Now we are ready to prove the comparison theorem:

Proof. Amazingly, we'll use the Čech-to-derived spectral sequence not just once, but several times! For an open set U in S, the exact functor $\mathcal{F} \rightsquigarrow \mathcal{F}_{\text{ét}}$ induces a morphism $\mathrm{H}^q(U, \mathcal{F}|_U) \to \mathrm{H}^q_{\text{ét}}(U, \mathcal{F}_{\text{ét}}|_U)$ (this is our comparison morphism Ψ_U). In particular, these morphisms are functorial with respect to restrictions on U. Thus, they induce a morphism of Zariski-presheaves $\underline{\mathrm{H}}^q(\mathcal{F}) \to \underline{\mathrm{H}}^q_{\text{ét}}(\mathcal{F}_{\text{ét}})$.

So take an affine open cover $\mathfrak{U}$ of S, and we get a morphism of the Čech-to-derived spectral sequences of $\mathcal{F}$ and $\mathcal{F}_{\text{ét}}$ corresponding to $\mathfrak{U}$.[1] To prove the theorem, it's enough to prove the induced morphism on any particular page is an isomorphism (because we have a morphism on the abutments compatible with the morphism on spectral sequences; cf. Theorem 5.2.12 in Weibel's *Homological Algebra*). Here comes the trick: the individual morphisms, say on $E_2^{p,q}$-terms, are induced from morphisms of the form $\Psi_{\widetilde{U}}$, where $\widetilde{U}$ is a finite intersection of elements of $\mathfrak{U}$. In particular, $\widetilde{U}$ is separated. So if the theorem holds for separated schemes, it holds for S.

Hence, we've reduced to showing the theorem for S separated. Now run the above argument again, and in this case the $\widetilde{U}$ are affine. Thus, it's enough to show the theorem in the case S is affine. Putting aside the spectral sequence (temporarily), we can see that Ψ_S in degree zero is the isomorphism $\mathcal{F}(S) \to \Gamma(S, \mathrm{id}_S^* \mathcal{F})$ on global sections. We claim that the higher cohomologies vanish for each. This is already known in the Zariski case and, in fact, one can give a proof along the same lines for the étale case.

[1] There's a subtlety here, which is that $\underline{\mathrm{H}}^q_{\text{ét}}(\mathcal{F}_{\text{ét}})$ is really an étale presheaf and we want to compute the étale version of the spectral sequence (because that's the one that abuts to the étale cohomology of $\mathcal{F}_{\text{ét}}$). But the opens in $\mathfrak{U}$ and the overlaps thereof are Zariski open sets, so we can just consider $\mathfrak{U}$ as an étale cover and get the same spectral sequence, hence the same abutment.

In particular, because S is affine, we can refine any étale cover by a finite one $\mathfrak{U}$ in which the open sets and their overlaps are all affine. We claim then that the Čech complex for $\mathcal{F}_{\text{ét}}$ with respect to the étale cover $\mathfrak{U}$ is exact in nonzero degrees. But by the lemma, the space of sections of $\mathcal{F}_{\text{ét}}$ on such an overlap $\widetilde{U} \xrightarrow{h} S$ is equal to $\Gamma(\widetilde{U}, h^*\mathcal{F})$. Or, more suggestively, if $S = \text{Spec } R$, $\widetilde{U} = \text{Spec } R'$, and M is the underlying R-module for the quasi-coherent sheaf $\mathcal{F}$, then $\mathcal{F}_{\text{ét}}(\widetilde{U}) = M \otimes_R R'$. Thus, by the proof of fpqc descent for quasi-coherent sheaves (cf. *Néron Models*, §6.1, Lemma 2), the Čech complex is indeed exact in higher degrees (in degree zero, we of course get global sections of $\mathcal{F}$, as described above).

Now, we will appeal to Cartan's lemma (cf. Grothendieck's Tôhoku paper, §3.8, Cor. 4) to see that the higher sheaf cohomology of $\mathcal{F}_{\text{ét}}$ vanishes. Recall that Cartan's lemma says that if we have a base $\mathfrak{B}$ of opens for the topology such that higher Čech cohomology vanishes for all finite collections in $\mathfrak{B}$ (not necessarily covering all of S), then Čech cohomology on S is in fact isomorpic to sheaf cohomology. In particular, if S itself is an element of $\mathfrak{B}$, this implies the vanishing of the higher sheaf cohomology. Note also that the proof of Cartan's lemma just has the Čech-to-derived spectral sequence as its main ingredient, so it is valid in our situation with the étale topology (using the same proof).

