THE SMOOTH BASE CHANGE THEOREM

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CONTENTS

1. Introduction

Our aim in these notes is to prove the smooth base change theorem. For the most part, we follow the exposition of [SGA4.5, Exp. V]. We also found [SGA4, Exp. XV], [Fu] and [Mi] to be useful references.

1.1. Statement of the smooth base change theorem.

Before stating the base change theorem, we recall how to define the "base change" morphism attached to a commutative square (as also came up in the formulation of the proper base change theorem):

Definition 1.1.1. For any commutative diagram of schemes (not necessarily Cartesian)

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd} \tag{1.1}$$

and an étale sheaf $\mathcal{F}$ on X, the *base change map*

$$g^* R^q f_* \mathcal{F} \to R^q f'_*(g'^* \mathcal{F}). \tag{1.2}$$

is the map of δ-functors from the erasable (hence universal) δ-functor $g^* R^\bullet f_*$ defined by the following procedure in degree 0. (Recall that pullback for abelian sheaves is exact, so $g^* R^\bullet f_*$ and $(R^\bullet f'_*) \circ g'^*$ are δ-functors.)

The desired natural map in degree 0,

$$g^* f_* \mathcal{F} \to f'_* \left(g'^* \mathcal{F}\right),$$

is (by adjointness) the "same" as a natural map

$$f_* \mathcal{F} \to g_* f'_* \left(g'^* \mathcal{F}\right)$$

to be defined. Since (1.1) commutes, we have $g_* f'_* = f_* g'_*$, and so it is equivalent to giving a natural map

$$f_* \mathcal{F} \to f_* g'_* \left(g'^* \mathcal{F}\right).$$

But we have a natural map $\mathcal{F} \to g'_* (g'^* \mathcal{F})$ and so composition with f_* does the job.

Remark 1.1.2. We now give a more concrete perspective on the base change map defined in Definition 1.1.1. By left-exactness of pushforward, it is easy to see that for any presheaf $\mathcal{G}$ on X with sheafification $\mathcal{G}^+$, we have naturally $(g_*\mathcal{G})^+ \to g_*(\mathcal{G}^+)$ via the universal property of sheafification. Ergo, the natural map $R^q f_* \to g_* R^q f'_* \circ g'^*$ adjoint to the base change morphism amounts to sheafification of the pullback maps

$$H^q(f^{-1}(U), \mathcal{F}) \to H^q(f'^{-1}(U'), \mathcal{F}')$$

for étale $U \to S$, $U' := g^{-1}(U)$, and $\mathcal{F}' := g'^*\mathcal{F}$.

Exercise 1.1.3. Check that the concrete procedure of Remark 1.1.2 produces a map of δ-functors which coincides with the more abstract construction of Definition 1.1.1. *Hint:* Verify that the two maps agree in degree 0.

It is useful to have both perspectives on this important construction, and this entire procedure is not specific to the étale topology.

Warning 1.1.4. Note that if one wants to work with sheaves of modules (for a ringed space or ringed topos) rather than with abelian sheaves then the pullback operation is *no longer exact* and so one has to proceed a bit differently than in Definition 1.1.1: first make the pullback on abelian sheaves, and then insert tensor products for *that* map against suitable sheaves of rings. For details in the context of ringed spaces (by completely general methods) see [EGA, 0_{III}, 12.1].

We can now state our main theorem:

Theorem 1.1.5 (Smooth base change)**.** *Consider a Cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array} \tag{1.3}$$

where $S = \varprojlim S_\lambda$ *with smooth* $g_\lambda : S_\lambda \to S$ *and affine transition maps* $S_\lambda \to S_{\lambda'}$. *Suppose further that* f *is quasi-compact and quasi-separated. Then, for any torsion abelian étale sheaf* $\mathcal{F}$ *on* X *with torsion orders invertible on* S*, the natural map* (1.2) *is an isomorphism.*

In order to handle various limit procedures that come up in the arguments below, the general formulation we have given is essential. Nonetheless, the first task in the proof is to show that such generality can be deduced from an affirmative result in situations that are geometrically more comforting (e.g., noetherian S, smooth g, and f of finite type.

Before discussing the proof of this theorem, we'll first discuss a topological analogue and then record a useful consequence for invariance under change of separably closed ground fields, as well as illustrate the necessity of the hypotheses on smoothness and torsion-orders. Note, the torsion order hypothesis was not needed for the proper base change theorem. We'll then outline the idea of the proof in § 1.5, and the remainder of these notes take up the full proof in detail.

1.2. **Topological smooth base change.** The core of the proof of the smooth base change theorem involves constant constructible $\mathcal{F}$. In the case of constant sheaves, the smooth base change theorem is an analogue of homotopy-invariance of singular cohomology. To explain this, we now prove a topological analogue:

Theorem 1.2.1. *Let $\mathcal{F}$ be an abelian sheaf on an arbitrary topological space X, and consider a topologically Cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd} \tag{1.4}$$

with g a "smooth" map (i.e., for each $s' \in S'$, there is an open neighborhood U of $g(s')$ in S such that s' admits an open neighborhood that is U-isomorphic to $U \times B$ for $B = (0,1)^n$). Then, the natural map (1.2) *is an isomorphism for any locally constant sheaf $\mathcal{F}$ on X.*

Proof. The statement is local on S and local on S', so we may assume that $S' = S \times B$. Further, since higher direct image sheaves are sheafifications of cohomology, we only need prove the pullback map on cohomology

$$H^i(X, \mathcal{F}) \to H^i(X \times B, g'^* \mathcal{F})$$

is an isomorphism.

The Čech to derived-functor spectral sequence makes this assertion local on X, so we can assume that $\mathcal{F}$ is the constant sheaf $\underline{A}$ associated to an abelian group A. Given any point $b \in B$, we have a map $X \times B \to X \times \{b\} \simeq X$ which induces a map

$$H^i(X \times B, g'^* \mathcal{F}) \to H^i(X, \mathcal{F}).$$

The map $X \times B \to X \times \{b\}$ is a deformation retract, and so admits an inverse up to homotopy.

Now it suffices to show that if $f_0, f_1 : Y' \rightrightarrows Y$ are homotopic maps between topological spaces then the induced maps $H^i(Y, \underline{A}) \rightrightarrows H^i(Y', \underline{A})$ coincide (for any abelian group A; recall that the pullback of a constant sheaf under any continuous map is canonically identified with the constant sheaf on the source associated to the same abelian group).

By definition, we have a continuous map $F : [0,1] \times Y' \to Y$ satisfying $F|_{\{0\}\times Y'} = f_0$ and $F|_{\{1\}\times Y} = f_1$. Letting $j_t : Y' \to [0,1] \times Y'$ be the inclusion $y' \mapsto (t, y')$, so $f_0 = F \circ j_0$ and $f_1 = F \circ j_1$, it suffices to show that for all i the maps $j_t^* : H^i([0,1] \times Y', \underline{A}) \to H^i(Y', \underline{A})$ coincide for all $t \in [0,1]$. Letting $\pi : [0,1] \times Y' \to Y'$ be the projection, so $\pi \circ j_t = \mathrm{id}_{Y'}$ for all y, it suffices to show that $\pi^* : H^i(Y', \underline{A}) \to H^i([0,1] \times Y', \underline{A})$ is an isomorphism (so then j_t-pullback in cohomology coincides with the inverse of π-pullback for all t).

Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(Y', R^q\pi_*(\underline{A})) \Rightarrow H^{p+q}([0,1] \times Y', \underline{A}).$$

Since π is topologically proper, by the topological proper base change theorem we see that the natural map

$$R^q\pi_*(\underline{A})_{y'} \to H^q([0,1], \underline{A})$$

is an isomorphism for all q and all $y' \in Y'$. But, by the comparison of derived-functor cohomology of constant sheaves and singular cohomology for locally contractible paracompact Hausdorff spaces (such as $[0,1]$!), we have $H^q([0,1], \underline{A}) = H^q([0,1], A) = 0$ for $q > 0$ and $H^0([0,1], \underline{A}) = A$. Ergo, the natural map $\underline{A} \to \pi_*(\underline{A})$ is an isomorphism and $R^q\pi_*(\underline{A}) = 0$ for $q > 0$. This implies that the Leray spectral sequence degenerates, with each edge map

$$\varphi_{p,A} : H^p(Y', \pi_*(\underline{A})) \to H^p([0,1] \times Y', \underline{A})$$

an isomorphism. We need to justify that this isomorphism has something to do with the π-pullback map in degree p that we want to prove is an isomorphism.

For any continuous map $f : Z' \to Z$ between topological spaces and an abelian sheaf $\mathcal{F}$ on Z', the edge map

$$E_2^{p,0} = H^p(Z, f_*\mathcal{F}) \to H^p(Z', \mathcal{F})$$

in the associated Leray spectral sequence is exactly the composite map

$$H^p(Z, f_*\mathcal{F}) \to H^p(Z', f^*f_*\mathcal{F}) \to H^p(Z', \mathcal{F}).$$

Exercise 1.2.2. Verify the above claim, making concrete meaning of certain edge maps in Grothendieck-Leray spectral sequences.

In the special case that $\mathcal{F} = \underline{A}$ on Z', there is an evident map $\underline{A} \to f_*\underline{A}$, and it is easy to check on stalks (or by abstract nonsense with adjunctions) that composing the f-pullback of this map with the natural map $f^*f_*\underline{A} \to \underline{A}$ is the *identity map* on the sheaf $\underline{A}$ on Z'. Consequently, the composition of the edge map $\varphi_{p,A}$ with $H^p(Y', \cdot)$ applied to the isomorphism $\underline{A} \simeq \pi_*\underline{A}$ on Y' really does *coincide* with $\pi^* : H^p(Y', \underline{A}) \to H^p([0,1] \times Y', \underline{A})$. This proves that these latter pullback maps are isomorphisms for all p, as desired. $\square$

Remark 1.2.3. The proof of Theorem 1.2.1 is somewhat "global", but here is an idea that can be adapted to construct a proof that is more local on X. This local method is the underlying inspiration for the proof we will give in the algebraic case.

As we will see in § 2.3, one can reduce to the case that $X \to S$ is an open immersion, and so we will now concentrate on the special case that $S = [0,1]$ and X is the half open interval $\eta := (0,1]$. We will rename S' as Y so that $g : Y \to [0,1]$ is smooth. Let $Y_0 := g^{-1}(0)$. Consider the diagram

$$\begin{tikzcd} Y_\eta \arrow[r, "g'"] \arrow[d, "f'"] & (0,1] \arrow[d, "f"] \\ Y \arrow[r, "g"] & [0,1] \\ Y_0 \arrow[r] \arrow[u] & \{0\} \arrow[u] \end{tikzcd} \tag{1.5}$$

We now focus on the case that the sheaf $\mathcal{F}$ on X is the constant sheaf $\underline{A}$ associated to an abelian group A. In this case, we claim that $R^q f_*\underline{A} = 0$ for $q > 0$ and that the natural map $\underline{A} \to f_*\underline{A}$ is an isomorphism. Ergo, to verify the smooth base change theorem in this case we would only need to verify that $\underline{A} \to f'_*\underline{A}$ is an isomorphism and that $R^q f'_*\underline{A} = 0$ for $q > 0$. The main point is that these assertions concern vanishing and isomorphism properties for sheaves in a form that is of very localizable nature, and these conditions essentially amount to acyclicity, or local acyclicity, as discussed later in the algebraic setting in § 2.2. Note that over the open subset $Y_\eta \subset Y$

there is nothing to be done, so the content is entirely for the stalks at points $y \in Y_0$.

First, we verify that condition that $\underline{A} \to f'_*\underline{A}$ is an isomorphism. For $y \in Y_0$, we only need to check that for a cofinal system of connected open neighborhood U of y in Y, the open subset $U - U_0$ is connected. But this problem is local on Y, so we can assume by the smoothness of g that $Y = B \times [0, t)$ for some $0 < t \leq 1$ and open ball B with origin at y. We can take U to vary through the neighborhoods $B' \times [0, \varepsilon)$ for $0 < \varepsilon \leq t$ and open balls $B' \subset B$ centered at y. Then $U - U_0 = B' \times (0, \varepsilon)$ is connected.

Next, we examine the case $q > 0$. For $y \in Y_0$, the stalk $(R^q f'_*\underline{A})_y$ is the limit of $H^q(U - U_0, \underline{A})$ as U ranges over a cofinal system of open neighborhoods of y in Y. But taking the cofinal system to be as in our treatment of the case $q = 0$, we see that $U - U_0 = B' \times (0, \varepsilon)$ for open balls B' in Euclidean spaces and $\varepsilon > 0$, so $H^q(U - U_0, \underline{A}) = H^q(U - U_0, A) = 0$.

Remark 1.2.4 (Aside on cospecialization)**.** As one further note, retaining the notation of Remark 1.2.3, we observe that one can use the discussion given in Remark 1.2.3 together with the Leray spectral sequence to show $H^\bullet(Y, \underline{A}) \simeq H^\bullet(Y_\eta, \underline{A})$ via restriction. This leads to the definition of the cospecialization map (see [SGA4.5, Exp. V, (1.6)] for an algebraic definition and [SGA4.5, Exp. V, pp. 52-53] for an analytic discussion) from the cohomology of the general fiber to the cohomology of the special fiber. This is an isomorphism in good situations (for example, see [SGA4.5, Exp. V, Thm. 1.7] and [SGA4.5, Exp. V, Thm. 3.1]).

1.3. **A useful case of smooth base change.** Here, we record a useful corollary of smooth base change, saying that cohomology groups of constructible sheaves with prime-to-characteristic torsion are invariant under arbitrary extension of separably closed ground fields. This comes in handy particularly in the case of $\overline{\mathbf{Q}} \to \mathbf{C}$.

Corollary 1.3.1. *Suppose* K/k *is an extension of separably closed fields,* X *is a qcqs* k*-scheme, and* $\mathcal{F}$ *is a torsion sheaf with torsion orders not divisible by the characteristic to* k*. Then, the base change map*

$$H^q(X, \mathcal{F}) \to H^q(X_K, \mathcal{F}_K)$$

is an isomorphism for $q \geq 0$*.*

Proof. Observe that by naturality of the base change map, we have a commutative diagram

$$\begin{tikzcd} H^q(X, \mathcal{F}) \arrow[r] \arrow[d] & H^q(X_K, \mathcal{F}_K) \arrow[d] \\ H^q(X_{\overline{k}}, \mathcal{F}_{\overline{k}}) \arrow[r] & H^q(X_{\overline{K}}, \mathcal{F}_{\overline{K}}). \end{tikzcd} \tag{1.6}$$

The vertical maps are isomorphisms due to the fact that radiciel integral surjections of schemes induce an equivalence of étale sites under pullback (with pushforward giving an inverse functor at the level of categories of étale sheaves of sets).

To show the top horizontal map is an isomorphism, we only need check the lower horizontal map is an isomorphism. But the extension $\overline{K}/\overline{k}$ can be written as a limit of smooth $\overline{k}$ algebras, since every finitely generated extension of an algebraically closed field admits a separating transcendence basis and so is the function field of a smooth algebra (so the result follows from Theorem 1.1.5 applied to $S' = \operatorname{Spec} \overline{K}$ and $S = \operatorname{Spec} \overline{k}$). □

1.4. **Necessity of hypotheses.** Now we describe the necessity of the hypotheses that the torsion orders of $\mathcal{F}$ be invertible on S and that g be smooth (or at least a suitable limit of smooth maps), though the latter hardly needs any motivation in view of the preceding discussion of topological analogues. We start with an example showing how cohomology may not be invariant under base change in positive characteristic.

Example 1.4.1. Let k be an algebraically closed field of characteristic $p > 0$. We will show that constant-coefficient p-torsion cohomology can fail to be invariant under change of algebraically closed fields in characteristic p (away from the proper case, of course!). Let K/k be a non-trivial field extension (the case of most interest being K also algebraically closed); this is a limit of smooth k-algebras since k is perfect. We claim the map

$$H^1(\mathbf{A}^1_k, \mathbf{Z}/(p)) \to H^1(\mathbf{A}^1_K, \mathbf{Z}/(p))$$

is not an isomorphism.

Recall the Artin-Schreier sequence

$$0 \longrightarrow \mathbf{Z}/(p) \longrightarrow \mathbf{G}_a \xrightarrow{\wp} \mathbf{G}_a \longrightarrow 0 \tag{1.7}$$

with $\wp : t \mapsto t^p - t$. This yields an exact sequence on cohomology (1.8)

$$H^0(\mathbf{A}^1, \mathbf{G}_a) \xrightarrow{\wp} H^0(\mathbf{A}^1, \mathbf{G}_a) \longrightarrow H^1(\mathbf{A}^1, \mathbf{Z}/(p)) \longrightarrow 0$$

since higher étale cohomology of quasi-coherent sheaves (such as for $\mathbf{G}_a$) coincides with the Zariski cohomology and so vanishes for affine schemes (i.e., $H^1(\mathbf{A}^1, \mathbf{G}_a) = 0$). Ergo, $H^1(\mathbf{A}^1, \mathbf{Z}/(p))$ is identified with the cokernel of $\wp : k[x] \xrightarrow{x^p - x} k[x]$, so the map on degree-1 cohomology is identified with the evident map

$$k\,[x] \,/\, \{f^p - f : f \in k\,[x]\} \to K\,[x] \,/\, \{f^p - f : f \in K[x]\}\,.$$

Exercise 1.4.2. Verify this is not an isomorphism when $k \neq K$. *Hint:* Show that the term ax^{p-1} for $a \in K \setminus k$ does not lie in the image by chasing degrees.

