Artin Approximation and Proper Base Change

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1 Proper base change theorem

We're going to talk through the proof of the Proper Base Change Theorem:

Theorem 1.1. *Let $f\colon X \to S$ be a proper map. For the Cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S \end{tikzcd}$$

and an abelian sheaf $\mathcal{F}$ of torsion abelian groups on X, the natural base change map

$$g^* \mathrm{R}^q f_* \mathcal{F} \xrightarrow{\sim} \mathrm{R}^q f'_* (g'^* \mathcal{F})$$

is an isomorphism.

By limit arguments and considerations with geometric stalks as discussed last time, we can arrange that $\mathcal{F}$ is a $\mathbf{Z}/n\mathbf{Z}$-sheaf for some $n > 0$ and it suffices to treat the case where $S = \mathrm{Spec}\,(A)$ for a strictly henselian local ring and $S' = \mathrm{Spec}\,(k')$ for a separably closed field k' over the separably closed residue field k of A. Next, we reduce to the "core case" $k' = k$ (i.e., S' is the closed point s of S):

Lemma 1.2. *If the core case is proved in general then for any proper scheme X over a separably closed field k and any torsion abelian sheaf $\mathcal{F}$ on X, the map*

$$\mathrm{H}^i(X, \mathcal{F}) \to \mathrm{H}^i(X_K, \mathcal{F}_K)$$

is an isomorphism for any separably closed extension K/k.

*Notes taken by Tony Feng

Proof. Since purely inseparable extension of the base field naturally has no effect on étale cohomology (in the precise sense discussed in an earlier lecture), we can first replace K by an algebraic closure and then replace k by its algebraic closure in K so that k is algebraically closed. Then we write K as a direct limit of finite-type k-subalgebras A_i and are led to consider the situation of a finite-type k-scheme S (such as some such $\mathrm{Spec}\,(A_i)$) and the base change morphism in $\mathrm{Ab}(S_{\text{ét}})$

$$\theta_S^i : \mathrm{H}^i(X, \mathcal{F})_S \to \mathrm{R}^i(f_S)_*(\mathcal{F}_S)$$

where $f_S : X_S \to S$ is the base change of $X \to \mathrm{Spec}\,(k)$ and $\mathcal{F}_S$ is the pullback of $\mathcal{F}$ to X_S. If these maps are isomorphisms for *all* such S, then by the limit formalism if such S vary through the $\mathrm{Spec}\,(A_i)$'s whose limit is the geometric point $\mathrm{Spec}\,(K)$ we'd get in the limit the map

$$\mathrm{H}^i(X, \mathcal{F}) \to \mathrm{H}^i(X_K, \mathcal{F}_K)$$

as an isomorphism, as desired.

So our original task of studying invariance under scalar extension through K/k is reduced to showing for finite-type k-schemes S that the maps θ_S^i are isomorphisms. Since S is finite type over a field, any étale map to S hitting all closed points is an étale surjections onto S. Thus, θ_S^i is an isomorphism if and only if it is so on stalks at all closed points. But k is algebraically closed, so the closed points of S all have residue field k (Nullstellensatz) and hence the map induced by θ_S^i between stalks at such a point is

$$\mathrm{H}^i(X, \mathcal{F}) \to \mathrm{H}^i(X_A, \mathcal{F}_A)$$

where $A = (\mathscr{O}_{S,s})^{\mathrm{sh}}$ is strictly henselian with residue field k (so the special fiber of $(X_A, \mathcal{F}_A)$ is identified with $(X, \mathcal{F})$!). ◻

Remark 1.3. Once the proof of proper base change is over, so we'll know that the formation of $\mathrm{H}^i(X, \mathcal{F})$ is insensitive to change of a separably closed ground field when X is a *proper* scheme over $k = k_s$ (and $\mathcal{F}$ is any torsion abelian sheaf on X), one would like to know the same holds when "proper" is relaxed to "separated of finite type" assuming (as is seen to be necessary by consideration of the Artin–Schreier sequence on the affine line) that $\mathcal{F}$ has its torsion-orders not divisible by $\mathrm{char}\,(k)$. Such invariance under such change of a separably closed ground field is used *all the time* to avoid any confusion about what is meant when we speak of computing cohomology "on a geometric fiber". (For example, applying it to $\mathbf{C}$ over $\overline{\mathbf{Q}}$, if X is any separated scheme of finite type over $\overline{\mathbf{Q}}$ then $\mathrm{H}^i(X, \mathcal{F})$ naturally coincides with $\mathrm{H}^i(X_{\mathbf{C}}, \mathcal{F}_{\mathbf{C}})$ which via the Artin comparison isomorphism to be discussed later may be computed via topological methods for constructible $\mathcal{F}$.)