Finishing up, we can take étale maps from affines as our base $\mathfrak{B}$. The hypothesis of Cartan's lemma is satisfied by the exactness we showed above (the descent proof we gave above is valid for all finite collections in $\mathfrak{B}$, replacing S by the disjoint union of the members of such a collection, which is still affine), so we are done. □

3 Cohomology of Curves

As described in Lectures 1 and 2, we get étale cohomology off the ground by starting with the case of curves, then inducting using the Leray spectral sequence, base change theorems, and fibrations. For the case of curves, we'll need to use the quasi-coherent comparison theorem of the previous section to get a handle on the cohomology. Let's state the main theorem in this direction:

Theorem 3.1. *Let X be a separated, finite-type scheme of dimension ≤ 1 over a separably closed field k. Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{\text{ét}}$. Then $\mathrm{H}^i_{\text{ét}}(X, \mathcal{F}) = 0$ if $i > 2$. If $\mathcal{F}$ is also constructible, then $\mathrm{H}^i_{\text{ét}}(X, \mathcal{F})$ is finite for $i \leq 2$.*

Moreover, if X is affine and sections of $\mathcal{F}$ are locally killed by an integer n not divisible by $\text{char}(k)$*, then $\mathrm{H}^2_{\text{ét}}(X, \mathcal{F}) = 0$. If X is instead proper and sections of $\mathcal{F}$ are locally p-torsion with $p = \text{char}(k) > 0$ then $\mathrm{H}^2_{\text{ét}}(X, \mathcal{F}) = 0$.*

Remark 3.2. (i) We won't give a full proof of this theorem. The reader can find a proof of the prime-to-p torsion case in Ch. I, §5 of the Freitag–Kiehl book and the core p-power torsion case near the end of SGA7 (see §2 of Exposé XXII, which rests on some constructions given in §1 there). The rest of these remarks sketch how one can reduce the theorem to a few more reasonable cases.

(ii) The theorem is also just a nice packaging of the several basic results we need into a single statement. That is to say, it's not proved all at once. Also, note that the curve case is the more significant one, since the étale site of a separably closed field is not so interesting.

(iii) The truth of the theorem is insensitive to modifying $\mathcal{F}$ at finitely many closed points of X. More precisely, if we have a map $\mathcal{F} \to \mathcal{G}$ (both constructible) which is an isomorphism away from these points then we may replace $\mathcal{F}$ by $\mathcal{G}$ in the statement of the theorem. Indeed the kernel and cokernel of this map are supported at finitely many copies of a separably closed field, so their higher cohomologies vanish.

(iv) Since the formation of étale sites is insensitive to integral radiciel surjections, such as scalar extension from a separably closed field to its algebraic closure or passing to the underlying reduced schemes, we can assume k is algebraically closed and X is reduced and connected; here we are using identifications such as $\mathrm{H}^{\bullet}_{\text{ét}}(X, \mathcal{F}) \simeq \mathrm{H}^{\bullet}_{\text{ét}}(X_{\overline{k}}, \pi^*\mathcal{F})$, where $\pi : X \otimes_{k_s} \overline{k} \to X$ and $\mathcal{F} \in \mathrm{Ab}(X)$. (Beware that $\pi^*\mathbf{G}_{m,k} \neq \mathbf{G}_{m,\overline{k}}$ in general, another example of Yoneda-type arguments failing for objects not in the category under consideration.)

(v) If $f : X' \to X$ is a finite map, then $f_* : \mathrm{Ab}(X') \to \mathrm{Ab}(X)$ is exact due to the behavior of strict henselization with respect to module-finite maps of rings [EGA IV$_4$, 18.6.8, 18.8.10] and Zariski's Main Theorem (which ensures that every étale cover of $\mathrm{Spec}\,(R)$ admits a section for any strictly henselian local ring R), so $R^q f_*$ vanishes for $q > 0$. (In contrast, the analogous vanishing for the Zariski topology is only valid for quasi-coherent sheaves!) By applying the Leray spectral sequence and (iii), we can then replace X by its normalization, which is smooth and connected (and separated).