Next we give an example showing that the map g must be smooth (or at least a limit of smooth maps).

Example 1.4.3. Let $S = \operatorname{Spec} A$ be the strict henselization A of the discrete valuation ring $k[t]_{(t)}$. Let X be the complement of $x = y = t = 0$ in $\operatorname{Spec} A[x, y]/(xy - t)$. Let $S' \to S$ be the inclusion of the closed point. (This is not flat, so it is certainly not a limit of smooth ring maps.) For any integer $n > 1$, we claim the sheaf $\mathbf{Z}/(n)$ does not satisfy the isomorphism (1.2) for the fiber square

(1.9)
$$\begin{array}{ccc} X \times_S S' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

Indeed, the special fiber of X over S is disconnected whereas the total space X is connected. More precisely, let $s' \in S'$ be the unique point of s'; it maps to the closed point $s \in S$. Using this, we see that the stalk of $g^* f_*(\mathbf{Z}/(n))$ at s' is $H^0(X, \mathbf{Z}/(n)) \simeq \mathbf{Z}/(n)$ since X is connected. However,

$$(f'_* g'^*(\mathbf{Z}/(n)))_{s'} = H^0(X \times_S S', \mathbf{Z}/(n)) \simeq \mathbf{Z}/(n) \oplus \mathbf{Z}/(n)$$

since the special fiber is disconnected.

1.5. **The idea of the proof: "Where's the beef?"** Before continuing with the proof of the smooth base change theorem, let us briefly outline the main idea of the proof. First, we make numerous standard reductions, to pass to the nice situation that the map $X \to S$ is a finite type map of affine noetherian schemes. Such maps factor as the composition of an open immersion and a proper map. Proper base change lets us verify the result for the proper map, so we only need check it for open immersions, using the composition of functors spectral sequence.

To check the result for open immersions, we will define a notion of "locally acyclic" morphism, and check that the result holds for $X \to S$ an open immersion an $S' \to S$ locally acyclic. Once we define locally acyclic morphisms, we can reduce to the case that $\mathcal{F}$ is the pushforward of a constant sheaf from a geometric point of X. To check this special case, we can further reduce to when A is normal but highly non-noetherian with the unusual property that its algebraic local rings at all primes are strictly henselian; in this setting we'll be able to conclude via a concrete calculation with stalks.

So, the only remaining (but also most substantial!) step is to verify that smooth morphisms are locally acyclic. To do so, we use the Zariski-local structure of smooth morphisms and considerations with spectral sequences to reduce to showing that $\mathbf{A}^1_S \to S$ is locally acyclic with $S = \operatorname{Spec} A$ for strictly henselian local normal noetherian domains A. This in turn will further be reduced to the case that the affine S-line is replaced with a more localized object: the strict henselization $\operatorname{Spec} A\{T\}$ of $A[T]$ at the origin $(\mathfrak{m}_A, T)$ in the special fiber.

Due our earlier work computing the cohomology of affine curves with coefficients in $\mathbf{Z}/(n)$ for n not divisible by the characteristic (such as the vanishing beyond degree 1), it will turn out that the key steps are to check an isomorphism result when $q = 0$ and $q = 1$ (with $\mathcal{F} = \mathbf{Z}/(n)$). The deep part is the case $q = 1$, which comes down to a study of finite étale $\mathbf{Z}/(n)$-torsors. The normality conditions permit us to focus attention in codimension 1, where our schemes are regular. Abhyankar's Lemma will ensure (with help from the Zariski–Nagata theorem on purity of the branch locus) that certain calculations at codimension-1 points only involve *tame* ramification (because of the divisibility condition imposed on n).

2. Proof of the smooth base change theorem

In this section, we prove the smooth base change theorem taking for granted that smooth morphisms satisfy a certain property called "local acyclicity" (to be defined in Definition 2.2.4), a fact proven in § 3.

2.1. Reduction to the finite type setting.

We now state a weaker version of smooth base change to which we reduce Theorem 1.1.5.

Proposition 2.1.1. *Suppose we have a Cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array} \tag{2.1}$$

where S' and S and X are affine and noetherian, g is smooth, and $X \to S$ is finite type. Then, the natural map (1.2) *is an isomorphism.*

Let us now explain why Proposition 2.1.1 implies Theorem 1.1.5. We will prove Proposition 2.1.1 later, in § 2.3.

Proof of Theorem 1.1.5, assuming Proposition 2.1.1. We reduce to the situation of Proposition 2.1.1, via the following sequence of reductions:

(1) We may first reduce to the case that S is strictly henselian as follows. It suffices to show (1.2) is an isomorphism over each stalk, and since f (and therefore f') is quasi-compact and quasi-separated we have for any geometric point s of S that

$$\begin{aligned}(R^p f_* \mathcal{F})_s &= \varinjlim_{\substack{\nu: U \to S, \\ s \in \nu(U)}} H^i(X \times_S U, \mathcal{F}) \\ &= H^i(\varinjlim_{\substack{\nu: U \to S, \\ s \in \nu(U)}} X \times_S U, \mathcal{F}) \\ &= H^i(X_{\mathcal{O}_s^{sh}}, \mathcal{F}),\end{aligned}$$

and a similar statement for higher direct images along f' at a geometric point $s' \in S'$ over $s \in S$. This reduces us to the case when S and S' are strictly henselian local and we are studying the cohomology groups on the total spaces X and X'. That is, our task is to show that $H^i(X, \mathcal{F}) \to H^i(X', g'^* \mathcal{F})$ is an isomorphism in this setting.

(2) By the compatibility of cohomology with direct limits (for qcqs schemes), it suffices to treat the case when S' is the strict henselization at a geometric point on the closed fiber of an affine scheme T that is smooth over S (since the coordinate ring for S' at the end of the preceding step is a direct limit of such strict henselizations).

(3) We may assume that $X \to S$ is separated. Indeed, granting that case, consider a finite cover $\mathcal{U}$ of X by affine open subschemes. Since S is affine, by Lemma A.3.1, every finite product over X of members of our cover is separated and quasi-compact. So, we may assume we know the statement for each finite overlap of our cover.

Define $\mathcal{F}' := \mathcal{F}_{X'}$, $\mathcal{U}'$ to be the base change of $\mathcal{U}$ in X', and

$$U_{\{i_1,\ldots,i_n\}} := U_{i_1} \times_X \cdots \times_X U_{i_n}.$$

Further define

$$H^q(\mathcal{U}, H^p(\mathcal{F})) := \prod_{\substack{I \subset J \\ |I| = q}} H^p(U_I, \mathcal{F}).$$

We have two Čech to derived spectral sequences

$$\begin{aligned} H^q(\mathcal{U}, H^p(\mathcal{F})) &\implies H^{p+q}(X, \mathcal{F}) \\ H^q(\mathcal{U}', H^p(\mathcal{F}')) &\implies H^{p+q}(X', \mathcal{F}') \end{aligned}$$

Defining $U_I' := U_I \times_S S'$, by our assumption for the separated case all maps

$$H^p(U_I, \mathcal{F}) \to H^p(U_I', g'^*\mathcal{F})$$

are isomorphisms. This implies that the maps

$$H^q(\mathcal{U}, H^p(\mathcal{F})) \to H^q(\mathcal{U}', H^p(\mathcal{F}'))$$

are also isomorphisms. Because spectral sequences are functorial, the maps

$$H^n(X, \mathcal{F}) \to H^n(X', \mathcal{F}')$$

are isomorphisms as desired.

(4) We may reduce to the case that X is affine. Indeed, this argument proceeds in exactly the same way as the previous step, once we note that a fiber product of affines over a *separated* scheme (such as X now) is affine, by Lemma A.3.2.

(5) We may assume that $X \to S$ is finitely presented, and, in particular, finite type. Indeed, the coordinate ring of the affine X as the limit over its finite type subalgebras over the coordinate ring of S, so X is the inverse limit of affine schemes X_λ of finite type over S. We saw in the lecture on the proper base change theorem that $\mathcal{F}$ is a direct limit of sheaves pulled back from the X_λ's, so by compatibility of cohomology with limits (first in the sheaves, and then in the schemes) we can assume X is finite type over S. But now by affineness we can use the same method to reduce to the finitely presented case (since for any ring A any quotient $A[t_1, \ldots, t_n]/J$ is the direct limit of quotients by finitely generated ideals contained in J).

(6) We can assume S is noetherian (even a strict henselization of a finite-type $\mathbf{Z}$-algebra). Indeed, since $X \to S =: \operatorname{Spec} A$ is finitely presented and A is the direct limit of strict henselizations of finitely generated $\mathbf{Z}$-subalgebras, we can find such a strict henselization A_0 and finite type A_0-scheme X_0 such that $X = X_0 \otimes_{A_0} A$. Now arguing as in the preceding step by expressing A is the direct limit of strict henselizations of finitely generated A_0-subalgebras allows us to do the job (by expressing $\mathcal{F}$ as a suitable compatible limit, etc.).

$\square$

So, to complete our proof of the smooth base change theorem, we only need verify the weaker statement in Proposition 2.1.1, even with $S = \operatorname{Spec} A$ with A a strict henselization of a finite type $\mathbf{Z}$-algebra (a refinement on the noetherian condition that will be technically important for excellence reasons when we need to ensure module-finiteness of certain normalizations later on).

2.2. **Locally acyclic morphisms.** In order to prove Proposition 2.1.1, we will reduce to the case that f is an open immersion in § 2.2, but to make this reduction we need to introduce the notion of a locally acyclic morphism. That smooth morphisms are locally acyclic will be proved in § 3.

Definition 2.2.1. A morphism of schemes $f : Y \to X$ is *weakly acyclic* if for all n invertible on Y the natural map $\mathbf{Z}/(n) \to f_*(\mathbf{Z}/(n))$ is an isomorphism and $R^q f_*(\mathbf{Z}/(n)) = 0$ for $q > 0$.

A morphism of schemes $f : Y \to X$ is *acyclic* if for all quasi-finite morphisms $g : X' \to X$ of finite type, the induced map $X' \times_X Y \to X'$ is weakly acyclic.

Warning 2.2.2. The notion of weakly acyclic morphisms is nonstandard, and we purely use it as a notational convenience. On the other hand, we shall not need the notion of acyclic morphisms, other than in Appendix B.

Remark 2.2.3. To define the concept of "locally acyclic" we will need to work with geometric points. To avoid getting caught up in the issue of how étale cohomology interacts with an extension between algebraically closed ground fields in these notes we shall use geometric point in the following sense: A *geometric point* of a scheme X is a map $\iota : t \to X$ with $t = \operatorname{Spec} k$ for k a separable closure of the residue field $\kappa(\iota(t))$. In the case we wish to refer to a geometric point for which the extension is not algebraic, we refer to it as a *nonalgebraic geometric point*. We will only need to do so in Appendix B.

Definition 2.2.4. For a map of schemes $f : Y \to S$, fix geometric points $\overline{s}$ of S and $\overline{y}$ of Y such that $f(\overline{y}) = \overline{s}$ (i.e., we specify an embedding of $k(\overline{s})$ into $k(\overline{y})$ over the inclusion $k(s) \hookrightarrow k(y)$ of residue fields at the actual points of S and Y at which these geometric points are supported.

For a geometric point η of $\operatorname{Spec}(\mathcal{O}^{sh}_{\overline{s}})$ we define the *vanishing cycles scheme* associated to $(f, \eta, \overline{s}, \overline{y})$ to be

$$\widetilde{Y}^{\overline{y}}_{\eta} := \operatorname{Spec} \mathcal{O}^{sh}_{\overline{y}} \times_{\operatorname{Spec} \mathcal{O}^{sh}_{\overline{s}}} \eta.$$

For $\mathcal{F}$ a sheaf on Y, a map $f : Y \to S$ is *locally acyclic* if for all $n \geq 1$ invertible on S and every vanishing cycle scheme $\widetilde{Y}^{\overline{y}}_{\eta}$ the natural map $\mathbf{Z}/(n) \to H^0(\widetilde{Y}^{\overline{y}}_{\eta}, \mathbf{Z}/(n))$ is an isomorphism and $H^q(\widetilde{Y}^{\overline{y}}_{\eta}, \mathbf{Z}/(n)) = 0$ for $q > 0$.

We now state some equivalent conditions for being a locally acyclic morphism. We give a proof, contenting ourselves with references for some parts, in Appendix B.

Proposition 2.2.5. *Let $g : Y \to X$ be a qcqs morphism of schemes. The following are equivalent.*

1. *The morphism g is locally acyclic.*
2. *For all geometric points y of Y and x of X with $y \mapsto x$, the induced map $\operatorname{Spec} \mathcal{O}^{sh}_{y} \to \operatorname{Spec} \mathcal{O}^{sh}_{x}$ is acyclic.*

(3) For all Cartesian diagrams

$$
\begin{array}{ccc}
Y'' & \xrightarrow{g''} & X'' \\
\downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\
Y' & \xrightarrow{g'} & X' \\
\downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\
Y & \xrightarrow{g} & X
\end{array}
\tag{2.2}
$$

with j *étale of finite type,* h *quasi-finite, and* $\mathcal{F}$ *a torsion sheaf on* X'' *with torsion orders invertible on* X*, the base change morphism*

$$g'^{*}R^{q}h_{*}(\mathcal{F}) \to R^{q}h'_{*}(g''^{*}\mathcal{F})$$

is an isomorphism for all q.

Remark 2.2.6. In particular, condition (2) below explains the reason for the name "locally acyclic:" locally acyclic morphisms are acyclic upon passing to stalks. We will not use (2) otherwise, but we will use the equivalence of (1) and (3) to show a composition of qcqs locally acyclic morphisms is locally acyclic in Proposition 3.1.2.

2.3. **Reduction to the open immersion case.** The special case when $X \to S$ is a (quasi-compact) open immersion and g locally acyclic amounts to:

Proposition 2.3.1. *Consider a Cartesian square*

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
S' & \xrightarrow{g} & S
\end{array}
\tag{2.3}
$$

with g *locally acyclic and* $X \to S$ *an open immersion, with* S *and* S' *noetherian. For any torsion abelian étale sheaf* $\mathcal{F}$ *on* X *with invertible torsion orders on* S*, the natural map* (1.2) *is an isomorphism.*

We will prove Proposition 2.3.1 in § 2.7. Assuming this and Theorem 3.3.1 (which says that smooth maps are locally acyclic), we now complete the proof of Proposition 2.1.1, which was all we needed to complete the proof of the smooth base change theorem.

The idea is to factor the morphism as the composition of an open immersion followed by a proper map (such as a projective space), use proper base change to handle with the proper map, and handle the open immersion via Proposition 2.3.1.

We now prove Proposition 2.1.1 assuming Proposition 2.3.1 and Theorem 3.3.1.

Proof. Since $X \to S$ is a finite type map between affine schemes, we can choose a locally closed immersion of X into projective space over S, and factor f through its projective closure $\overline{X}$. Ergo, we have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & \overline{X} \arrow[dl, "\overline{f}"] \\ & S & \end{tikzcd} \tag{2.4}$$

with j open and $\overline{f}$ proper. Consider the resulting diagram of Cartesian squares

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "j'"] & X \arrow[d, "j"] \\ \overline{X}' \arrow[r, "\overline{g}'"] \arrow[d, "\overline{f}'"] & \overline{X} \arrow[d, "\overline{f}"] \\ S' \arrow[r, "g"] & S. \end{tikzcd} \tag{2.5}$$

By the proper base change theorem, the result holds for $\overline{f}$. By Proposition 2.3.1 and Theorem 3.3.1, the result also holds for j. The composition of functors spectral sequence

$$R^q\overline{f}_*(R^p j_* \mathcal{F}) \Rightarrow R^{p+q}(\overline{f} \circ j)_*(\mathcal{F})$$

is compatible with base change morphisms (as we see from the construction of the spectral sequence), so it is base-change compatible with the analogous spectral sequence for the factorization $f' = \overline{f}' \circ j'$. Ergo, the isomorphism property for the base change morphisms for each of the two Cartesian squares yields the result for the composite maps. (Note that it is essential we permit extreme generality in the torsion sheaf on $\overline{X}$ since we have no real control yet on properties of the higher direct images $R^p j_*(\mathcal{F})$ even if $\mathcal{F}$ is constructible.) $\square$

To complete the proof of the smooth base change theorem, it remains to prove Proposition 2.3.1 and Theorem 3.3.1.

2.4. **Exactness of finite pushforwards.** Before continuing, we will need a useful criterion that describes the stalks of the pushforward of a sheaf. We do this in Lemma 2.4.2, but in order to do so, we first recall how the formation of strict henselization interacts with finite extension of scalars:

Lemma 2.4.1. *For any finite map* $f : X \to Y$ *and geometric point* y *of* Y, *the diagram*

$$\begin{tikzcd} \coprod_{x \in X_y} \operatorname{Spec} \mathcal{O}_x^{sh} \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} \mathcal{O}_y^{sh} \arrow[r] & Y \end{tikzcd} \tag{2.6}$$

is Cartesian, where x *varies through geometric points of the fiber* X_y.

Proof. This is [EGA, IV_4, 18.8.10]. □

Using Lemma 2.4.1, we can now prove that pushforwards along finite maps are exact, and further describe their stalks.