But why should this be true? After passing to algebriacally closed k as above, it rests on the *smooth base change theorem* that will be discussed next week (via the fact that if K/k is a finitely generated extension of such a k then K is the function field of a smooth k-scheme, thanks to the existence of a separating transcendence basis over the perfect k). This further base change theorem requires the torsion-orders to be invertible on the base.

To summarize our setup, we are considering a cartesian diagram of the form

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d] \\ s \arrow[r] & \mathrm{Spec}\ A \end{tikzcd}$$

with $s = \mathrm{Spec}\,(k)$ for $k = A/m_A$. The claim is that the restriction map

$$\mathrm{H}^i(X, \mathcal{F}) \to \mathrm{H}^i(X_s, \mathcal{F}_s)$$

is an isomorphism for all $i \geq 0$.

In the special case of proper flat map $f : X \to C$ to a smooth curve C over $\mathbf{C}$ with f smooth away from $c_0 \in C(\mathbf{C})$ and constant $\mathcal{F}$, the topological analogue is that the inclusion of $X^{\mathrm{an}}_{c_0} \hookrightarrow X^{\mathrm{an}}$ is a homotopy equivalence over a small neighborhood of c_0 in C^{an}. [1]

1.1 Reductions

We next reduce to the case $\dim X_s \leq 1$. First we reduce to the case that X is projective. Grant the case of projective A-schemes for all strictly henselian A (which would settle the case of projective f in general). We can assume A is noetherian, and then Chow's Lemma provides a projective map $h : Y \to X$ that is an isomorphism over a dense open $U \subset X$ with Y projective over A. We will argue by noetherian induction on closed subsets of X on which $\mathcal{F}$ is supported. The map $\mathcal{F} \to h_*(h^*\mathcal{F})$ has kernel and cokernel supported in $X - U$ (using the precise sense in which étale sheaf theory on a scheme depends only on the underlying reduced scheme, it doesn't matter what closed subscheme structure we impose on $X - U$), so the long exact cohomology sequence associated to a short exact sequence of sheaves reduces our task for $\mathcal{F}$ (and all $i \geq 0$) to that of $h_*(h^*\mathcal{F})$. In other words, we can assume $\mathcal{F} = h_*(\mathcal{F}')$ for $\mathcal{F}'$ on Y.

The higher direct images $\mathrm{R}^j h_*(\mathcal{F}')$ are supported in $X - U$ for $j > 0$, so in the Leray spectal sequence

$$E_2^{ij} = \mathrm{H}^i(X, \mathrm{R}^j h_*(\mathcal{F}')) \Rightarrow \mathrm{H}^{i+j}(Y, \mathcal{F}')$$

whose formation is compatible with base change morphisms throughout, the projective case we are granting in general (applied to h and $Y \to \mathrm{Spec}\,(A)$) implies that the base change maps on the abutments are isomorphisms as are those on E_2^{ij} for all $j > 0$. Thus, it follows that the same holds for each $E_2^{i0} = \mathrm{H}^i(X, h_*(\mathcal{F}'))$, as desired.

[1] What is a genuine reference for an actual proof of this statement?

Now we may and do assume X is projective over A, or more generally that there exists a closed immersion $X \hookrightarrow \mathbf{P}^n_S$. Aftering pushing $\mathcal{F}$ forward, we can assume $X = \mathbf{P}^n_S$. Consider the finite map

$$p : (\mathbf{P}^1_S)^N \to \mathbf{P}^N_S$$

defined by $([a_1, b_1], \ldots, [a_N, b_N]) \mapsto [h_0(a, b), \ldots, h_N(a, b)]$ where

$$\prod_{i=1}^{N}(a_i T + b_i) = h_0(a, b) + h_1(a, b)T + \cdots + h_N(a, b)T^N$$

as universal polynomials in a variable T. It is easy to check that p is quasi-finite and surjective by considering geometric points, so it is a finite surjection by properness. Thus, we have a short exact sequence