(vi) One can also reduce the general torsion case to the case of constructible $\mathcal{F}$ by taking direct limits; the good behavior of étale cohomology of noetherian schemes with respect to direct limits of sheaves will be discussed in Evan's talk next week. By noetherianity of S, we may assume $\mathcal{F}$ is a $\underline{\mathbf{Z}/n\mathbf{Z}}$-module for some n (or similarly for the p-power torsion case). By constructibility, there is a dense open $U \subset X$ and a connected finite étale cover $U' \to U$ over which $\mathcal{F}$ becomes constant. Letting $h : X' \to X$ be the normalization of X in U', we have that $\mathcal{F}' := h^*\mathcal{F}$ is constant over U', and the evident map $\mathcal{F} \to h_*(\mathcal{F}')$ is *monic* (since h is surjective!) with constructible cokernel.

Since $\mathrm{H}^i_{\text{ét}}(X, h_*(\mathcal{F}')) \simeq \mathrm{H}^i_{\text{ét}}(X', \mathcal{F}')$ by finiteness of h, by standing descending induction considerations (check!) our task reduces to that of $\mathcal{F}'$ on X'; i.e., we can assume there is a finite abelian group M such that $\mathcal{F}|_U \simeq \underline{M}|_U$ for some dense open $j : U \hookrightarrow X$. The constructible sheaf $j_!(\mathcal{F}|_U) = j_!(\underline{M}|_U)$ is a subsheaf of each of $\mathcal{F}$ and $\underline{M}$ with cokernel supported at finitely many points, so by (iii) we can replace $\mathcal{F}$ with $\underline{M}$! In this way, we reduce to the case of the constant sheaf $\underline{\mathbf{Z}/n\mathbf{Z}}$ (or, similarly, $\underline{\mathbf{Z}/p^r\mathbf{Z}}$).

(vii) One can also the preceding argument and variants to compare the theorem for X and a dense open subset U (especially an affine open). For example, one deduces the extra vanishing for H^2 of an affine curve by embedding it in a smooth projective curve and comparing their Picard groups (especially, surjectivity of the multiplication-by-n map on Pic^0 of the projective curve leads to surjectivity on Pic of the affine curve).

The upshot of the above remarks is that the fundamental case of interest is X smooth, projective, and connected over $k = \overline{k}$, and $\mathcal{F}$ is the constant sheaf coming from $\mathbf{Z}/n\mathbf{Z}$ with $\mathrm{char}(k) \nmid n$ or $\mathbf{Z}/p\mathbf{Z}$ with $p = \mathrm{char}(k) > 0$. Noting that $\mu_n \simeq \underline{\mathbf{Z}/n\mathbf{Z}}$, the key statement we need is the following:

Proposition 3.3. *Let X be a connected, smooth, projective curve over an algebraically closed field k of characteristic $p \geq 0$. Also, let $n > 0$ be an integer not divisible by p. Then canonically $\mathrm{H}^1_{\text{ét}}(X, \mu_n) \simeq \mathrm{Pic}(X)[n]$ and $\mathrm{H}^2_{\text{ét}}(X, \mu_n) \simeq \mathbf{Z}/n\mathbf{Z}$. Also, $\mathrm{H}^0_{\text{ét}}(X, \underline{M}) \simeq M$ for any finite abelian group M, $\mathrm{H}^1_{\text{ét}}(X, \underline{\mathbf{Z}/p\mathbf{Z}})$ is finite, and $\mathrm{H}^2(X, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$. Finally, $\mathrm{H}^i_{\text{ét}}(X, \underline{\mathbf{Z}/p\mathbf{Z}})$ and $\mathrm{H}^i_{\text{ét}}(X, \underline{\mathbf{Z}/n\mathbf{Z}})$ vanish for $i \geq 3$.*

As alluded to above, the proof of the proposition relies on relating the étale cohomology of finite constant sheaves to that of quasi-coherent sheaves. This is where the Kummer and Artin-Schreier sequences enter.