Lemma 2.4.2. *For a finite map* $f : X \to Y$, *étale sheaf of sets* $\mathcal{F}$ *on* X, *and geometric point* $y \in Y$, *the natural map of sets*

$$\prod_{x \in f^{-1}(y)} \mathcal{F}_x \simeq (f_* \mathcal{F})_y$$

is an isomorphism. In particular, f_* *is exact as a functor from the category of abelian étale sheaves on* X *to the category of abelian étale sheaves on* Y.

Warning 2.4.3. Lemma 2.4.2 does not hold in the Zariski topology! The reason is that a module-finite algebra over a general local ring is semi-local but not necessarily a direct product of local algebras. More specifically, if $f : C' \to C$ is a degree-2 generically étale cover between smooth affine curves over an algebraically closed field then for the Zariski topology $f_*(\mathbf{Z}/(n))$ has stalk $\mathbf{Z}/(n)$ at every point of C since C' is irreducible whereas a typical fiber of f consists of a disjoint union of two points.

However, for any geometric point $c \in C$ over which f is (finite) étale the stalk $f_*(\mathbf{Z}/(n))_c$ for the étale topology is $\mathbf{Z}/(n) \oplus \mathbf{Z}/(n)$ because "spreading out" for the decomposition in Lemma 2.4.1 provides an étale neighborhood U of (C, c) over which the pullback of C' becomes a disjoint union of two copies of U.

Proof of Lemma 2.4.2. Our problem is étale-local around (Y, y), so by Lemma 2.4.1 we can pass to a suitable neighborhood so that $X = \coprod_{x \in X_y} W(x)$ with each $W(x) \to Y$ having x as its unique geometric point over y. That is, we may reduce to the case where X_y consists of a single geometric point $\{x\}$. Hence, for any étale neighborhood

(U,u) of (Y,y), the base change $U' := U \times_Y X = f^{-1}(U)$ has a *unique* point over u, namely $u' = (u,x)$. Ergo, it suffices to show that the pointed étale neighborhoods (U',u') of (X,x) are cofinal.

In other words, if (W,w) is an étale neighborhood of (X,x) we seek such (U,u) so that $(f^{-1}(U),u')$ maps to (W,w) over (X,x). We may assume Y (and hence X) is affine and then by shrinking that W is affine. Ergo, by "spreading out" it suffices to make the desired factorization after base change to $\operatorname{Spec} \mathcal{O}_y^{sh}$. By Lemma 2.4.1 and our preliminary arrangements concerning x, we may thereby assume that Y and X are *local* and strictly henselian. But then any étale neighborhood of (X,x) admits a section (by the characterization of strictly henselian local rings). In particular, $W \to X$ admits a section carrying x into w, so we are done. □

2.5. **A dévissage technique.** In order or prove Proposition 2.3.1, we require a general homological technique to make systematic and efficient certain arguments for bootstrapping from special cases to a general situation:

Proposition 2.5.1. *Suppose $\mathcal{C}$ is an abelian category which has direct limits. Suppose $\phi^\bullet : T^\bullet \to R^\bullet$ is a map of δ-functors from $\mathcal{C}$ to* Ab *that commutes with direct limits and vanishes in negative degrees. Further, suppose there are full subcategories $\mathcal{D}, \mathcal{E}$ of $\mathcal{C}$ so that*

(a) Every object in $\mathcal{C}$ is a direct limit of objects in $\mathcal{D}$.
(b) Every object in $\mathcal{D}$ is a subobject of an object in $\mathcal{E}$.

The following two conditions are equivalent:

(i) $\phi^q(A)$ is bijective for all $q \geq 0$ and all $A \in \mathcal{C}$.
(ii) $\phi^q(M)$ is bijective for all $q \geq 0$ and all $M \in \mathcal{E}$.

Proof. Clearly $(i) \implies (ii)$, since every object in $\mathcal{E}$ lies in $\mathcal{C}$. For the converse we need the following refined version of the five-lemma, which can be proven using a diagram chase that we leave to the reader:

Lemma 2.5.2 (Refined Five Lemma)**.** *For any map of exact sequences in an abelian category*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \\ \downarrow m & & \downarrow n & & \downarrow p & & \downarrow q \\ E & \longrightarrow & F & \longrightarrow & G & \longrightarrow & H \end{array} \tag{2.7}$$

the following hold:

(1) If n and q are injective, and m is surjective, then p is injective.

(2) If m and p are surjective and q is injective, then n is surjective.

Exercise 2.5.3. Prove Lemma 2.5.2, either via a routine diagram chase or a spectral sequence argument.

We first check that if $\phi^j(M)$ is bijective for all j and all $M \in \mathcal{E}$ then $\phi^q(A)$ is injective for each q and all $A \in \mathcal{C}$. The conclusion trivially holds for all $q \leq -1$. Now assume $q \geq 0$ and that the assertion holds for $q-1$. To show $\phi^q(A)$ is an injection for all objects A of $\mathcal{C}$, it suffices to show it is an injection on all objects of $\mathcal{D}$ since every object in $\mathcal{C}$ is a direct limit of objects in $\mathcal{D}$ and the given δ-functors commute with the formation of direct limits.

Take any $A \in \mathcal{D}$. We can find an injection of A into some $M \in \mathcal{E}$, so we have an exact sequence

$$0 \longrightarrow A \longrightarrow M \longrightarrow B \longrightarrow 0 \tag{2.8}$$

with $B \in \mathcal{E}$. By δ-functoriality, we get a map of exact sequences

$$\begin{tikzcd} T^{q-1}(M) \arrow[r] \arrow[d, "m"] & T^{q-1}(B) \arrow[r] \arrow[d, "n"] & T^q(A) \arrow[r] \arrow[d, "p"] & T^q(M) \arrow[d, "q"] \\ R^{q-1}(M) \arrow[r] & R^{q-1}(B) \arrow[r] & R^q(A) \arrow[r] & R^q(M) \end{tikzcd} \tag{2.9}$$

where the vertical maps are ϕ's. By our inductive hypothesis, the map $\phi^{q-1}(B)$ is injective, and by assumption the maps $\phi^{q-1}(M)$ and $\phi^q(M)$ are bijective. Therefore, by Lemma 2.5.2(1), $\phi^q(A)$ is also injective.

To conclude the proof, we may assume that $\phi^\bullet$ is injective on all objects of $\mathcal{C}$, bijective on all objects in $\mathcal{D}$, and we wish to deduce it is bijective on all objects of $\mathcal{C}$. To show this, it suffices (by passage to direct limits) to show $\phi^\bullet$ is bijective on all objects of $\mathcal{D}$. Again, we induct on q, the base case $q \leq -1$ being clear. Choose any $A \in \mathcal{D}$ and choose $M \in \mathcal{E}$ fitting into an exact sequence as in (2.8). Consider the map of exact sequences

$$\begin{tikzcd} T^{q-1}(B) \arrow[r] \arrow[d, "m"] & T^q(A) \arrow[r] \arrow[d, "n"] & T^q(M) \arrow[r] \arrow[d, "p"] & T^q(B) \arrow[d, "q"] \\ R^{q-1}(B) \arrow[r] & R^q(A) \arrow[r] & R^q(M) \arrow[r] & R^q(B). \end{tikzcd} \tag{2.10}$$

By our inductive hypothesis, $\phi^{q-1}(B)$ is surjective. By assumption $\phi^q(M)$ is bijective and by the preceding work $\phi^q(B)$ is injective. Ergo, by Lemma 2.5.2(2), $\phi^q(A)$ is surjective. $\square$

We want to verify the hypotheses of Proposition 2.5.1 in particular cases of interest that will be used to prove Proposition 2.3.1 in § 2.7. This rests on the following lemma.

Lemma 2.5.4. *Let $\mathcal{F}$ be any abelian étale sheaf of $\mathbf{Z}/(n)$-modules on a noetherian scheme X. There exists an étale cover $\{U_i \to X\}$ with each U_i affine and connected such that $\mathcal{F}$ is a quotient of $\oplus_i (h_i)_!(\mathbf{Z}/(n))$. The sheaf $\mathcal{F}$ is constructible if and only if we can achieve this using finitely many U_i's.*

Proof. To see we can find such a surjection, it suffices for each geometric point x of X and $s \in \mathcal{F}_x$ to find such a connected affine U_x^s and étale $h : U_x^s \to X$ so that there is a map $(h_x)_!(\mathbf{Z}/(n)) \to \mathcal{F}$ whose image on x-stalks hits s; we can then take the disjoint union over all $x \in X$, and in the constructible case a finite collection of such U_x^s's would suffice (by noetherian induction and the definition of constructibility for abelian étale sheaves). Also, if a finite collection of such U_x^s's covering X does the job then we will have expressed $\mathcal{F}$ as a quotient of a constructible abelian sheaf, so $\mathcal{F}$ will be also be constructible.

We will now produce such a U_x^s. Since $\mathcal{F}$ is n-torsion, s arises from an element of the $\mathbf{Z}/(n)$-module $\mathcal{F}(V)$ for some étale neighborhood $V \to X$ of x. Choose $U_x^s \subset V$ an affine open containing some preimage of x in V, we obtain a map

$$\begin{aligned} \mathbf{Z}/(n) &\to h^*\mathcal{F} \\ 1 &\mapsto s \end{aligned}$$

The corresponding map $h_!(\mathbf{Z}/(n)) \to \mathcal{F}$, via the adjunction of $h_!$ and h^*, does the job. □

We now use Lemma 2.5.4 to prove Lemma 2.5.5 below, which will allow us to apply Proposition 2.5.1 in the proof of Proposition 2.3.1.

Lemma 2.5.5. *Let S be a noetherian scheme and choose $n \in \mathbf{Z}_{>0}$. Then:*

(1) Every sheaf of $\mathbf{Z}/(n)$-modules on S is a direct limit of constructible subsheaves.

(2) Any constructible sheaf $\mathcal{F}$ of $\mathbf{Z}/(n)$-modules injects into a sheaf of the form $\prod_{\lambda=1}^m (i_\lambda)_* C_\lambda$, where t'_λ is the spectrum of a field, $i_\lambda : t'_\lambda \to S$ is a map such that the induced map $t'_\lambda \to i_\lambda(t'_\lambda)$ is finite Galois, and each C_λ a finite free $\mathbf{Z}/(n)$-module.

(3) Any constructible sheaf $\mathcal{F}$ of $\mathbf{Z}/(n)$-modules injects into a sheaf of the form $\prod_{\lambda=1}^m (i_\lambda)_* C_\lambda$, where $i_\lambda : \bar{t}_\lambda \to S$ are geometric points and each C_λ is a free $\mathbf{Z}/(n)$-module of finite rank.

(4) *Any constructible sheaf* $\mathcal{F}$ *of* $\mathbf{Z}/(n)$*-modules injects into a sheaf of the form* $\prod_{\lambda=1}^{m}(h_\lambda)_* C_\lambda$*, where* $h_\lambda : X_\lambda \to S$ *is a finite morphism and each* C_λ *a finite free* $\mathbf{Z}/(n)$*-module.*

We refer the reader to Remark 2.2.3 for our convention in these notes concerning the meaning of "geometric point".

Proof. Note first that (1) follows from Lemma 2.5.4 by exhausting $\mathcal{F}$ by the images of finite direct sums.

To prove (2) we will show that any constructible sheaf injects into a finite product of pushforwards of skyscraper sheaves.

We first reduce to the case that $\mathcal{F}$ is lcc and S is irreducible. Let $j : U \hookrightarrow X$ be a nonempty irreducible open subscheme on which $\mathcal{F}$ is lcc. By shrinking U around its generic point, we can arrange (by "spreading out" from the generic point, over which all constructible sheaves are split by a finite étale cover) that there exists a *finite* étale cover $h : V \to U$ by an irreducible V so that V is Galois over U and $h^*\mathcal{F}$ is constant. Suppose we can construct a sheaf $\mathcal{H}$ of the form $\prod_{\lambda=1}^{q}\left(t'_\lambda\right)_* C_\lambda$ and an injection $\mathcal{F}|_U \to \mathcal{H}$. By factoring each t'_λ through j we get a map $\mathcal{F} \to j_*(\mathcal{F}|_U) \to \prod_{\lambda=1}^{q}\left(t'_\lambda\right)_* C_\lambda$ that has vanishing kernel over U. Then, by noetherian induction, we can find points $t'_{q+1}, \dots, t'_n$ of $X - U$ and suitable C_λ on t'_λ for $m+1 \le \lambda \le n$ such that $\mathcal{F}|_{X-U}$ is a subsheaf of $\prod_{\lambda=q+1}^{m}(t'_\lambda)_* C_\lambda$. Ergo, for the inclusion $i : X - U \to X$ the injection $\mathcal{F} \to j_*(\mathcal{F}|_U) \oplus i_*(\mathcal{F}|_{X-U})$ yields the desired result for $\mathcal{F}$ on X.

Now we may assume S is irreducible say with generic point η, and there exists a finite étale cover $h : V \to S$ so that $\mathcal{F}|_V$ is constant and V is irreducible and Galois over U. By finiteness and surjectivity of h it is clear that the natural map $\mathcal{F} \to h_* h^* \mathcal{F}$ is an injection (use Lemma 2.4.2). It suffices to treat $(V, h^*\mathcal{F})$ in place of $(S, \mathcal{F})$ (as we can then compose the points t'_λ of V with h to get points of S that do the job for $\mathcal{F}$), so we can assume $\mathcal{F}$ is the constant sheaf associated to a finite $\mathbf{Z}/(n)$-module M. Let η_V denote the generic point of V. But any finite $\mathbf{Z}/(n)$-module is easily seen to be a submodule of a finite free $\mathbf{Z}/(n)$-module, so we can assume M is free. Letting $i_{\eta_V} : \eta_V \to V$ denote the inclusion, it is elementary to check that $\mathcal{F} \to i_{\eta_V *}(i^*_{\eta_V}\mathcal{F})$ is injective (using that any étale S-scheme has all generic points over η, and that the pullback of a constant sheaf is constant), so we are done.

Assertion (3) is immediate from (2) by taking $\bar{t}_\lambda$ to be a geometric point over t_λ corresponding to a separable closure of the residue field at t_λ. The main point is that if k'/k is any extension of fields and

$h : \operatorname{Spec} k' \to \operatorname{Spec} k$ is the corresponding map then $\mathcal{G} \to h_*(h^*\mathcal{G})$ has trivial kernel for any abelian étale sheaf $\mathcal{G}$ on $\operatorname{Spec} k$.

Next, we prove (4). In the case that S is excellent (so normalization is finite) this follows from part (2) by taking X_λ to be normalization of S in t_λ. (See also [FK, I, Proposition 4.12] for another proof in the case that S is excellent.) In the general case, this is shown in By [SGA4, Exp. IX,Proposition 2.14] we obtain an injection $\mathcal{F} \to \prod_{\lambda=1}^m (h_\lambda)_* G_\lambda$ for G_λ some constant constructible sheaf of $\mathbf{Z}/(n)$-modules. Upon choosing an injection $G_\lambda \to C_\lambda$ for C_λ a finite free $\mathbf{Z}/(n)$ module, we obtain the desired injection $\mathcal{F} \to \prod_{\lambda=1}^m (h_\lambda)_* G_\lambda \to \prod_{\lambda=1}^m (h_\lambda)_* C_\lambda$. □

2.6. **Local acyclicity is preserved under quasi-finite base change.** To establish Proposition 2.3.1, we first show (in Lemma 2.6.3) that local acyclicity is stable under quasi-finite base change. This requires the following two lemmas recording useful properties of strict henselization and of strictly henselian local rings.

Lemma 2.6.1. *Let $\{S_\lambda\}$ be an inverse system of affine schemes and define $S' := \varprojlim_\lambda S_\lambda$. For $s' \in S'$ and corresponding images $s_\lambda \in S_\lambda$, we have $\varinjlim_\lambda \mathcal{O}^{sh}_{s_\lambda} = \mathcal{O}^{sh}_{s'}$.*

Proof. To verify this, we only need show that the strictly henselian property is preserved under direct limits of local rings with local transition maps. But among all local rings, the strictly henselian ones are *precisely* those admitting a section to any étale cover (due to [EGA, IV$_4$, 18.5.11]). Ergo, by descent of affine étale maps through direct limits and the fact that an étale map to a local scheme is a cover if and only if it hits the closed point, we are done. □

Lemma 2.6.2. *Suppose we have a Cartesian diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd} \tag{2.11}$$

with $X \to S$ finite. For any point $x' \in X'$ over points s', s, x of S', S, X respectively, the natural map

$$\mathcal{O}^{sh}_x \otimes_{\mathcal{O}^{sh}_s} \mathcal{O}^{sh}_{s'} \to \mathcal{O}^{sh}_{x'}$$

is an isomorphism. In particular, if S, S', X' are strictly henselian then their fiber product X' is strictly henselian.

Proof. The formation of $\mathcal{O}^{\mathrm{sh}}_{x'}$ is unaffected by base change along the map $\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{s'}) \to S'$, and likewise for $\mathcal{O}^{\mathrm{sh}}_{x}$ along $\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{s}) \to S$, so we may assume $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(A')$ for strictly henselian local rings A and A' with respective closed points s and s'. By finiteness, we then have $X = \mathrm{Spec}(B)$ for a module-finite A-algebra B, and $X' = \mathrm{Spec}(B')$ for $B' = A' \otimes_A B$.

The A-finiteness of B and the henselian property of A imply that B is a product of finitely many henselian local A-algebras (see Proposition A.1.2 and Corollary A.1.3). Ergo, by passage to connected components we can assume X is local henselian; its unique closed point must be x. The residue field of X at its closed point is finite over that of S and hence is separably closed, so X is strictly henselian.