$$0 \to \mathcal{F} \to p_*(p^*\mathcal{F}) \to \mathcal{Q} \to 0.$$

Granting that the base change map is an isomorphism for all sheaves of the form $p_*(\mathcal{F}')$ with every $i \geq 0$, the base change map for $\mathcal{F}$ in degree i is injective provided that the isomorphism assertion for degree $i - 1$ is settled for *all* $\mathcal{F}$ (as holds for $i = 0$!). Feeding general injectivity in degree i into the same diagram-chasing gives the isomorphism property in general for degree i. In this way, via induction on i we reduce to the case $\mathcal{F} = p_*(\mathcal{F}')$. But p is *finite*, so we can rename $\mathcal{F}'$ as $\mathcal{F}$ and replace $\mathbf{P}^N_S$ with $(\mathbf{P}^1_S)^N$.

The map $(\mathbf{P}^1_S)^N \to S$ factors as a composition of relative projective lines (by projecting away from more and more factors). But using a Leray spectral sequence argument we see easily that if f factors as $f_1 \circ f_2$ for proper $f_2 : X \to Z$ and proper $f_1 : Z \to S$ then the assertion for f and *all* $\mathcal{F}$ reduces to the same for each of f_1 and f_2 separately for *general* abelian sheaves (say killed by a fixed integer $n \geq 1$). This argument involves higher direct images under f_1 evaluated on higher direct images under f_2 (and the behavior of the latter with respect to constructibility is not yet known, so it is essential that our method does not yet impose any constructibility hypotheses on $\mathcal{F}$). To summarize, we are reduced to the case $X = \mathbf{P}^1_S$ with general S, or equivalently $X = \mathbf{P}^1_A$ for strictly henselian local A, and more generally that $\dim X_s \leq 1$ with strictly henselian local S.

This latter formulation is more robust because we saw in Evan's talk last time that *now* after we pass to the case of constructible $\mathcal{F}$ we can pass to finite X-schemes (preserving the condition $\dim X_s \leq 1$!) to arrange furthermore than $\mathcal{F} = \mathbf{Z}/(n)$ for some $n > 0$. Some cleverness with homological algebra on pp. 65-66 of Freitag-Kiehl shows that if in this general setting the base change map $\mathrm{H}^i(X, \mathbf{Z}/(n)) \to \mathrm{H}^i(X_s, \mathbf{Z}/(n))$ is bijective for $i = 0$ and surjective for *all* $i > 0$ then the general base change isomorphism property for such X and any n-torsion $\mathcal{F}$ and i follows. Thus, it remains to prove that if A is a strictly henselian local ring and X is a proper A-scheme satisfying $\dim X_s \leq 1$ then the natural map

$$\mathrm{H}^i(X, \mathbf{Z}/n\mathbf{Z}) \to \mathrm{H}^i(X_s, \mathbf{Z}/n\mathbf{Z})$$

is an isomorphism for $i = 0$ and surjective for $i \geq 1$. Since $k(s)$ is separably closed, our work with cohomology of proper curves over algebraically closed fields implies that $\mathrm{H}^i(X_s, \mathbf{Z}/(n))$ vanishes for $i > 2$ and also for $i = 2$ if n is a power of $\mathrm{char}(s)$. Thus, for $i > 1$ we can assume (by passage to primary parts) that n is a power of a prime $\ell \neq \mathrm{char}\,(s)$, so more generally that n is a unit on S and hence $\mathbf{Z}/(n)$ can be replaced with μ_n if we wish (as A is strictly henselian). By limit considerations, we may and do also assume A is noetherian, even the strict henselization at a point on a $\mathbf{Z}$-scheme of finite type. With further limit considerations we can relax "strict henselization" to "henselization", which will be useful later.

1.2 The case $i = 0$

Upon decomposing X into a finite disjoint union of its connected components, the content here is that if X is connected then X_s is connected. The connected components of X_s are in bijection with primitive idempotents in $\Gamma(X_s, \mathcal{O}_{X_s})$, and similarly for X. So it suffices to show that we can *lift idempotents* from the special fiber.