3.1 Kummer and Artin-Schreier sequences

Now, let S be any k-scheme, where $\mathrm{char}(k) = p$ (we may have $p = 0$). For this section, n denotes a positive integer not divisible by p. Then we have the (commutative) S-group schemes $\mathbf{G}_{m,S}$ and $\mathbf{G}_{a,S}$, representing $U \mapsto \mathrm{H}^0(U, \mathscr{O}_U^\times)$ and $U \mapsto \mathrm{H}^0(U, \mathscr{O}_U)$, respectively. Also, we have the group homomorphisms $\mathbf{G}_m \xrightarrow{x^n} \mathbf{G}_m$ and $\mathbf{G}_a \xrightarrow{t^p - t} \mathbf{G}_a$. These have scheme-theoretic kernels μ_n and $\mathbf{F}_p$, respectively. We claim that, in fact, each homomorphism is surjective as a matter of étale sheaves:

Proposition 3.4. *The Kummer sequence*

$$1 \to \mu_n \to \mathbf{G}_m \xrightarrow{x^n} \mathbf{G}_m \to 1$$

is exact for the étale topology on S, and if $p > 0$ then so is the Artin–Schreier sequence

$$0 \to \underline{\mathbf{F}_p} \to \mathbf{G}_a \xrightarrow{t^p - t} \mathbf{G}_a \to 0.$$

Proof. The only part that remains to check is surjectivity. But we can check surjectivity étale-locally. So, for the Kummer sequence, let U an étale S-scheme. We may assume $U = \mathrm{Spec}\ A$ is affine, and consider a unit $u \in A^\times$. Then we'd like to show that u locally has an nth root. So consider $A' = A[T]/(T^n - u)$. Because n is a unit in k, Spec A' is an étale cover of U (so it is also étale over S). By construction, u is an nth power on Spec A'.

For the Artin-Schreier sequence (with $p > 0$), we apply the same general framework. Thus, given $b \in B$, where Spec $B \to S$ is étale, we claim that b is étale-locally of the form $t^p - t$. Indeed, set $B' = B[T]/(T^p - T - b)$, and we see that Spec B' is an étale cover of Spec B (because char$(k) = p$). $\square$

In general, we can rewrite the constant sheaf $\underline{\mathbf{F}_p}$ as $\underline{\mathbf{Z}/p\mathbf{Z}}$, and the étale sheaf $\mathbf{G}_a$ on S is just $\mathscr{O}_{S_{\text{ét}}}$ from the previous section. In particular, it has the form $(\mathscr{O}_S)_{\text{ét}}$, where $\mathscr{O}_S$ is quasi-coherent on S. So we can immediately combine the long exact sequence for Artin-Schreier with the previous section's comparison theorem to get our desired information about étale cohomology with $\mathbf{Z}/p\mathbf{Z}$-coefficients. We'd like to do the same for the Kummer sequence and $\mathbf{Z}/n\mathbf{Z}$-coefficients, using that $\mu_n \simeq \underline{\mathbf{Z}/n\mathbf{Z}}$ upon choosing a primitive nth root of unity in the separably closed ground field k, except for a serious problem: $\mathbf{G}_m$ isn't a quasi-coherent sheaf.

Now we use our description of $\mathrm{H}^1_{\text{ét}}(X, \mathbf{G}_m)$ in terms of Čech cohomology; i.e., in terms of the Picard group of X. There's a slight subtletly here: we know that $\mathrm{H}^1_{\text{ét}}(X, \mathbf{G}_m)$ classifies étale $\mathbf{G}_m$-torsors. We can view these as étale-locally trivial $\mathscr{O}_{X_{\text{ét}}}$-modules. And since $\mathscr{O}_X$ is quasi-coherent (unlike $\mathbf{G}_m$!), descent theory implies that the classes of such torsors actually arise as $\mathscr{L}_{\text{ét}}$ for conventional line bundles $\mathscr{L}$. So $\mathrm{H}^1_{\text{ét}}(X, \mathbf{G}_m)$ really is isomorphic to Pic(X). For degrees 2 and higher, our foothold is Tsen's theorem in Galois cohomology:

Theorem 3.5 (Tsen's Theorem)**.** *Let K be a field of transcendence degree* 1 *over an algebraically closed field k (i.e., the function field field of a smooth, connected k-curve). Then the higher Galois cohomology vanishes:* $\mathrm{H}^i(K_s/K, \mathbf{G}_m) = 0$*, $i > 0$.*