By similar reasoning, X' is a disjoint union of finitely many strictly henselian local schemes. It remains to show that X' is local, or equivalently (by S'-finiteness) has a unique point over s'. Since x' is the unique point in X_s, the fiber $X'_{s'}$ is topologically the same as the spectrum of the ring $\kappa(s') \otimes_{\kappa(s)} \kappa(x)$. But $\kappa(x)/\kappa(s)$ is purely inseparable since X is S-finite and $\kappa(s)$ is separably closed, so we are done. $\square$

The preservation of local acyclicity under quasi-finite base change will be needed later in a form involving limits of schemes (and taking the inverse system of schemes to be constant gives the desired quasi-finite case), so we record it in this generality:

Lemma 2.6.3. *Suppose $g : Y \to S$ is a locally acyclic morphism. Let $(S_\lambda, s_{\mu\lambda})$ be an inverse system of quasi-finite S-schemes with transition maps $s_{\mu\lambda} : S_\mu \to S_\lambda$ that are affine. For $S' := \varprojlim_\lambda S_\lambda$ and $Y' := Y \times_S S'$, the induced map $g' : Y' \to S'$ is locally acyclic.*

In particular, the base change of a locally acyclic map along a quasi-finite map is locally acyclic.

Proof. Choose $y' \in Y'$ mapping to $s' \in S, y \in Y$, and $s \in S$, and let s_λ be the image of s' in S_λ and y_λ be the image of y' in $S_\lambda \times_S Y$. We first want to show that $\mathcal{O}^{\mathrm{sh}}_{s'}$ is integral over $\mathcal{O}^{\mathrm{sh}}_{s}$ and that the natural map

$$\mathcal{O}^{\mathrm{sh}}_{s'} \otimes_{\mathcal{O}^{\mathrm{sh}}_{s}} \mathcal{O}^{\mathrm{sh}}_{y} \to \mathcal{O}^{\mathrm{sh}}_{y'}$$

is an isomorphism. By Lemma 2.6.1, we have naturally $\varinjlim_\lambda \mathcal{O}^{\mathrm{sh}}_{y_\lambda} \simeq \mathcal{O}^{\mathrm{sh}}_{y'}$ and

$$\varinjlim_\lambda (\mathcal{O}^{\mathrm{sh}}_{s_\lambda} \otimes_{\mathcal{O}^{\mathrm{sh}}_{s}} \mathcal{O}^{\mathrm{sh}}_{y}) = (\varinjlim_\lambda \mathcal{O}^{\mathrm{sh}}_{s_\lambda}) \otimes_{\mathcal{O}^{\mathrm{sh}}_{s}} \mathcal{O}^{\mathrm{sh}}_{y} = \mathcal{O}^{\mathrm{sh}}_{s'} \otimes_{\mathcal{O}^{\mathrm{sh}}_{s}} \mathcal{O}^{\mathrm{sh}}_{y},$$

so our task reduces to showing that $\mathcal{O}^{\mathrm{sh}}_{s_\lambda}$ is finite over $\mathcal{O}^{\mathrm{sh}}_{s}$ and the natural map

$$\mathcal{O}^{\mathrm{sh}}_{s_\lambda} \otimes_{\mathcal{O}^{\mathrm{sh}}_{s}} \mathcal{O}^{\mathrm{sh}}_{y} \to \mathcal{O}^{\mathrm{sh}}_{y_\lambda}$$

is an isomorphism for each λ.

For the purpose of verifying these assertions for each λ, we may now fix a choice of λ and rename S_λ as S' so that $f : S' \to S$ is quasi-finite. Our problem is also local on S', so we may assume that S is affine and then that S' is affine. Now S' is quasi-finite and separated over S, so by Zariski's Main Theorem [EGA, IV_4, 18.12.13] we can factor f as an open immersion followed by a finite map. This reduces our task to the separate cases when f is an open immersion or finite. The case when f is an open immersion is trivial, so we may assume f is finite. The isomorphism aspect in this case is settled by Lemma 2.6.2, and the module-finiteness is due to Lemma 2.4.1.

Returning to our initial situation, pick a geometric point $t' \to \mathcal{O}^{\mathrm{sh}}_{s'}$. Consider the diagram

$$\begin{tikzcd}
(\operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{y'})_{t'} \arrow[r] \arrow[d] & \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{y'} \arrow[r] \arrow[d] & \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{y} \arrow[d] \\
t' \arrow[r] & \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{s'} \arrow[r] & \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{s}
\end{tikzcd} \tag{2.12}$$

The integrality of $\mathcal{O}^{\mathrm{sh}}_{s'}$ over $\mathcal{O}^{\mathrm{sh}}_{s}$ implies that the separably closed field $\kappa(t')$ is algebraic over the residue field at its image in $\operatorname{Spec}(\mathcal{O}^{\mathrm{sh}}_{s})$, so $\kappa(t')$ is purely inseparable over the residue field of a "geometric point" of $\operatorname{Spec}(\mathcal{O}^{\mathrm{sh}}_{s})$ in the sense of Remark 2.2.3. Ergo, we are done due to the local acyclicity of $Y \to S$ and the invariance of the étale site for an arbitrary scheme (such as a typically non-noetherian vanishing cycles scheme) over a field with respect to purely inseparable extension of the ground field. $\square$

2.7. **The proof in the open embedding case.** In this subsection we prove Proposition 2.3.1, the open embedding version of smooth base change, assuming the following criterion for checking local acyclicity on sheaves which are the pushforward of a sheaf supported at a point.

Proposition 2.7.1. *Suppose $g : Y' \to Y$ is a locally acyclic morphism. Let $\varepsilon : t \to Y$ be an arbitrary geometric point and consider the Cartesian* *diagram*

$$
\begin{tikzcd}
Y'_t \arrow[r, "g'"] \arrow[d, "\varepsilon'"] & t \arrow[d, "\varepsilon"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}
\tag{2.13}
$$

Then $g^*\varepsilon_*(\mathbf{Z}/(n)) \simeq \varepsilon'_*(\mathbf{Z}/(n))$ *via the natural map and for each* $q > 0$ *the sheaf* $R^q\varepsilon'_*(\mathbf{Z}/(n))$ *vanishes.*

We will prove Proposition 2.7.1 in § 2.9, and now complete the proof of Proposition 2.3.1 assuming Proposition 2.7.1. In conjunction with Theorem 3.3.1, this will complete the proof of the smooth base change theorem.

Proof of Proposition 2.3.1 assuming Proposition 2.7.1. Let $\varepsilon : t \to X$ be a geometric point, and form the commutative diagram of Cartesian squares

$$
\begin{tikzcd}
X'_t \arrow[r, "g_t"] \arrow[d, "\varepsilon'"] \arrow[dd, bend right, "\delta'"'] & t \arrow[d, "\varepsilon"] \arrow[dd, bend left, "\delta"] \\
X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}
\tag{2.14}
$$

First, we treat the case that $\mathcal{F} = \varepsilon_*C$ with C a free $\mathbf{Z}/(n)$-module of finite rank on the geometric point t. We may and do assume C has rank 1.

To check that that the base change map (1.2) is an isomorphism when $q = 0$ for this $\mathcal{F}$, apply Proposition 2.7.1 to the outer fiber square and the upper fiber square of (2.14) (we may apply this for the upper fiber square since g' is locally acyclic due to the trivial case of Lemma 2.6.3 with f an open immersion). Then we get a composite isomorphism

$$
\begin{aligned}
g^*f_*\varepsilon_*(\mathbf{Z}/(n)) &= g^*\delta_*(\mathbf{Z}/(n)) \\
&= \delta'_*g_t^*(\mathbf{Z}/(n)) \\
&= f'_*\varepsilon'_*g_t^*(\mathbf{Z}/(n)) \\
&= f'_*g'_*\varepsilon_*(\mathbf{Z}/(n))
\end{aligned}
$$

that is easily checked to coincide with the base change morphism for $\mathcal{F}$ (this just expresses a general functorial behavior of base change morphisms under concatenation of commutative squares).

To treat the case $q > 0$, it suffices to show that $R^q f_* \varepsilon_*(\mathbf{Z}/(n))$ and $R^q f'_* g'^* \varepsilon_*(\mathbf{Z}/(n))$ vanish for $q > 0$. We first show $R^q f_* \varepsilon_*(\mathbf{Z}/(n)) = 0$ for $q > 0$. By Proposition 2.7.1 with $Y' \to Y$ equal to the identity map for each of X and S, $R^q \varepsilon_*(\mathbf{Z}/(n)) = 0$ for $q > 0$ and likewise for δ. Hence, in the Leray spectral sequence

$$R^q f_* R^p \varepsilon_*(\mathbf{Z}/(n)) \implies R^{p+q} \delta_*(\mathbf{Z}/(n))$$

we have $R^{p+q}\delta_*(\mathbf{Z}/(n)) = 0$ for $p + q > 0$ and $R^p \varepsilon_*(\mathbf{Z}/(n)) = 0$ for $p > 0$. Therefore, the spectral sequence degenerates to the $p = 0$ column and also converges to 0 in positive degrees, so all terms in the column for $p = 0$ vanish above the $(0, 0)$-term. That is, $R^q f_* \varepsilon_*(\mathbf{Z}/(n)) = 0$ for $q > 0$.

Next, we verify $R^q f'_* g'^* \varepsilon_*(\mathbf{Z}/(n)) = 0$ for $q > 0$. By Proposition 2.7.1 applied to the upper Cartesian square,

$$R^q f'_* g'^* \varepsilon_*(\mathbf{Z}/(n)) = R^q f'_* \varepsilon'_*(\mathbf{Z}/(n))$$

via the natural map. Hence, it suffices to show $R^q f'_* \varepsilon'_*(\mathbf{Z}/(n)) = 0$ for $q > 0$. The proof of this is quite similar to our proof above that $R^q f_* \varepsilon_*(\mathbf{Z}/(n)) = 0$. Indeed, in the spectral sequence

$$R^q f'_* R^p \varepsilon'_*(\mathbf{Z}/(n)) \implies R^{p+q} \delta'_*(\mathbf{Z}/(n))$$

we have $R^{p+q}\delta'_*(\mathbf{Z}/(n)) = 0$ for $p + q > 0$ by Proposition 2.7.1 applied to the outer Cartesian square, and likewise $R^p \varepsilon'_*(\mathbf{Z}/(n)) = 0$ for $p > 0$ by Proposition 2.7.1 applied to the upper Cartesian square. Therefore the spectral sequence once again degenerates to the column $p = 0$ yet has vanishing convergent term for $q > 0$, so $R^q f'_* \varepsilon'_*(\mathbf{Z}/(n)) = 0$ for $q > 0$. This finishes the proof of our two general vanishing claims in positive degrees, and so establishes the isomorphism property for the base change maps associated to $\mathcal{F} := \varepsilon_* C$ with C free over $\mathbf{Z}/(n)$ for n invertible on S.

To conclude the result for all sheaves, we use Proposition 2.5.1. First, by the compatibility of higher direct images with direct limits of sheaves we can reduce to the case that $\mathcal{F}$ is n-torsion for a fixed integer $n > 0$ invertible on S since every abelian sheaf with invertible torsion orders can be written as a limit of its n-torsion subsheaves as n ranges over all integers invertible on S. In the statement of Proposition 2.5.1, we take $\mathcal{C}$ to be all sheaves of $\mathbf{Z}/(n)$-modules on X, $\mathcal{D}$ to be all constructible sheaves of $\mathbf{Z}/(n)$-modules on S, and $\mathcal{E}$ to be all sheaves which are finite products of sheaves of the form $\varepsilon_* C$ for ε

as above and C a free $\mathbf{Z}/(n)$-module (on the geometric point t). By Lemma 2.5.5, the hypotheses of Proposition 2.5.1 are satisfied, so the map (1.2) is an isomorphism for all sheaves in $\mathcal{C}$. $\square$

2.8. **Two final lemmas.** To prove Proposition 2.7.1, we will need two final technical lemmas. The first concerns a sufficient criterion for when a normal local ring is henselian; this criterion essentially only applies to non-noetherian rings.

Lemma 2.8.1. *Suppose* $\operatorname{Spec} A$ *is a normal local domain with fraction field* K. *If* K *is separably closed, then* A *is strictly henselian.*

Proof. It suffices to show that any étale map $g : \operatorname{Spec} B \to \operatorname{Spec} A$ with nontrivial fiber over the closed point has a section (by the equivalent characterization of strictly henselian local rings as those for which every étale cover with nonempty fiber over the closed point has a section; see Proposition A.1.2). Note that g must be surjective since its image is open and contains the unique closed point of the local target.

Since $\operatorname{Spec} B \to \operatorname{Spec} A$ is an étale map of affine schemes, it is quasi-finite. Further, since normality is local in the étale topology, $\operatorname{Spec} B$ is normal. In particular, all local rings of B are domains, so no distinct irreducible components of $\operatorname{Spec} B$ can meet (as the local ring at such an intersection point would have at least two minimal primes and hence could not be a domain). Also, $\operatorname{Spec} B$ has only finitely many generic points since they all lie over the generic point of $\operatorname{Spec} A$ (as flat maps carry generic points to generic points, and $\operatorname{Spec} A$ has a unique generic point because A is a domain).

Since $\operatorname{Spec} B$ has only finitely many irreducible components and they are pairwise disjoint, each irreducible component is both open and closed and each is even a connected component. We may therefore assume that $\operatorname{Spec} B$ has a unique connected component; i.e., it is irreducible and thus B is a domain (as it is reduced, due to normality). That is, now B is a normal domain. In this setting it suffices to prove that g is an isomorphism.

By Zariski's Main Theorem [EGA, IV_4, 18.12.13], we can factor $g : \operatorname{Spec} B \to \operatorname{Spec} A$ as the composition

$$\begin{tikzcd} \operatorname{Spec} B \arrow[rr, "h"] \arrow[dr, "g"'] & & T \arrow[dl, "\overline{g}"] \\ & \operatorname{Spec} A & \end{tikzcd} \tag{2.15}$$

with $T \to \operatorname{Spec} A$ a finite map and $h : \operatorname{Spec} B \to T$ a schematically dense open immersion. In particular, T inherits reducedness and irreducibility from $\operatorname{Spec} B$. We know T is affine as $\overline{g}$ is finite, so say $T = \operatorname{Spec} C$.

By reducedness and irreducibility of T, it follows that C must be a domain. The dominance of the open immersion h makes C an A-subalgebra of B with the same fraction field. But g is étale, so this common fraction field is separable over the fraction field K of A. By hypothesis K is separably closed, so $C \subset K$. But C is A-finite by design, and A is normal, so $C = A$. This says that $\overline{g}$ is an isomorphism, so g is identified with h; in particular, g is an open immersion. But g hits the closed point of the local $\operatorname{Spec} A$, so g is an isomorphism. $\square$

Next, we address an exactness result that generalizes the exactness of finite pushforwards at the cost of a torsion hypothesis. The role of the torsion condition is to permit the use of limit arguments to reduce back to the finite case:

Lemma 2.8.2. *Let $f : X \to Y$ be an integral map. The functor f_* is exact between categories of torsion abelian sheaves.*

Proof. We want to show that $R^q f_*(\mathcal{F}) = 0$ for any $q > 0$ and torsion abelian sheaf $\mathcal{F}$ on X. This question is local on Y, so we can assume $Y = \operatorname{Spec} A$ is affine; then also $X = \operatorname{Spec} B$ is affine. The compatibility of higher direct images with direct limits of sheaves allows us to reduce to the case when $\mathcal{F}$ is n-torsion for some $n > 1$. By Lemma 2.5.4, we can exhaust $\mathcal{F}$ by a directed system of subsheaves that are images of maps $h_!(\mathbf{Z}/(n)) \to \mathcal{F}$ for étale maps $h : U \to X$ with affine U. Hence, by replacing $\mathcal{F}$ by such an image subsheaf (as we may do by the compatibility of each $R^q f_*$ with direct limits of sheaves) we may assume that there is an exact sequence

$$0 \to \mathcal{K} \to h_!(\mathbf{Z}/(n)) \to \mathcal{F} \to 0.$$

If $\mathcal{K}$ could be presented as a direct limit of subsheaves $\mathcal{K}_\lambda$ then $\mathcal{F} = \varinjlim \mathcal{F}_\lambda$ for

$$\mathcal{F}_\lambda := (h_!(\mathbf{Z}/(n)))/\mathcal{K}_\lambda,$$

so it would then suffice to prove $R^q f_*(\mathcal{F}_\lambda) = 0$ for all $q > 0$ and each λ separately. In fact, $\mathcal{K}$ can be presented in this way: run through the same arguments for $\mathcal{K}$ that we initially did for $\mathcal{F}$ to find such $\mathcal{K}_\lambda$'s that are quotients of sheaves of the form $g_!(\mathbf{Z}/(n))$ for affine étale maps $g : V \to X$. So to summarize, this finally reduces us to the setting where $\mathcal{F}$ is presented as a cokernel

$$g_!(\mathbf{Z}/(n)) \xrightarrow{\varphi} h_!(\mathbf{Z}/(n)) \to \mathcal{F} \to 0.$$

Since $X = \operatorname{Spec} B$ with B an integral A-algebra, we can write $B = \varinjlim B_\mu$ for the A-finite subalgebras B_μ of B. Since étale maps are finitely presented, for some large μ_0 the affine étale maps $g : V \to X$ and $h : U \to X$ descend to affine étale maps

$$g_0 : V_0 \to X_0, \ \ h_0 : U_0 \to X_0$$

for the Y-finite $X_0 := \operatorname{Spec} B_{\mu_0}$ and the sheaf map $\varphi : g_!(\mathbf{Z}/(n)) \to h_!(\mathbf{Z}/(n))$ (which corresponds to a global section of the g-pullback of $h_!(\mathbf{Z}/(n))$) likewise is identified with the base change of a sheaf map

$$\varphi_0 : (g_0)_!(\mathbf{Z}/(n)) \to (h_0)_!(\mathbf{Z}/(n))$$

over X_0. Ergo, $\mathcal{F}_0 := \operatorname{coker} \varphi_0$ is an abelian étale sheaf on X_0 that descends $\mathcal{F}$.