Let $X_m \to \mathrm{Spec}\ A/\mathfrak{m}^{m+1}$ be the infinitesimal special fibers. An idempotent in $\Gamma(X_s, \mathcal{O}_{X_s})$ *uniquely* lifts to $\Gamma(X_m, \mathcal{O}_{X_m})$ for all $m \geq 0$, since the topological spaces are the same. The Theorem on Formal Functions identifies $\Gamma(X_{\widehat{A}}, \mathcal{O}_{X_{\widehat{A}}}) = \Gamma(X, \mathcal{O}_X) \otimes_A \widehat{A}$ with the inverse limit of the rings $\Gamma(X_m, \mathcal{O}_{X_m})$, so it suffices to show that the idempotents in the finite A-algebra $\Gamma(X, \mathcal{O}_X)$ naturally match those in its faithfully flat extension $\Gamma(X, \mathcal{O}_X) \otimes_A \widehat{A}$. But one of the fundamental equivalent characterizations of the henselian property for local rings (such as A and $\widehat{A}$) in [EGA, IV_4, 18.5.11] is that if R is a henselian local ring and R' is a *module-finite* R-algebra then R' is a direct product of *local* rings (so the idempotents in R' and $R'/\mathfrak{m}_R R'$ match). In particular, if R is noetherian then idempotents in R' and $R' \otimes_R \widehat{R}$ match; taking $R = A$ and $R' = \Gamma(X, \mathcal{O}_X)$ then gives what we need.

1.3 The case $i = 1$

The group $\mathrm{H}^1(X_s, \mathbf{Z}/(n))$ parametrizes the set of isomorphism classes of finite étale $\mathbf{Z}/(n)$-torsors over X_s, and similarly for X. Therefore, the surjectivity of

$$\mathrm{H}^1(X, \mathbf{Z}/(n)) \twoheadrightarrow \mathrm{H}^1(X_s, \mathbf{Z}/(n))$$

comes down to the assertion that any finite étale $\mathbf{Z}/(n)$-torsor

$$Y_0 \xrightarrow{\mathbf{Z}/n\mathbf{Z}} X_s$$

can be lifted to a finite étale $\mathbf{Z}/(n)$-torsor over X.

[For $i = 2$, in which case we have seen that we may assume n is invertible on S and may work with μ_n-coefficients, one can proceed in a related manner with line bundles as follows. The cohomology group $\mathrm{H}^2(X_s, \mu_n)$ is identified with $\mathrm{H}^1(X_s, \mathbf{G}_m)/(n)$ due to the Kummer sequence provided that $\mathrm{Br}(X_s)[n] = 0$. The insensitivity of such n-torsion with respect to scalar extension to $\overline{k(s)}$ and general techniques with Brauer groups of curves reduces this vanishing to the well-known vanishing in the *smooth* case over an algebraically closed field. In this way, the case $i = 2$ reduces to proving the surjectivity of $\mathrm{Pic}(X) \to \mathrm{Pic}(X_s)$ that can be handled by a variant of the method below with Artin approximation since the infinitesimal deformation theory of line bundles on arbitrary proper curves over an artin local ring is unobstructed. An alternative method avoiding Artin approximation is given in [SGA4, Exp. XIII, §3] in the projective case, which suffices for us, by hands-on work with relative Cartier divisors.]

The "topological invariance" of the étale site of a scheme (i.e., the equivalence of categories via passage to underlying reduced schemes) shows that $Y_0 \to X_s$ uniquely lifts to a finite étale $\mathbf{Z}/(n)$-torsor over each infinitesimal special fiber of X over A and hence over the formal scheme $\widehat{X}$ given by completion of X along X_s. By formal GAGA from [EGA III_1, §5], the categories of coherent sheaves on the proper formal $\widehat{A}$-scheme $\widehat{X}$ and the proper $\widehat{A}$-scheme $X \otimes_A \widehat{A}$ are equivalent, and so likewise for coherent sheaves of algebras and hence for finite morphisms to the formal scheme $\widehat{X}$ and to the scheme $X \otimes_A \widehat{A}$. (In general, this algebraization step from proper formal schemes back to proper schemes is hard to "see" by hand. To appreciate the essential role of the properness hypothesis, observe that the m_A-adic completion of the coordinate ring of the non-proper $\mathbf{A}^1_A = \mathrm{Spec}\,(A[t])$ is *not* $\widehat{A}[t]$ but rather is the ring $\widehat{A}\{\{t\}\}$ of formal power series $\sum a_n t^n \in \widehat{A}[\![t]\!]$ such that $a_n \to 0$ for the max-adic topology of $\widehat{A}$.)