Strictly speaking, Tsen's theorem is the case $i = 2$; the case $i = 1$ is, of course, multiplicative Hilbert 90, and we can deduce $i > 2$ from these two by considerations involving cohomological dimension (cf. Serre's *Galois Cohomology*, Ch. II, §3, Prop. 5). A nice geometric proof of Tsen's theorem is given in §5.2 of Chapter 4 in Part I (by Danilov) of the book *Algebraic Geometry II* (volume 35 of the Encyclopedia of Math series), the key point being that in an affine space $\mathbf{A}^M_k$ any collection of $< M$ hypersurfaces has intersection with positive dimension (and hence at least 2 rational points, since $k = \overline{k}$) if the intersection is non-empty.

We finish by briefly indicating how to apply Tsen's theorem (vanishing for the cohomology of the generic point) to the cohomology of the curve itself. The following argument is adapted from Lemma 5.2 in Ch. I of Freitag–Kiehl. We shall use Tsen's theorem to show that $\mathrm{H}^i_{\text{ét}}(X, \mathbf{G}_m) = 0$ for $i \geq 2$, where X is the smooth projective curve with function field K and $j : \eta = \text{Spec } K \to X$ the inclusion of the generic point. To see this, we embed $\mathscr{O}^{\times}_{X_{\text{ét}}} := \mathbf{G}_{m,X}$ inside $j_*(\mathbf{G}_{m,\eta})$. This pushforward coincides with the sheaf $\mathscr{R}^{\times}_X$ of units in the sheaf of rational functions: its value on any étale X-scheme U is the group $k(U_\eta)^\times$ of global units on the scheme U_η of generic points of U. The crucial geometric observation is that the cokernel $\mathscr{R}_X^\times/\mathscr{O}_{X_{\text{ét}}}^\times$ coincides with the sheaf $\mathscr{D}_X$ of "Weil divisors" on $X_{\text{ét}}$, defined to be $\bigoplus_{x\in X^0} x_*(\underline{\mathbf{Z}})$ with X^0 the set of closed points of X.

Now consider the associated long exact cohomology sequence. Our goal is to show that the higher cohomology of $\mathscr{O}_{X_{\text{ét}}}^\times$ vanishes. The first key point is that the higher cohomology of $\mathscr{R}_X^\times$ vanishes. This comes down to Tsen's theorem as follows. Consider the Leray spectral sequence

$$E_2^{n,m} = \mathrm{H}^n_{\text{ét}}(X, \mathrm{R}^m(j_*)(\mathbf{G}_{m,\eta})) \Rightarrow \mathrm{H}^{n+m}_{\text{ét}}(\eta, \mathbf{G}_{m,\eta}).$$

The abutment vanishes when $n+m>0$ by Tsen's theorem, and $E_2^{n,0}$ is the cohomology of $\mathscr{R}_X^\times$, so it suffices to prove the vanishing of $E_2^{n,m}$ for all $m>0$ (allowing $n=0$!). More specifically, we claim that the higher direct image sheaves $\mathrm{R}^m(j_*)(\mathbf{G}_{m,\eta}))$ vanish for $m>0$. The vanishing of the η-stalk is elementary (why?), and at a geometric point $\overline{x}$ over a closed point $x\in X$ it coincides with $\mathrm{H}^m(K_x^{\text{sh}}, \mathbf{G}_m)$ where K_x^{sh} is the fraction field of the strict henselization of $\mathscr{O}_{X,x}$ with respect to $\overline{x}$ (check!). This vanishes for $m>0$ by Tsen's theorem.

It remains to show that the higher cohomology of $\mathscr{D}_X$ also vanishes. But $\mathscr{D}_X$ is an (infinite) direct sum of pushforwards of the constant sheaf $\underline{\mathbf{Z}}$ from the geometric closed points x of X, and étale cohomology on a noetherian scheme commutes with direct limits of sheaves (such as arbitrary direct sums). So it's enough to see that the higher cohomology of each one of these pushforwards $x_*\underline{\mathbf{Z}}$ vanishes. But the geometric point x is Spec of an algebraically closed field, so the higher cohomology of the constant sheaf vanishes, implying the same for $x_*\underline{\mathbf{Z}}$ since this is pushforward under a *finite* map (even a closed immersion).