For each $\mu \geq \mu_0$, let $X_\mu = \operatorname{Spec} B_\mu$ and define $\mathcal{F}_\mu$ to be the pullback of $\mathcal{F}_0$. For any affine étale $W \to Y$, the natural map

$$\varinjlim H^q(X_\mu \times_Y W, \mathcal{F}_\mu) \to H^q(X \times_Y W, \mathcal{F})$$

is an isomorphism for all q due to the compatibility of étale cohomology with the formation of limits of qcqs schemes (with affine transition maps). Viewing the above two objects as presheaves taking input W, and sheafifying, it follows that we have naturally

$$R^q f_*(\mathcal{F}) \simeq \varinjlim R^q(f_\mu)_*(\mathcal{F}_\mu)$$

for the maps $f_\mu : X_\mu \to Y$. But each f_μ is *finite*, so $R^q(f_\mu)_* = 0$ for all $q > 0$ (even on the category of all abelian sheaves on X_μ). □

2.9. **Proof of local acyclicity of sheaves pushed forward from a point.** In this subsection, we complete the proof of Proposition 2.7.1, and so aside from proving Theorem 3.3.1 in the next section this will finally complete the proof of the smooth base change theorem!

Proof of Proposition 2.7.1. Let $\overline{Y}$ denote the normalization in the field $\kappa(t)$ of the closure of $\varepsilon(t)$. In the Cartesian diagram

$$\begin{tikzcd} Y'_t \arrow[r, "i'"] \arrow[d, "g'"] \arrow[rr, bend left, "\varepsilon'"] & \overline{Y}' \arrow[r, "\alpha'"] \arrow[d, "\overline{g}"] & Y' \arrow[d, "g"] \\ t \arrow[r, "i"] \arrow[rr, bend right, "\varepsilon"'] & \overline{Y} \arrow[r, "\alpha"] & Y \end{tikzcd} \tag{2.16}$$

observe that $\overline{g}$ inherits local acyclicity from g by working locally on Y (e.g., over affine opens in Y) due to Lemma 2.6.3 since normalization maps are integral and an integral map between affine schemes is a limit of finite maps. By Lemma 2.8.1, every local ring on $\overline{Y}$ is strictly henselian (because such local rings are normal domains with fraction field $\kappa(t)$ that is separably closed, as $\kappa(t)$ is *algebraic* over the residue field at its image point in Y due to our convention on the meaning of "geometric point" in Remark 2.2.3).

We claim that the natural map $\overline{g}^* i_*(\mathbf{Z}/(n)) \to i'_*(\mathbf{Z}/(n))$ is an isomorphism and that $R^q i'_*(\mathbf{Z}/(n)) = 0$ for $q > 0$. To check this, we consider stalks at geometric points $\overline{y}'$ over physical points $y' \in \overline{Y}'$ for $q \geq 0$:

$$(R^q i'_*(\mathbf{Z}/(n))_{\overline{y}'} = H^q(t \times_{\overline{Y}} \operatorname{Spec}(\mathcal{O}^{\mathrm{sh}}_{y'}), \mathbf{Z}/(n)),$$

The description on the right gives the desired vanishing for $q > 0$ due to local acyclicity of $\overline{g}$, and identifies this with $\mathbf{Z}/(n)$ for $q = 0$ (via the natural map). Ergo, the problem for $q = 0$ reduces to proving that the natural map $\mathbf{Z}/(n) \to (i_*(\mathbf{Z}/(n)))_{\overline{y}'}$ is an isomorphism. This stalk is identified with $(i_*(\mathbf{Z}/(n)))_{\overline{z}}$ for a geometric point $\overline{z}$ over $\overline{g}(y') \in \overline{Y}$. But all Zariski-local rings on $\overline{Y}$ are already strictly henselian! Hence, this final stalk coincides with the Zariski stalk, and that in turn coincides with $\mathbf{Z}/(n)$ (via the natural map!) because the Zariski-local ring of $\overline{Y}$ at any point is a domain with fraction field $\kappa(t)$.

Finally, we use our result for $\overline{g}$ to prove the desired analogous result for g. By Lemma 2.8.2, α_* and α'_* are exact on n-torsion sheaves. Hence, $R^q\alpha_* = 0$ and $R^q\alpha'_*$ on n-torsion sheaves for $q > 0$. Therefore, by the composition of functors spectral sequence, we have

$$\alpha_* R^q i_*(\mathbf{Z}/(n)) \simeq R^q \varepsilon_*(\mathbf{Z}/(n)), \quad \alpha'_* R^q i'_*(\mathbf{Z}/(n)) \simeq R^q \varepsilon'_*(\mathbf{Z}/(n))$$

for all $q \geq 0$ (via the natural maps). In particular, $R^q\varepsilon'_*(\mathbf{Z}/(n)) = 0$ for $q > 0$. To handle the case $q = 0$, since $\overline{g}^* i_*(\mathbf{Z}/(n)) \to i'_*(\mathbf{Z}/(n))$ is an isomorphism it suffices to show that the natural map

$$g^*\alpha_* \to \alpha'_*\overline{g}^*$$

is an isomorphism when evaluated on $i_*(\mathbf{Z}/(n))$.

If α were finite then this would be an instance of the compatibility of finite pushforward with arbitrary base change (see Lemma 2.4.2). However, α is instead an integral map and virtually never finite. By working locally on Y to pass to the case of affine Y, the integral α can be expressed as a limit of finite maps, and we will bootstrap from the isomorphism property when α is replaced with a finite map.

The first crucial thing to observe is that the natural map $\mathbf{Z}/(n) \to i_*(\mathbf{Z}/(n))$ of abelian étale sheaves on $\overline{Y}$ is an isomorphism. Indeed, since $\overline{Y}$ has strictly henselian Zariski-local ring at every point, when computing stalks at each geometric point it is the same to compute Zariski stalks at actual points of $\overline{Y}$. But those Zariski-local rings are *domains* with the same fraction field $\kappa(t)$, so the conclusion follows. Hence, we can recast our problem as that of showing that the natural map $g^*(\alpha_*(\mathbf{Z}/(n))) \to \alpha'_*(\mathbf{Z}/(n))$ is an isomorphism. (It is crucial for the argument that follows that the abelian étale sheaf we are using on $\overline{Y}$ is the sheaf $\mathbf{Z}/(n)$ that is actually a pullback from Y and is not just some arbitrary uncontrollable n-torsion sheaf on $\overline{Y}$.)

For a geometric point y' of Y', the y'-stalk of $g^*\alpha_*(\mathbf{Z}/(n))$ is identified with global sections of $\mathbf{Z}/(n)$ over $\overline{Y} \times_Y \mathrm{Spec}(\mathcal{O}_y^{\mathrm{sh}})$ for the corresponding geometric point y of Y induced by y'. On the other hand, the y'-stalk of $\alpha'_*(\mathbf{Z}/(n))$ is identified with with global sections of $\mathbf{Z}/(n)$ over $\overline{Y}' \times_{Y'} \mathrm{Spec}(\mathcal{O}_{y'}^{\mathrm{sh}})$. To summarize, for the natural map of schemes

$$\overline{Y}' \times_{Y'} \mathrm{Spec}(\mathcal{O}_{y'}^{\mathrm{sh}}) \to \overline{Y} \times_Y \mathrm{Spec}(\mathcal{O}_y^{\mathrm{sh}})$$

we get an induced map in the opposite direction between the global sections of the constant sheaf $\mathbf{Z}/(n)$ and we want this latter map to be an isomorphism. The only property of $\overline{Y} \to Y$ that we shall use is that it is integral (nothing else about the specifics of its construction will be relevant).

Our task only depends on Y through a Zariski-neighborhood of y and likewise for Y' around y', so we can assume $Y = \mathrm{Spec}\, A$ and $Y' = \mathrm{Spec}\, A'$ are affine. Then $\overline{Y} = \mathrm{Spec}\, B$ for an integral A-algebra B. We may write $B = \varinjlim B_\lambda$ for A-finite subalgebras $B_\lambda \subset B$. The compatibility of global sections with limits of qcqs schemes allows us to identify the global sections of $\mathbf{Z}/(n)$ on $\overline{Y} \times_Y \mathrm{Spec}(\mathcal{O}_y^{\mathrm{sh}})$ with the direct limit of the analogous calculation when $\overline{Y} \to Y$ is replaced with the Y-schemes $\mathrm{Spec}(B_\lambda) \to Y$, and likewise for $\overline{Y}' \times_{Y'} \mathrm{Spec}(\mathcal{O}_{y'}^{\mathrm{sh}})$ when $\overline{Y}' \to Y'$ is replaced with the Y'-schemes $\mathrm{Spec}(B_\lambda) \times_Y Y'$. In this manner, our comparison problem is passage to the direct limit on the analogous comparison problem when $\overline{Y} \to Y$ is *finite*. That is, it would be sufficient to affirmatively settle this problem in the case of such finite maps. But that in turn is exactly the stalk comparison question for the formation of finite pushforwards, which we have already addressed above. $\square$

3. Smooth maps are locally acyclic

3.1. **Basic properties of locally acyclic morphisms.** Let us begin by developing some basic properties of locally acyclic morphisms: étale maps are locally acyclic and the composition of locally acyclic maps is locally acyclic. First, we show étale maps are locally acyclic.

Lemma 3.1.1. *If* $f : X \to Y$ *is étale, then* f *is locally acyclic.*

Proof. In this case, a geometric point x of X is also a geometric point y of Y and the natural induced map between strict henselizations for X and Y at this geometric point is an isomorphism. Therefore, for any geometric point $t \in \mathcal{O}_y^{sh}$, the scheme of vanishing cycles $\widetilde{X}_t^x$ is isomorphic to t via its structure map. Ergo, the local acyclicity is a trivial verification in this case. □

Next, we address local acyclicity for the composition of locally acyclic morphisms.

Proposition 3.1.2. *Suppose* $f \colon Z \to Y$, $g \colon Y \to X$ *are qcqs locally acyclic morphisms. Then, the composition* $g \circ f : Z \to X$ *is locally acyclic.*

Proof. The proof is fairly straightforward, granting that being locally acyclic is equivalent to satisfying the second equivalent property of Proposition 2.2.5. Indeed, for maps

$$X'' \xrightarrow{h} X' \xrightarrow{j} X$$

with j étale of finite type and h quasi-finite, consider the fiber squares

$$\begin{tikzcd} Z'' \arrow[r, "f''"] \arrow[d, "h''"] & Y'' \arrow[r, "g''"] \arrow[d, "h'"] & X'' \arrow[d, "h"] \\ Z' \arrow[r, "f'"] \arrow[d, "j''"] & Y' \arrow[r, "g'"] \arrow[d, "j'"] & X' \arrow[d, "j"] \\ Z \arrow[r, "f"] & Y \arrow[r, "g"] & X \end{tikzcd} \tag{3.1}$$

By Proposition 2.2.5 applied to $g \circ f$, we only need show that if $\mathcal{F}$ is a torsion sheaf on X'' with torsion-orders invertible on X then the base change map

$$(g' \circ f')^* R^q h_* (\mathcal{F}) \to R^q h''_* ((g'' \circ f'')^* \mathcal{F}))$$

is an isomorphism.

By the *functoriality* of base change morphisms, this map is equal to the composition

$$\begin{aligned}(g' \circ f')^* R^q h_*(\mathcal{F}) &\simeq f'^* g'^* R^q h_*(\mathcal{F}) \\ &\to f'^* R^q h'_*(g''^* \mathcal{F}) \\ &\to R^q h''_*(f''^*(g''^* \mathcal{F})) \\ &\simeq R^q h''_*((g'' \circ f'')^* \mathcal{F})),\end{aligned}$$

so it suffices to show that the base change maps

$$\begin{aligned}g'^* R^q h_*(\mathcal{F}) &\to R^q h'_*(g''^* \mathcal{F}) \\ f'^* R^q h'_*(\mathcal{G}) &\to R^q h''_*(f''^* \mathcal{G})\end{aligned}$$

for the two upper squares are isomorphisms if $\mathcal{F}$ is a torsion sheaf on X'' with torsion-orders invertible on X' and $\mathcal{G}$ is a torsion sheaf on Y'' with torsion-orders invertible on Y'. But these are each special cases of Proposition 2.2.5 applied to f and g (taking j there to be the identity map in each case). □

3.2. **Computations in special cases.** The core calculations that underlie the proof of local acyclicity of smooth maps concern the cohomology over strict henselizations on the affine line over an excellent ring. There are two such calculations we need: in degree 0 and in degree 1.

The role of excellence is to ensure module-finiteness of some normalizations that arise in the proofs (and later, when we need to invoke these vanishing results we will carry out a standard limit argument to reduce ourselves into the excellent setting). We begin with the degree-0 result that expresses a connectedness property:

Proposition 3.2.1. *Let A be a strictly henselian excellent local ring, $S = \operatorname{Spec} A$, and $X = \operatorname{Spec} A\{T\}$, where $A\{T\}$ is by definition the (necessarily strict) henselization of $A[T]$ at the origin $(\mathfrak{m}_A, T)$ in the special fiber. Then, $H^0(X_t, \mathbf{Z}/(n)) = \mathbf{Z}/(n)$ for any geometric point $t : \operatorname{Spec}(K) \to S$.*

Proof. To show X_t is connected, say the geometric point t has image point $s \in S$, so K is a separable closure of $\kappa(s)$. Expressing K as the direct limit of its subfields k' that are finite separable over $\kappa(s)$, the coordinate ring of X_t is the direct limit of those of the fibers $X_{k'}$. Hence, if the (visibly non-empty) fiber X_t is not connected then it has a non-trivial idempotent, which in turn must arise from some $X_{k'}$. It therefore suffices (arguing by contradiction) to show that each such $X_{k'}$ is connected.

Fix a finite separable extension k' of $\kappa(s)$ and let $t' = \operatorname{Spec} k'$. We aim to prove that $X_{t'}$ is connected. To do this we first reduce to the case where t' is the generic point of S (rather than merely finite étale over the generic point). By excellence of A, the normalization A' of A in k' is A-finite (and so is noetherian). The henselian property of A ensures that the A-finite A' is a finite product of *local* rings, which are necessarily henselian by Proposition A.1.2. But A' is a *domain* (as it is built as a subring of K), so it must therefore be local! The residue field of A' is finite over that of A, hence is separably closed, so A' is even strictly henselian, and A' also inherits excellence from A.

We next justify replacing A with A'. By design, $X_{t'}$ is the generic fiber of $X_{A'} = \operatorname{Spec}(A' \otimes_A A\{T\})$. Ergo, we just need to identify $A' \otimes_A A\{T\}$ with $A'\{T\}$ as A'-algebras to permit replacing A' with A to gain the properties that A is normal and t' is the generic point of S (without losing any of our running hypotheses). Indeed, this follows from Lemma 2.6.2.

We now rename A' as A so that A is a normal domain and t' is the generic point of the irreducible S. To show that the generic fiber $X_{t'}$ is connected we will show the stronger fact that it is irreducible. In fact, we will prove that the local scheme X is normal, so *any* localization of its coordinate ring at a multiplicative set not containing 0 is a domain. The normality property of the henselization $A\{T\}$ of $R := A[T]_{(\mathfrak{m}_A, T)}$ reduces to that of the local ring R by Lemma 2.4.1, and R is normal because $A[T]$ inherits normality from A. $\square$

Here is a degree-1 companion to the preceding result:

Proposition 3.2.2. *Let $S = \operatorname{Spec} A$ be a strictly henselian excellent local ring, $X := \operatorname{Spec}(A\{T\})$, $X \to S$ the natural map, and t a geometric point of S. For $n > 0$ a unit at t we have $H^1(X_t, \mathbf{Z}/(n)) = 0$.*

Remark 3.2.3. This crucially depends on n being nonzero in $\kappa(t)$: it fails whenever $A = \kappa$ is a separably closed field of characteristic $p > 0$ and $n = p$. Indeed, $\operatorname{Spec}(\kappa\{T\}[x]/(x^p - x - T)) \to \operatorname{Spec}(\kappa\{T\})$ is a $\mathbf{Z}/(p)$-torsor, so it is a counterexample provided that it is connected. By Gauss' Lemma it suffices to show that $x^p - x - T$ is irreducible over the fraction field of the discrete valuation ring $\kappa\{T\}$. In fact, it is even irreducible over the fraction field $\kappa((T))$ of the max-adic completion $\kappa[\![T]\!]$ of $\kappa\{T\}$ (recall that the henselization of a local noetherian ring has the same completion, by consideration of universal properties of completion and henselization [EGA, IV_4, 18.6.6(iv)(v)]).