By properness of X and infinitesimal considerations along the special fiber, the "finite étale" property is inherited under algebraization (exercise!), so by using the proper formal scheme $\widehat{X}$ as an intermediate device as above we get an equivalence of categories between that of finite étale X_s-schemes and finite étale schemes over $X \otimes_A \widehat{A}$. The "cover" aspect passes through this equivalence in both directions since finite étale maps are open and closed (and X is A-proper), as does the data of being a $\mathbf{Z}/(n)$-torsor due to the equivalence condition. We have succeeded to lift Y_0 to a finite étale $\mathbf{Z}/(n)$-torsor $Y \to X \otimes_A \widehat{A}$.

There are two approaches to getting rid of the intervention of $\widehat{A}$:

1. Build a finite étale $\mathbf{Z}/(n)$-torsor of X with the same pullback over X_s as for Y (thereby recovering Y_0). This will rest on Artin approximation.

2. Follow a proof in [SGA4, Exp. XIII] that is more elementary and direct via smoothness at a certain point on a moduli scheme. This is tied up very much with the specific task of lifting finite étale covers (whereas arguments based on Artin approximation, which came after SGA4, are much more widely applicable and robust in practice).

2 Proof using Artin approximation

We first give the proof via the Artin approximation theorem. We next want to show that if A is any henselian local noetherian ring and X is a proper A-scheme, then for a given finite étale $\mathbf{Z}/(n)$-torsor $Y \to X \otimes_A \widehat{A}$ we can make a finite étale $\mathbf{Z}/(n)$-torsor $Y' \to X$ with the *same special fiber.* (It will be convenient below that we have relaxed "strictly henselian" to "henselian".) Artin approximation is a tool which is made for answering such "approximate from the completion" questions in very general settings.

2.1 Statement of the theorem

Definition 2.1. A Noetherian local ring B has the *approximation property* if, given some system of polynomials $P_i \in B[Y_1, \dots, Y_m]$ and a solution $(\widehat{b}_1, \dots, \widehat{b}_m) \in \widehat{B}^m$, and some $N \geq 1$, then one can find a solution $(b_1, \dots, b_n) \in B^m$ such that

$$b_j \equiv \widehat{b}_j \mod \mathfrak{m}^N.$$

Theorem 2.2 (Artin Approximation)**.** *The henselization of a finitely generated algebra over a field or excellent Dedekind domain* (*such as any Dedekind domain with generic characteristic 0; e.g.,* $\mathbf{Z}$) *has the approximation property.*

The "approximation" aspect is essential for applications, but is a red herring in the proof: given that we're allowing ourselves to consider all finite systems of polynomial equations, we can enforce any desired level of approximation by imposing some auxiliary equations encoding a congruence condition (e.g., if one imposes a new equation $x = y + bt + b't'$ we force $x \equiv y \bmod (b, b')$). Thus, it is equivalent to prove the seemingly weaker condition that for any such B, if a finite system of polynomial equations over B has a solution in $\widehat{B}$ then it has a solution in B.

Let's accept this deep result for now and see how to finish the argument. Later we'll make some comments about its proof. First, we give an (unrelated) application which demonstrates the power of this theorem.

Application. Let X, Y be schemes of finite type over a field k. Let $x \in X(k), y \in Y(k)$ such that

$$\widehat{\mathcal{O}}_{X,x} \simeq \widehat{\mathcal{O}}_{Y,y}.$$

as k-algebras. (The interesting case is when x and y are not smooth points.)

Proposition 2.3. *In this situation, there is a common residually trivial étale neighborhood of x and of y: there exists V and $z \in V(k)$ admitting étale maps $(V, z) \rightrightarrows (X, x), (Y, y)$.*

This is a really remarkable result. It tells us that the étale topology is fine enough to distinguish the formal singularity type of a rational point.

Proof. We have an evident local k-algebra map $\mathcal{O}_{X,x} \to \widehat{\mathcal{O}}_{Y,y}$. Writing a presentation for an affine open neighborhood of x in X in terms of generators and relations over k, this is saying that a certain sytem of equations over k has solutions in $\widehat{\mathcal{O}}_{Y,y}$ lifting a specified residual solution. We can approximate this by a solution in the henselization of $\mathcal{O}_{Y,y}$ by Artin approximation, say approximating to 2nd order for the max-adic topology. This henselization is a limit of residually trivial pointed étale neighborhoods, so we can spread out this solution to an actual residually trivial étale neighborhood (V, z) of (Y, y). Approximating an isomorphism between formal completions at least to order 2 is again an isomorphism (by successive approximation, since a surjective endomorphism of a noetherian ring is an *automorphism*), so we infer that the induced map $(V, z) \to (X, x)$ is étale as well. $\square$

To use Artin approximation, here is a useful general framework:

Definition 2.4. A (covariant) functor $\mathcal{F}$ from B-algebras to Sets is *locally of finite presentation* if

$$\varinjlim F(B_i) \to F(\varinjlim B_i)$$

is bijective for any directed system of B-algebras $\{B_i\}$.