In order to prove Proposition 3.2.2, we require a result on extending finite étale covers:

Proposition 3.2.4. *Let $S = \operatorname{Spec} R$ for an excellent discrete valuation ring R with uniformizer π, generic point η, and suppose $X \to S$ is a smooth map with geometrically connected generic fiber X_η of dimension 1. Let $\widetilde{X_\eta}$ be a connected finite étale Galois cover of X_η with degree n invertible on S.*

Choose $m \in n \cdot \mathbf{Z}_{>0}$ invertible on S. For $S_1 := \operatorname{Spec} R\left[\pi^{1/m}\right]$, let $X_1 := X \times_S S_1$ and $\eta_1 := \operatorname{Spec}(K(R[1/\pi^{1/m}]))$. The normalization $\widetilde{X}_1$ of X_1 in the (possibly disconnected!) finite étale cover $\widetilde{X_\eta} \times_\eta \eta_1$ of the smooth connected η_1-curve $(X_1)_\eta = X_\eta \times_\eta \eta_1$ is finite étale over X_1; equivalently, $\widetilde{X_\eta} \times_\eta \eta_1 \to (X_1)_\eta$ extends to a finite étale cover of X_1.

We now prove Proposition 3.2.2 using Proposition 3.2.4 (which is invoked at the end of the proof of the intermediate Lemma 3.2.5); this latter Proposition will be proved in § 3.4.

Proof. Recall that $H^1(X_t, \mathbf{Z}/(n))$ is identified with the group of isomorphism classes of $\mathbf{Z}/(n)$-torsors over X_t in the étale topology. Ergo, we want to show there are no non-split $\mathbf{Z}/(n)$-torsors E over X_t. As for a finite étale cover of any connected scheme (such as X_t), E is a finite disjoint union of connected finite étale covers of X_t that are torsors for quotients of $\mathbf{Z}/(n)$. Ergo, to show $H^1(X_t, \mathbf{Z}/(n)) = 0$, it suffices to show that there are no Galois *connected* finite étale covers of X_t with degree $n > 1$ that is a unit on S.

Let ξ denote the image of the geometric point $t = \operatorname{Spec} K$ in $S = \operatorname{Spec} A$, so K is a separable closure of $\kappa(\xi)$. We first reduce to the case that ξ is the generic point of S. Let $S' = \operatorname{Spec}(A')$ denote the closure of ξ in S. Since $S' \to S$ is a closed immersion, S' is also local noetherian and excellent. Further, by Lemma 2.4.1, S' is strictly henselian and $X' := S' \times_S X$ coincides with $\operatorname{Spec}(A'\{T\})$ by Lemma 2.6.2. We may therefore replace $X \to S$ with $X' \to S'$ so that ξ is now the generic point of S; it shall now be denoted as η.

Next, we show it suffices to prove that when S is also *normal*, the generic fiber X_η has no Galois connected finite étale cover E_η of degree $n > 1$ that is a unit on S with $E_\eta \otimes_{\kappa(\eta)} K$ also connected. Assuming this property is proved for normal S, we shall deduce a contradiction in the general case when there exists a Galois connected finite étale cover $E \to X_t$ with degree $n > 1$, for n a unit on S.

Since K is the direct limit of its $\kappa(\eta)$-finite subfields k', for some such k' there exists a Galois connected finite étale cover $E' \to X_{\operatorname{Spec} k'}$ with degree $n > 1$ such that $E' \otimes_{k'} K = E$ over X_t. Hence, to get a contradiction, it suffices to show that for any finite separable $k'/\kappa(\eta)$ there is no Galois connected finite étale cover $E' \to X_{\operatorname{Spec} k'}$ with degree $n > 1$ that is a unit on S such that $E' \otimes_{k'} K$ is connected (upon choosing a $\kappa(\eta)$-embedding of k' into K, identifying K as a separable closure of k').

The extension $k'/\kappa(\eta)$ of the function field of $S = \operatorname{Spec}(A)$ is finite, so we can find an A-finite A-subalgebra $B \subset k'$ with fraction field k'. The A-algebra B is a domain but also a finite product of henselian local rings by Proposition A.1.2, so it is a henselian local ring, and even strictly henselian local ring because its residue field is a finite extension of that of A. Note that $B \otimes_A A\{T\} = B\{T\}$ by Lemma 2.4.1 and Lemma 2.6.2 (with reasoning analogous to that given earlier in this proof). Hence, we may replace A with B and X with $X \times_{\operatorname{Spec} A} \operatorname{Spec} B$ to reduce to the case that $k' = \kappa(\eta)$. By excellence of A even its normalization in k' is module-finite, so we could choose B to be that normalization so that S is even normal. This completes the desired reduction step to considering normal S and certain Galois connected finite étale covers E_η of X_η as described above.

Now with S normal we assume there exists a Galois connected finite étale cover $E \to X_\eta$ with degree $n > 1$ that is a unit on S such that $E \otimes_{\kappa(\eta)} K$ is connected, and we seek a contradiction. By spreading-out considerations we can certainly finite a dense open $U \subset S$ and a Galois connected finite étale cover $\widetilde{E}$ over $X_U := X \times_S U$ with generic fiber E. Here is a crucial construction:

Lemma 3.2.5. *At the cost of scalar extension to the normalization of S in a finite Galois extension of its function field, it can be arranged that $S - U$ has codimension at least 2 in S.*

This is the key step; its proof uses Proposition 3.2.4 at the end.

Proof. The complement $S - U$ is a proper closed subset of the noetherian S and so has at most finitely many codimension-1 points in S (as they must be among the finitely many generic points of $S - U$). If there are no codimension-1 points of S in $S - U$ then there is nothing to do, so suppose there are such points, say $s_1, \ldots, s_N$. Each local ring $\mathcal{O}_{S,s_j}$ is a discrete valuation ring since S is normal.

For a uniformizer π_j of $\mathcal{O}_{S,s_j}$, the monic $X^n - \pi_j$ is certainly irreducible over $\kappa(\eta)$ and its splitting field $L_j/\kappa(\eta)$ is a cyclic extension of degree n since $\kappa(\eta)$ contains primitive nth roots of unity (as A is normal with separably closed residue field in which n is a unit). Ergo, the compositum L of the extensions L_j of $\kappa(\eta)$ is an finite abelian extension whose Galois group is n-torsion, so $[L : \kappa(\eta)]$ is a unit on S and $L/\kappa(\eta)$ has ramification degree divisible by n at each s_j.

Let $W = \operatorname{Spec}(B)$ be the normalization of S inside $\operatorname{Spec} L$ and let $X_W := X \times_S W = \operatorname{Spec}(B\{T\})$. Define $W_U := W \times_S U$ and $X_U :=$

$X \times_S U$. Note that the base change $\widetilde{E} \times_U W_U$ is finite étale over W_U and so is *connected* because its generic fiber is $E \times_\eta \operatorname{Spec} L$, which is connected by the hypotheses on E. Normality of X (inherited from that of S) ensures that the connected noetherian $\widetilde{E} \times_U W_U$ is normal and thus *irreducible*. Hence, $\widetilde{E} \times_U W_U$ has a function field. The finite étale map $\widetilde{E} \times_U W_U \to X_U \times_U W_U$ thereby extends to a finite surjection $Y \to X_W$ where Y is the integral normalization of $X_U \times_U W_U$ in the function field of $\widetilde{E} \times_U W_U$. (The restriction of Y over $X_U \times_U W_U$ recovers the W_U-finite $\widetilde{E} \times_U W_U$ because this latter scheme is normal due to étaleness over the normal W_U.)

We claim that Y is étale over all codimension-1 points of W. It suffices to check that Y is étale over all generic points of $W \setminus W_U$ that are codimension-1 in W. That is, we only need check Y is étale over all points of $W \setminus W_U$ lying over some s_j. Once that is proved, then by replacing S with W and X with X_W (as we may certainly do!) the open locus in W over which $Y \to W$ is étale not only contains W_U but even has complementary codimension at least 2 in W. This would complete the proof of lemma.

Let B_j denote the module-finite integral closure of the discrete valuation ring $\mathcal{O}_{S,s_j}$ in the finite Galois extension L of $\kappa(\eta)$. Our problem now comes down to proving that when the connected finite étale Galois cover $E \times_\eta \operatorname{Spec} L \to X_\eta \times_\eta \operatorname{Spec} L$ is extended by normalization to a finite integral scheme Y_j over $X \times_S \operatorname{Spec}(B_j)$ then it is étale. The formation of such normalization commutes with scalar extension to the strict henselization of $\mathcal{O}_{S,s_j}$ (which causes B_j to be replaced with the direct product of its strict henselizations at its finitely many maximal ideals, and L with the direct product of their fraction fields).

We can use Lemma 3.4.1 to describe the strict henselization of B_j at each maximal ideal as a root extraction over the strict henselization of $\mathcal{O}_{S,s_j}$ (this is where we finally use that n is a unit on S, as that ensured the Galois extension $L/\kappa(\eta)$ has degree that is a unit on S!). Ergo, we conclude by applying Proposition 3.2.4 upon expressing $X = \operatorname{Spec}(A\{T\})$ as a limit of smooth affine A-curves X_α (and realizing our setup with $\widetilde{E} \to X_U$ as a base change from some $X_{\alpha_0} \times_S U$). $\square$

We next reduce to the setting in which the restriction $\widetilde{E}|_{T=0}$ of $\widetilde{E}$ over $\{T = 0\} \cap X_U = U$ is split as a finite étale over of U. To make this reduction, we will need to make a preliminary base change on S to its S-finite normalization in a finite Galois extension of $\kappa(\eta)$, an operation that we have seen is harmless for our purposes. As for finite étale covers of connected schemes in general, the finite étale cover

$\widetilde{E}|_{T=0} \to U$ is split by a *connected* finite étale cover $U' \to U$ (so U' is normal, hence irreducible, due to normality of U). Ergo, by applying base change from S to its normalization S' in the function field of U' and replacing U with its preimage U' in S', we have arranged that $\widetilde{E}|_{T=0}$ is split over U.

We have reduced to the setting that S is normal, $S - U$ has codimension at least 2 in S, and the Galois connected finite étale cover $\widetilde{E} \to X_U$ becomes split upon restriction over $\{T = 0\} \cap X_U = U$. The following general lemma, Lemma 3.2.6, thereby applies to conclude that $\widetilde{E} \to X_U$ is split, so its generic fiber $E \to X_\eta$ is split, contradicting that E is connected with degree $n > 1$ over X_η! The proof of Proposition 3.2.2 is therefore done, conditional on Proposition 3.2.4 (which we used crucially at the end of the proof of Lemma 3.2.5). $\square$

At the very end of the preceding proof, we required the following general result:

Lemma 3.2.6. *Suppose A is a strictly henselian excellent normal noetherian local ring and $U \subset S := \operatorname{Spec} A$ is a dense open subset such that $S - U$ has codimension at least 2 in S. Let $X := \operatorname{Spec} A\{T\}$ and $V := U \times_S X$. If $V' \to V$ is a finite étale cover such that $V' \times_X \{T = 0\} \to V \times_X \{T = 0\} = U$ is split then $V' \to V$ is split.*

The idea is to use fpqc descent to reduce to the study of complete noetherian rings, because over the T-adic completion $A[\![T]\!]$ of $A\{T\}$ it is easier to lift infinitesimal information. However, this idea requires some care because the formation of completion does not commute with localization (such as passage to an open subscheme).

Proof. Since V' is normal, it is the normalization of V inside the fraction field of V'. Ergo, since normalization commutes with localization, it follows that we have a fiber square

$$\begin{tikzcd} V' \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ V \arrow[r] & X \end{tikzcd} \tag{3.2}$$

where B is the module-finite normalization of the excellent normal noetherian domain $A\{T\}$ in the generic fiber of V' over the connected normal V.

If we knew that the finite map $\operatorname{Spec} B \to \operatorname{Spec} A\{T\} = X$ were étale then we would be done because X is strictly henselian (and so every finite étale cover of X is split). By fpqc descent, in order to check

$\operatorname{Spec} B \to \operatorname{Spec} A\{T\}$ is étale it suffices to check $\operatorname{Spec}\widehat{B} \to \operatorname{Spec} A[\![T]\!]$ is étale. Let $\widehat{X} := \operatorname{Spec} A[\![T]\!]$, and respectively define $\widehat{V}, \widehat{V'}, \operatorname{Spec}\widehat{B}$ to be the base change of $V, V', \operatorname{Spec} B$ along $\widehat{X} \to X$. (Warning: although $\widehat{B}$ is the T-adic completion of B due to the finiteness of $\operatorname{Spec} B \to X$, $\widehat{V}$ is generally not the T-adic completion of V since the open immersion $V \to X$ is generally not finite; i.e., localization and completion do not generally commute.)

Since B inherits excellence from X, its T-adic completion inherits normality from B, as the completion of a normal excellent noetherian ring along any ideal is excellent [EGA, IV_3, 7.8.3(v)]. Recall that sections of the structure sheaf of a normal noetherian scheme over an open set with complement of codimension ≥ 2 uniquely extend to global sections. By design the open subscheme $V' \subset \operatorname{Spec}(B)$ has complement with codimension at least 2. Therefore,

$$\Gamma(\widehat{V'}, \mathcal{O}_{\widehat{V'}}) = \Gamma(V', \mathcal{O}_{V'}) \otimes_{A\{T\}} A[\![T]\!] = B \otimes_{A\{T\}} A[\![T]\!] = \widehat{B}$$

(where the first equality uses the compatibility of the formation of global sections of quasi-coherent sheaves with flat base change).

We now set up notation to show that the finite map $\operatorname{Spec}\widehat{B} \to \widehat{X}$ is étale. Let V_m denote the closed subscheme $\{T^{m+1} = 0\} \subset \widehat{V}$, and let V'_m denote the same in $\widehat{V'}$. Note that $V_0 = U$ is connected. The hypothesis of the lemma says that the finite étale cover $V'_0 \to V_0$ is split; its degree coincides with the degree n of the finite étale cover $V' \to V$ of the integral normal noetherian scheme $V = X \times_S U$ (which is a dense open in the connected normal noetherian X). The infinitesimal invariance of the étale site then implies that the finite étale covers $V'_m \to V_m$ are *compatibly split* of degree n for all $m \geq 0$, so we obtain a map of $A\{T\}$-algebras

$$\begin{aligned} \phi : \Gamma(\widehat{V'}, \mathcal{O}_{\widehat{V'}}) &\to \varprojlim \Gamma(V'_m, \mathcal{O}_{V'_m}) \\ &= \left(\varprojlim \Gamma(V_m, \mathcal{O}_{V_m})\right)^n \\ &= A[\![T]\!]^n. \end{aligned}$$

To summarize, we have identifications

$$\begin{array}{ccc} \Gamma(\widehat{V'}, \mathcal{O}_{\widehat{V'}}) & \xrightarrow{\phi} & \left(\varprojlim \Gamma(V_m, \mathcal{O}_{V_m})\right)^n \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ \widehat{B} & \xrightarrow{\widetilde{\phi}} & A[\![T]\!]^n \end{array} \tag{3.3}$$

and it suffices to show that the $A[\![T]\!]$-algebra map $\widetilde{\phi}$ is an isomorphism (as that would ensure $\widehat{B}$ is finite étale over $A[\![T]\!]$, which is all we need to prove). By commutativity of the diagram, it is the same to show that ϕ is an isomorphism.

Upon restricting $\widetilde{\phi}$ over U, we obtain a map

$$\begin{tikzcd} \coprod_{i=1}^{n} \widehat{V} \arrow[rr, "\nu"] \arrow[dr] & & \widehat{V'} \arrow[dl] \\ & \widehat{V} & \end{tikzcd} \tag{3.4}$$

and it suffices to show that ν is an isomorphism. Since both $\coprod_{i=1}^{n} \widehat{V}$ and $\widehat{V'}$ are finite étale over $\widehat{V}$, the map ν is finite étale. Therefore, the image in $\widehat{V'}$ of each connected component $\widehat{V}$ in the source of ν is open and closed. We know that no two of the maps $\widehat{V} \to \widehat{V'}$ induced by ν agree because ν is an isomorphism upon restriction over $\{T = 0\}$, so $\widehat{V'}$ has at least n connected components. But $\widehat{V'}$ is a degree-n finite étale cover of the connected scheme $\widehat{V}$, so it must be a union of those n copies of $\widehat{V}$, and ν must be an isomorphism as desired. $\square$

3.3. **Proof that smooth morphisms are locally acyclic.** We have everything we need to prove that smooth maps are locally acyclic (and so complete the proof of the smooth base change theorem), granting Proposition 3.2.4 (which we will prove in § 3.4):

Theorem 3.3.1. *A smooth map* $f : X \to S$ *is locally acyclic.*

The idea is as follows. Since smooth maps are locally the composition of an étale map with a map of the form $\mathbf{A}^n_S \to S$, we can reduce to checking maps of the form $\mathbf{A}^1_S \to S$ are locally acyclic. Verifying the latter amounts to checking the cohomology of $\mathbf{A}^1_S$ is as expected in degrees 0 and 1 (essentially the content of Proposition 3.2.1 and Proposition 3.2.2, the latter resting on Proposition 3.2.4 that has not yet been proved).

Proof. We first reduce to the case that the map is of the form $\mathbf{A}^1_S \to S$. Since the properties of being smooth locally acyclic are local on the source and target, we can assume that X and S are both affine.