Example 2.5. If $\mathcal{F} = \mathrm{Hom}_B(B[x_1, \ldots, x_r]/(f_1, \ldots, f_s), -)$ then $\mathcal{F}$ is locally of finite presentation. In fact, for representable functors it is equivalent to the usual notion of locally finitely presented for scheme morphisms (by [EGA IV$_4$, 8.14.1]).

Example 2.6. If $X \to B$ is a quasicompact map, then the functor

$$R \rightsquigarrow \{\text{finite étale covers of } X \otimes_B R\}/\simeq$$

is locally of finite presentation. The point is that a finite étale map is finitely presented and isomorphisms between them are described by finitely many equations, which can be spread out appropriately to a finite stage in any filtered direct limit of R's.

Corollary 2.7. *If B has the approximation property and $\mathcal{F}$ is a functor locally of finite presentation, then*

$$\mathcal{F}(B) \to \mathcal{F}(\widehat{B})$$

has dense image; i.e., for any $\xi' \in \mathcal{F}(\widehat{B})$ and $n > 0$ there exists $\xi \in \mathcal{F}(B)$ which coincides modulo the nth power of the maximal ideals of B and $\widehat{B}$.

Proof. Write $\widehat{B} = \varinjlim B_i$ where the B_i are the finitely generated B-subalgebras of $\widehat{B}$. Any element in $\mathcal{F}(\widehat{B})$ comes from some $\mathcal{F}(B_i)$, by the locally finite presentation assumption. This B_i has a finite presentation (recall that B is noetherian)

$$B_i = B[X_1, \ldots, X_r]/(f_1, \ldots, f_s).$$

Since B_i is inside $\widehat{B}$, the equations $f_j = 0$ admit a simultaneous solution in $\widehat{B}$. By Artin approximation, we can thereby find a homomorphism $h : B_i \to B$ approximating the inclusion $B_i \hookrightarrow \widehat{B}$ as well as we wish. Using $\mathcal{F}(h)$ does the job. $\square$

Combining Example 2.6 (adapted to torsors for a fixed finite group, such as $\mathbf{Z}/(n)$), Corollary 2.7, and a limit argument (to reduce general henselian local noetherian A to the henselization of a finitely generated $\mathbf{Z}$-algebra at some prime ideal) we find that for a finite étale $\mathbf{Z}/(n)$-torsor $Y \to X \otimes_A \widehat{A}$, there is a finite étale $\mathbf{Z}/(n)$-torsor over X which approximates it in the sense of inducing the same special fiber over X_s.

3 The SGA4 argument

We give an alternate argument to lift finite étale covers from X_s to X when X projective and flat over a henselian noetherian ring A. (There is an additional argument in [SGA4, Exp. XIII, §1] to remove the flatness assumption.) The same method can be applied to $\mathbf{Z}/(n)$-torsors (as we require) at the cost of extra notation.

The real power of Artin approximation is in application to "singular" systems of equations. However, the functor of finite étale covers that we applied it to turns out to be very close to being "smooth". More precisely, we will cook up a smooth moduli space which maps to the "space of finite étale covers" (and then exploit the fact that for a *smooth* scheme over a *henselian* local ring R, any rational point in the special fiber – such as arising from a point over $\widehat{R}$ – lifts to an R-point, by using that a smooth map factors Zariski-locally on the source as an étale map to an affine space).

Using that X is projective and A-flat, Grothedieck's work on Quot schemes yields:

Theorem 3.1. *For a fixed vector bundle V on X, the functor sending an A-scheme T to*

$$\{T\text{-flat quotients of } V|_{X\otimes_A T},\ \text{equipped with étale algebra structure}\}$$

is representable, by a scheme $\mathcal{QE}$ locally of finite type over A.

Why is this true? Granting Grothendieck's existence result on Quot schemes, to give an étale algebra structure is to write down some diagrams, which is structure that can be parametrized by coordinates. So it is not too surprising that this enhanced moduli problem is also representable.