Furthermore, recall that smooth morphisms $X \to S$ factor Zariski-locally on the source and target as

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathbf{A}^n_S \arrow[dl] \\ & S & \end{tikzcd} \tag{3.5}$$

with $X \to \mathbf{A}^n_S$ étale. By Proposition 3.1.2, in order to show smooth morphisms are locally acyclic, it suffices to show étale morphisms are locally acyclic and $\mathbf{A}^n_S \to S$ is locally acyclic. The case of étale maps follows from Lemma 3.1.1. It only remains to show $\mathbf{A}^n_S \to S$ is locally acyclic. By induction on n, and using Proposition 3.1.2, we can further reduce to the case of showing that $\mathbf{A}^1_S \to S$ is locally acyclic.

We want to show that for any geometric point $s \to S$, geometric point x of $X := \mathbf{A}^1_S$ over s, and geometric point t of $\operatorname{Spec} \mathcal{O}^{sh}_{S,s}$, the $\mathbf{Z}/(n)$-cohomology of the vanishing cycles scheme $\widetilde{X}^x_t$ vanishes in positive degree and coincides with $\mathbf{Z}/(n)$ in degree 0. That is, we want to show $\widetilde{X}^x_t$ is weakly acyclic. We may rename $\operatorname{Spec} \mathcal{O}^{sh}_{S,s}$ as S so that S is local and strictly henselian with closed point s, and we can assume that S is integral with t over its generic point η by replacing S with the closure of the image of t in S.

The fiber $\widetilde{X}^x_t$ is a limit of étale affine schemes E_α over the smooth affine curve $\mathbf{A}^1_t$ over the separably closed field $\kappa(t)$. Our study of the étale cohomology of curves gives that $H^q(E_\alpha, \mathbf{Z}/(n)) = 0$ for all $q > 1$ since n is invertible on S, so passage to the limit gives that $H^q(\widetilde{X}^x_t, \mathbf{Z}/(n)) = 0$ for all $q > 1$.

It remains to consider $q = 0, 1$. If x is the origin in $\mathbf{A}^1_s$ then Proposition 3.2.1 and Proposition 3.2.2 do the job when A is excellent. More generally, still with x the origin in $\mathbf{A}^1_s$, we can infer the case of general A (for $q = 0, 1$) from the excellent case by using the compatibility of cohomology with limits. Indeed, A is the direct limit of the henselizations of localizations of finitely generated $\mathbf{Z}$-subalgebras at contractions of $\mathfrak{m}_A$, and all such henselizations are excellent since finitely generated $\mathbf{Z}$-algebras inherit excellent from $\mathbf{Z}$ and henselization preserves excellence. This establishes the desired results for $q = 0, 1$ when x is the origin in $\mathbf{A}^1_s$. Now apply the lemma below. $\square$

Lemma 3.3.2. *Let* A *be a strictly henselian local ring,* $S := \operatorname{Spec} A$*,* s *the closed point of* S*,* $X := \mathbf{A}^1_S$*,* t *a geometric point of* S*. For all geometric points* y *of* X_s*, the vanishing cycles scheme* $\widetilde{X}^y_t$ *is weakly acyclic.*

Proof. We proceed by analyzing the residue field at y, letting x be the origin in $\mathbf{A}^1_s$. First, suppose that the residue field of y in $\mathbf{A}^1_s$ is $\kappa(s)$, so y corresponds to a prime ideal $(T - a, \mathfrak{m}_A)$ for some $a \in A$. Then the A-automorphism of $A[T]$ given by $T \mapsto T - a$ induces an A-algebra isomorphism between the strict henselizations of $\mathbf{A}^1_A$ at x and y, so $\widetilde{X}^y_t \simeq \widetilde{X}^x_t$, giving the desired weak acyclicity.

Now we consider y a general closed point of $\mathbf{A}^1_s$ (a geometric point since $\kappa(s)$ is separably closed). By expressing $\kappa(y)/\kappa(s)$ is a tower of finitely many primitive extensions, we can clearly build a finite faithfully flat local A-algebra B such that the residue field of B is $\kappa(s)$-isomorphic to $\kappa(y)$.

Let $Z := \mathbf{A}^1_{\operatorname{Spec} B}$, and s' the closed point of $\operatorname{Spec} B$. Let W denote the fiber $Z \times_X y$ in the fiber cube

$$\begin{tikzcd}
 & W \arrow[rr] \arrow[dl] \arrow[dd] & & s' \arrow[dl] \arrow[dd] \\
Z \arrow[rr] \arrow[dd] & & \operatorname{Spec} B & \\
 & y \arrow[rr] \arrow[dl] & & s \arrow[dl] \\
X \arrow[rr] & & \operatorname{Spec} A \arrow[uu] &
\end{tikzcd}$$

in which all squares are Cartesian. Since $s' \to s$ is purely inseparable of finite degree on residue fields, W is y-finite and infinitesimal. In other words, there is a unique point z of Z in the fiber over y. The key point is that by design, $\kappa(z)$ has the same residue field as B because if k'/k is a purely inseparable finite extension of fields (such as $\kappa(y)/\kappa(s)$) then $k' \otimes_k k'$ is an Artin local ring with the natural map $k' \otimes_k k' \twoheadrightarrow k'$ identifying its residue field with k'.

By Lemma 2.6.2, we see that $\operatorname{Spec} \mathcal{O}^{sh}_z \simeq \operatorname{Spec} \mathcal{O}^{sh}_y \times_S \operatorname{Spec} B$. We can clearly construct a geometric point t' of $\operatorname{Spec} B$ mapping to the geometric point t in $\operatorname{Spec} A$ as in the following diagram:

$$\begin{tikzcd} t' \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ t \arrow[r] & \operatorname{Spec} A \end{tikzcd} \tag{3.6}$$

The vanishing cycles scheme $\widetilde{Z}_z^{t'} := \operatorname{Spec}(\mathcal{O}_z^{sh}) \times_{\operatorname{Spec} B} t'$ is weakly acyclic by the case treated at the start of this proof because z has the same residue field as B, and this scheme is identified with

$$\operatorname{Spec}(\mathcal{O}_y^{sh}) \times_S \operatorname{Spec} B \times_{\operatorname{Spec} B} t' \simeq \operatorname{Spec}(\mathcal{O}_y^{sh}) \times_S t'.$$

We conclude that $\operatorname{Spec}(\mathcal{O}_y^{sh}) \times_S t'$ is weakly acyclic.

Recall that our aim has been to prove that $\operatorname{Spec} \mathcal{O}_y^{sh} \times_S t$ is weakly acyclic. But this follows from topological invariance of the étale site since we have a fiber square

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_y^{sh} \times_S t' \arrow[r] \arrow[d] & t' \arrow[d] \\ \operatorname{Spec} \mathcal{O}_y^{sh} \times_S t \arrow[r] & t \end{tikzcd} \tag{3.7}$$

with $t' \to t$ corresponding to a purely inseparable finite extension since t is a geometric point.

The case when y is over a closed point in $\mathbf{A}_s^1$ is now settled, so it remains to treat the case that y is over the generic point η of $\mathbf{A}_s^1$. Rather generally, if $X \to S$ is any map of schemes, $t \to S$ is a geometric point, and $y \to X$ is a geometric point then for the natural map $i : X_t \to X$ and any abelian group A the stalk $(R^q i_*(A))_y$ is identified with

$$H^q(X_t \times_X \operatorname{Spec}(\mathcal{O}_y^{sh}), A) = H^q(t \times_S \operatorname{Spec}(\mathcal{O}_y^{sh}), A) = H^q(\widetilde{X}_t^y, A).$$

Hence, the cohomologies on the right vanish for a fixed q, A, t with varying y over some fiber X_s precisely when $R^q i_*(A)|_{X_s}$ has vanishing stalks at all geometric points of X_s. Thus, we are reduced to checking that if V is a scheme of locally of finite type over a field (such as an affine line over a field), or more generally a Jacobson scheme (see [EGA, IV$_4$, §10]), and an abelian étale sheaf $\mathcal{F}$ on V has vanishing stalks at all geometric closed points then $\mathcal{F} = 0$.

For any étale $U \to V$ and $f \in \mathcal{F}(U)$ we have to show $f = 0$. Each geometric closed point $\overline{u}$ of U lies over a geometric closed point $\overline{v}$ of V, so f has vanishing image in the stalk $(\mathcal{F}|_U)_{\overline{u}} \simeq \mathcal{F}_{\overline{v}}$. Thus, there exists an étale map $W(\overline{u}) \to U$ hitting the image u of $\overline{u}$ such that $f|_{W(\overline{u})} = 0$. For $W := \coprod_{\overline{u}} W(\overline{u}) \to U$ clearly $f|_W \in \mathcal{F}(W)$ vanishes. But $W \to U$ is an étale map whose *open* image contains all closed points, and the only such open subset of U is the entire space since U inherits the Jacobson property from V, so $W \to U$ is an *étale cover*. It follows that $\mathcal{F}(U) \to \mathcal{F}(W)$ is injective, so $f = 0$. $\square$

3.4. **Extending étale covers.** We now prove Proposition 3.2.4, completing the proof of Theorem 3.3.1. This requires a weak version of Abhyankar's Lemma, which tells us that after a finite base change we can extend covers over codimension-1 points:

Lemma 3.4.1 (Abhyankar's Lemma)**.** *Suppose A is a discrete valuation ring with fraction field K and L a finite Galois extension of K, of degree that is a unit in A. Let B be the A-finite normalization of A in L, and let A' and B' respectively denote the strict henselization of A and the strict henselization of B at a maximal ideal. Let π denote the generator of the maximal ideal of A'. Then*

$$B' = A'[\pi^{1/e}],$$

where e is the common ramification degree over A at maximal ideals of B.

The general form of Abhyankar's Lemma allows A to be regular local of any dimension and assumes B is regular (a strong hypothesis beyond the 1-dimensional case).

Proof. First note that B' and A' have isomorphic residue fields. Indeed, by hypothesis $n = [L : K]$ is a unit in A, so the Galois property for L/K implies that the residue field extension for B over A is normal with degree f satisfying $n = efg$ where g is the number of maximal ideals of B. In particular, f is not divisible by the residue characteristic, so the normal residue field extension is separable (and hence Galois). Thus, the extension of separable closures of the residue fields is trivial, so each of the g local factor rings of $A' \otimes_A B$ is totally tame of degree e over A'. One of those local factors is B', by Corollary A.1.3.

Now we can rename A' as A and B' as B to reduce to the case that A and B are strictly henselian discrete valuation rings, with $[L : K] = e$ a unit in A. We want to show that $B = A[\pi^{1/e}]$. To see this, let t be a uniformizer for B, so $t^e = u\pi$ for some $u \in B^\times$. The equation $X^e - u \in B[X]$ has reduction that is split with all roots simple over the residue field of B since $e \in B^\times$ and the residue field of B is separably closed. Thus, by the henselian condition any of those residual roots lifts to a root in B.

We have shown that $u = v^e$ for some $v \in B^\times$, so $(t/v)^e = \pi$. Replace t with t/v, so $t^e = \pi$. Now B contains $A[t] = A[X]/(X^e - \pi)$. But $X^e - \pi$ is Eisenstein, so $A[t]$ is a discrete valuation ring of degree e over A. Hence, $A[t]$ has the same fraction field as B, so the module-finite inclusion of normal domains $A[t] \hookrightarrow B$ is an equality. □

We next prove that if we can extend our cover over codimension-1 points then we can extend it over codimension-2 points. The idea is to show that in favorable situations non-étaleness must occur in codimension-1 if it occurs at all by verifying that the finite ring extensions of interest are flat with non-étale locus in the base given by the vanishing of a single regular element (namely, the discriminant).

Lemma 3.4.2 (Zariski-Nagata purity of the branch locus in dimension 2)**.** *Let $(C, \mathfrak{m})$ be a regular local ring of dimension 2 and C' a domain that is a finite normal extension of C étale over the complement of the closed point of* Spec C*. Then* Spec $C' \to$ Spec C *is étale.*

Proof. The ring C' is normal and is 2-dimensional at all maximal ideals, so it is CM at each maximal ideal due to Serre's homological criterion for normality. Thus, by the miracle flatness theorem [Ma2, 23.1], C' is flat over the regular local C.

Since C' is finite flat over the local C, so it is free, we can write $C' = C^n$ as C-modules. For the corresponding C-basis $\{e_1, \ldots, e_n\}$ of C', let

$$\delta = \det(\mathrm{Tr}_{C'/C}(e_i e_j)) \in C.$$

It is well-known from the theory of étaleness for finite algebras over fields that for any prime $\mathfrak{p}$ of C, the finite $\kappa(\mathfrak{p})$-algebra $C' \otimes_C \kappa(\mathfrak{p})$ is étale if and only if $\delta \in \kappa(\mathfrak{p})^\times$. The hypotheses thereby imply that δ is a unit at all height-1 primes of the normal domain C, so $\delta \in C^\times$ and hence $C'/\mathfrak{m}C'$ is étale over $C/\mathfrak{m}$. Thus, the 2-dimensional semi-local C' is C-étale at its maximal ideals too, so C' is C-étale. □

Before completing the proof of Proposition 3.2.4, we will need the following basic fact about the interaction between strict henselization and normalization:

Lemma 3.4.3. *Let X be an excellent noetherian scheme and $x \in X$ a point. Let $\widetilde{X}$ denote the normalization of X. Then, $\widetilde{X} \times_X$ Spec $\mathcal{O}_x^{sh}$ is the normalization of* Spec $\mathcal{O}_x^{sh}$.

Proof. We will first check that $\widetilde{X} \times_X \mathcal{O}_x^{sh}$ is normal. By the construction of strict henselizations, $\widetilde{X} \times_X$ Spec $\mathcal{O}_x^{sh}$ is a limit of schemes affine étale over the normal $\widetilde{X}$ (so the affine transition maps in the limit are étale and hence flat). Normality of a scheme means that each of its local rings are integrally closed domains, and clearly each local ring R on $\widetilde{X} \times_X \operatorname{Spec} \mathcal{O}_x^{sh}$ is a direct limit with local flat (hence injective!) transition maps of local rings on étale $\widetilde{X}$-schemes. This exhibits each R as a direct limit of integrally closed domains with injective transition maps, so such R are clearly also integrally closed domains, as desired.

Since the formation of the strict henselization and normalization commutes with passing to the underlying reduced scheme, we may and do assume X is reduced. Excellence of X implies the normalization map $f : \widetilde{X} \to X$ is finite, so $\widetilde{X}$ is noetherian and $\widetilde{X} \times_X \operatorname{Spec} \mathcal{O}_x^{sh}$ is $\mathcal{O}_x^{sh}$-finite (thus noetherian, since $\mathcal{O}_x^{sh}$ is noetherian).

By excellence of the reduced noetherian X (hence openness of its normal locus) and the construction of the normalization, there is a dense open subscheme $U \subset X$ such that $f : \widetilde{X} \to X$ restricts to an isomorphism over U. By flatness of $\mathcal{O}_x^{sh} \to X$, the open subscheme $f^{-1}(U) \times_X \mathcal{O}_x^{sh}$ in $\widetilde{X} \times_X \mathcal{O}_x^{sh}$ is dense and maps isomorphically onto the dense open subscheme of $U \times_X \mathcal{O}_x^{sh} \subset \mathcal{O}_x^{sh}$. Hence, by the universal property of normalization of reduced noetherian schemes, we obtain a unique $\mathcal{O}_x^{sh}$-morphism from $\widetilde{X} \times_X \operatorname{Spec} \mathcal{O}_x^{sh}$ to the normalization of $\operatorname{Spec} \mathcal{O}_x^{sh}$. But this map between normal noetherian schemes is finite and an isomorphism between dense opens, so it is an isomorphism. $\square$

Now we finally complete the proof of Proposition 3.2.4, and so the proof of Theorem 3.3.1. The idea of the proof is that we can use Lemma 3.4.1 to extend the étale cover over all codimension-1 points, and then Lemma 3.4.2 to extend the map over codimension-2 points.

Proof of Proposition 3.2.4. We first reduce to the case that $m = n$. Define $S_0 := \operatorname{Spec} R[\pi^{1/n}]$ with generic point η', $X_0 := X \times_S S_0$, and let $\widetilde{X_0}$ be the normalization of X_0 in the function field of $\widetilde{X_\eta} \times_\eta \eta'$. Granting the case $m = n$, $\widetilde{X}_0 \to X_0$ is an étale cover. Thus, $\widetilde{X}_0 \times_{X_0} X_1 \to X_1$ is a finite étale cover, so it suffices to check that $\widetilde{X}_0 \times_{X_0} X_1$ has η-fiber $\widetilde{X_\eta} \times_\eta \eta_1$ over $(X_1)_\eta = X_\eta \times_\eta \eta_1$. The normalization $(\widetilde{X_0})_\eta$ of $(X_0)_\eta = X_\eta \times_\eta \eta'$ in the function field of $\widetilde{X_\eta} \times_\eta \eta_1$ is $\widetilde{X_\eta} \times_\eta \eta'$, so

$$(\widetilde{X}_0 \times_{X_0} X_1)_\eta = (\widetilde{X_\eta} \times_\eta \eta') \times_{X_\eta \times_\eta \eta'} (X_\eta \times_\eta \eta_1) = \widetilde{X_\eta} \times_\eta \eta_1$$

over $(X_1)_\eta$ as desired.