Let $\pi : Y_0 \to X_s$ be a finite étale map. Fix a vector bundle V on X and a surjection

$$V|_{X_s} \twoheadrightarrow \pi_* \mathcal{O}_{Y_0}.$$

(Many V can be found since X is A-projective, using direct sums of copies of $\mathcal{O}(-r)$ for sufficiently big r.) Then we have a point of $\mathcal{QE}(A/\mathfrak{m})$. If $\mathcal{QE}$ were A-*smooth* at this point then we could lift it to $\mathcal{QE}(A)$, by the henselian property of A.

To show smoothness, one uses the infinitesimal criterion: the aspect concerning lifting finite étale covers through nilpotent thickenings is clear, and to lift the surjection from V we can succeed if V is anti-ample enough.

4 Comments on the proof of Artin approximation

An elegant complete discussion of the proof of Artin approximation is given in §3.6 of the book "Néron Models", building on some earlier results in Chapter 3 of that book. Artin proved an approximation property for the following two classes of henselian local noetherian rings:

- the henselization of a finitely generated algebra over a field or excellent Dedekind domain,
- $\mathbf{C}\{x_1, \ldots, x_n\}$ (ring of convergent power series near 0);

i.e., given finitely many $P_i \in A[Y_1, \ldots, Y_m]$ and a solution $(\widehat{y}_1, \ldots, \widehat{y}_m) \in \widehat{A}$, we can find a nearby solution $(a_1^*, \ldots, a_m^*) \in A$ (or equivalently merely *some* solution, since we're allowed to enlarge the system of equations in any fixed desired way). .

We'll discuss the proof of the second case, which involves a lot of the same steps as far as gritty formal power series manipulations are concerned. The first case has a crucial further ingredient called "Néron desingularization" that we'll come back to at the end.

Step 1. Reduce to the case where the numbers of equations and variables are the same, and the Jacobian

$$\Delta := \det\left(\frac{\partial P_i}{\partial y_j}\right)$$

satisfies $\Delta(\widehat{a}_1, \ldots, \widehat{a}_m) \neq 0$.

Step 2. Show: given $\vec{a} = (a_1, \ldots, a_m) \in A^m$ such that $P_i(\vec{a}) \in \Delta(\vec{a})^2\mathfrak{m}^N$, then there exists $\vec{a'} = (a_1', \ldots, a_m') \in A^m$ with $\vec{a} \equiv \vec{a}' \mod \mathfrak{m}^N$ and $P_i(\vec{a}') = 0$.

Proof. Set $\vec{a'} = \vec{a} + \Delta(\vec{a})\vec{u}$ for some $\vec{u} = (u_1, \ldots, u_m)$ with $u_i \in \mathfrak{m}^N$. Then

$$\vec{P}(\vec{a}') = \vec{P}(\vec{a}) + \Delta(\vec{a}) \cdot J\vec{u} + [\Delta^2 O(\vec{u}^2)]. \tag{4.1}$$

By assumption, $\vec{P}(\vec{a})$ is $\Delta(\vec{a})^2$ times something in $\mathfrak{m}^N$. Note that

$$\Delta \cdot \mathrm{Id}_m = J \cdot J^{\mathrm{ad}},$$

so you can "divide out" $\Delta \cdot J$ in (4.1). The point is then that reducing the right side modulo the maximal ideal just gives the reduction of $\vec{u}$, which we can solve by the Henselian property. □

Step 3. We show by induction on r (recall $A = \mathbf{C}\{x_1, \ldots, x_r\}$) that you can apply Step 2. We claim that given $g \in A[Y_1, \ldots, Y_m]$ and $\widehat{a}_1, \ldots, \widehat{a}_m \in \widehat{A}$ such that

$$g(\widehat{a}_1, \ldots, \widehat{a}_m) \mid P_i(\widehat{a}_1, \ldots, \widehat{a}_m) \text{ for all } i,$$

we can approximate $\widehat{a}_i$ by $a_i \in A$ with the same divisibility. To show this, use Weierstrass preparation on g to reduce the number of variables. This involves a kind of "division algorithm", so it is used to reduce divisibility to a collection of polynomial equations in dimension less by 1.

Remarks on Néron desingularization. We now make some remarks on the proof of Artin approximation for $(A, \widehat{A})$ where A is the henselization of a finitely generated algebra over an excellent discrete valuation ring R with fraction field K. (For discrete valuation rings, the definition of excellence amounts to the separability of $\widehat{K}/K$.) We want to relate solving equations in $R[T_1, \dots, T_n]^{\mathrm{h}}_{(T_1,\dots,T_n)}$ and $\widehat{R}[\![T_1, \dots, T_n]\!]$.

We will focus on the key case $n = 0$, since the case $n > 0$ is handled via induction on n (which rests on calculations with formal power series akin to what was done in the preceding context over $\mathbf{C}$). For $n = 0$, the key idea of Néron desingularization appears. We want to show that if $X(R)$ is non-empty for an R-scheme X of finite type (e.g., encoding a finite system of polynomial equations over R) then $X(\widehat{R})$ is non-empty. (We recall that we don't need to keep track of approximations, since the approximation can be encoded by including additional equations at the outset.)

Suppose that the map $X \to \mathrm{Spec}\ R$ can be dominated by a smooth $Y \to \mathrm{Spec}\ R$:

$$\begin{tikzcd} & Y \arrow[d, dotted, "?"] \\ \mathrm{Spec}\ \widehat{R} \arrow[ur, "\exists"] \arrow[r, "\widehat{a}"] \arrow[dr] & X \arrow[d] \\ & \mathrm{Spec}\ R \end{tikzcd}$$

Then $Y(R) \twoheadrightarrow Y(k)$ since Y is R-smooth and R is henselian, so we could find an R-point of Y lifting any rational point of the special fiber, such as arising from an $\widehat{R}$-point (!), and so would be done. Making such a Y is a weak version of resolution of singularities over R because we demand nothing about the R-map $Y \to X$. The technique of Néron desingularization will amount to a very weak version of such resolution around a single point of the special fiber.

Let $Z \subset X$ be the schematic closure of $\widehat{a}(\widehat{\eta})$, where $\widehat{\eta}$ is the generic point. Then Z_K is dominated by Spec $\widehat{K}$, with $\widehat{K}/K$ is separable. So Z_K is geometrically reduced over K and hence is generically smooth over K. Since $\widehat{R}[1/\pi] = \widehat{K}$ for a uniformizer π of R, we can then shrink Z around the image $\widehat{a}_0 \in Z(k)$ of the special point under $\widehat{a}$ so that Z_K is K-smooth, and rename Z as X; in this way we have arranged that X_K is smooth over K.

Now comes "Néron desingularization". For any scheme W of finite type over a discrete valuation ring A with fraction field F such that W_F is smooth, Néron defined the "defect of smoothness" $\delta(w)$ of W at a point $w \in W(A)$: it is the length of the torsion submodule of the finite A-module $w^*(\Omega^1_{W/A})$. This length obviously vanishes if W is A-smooth near w (as then $\Omega^1_{W/A}$ is a vector bundle), and Néron proved the converse by a clever short argument: if $\delta(w) = 0$ then W is A-smooth (in particular, A-flat!) at w. We now apply this to $X_{\widehat{R}}$ with its $\widehat{R}$-point $\widehat{a}$: since X is R-smooth at $\widehat{a}_0$ if and only if $X_{\widehat{R}}$ is $\widehat{R}$-smooth at $\widehat{a}_0$, the invariant $\delta(\widehat{a}) \geq 0$ detects if X is R-smooth or not at $\widehat{a}_0$. If such smoothness holds then it holds in a Zariski-open neighborhood of $\widehat{a}_0$ so we can shrink X to conclude. So now assume $\delta(\widehat{a}) > 0$.

Consider the affine open locus $X_\pi \subset \mathrm{Bl}_{\widehat{a}_0}(X)$ on which π generates the ideal of the exceptional divisor (i.e., the pullback of the maximal ideal of $\widehat{a}_0$). By the universal property of blow-ups there is a unique lift of $\widehat{a}$ to the blow-up at $\widehat{a}_0$, and this factors through X_π since π generates the maximal ideal of $\widehat{R}$. Now the miracle occurs: by a remarkable calculation in commutative algebra (which requires a few pages of work), Néron's measure of defect of smoothness at the lifted point on X_π is *strictly smaller* than at the point on X! So by iterating several times we reach the desired smooth R-scheme over the original X to which our $\widehat{R}$-point lifts, so as we have seen we can conclude the existence of the desired R-point due to the henselian condition on R (which is finally used at this step!).