Now we may and do assume $m = n$. We want to show $\widetilde{X}_1 \to X_1$ is finite étale. Finiteness follows because the normalization is finite by the excellence of X_1 (which follows from that of R over which X_1 is finite type). By Lemma 3.4.2 (applied over local rings at codimension-2 points of the smooth R-curve X_1) it suffices to check étaleness over the codimension-1 points of X_1.

By design, over η the map $\widetilde{X}_1 \to X_1$ restricts to $\widetilde{X_\eta} \times_\eta \eta_1 \to X_\eta \times_\eta \eta_1$ that is clearly (finite and) étale. Thus, we just have to check étaleness over the generic points of the smooth curve $(X_1)_s$ over the closed point s of S_1. We shall deduce such étaleness from Lemma 3.4.1.

Consider the normalization X' of X in $\widetilde{X_\eta}$ and any point $x' \in X'$ over a generic point w of X_s, so a uniformizer π of R is also a uniformizer of the discrete valuation ring $\mathcal{O}_w$ (since X is an S-curve). By Lemma 3.4.1 applied to the degree-n extension of function fields $K(X')/K(X)$, we have $\mathcal{O}_{x'}^{\mathrm{sh}} \simeq \mathcal{O}_w^{\mathrm{sh}}[\pi^{1/e}]$ over $\mathcal{O}_w^{\mathrm{sh}}$ with $e|n$ the ramification degree of X' over X at w. Letting R' be the strict henselization of R, by the compatibility of strict henselization and finite base change we see that the local ring at the unique point of $X' \times_S S_1$ over x' has strict henselization equal to the reduced local ring

$$\begin{aligned} \mathcal{O}_{x'}^{\mathrm{sh}} \otimes_R R[\pi^{1/n}] &= \mathcal{O}_w^{\mathrm{sh}} \otimes_{R'} (R'[\pi^{1/e}] \otimes_R R[\pi^{1/n}]) \\ &= \mathcal{O}_w^{\mathrm{sh}} \otimes_{R'} (R'[\pi^{1/e}] \otimes_{R'} R'[\pi^{1/n}]) \end{aligned} \tag{3.8}$$

Further, the normalization of the ring in Equation 3.8 is equal to $\prod_{\zeta \in \mu_e(R')} \mathcal{O}_w^{\mathrm{sh}}[\pi^{1/n}]$ since the ring

$$\begin{aligned} R'[\pi^{1/e}] \otimes_R R[\pi^{1/n}] &= R'[\pi^{1/e}] \otimes_{R'} R'[\pi^{1/n}] \\ &= R'[\pi^{1/n}][X]/(X^e - (\pi^{1/n})^n) \end{aligned}$$

has normalization $\prod_{\zeta \in \mu_e(R')} R'[\pi^{1/n}]$.

We next claim that $\mathcal{O}_w^{\mathrm{sh}}[\pi^{1/n}]$ is naturally identified with the strict henselization at every point of X_1 over any generic point w of the special fiber of $X \to S$. Indeed, this follows from the above computations, together with the facts that finite base change commutes with the formation of strict henselizations, and Lemma 3.4.3 (strict henselization interacts well with normalization).

Thus, for any generic point w_1 in the special fiber of $X_1 = X \times_S S_1$ and any point $x_1 \in \widetilde{X}_1$ over w_1, the preceding identification of $\mathcal{O}_w^{\mathrm{sh}}[\pi^{1/n}]$ with each factor ring of the normalization of $\mathcal{O}_{x'}^{\mathrm{sh}} \otimes_R R[\pi^{1/n}]$ for the image x' of x_1 in X' implies that the induced map $\mathcal{O}_{w_1}^{\mathrm{sh}} \to \mathcal{O}_{x_1}^{\mathrm{sh}}$ is an *isomorphism*. This establishes that $\widetilde{X}_1 \to X_1$ is étale over generic points of the special fiber of $X_1 \to S_1$, which is what we needed to show. $\square$

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Appendix A. Some Commutative Algebra

Here, for the reader's convenience, we collect a few statements of commutative algebra results that we chose not to prove; we content ourselves with giving references.

A.1. **Basics of henselian local rings.** Here, we review some basics of henselian local rings. First, recall:

Definition A.1.1. A local ring $(R, \mathfrak{m})$ with residue field k is *henselian* if for every monic $f \in R[T]$ and every simple root $a_0 \in k$ of f mod $\mathfrak{m}$, there exists $a \in R$ with $f(a) = 0$ and a mod $\mathfrak{m} = a_0$.

Here are some useful equivalent definitions of a henselian local ring, part of [EGA, IV_4, 18.5.11]:

Proposition A.1.2. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. The following are equivalent*

(1) R is henselian
(2) any finite R-algebra is a product of local rings
(3) if $\operatorname{Spec} S \to \operatorname{Spec} R$ is an étale map, with nonempty fiber over $\mathfrak{m}$, then there is a section $\operatorname{Spec} R \to \operatorname{Spec} S$.

Finally, we note that a local ring which is finite over a henselian ring is henselian:

Corollary A.1.3. *If $A \to B$ is a finite map of local rings with A henselian, then B is henselian.*

Proof. By Proposition A.1.2, to show B is henselian, it suffices to show every finite B-algebra is a product of local rings. Take R and finite B-algebra. Then, R is also a finite A-algebra, and so it is a product of local rings, by Proposition A.1.2. □

A.2. **Basics of excellent rings.** Here, we record some key properties of excellent rings (for which [Ma1, Ch. 13] is an excellent first reference). We start with the definition from [EGA, IV_2, 7.8.2]:

Definition A.2.1. A ring A is *excellent* if it is noetherian and satisfies the following three conditions:

(1) A is universally catenary.

(2) For all prime ideals $\mathfrak{p} \subset A$, the formal fibers of $A_\mathfrak{p}$ are geometrically regular (i.e., every fiber algebra of $\operatorname{Spec} \widehat{A_\mathfrak{p}} \to \operatorname{Spec} A_\mathfrak{p}$ is regular and remains so after any finite extension of its ground field).

(3) For all quotients $A \to B$ that are domains and all finite purely inseparable extensions K' over $K(B)$, there is a module-finite B-subalgebra B' of K' with fraction field K' such that the set of regular points of $\operatorname{Spec} B$ contains a nonempty open set.

Here are some useful results on excellence from [EGA, IV_2, 7.8.3]:

Lemma A.2.2. *Let A be an excellent ring.*

(1) Any localization of A and any ring of finite type over A is excellent.
(2) The ring $\mathbf{Z}$ is excellent.
(3) The completion of A at any ideal contained in the radical of A is excellent. In particular, if A is a local ring, the completion of A at its maximal ideal is excellent.
(4) If A is integral, then the normalization of A in any finite extension of $K(A)$ is a finite A algebra.

We now briefly discuss the ideas behind the proof of Lemma A.2.2. In proving (1), the crucial aspect is preservation under passage to finite-type algebras, and that is shown in [EGA, IV_2, 7.8.3(ii)]. (The essential aspect of this is also shown in [Ma1, 33.G]). For (2), the excellence of $\mathbf{Z}$ is immediate from the definition. For (3), the behavior under completions lies deeper and is [EGA, IV_2, 7.8.3(v)]. Finally, (4), the module-finiteness of normalization is [EGA, IV_2, 7.8.3(vi)] (or alternatively [Ma1, 33.H]).

We next note that passage to the henselization and strict henselization preserve the noetherian property (due to [EGA, 0_{IV}, 10.3.1.3]), and also preserve the excellence property. In the case of henselization the preservation of excellence is [EGA, IV_4, 18.7.6], but curiously the case of strict henselization was missed in [EGA]; a proof in that case is given in [Ga, Cor. 5.6].

Finally, we recall that the henselization and strict henselization of a normal ring are normal (essentially because normality in the noetherian case is local for the étale topology by Serre's homological criterion, and in the general case we can pass to direct limits with care):

Lemma A.2.3. *Let R be a local ring. The following are equivalent: R is a normal domain, the henselization R is a normal domain, and the strict henselization R is a normal domain.*

The equivalence for henselization is [EGA, IV_4, 18.6.9(i)], and for strict henselization is [EGA, IV_4, 18.8.12(i)].

A.3. **Separated and affineness properties.** Here, we collect two elementary, but useful facts about fiber products about separated and affine maps, which we use for a spectral sequence trick to reduce to proving smooth base change in the case that X is affine.

Lemma A.3.1. *Suppose we have a morphisms $X \to S$ and two X schemes $U \to X, V \to X$ so that the compositions $U \to S$ and $V \to S$ are separated. Then, $U \times_X V$ is S-separated. In particular, if S is affine, a fiber product of affines over X is separated.*

Proof. The last statement holds assuming the first, because any map of affine schemes is separated. We have a commuting triangle

$$\begin{tikzcd} U \times_X V \arrow[rr] \arrow[dr] & & U \times_S V \arrow[dl] \\ & S & \end{tikzcd} \tag{A.1}$$

so to show $U \times_X V \to S$ is separated, it suffices to show $U \times_X V \to U \times_S V$ and $U \times_S V \to S$ is separated. The latter holds because U and V are both separated. To show the first statement, observe that we have a fiber square

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & U \times_S V \arrow[d] \\ X \arrow[r] & X \times_S X \end{tikzcd} \tag{A.2}$$

The bottom rightward map is a locally closed immersion. Therefore, the top rightward map is also a locally closed immersion, hence separated. □

Lemma A.3.2. *Suppose $X \to S$ is a separated morphism with S affine and $U \to X, V \to X$ are two affine X schemes. Then $U \times_X V$ is also affine.*

Proof. We have a fiber square

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & U \times_S V \arrow[d] \\ X \arrow[r] & X \times_S X \end{tikzcd} \tag{A.3}$$

Since X is separated, the lower rightward map is a closed immersion, and hence the upper rightward map is also a closed immersion.

Therefore, $U \times_X V$ has a closed immersion into an affine scheme, and is therefore affine. $\square$

APPENDIX B. THE SPECTERS OF SEQUENCES ARE HAUNTING EUROPE

In this section, we prove Proposition B.1.2. In to do so, we will next state a longer version with more equivalent conditions, Proposition B.1.2, and prove that instead. First, in § B.1 we state this main result of the section. Then, in § B.2 we introduce various tools we need for the proof of Proposition B.1.2. Finally, we prove Proposition B.1.2 in § B.3.

B.1. **Statement of equivalent conditions.** In this section, we state Proposition B.1.2. The main thrust of its proof is well exposited in the mostly self-contained [Mi, pp. 230-pp. 238] (see in particular [Mi, IV §4 Lem. 4.8, Cor. 4.13, Cor. 4.9]). The original reference for the proof is [SGA4, XV] (in particular [SGA4, XV, Cor. 1.16; XV, Prop. 1.10]) though the material there is less self-contained. For the reason, we do not reproduce the proofs in [Mi], although we do summarize the techniques going into them in Remark B.3.1. We state an expanded form of Proposition 2.2.5, namely Proposition B.1.2, including additional equivalent conditions which we will use to relate the three appearing in Proposition 2.2.5 with those appearing in [Mi].

Before continuing, we introduce some notation, which is relegated to this section. While previously we had been using the notational convention that for x a geometric point of X, $\mathcal{O}_x^{sh}$ denotes the strict henselization of X at x, in this section we shall notate it as $\mathcal{O}_{X,x}^{sh}$. The reason for this is that we shall need to refer to geometric points of two schemes corresponding to maps from the spectrum of the same separably closed field (see, for example Proposition B.1.2(1C) and (2C)). In order to state the expanded version, we introduce the temporary nonstandard (!) notion of a morphism being "strongly acyclic." (In [Mi] this is just called acyclic.) We will see this is equivalent to the morphism being acyclic in the case that the map is qcqs.

Definition B.1.1. Let $g : Y \to X$ be a morphism, $X' \to X$ be a morphism, $\mathcal{F}$ be a sheaf on X'. Define $Y' := Y \times_X X'$ and $g' : Y' \to X'$. We say $g : Y \to X$ is *strongly acyclic for* $(X', \mathcal{F})$ if the following two conditions hold:

(1) The map $g'_* g'^* \mathcal{F} \to \mathcal{F}$ is an isomorphism.
(2) We have $R^i g'_*(g'^* \mathcal{F}) = 0$ for $i > 0$.

We say $g : Y \to X$ is *strongly acyclic* if for all finite type étale maps $X' \to X$ and all torsion sheaves $\mathcal{F}$ with invertible torsion orders on X, g is strongly acyclic for $(X', \mathcal{F})$.

We now state the main result of this section, which is proven in § B.3.

Proposition B.1.2 (An expanded version of Proposition 2.2.5)**.** *Let $g : Y \to X$ be a qcqs morphism of schemes. The following are equivalent.*

(1A) The morphism g is locally acyclic.

(1B) Let y be a geometric point of Y mapping to a geometric point x of X. Then, for any geometric point z of $\operatorname{Spec} \mathcal{O}^{sh}_{X,y}$ the geometric fiber $\operatorname{Spec} \mathcal{O}^{sh}_{Y,y} \times_{\operatorname{Spec} \mathcal{O}^{sh}_{X,x}} z \to z$ is strongly acyclic.

(1C) Let y be a geometric point of Y and consider y as a nonalgebraic geometric point of X via the composition $y \to Y \to X$. Then, for any (algebraic) geometric point z of $\operatorname{Spec} \mathcal{O}^{sh}_{X,y}$ the geometric fiber $\operatorname{Spec} \mathcal{O}^{sh}_{Y,y} \times_{\operatorname{Spec} \mathcal{O}^{sh}_{X,y}} z \to z$ is strongly acyclic.

(2A) For all geometric points y of Y mapping to a geometric point x of X, the map $\operatorname{Spec} \mathcal{O}^{sh}_{Y,y} \to \operatorname{Spec} \mathcal{O}^{sh}_{X,x}$ is acyclic.

(2B) For all geometric points x of X and y of Y with $y \mapsto x$, the induced map $\operatorname{Spec} \mathcal{O}^{sh}_{Y,y} \to \operatorname{Spec} \mathcal{O}^{sh}_{X,x}$ is strongly acyclic.

(2C) Let y be a geometric point of Y consider y also as a nonalgebraic geometric point of X via the composition $y \to Y \to X$. Then, for all geometric points y of Y, the map $\operatorname{Spec} \mathcal{O}^{sh}_{Y,y} \to \operatorname{Spec} \mathcal{O}^{sh}_{X,y}$ is strongly acyclic.

(3) For all Cartesian diagrams

(B.1)
$$\begin{tikzcd} Y'' \arrow[r, "g''"] \arrow[d, "h'"] & X'' \arrow[d, "h"] \\ Y' \arrow[r, "g'"] \arrow[d, "j'"] & X' \arrow[d, "j"] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

with j étale of finite type, h quasi-finite, and $\mathcal{F}$ a torsion sheaf on X'' with torsion orders invertible on X, the base change morphism

$$g'^* R^q h_*(\mathcal{F}) \to R^q h'_*(g''^* \mathcal{F})$$

is an isomorphism for all q.

Furthermore, if g is locally of finite type, then the above equivalent conditions are also equivalent to the following one.

(4) *For all geometric points* y *of* Y *closed in a geometric fiber* Y_x*, the map* $\operatorname{Spec} \mathcal{O}_y^{sh} \to \operatorname{Spec} \mathcal{O}_x^{sh}$ *is strongly acyclic.*

B.2. **Preparatory Equivalences.** The point of this section is develop tools to prove the equivalence of conditions (1A), (1B), (1C) as well as (2A), (2B), (2C) of Proposition B.1.2. The equivalences (1B) $\iff$ (1C) and (2B) $\iff$ (2C) follow from Lemma B.2.1 while the equivalence of (1A) $\iff$ (1B) and (2A) $\iff$ (2B) follow from Proposition B.2.4.

We begin with a lemma relating strict henselizations at nonalgebraic geometric points to strict henselizations at geometric points.

Lemma B.2.1. *Let* S *be a scheme and let* t *be a nonalgebraic geometric point lying over* $s \in S$ *factoring through some (algebraic) geometric point* $\overline{s}$ *with* $\kappa(\overline{s})$ *isomorphic to an algebraic separable closure of* $\kappa(s)$*. Then, there is a canonical identification* $\mathcal{O}_{\overline{s}}^{sh} \simeq \mathcal{O}_t^{sh}$*.*

The proof of the above lemma is essentially that both $\mathcal{O}_{\overline{s}}^{sh}$ and $\mathcal{O}_t^{sh}$ are defined as limits of étale covers, and so the residue fields of all such covers at points over s factor through $\kappa(\overline{s})$. For full details, see [Stacks, Tag 04HX].

For the remainder of the section, we build to proving Proposition B.2.4. Before doing so, we establish Lemma B.2.2, which gives an equivalent condition for a morphism being acyclic. We also establish Lemma B.2.3, which verifies strong acyclicity of the pushforward of a constant sheaf along a finite map. The following lemma is casually stated in [Mi, pp. 232], but for completeness, we give a proof.

Lemma B.2.2. *Let* $g : Y \to X$ *be a qcqs morphism* $X' \to X$ *be finite type and étale, with Cartesian square*

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

Let $\mathcal{F}$ *be an* n*-torsion sheaf on* X'*. Then, for all such* $(X', \mathcal{F})$*, the natural map* $\phi^q_{X',\mathcal{F}} : H^q(X', \mathcal{F}) \to H^q(Y', g'^*\mathcal{F})$ *is a bijection for all* q *if and only if for all such* $(X', \mathcal{F})$ *we have that* $\mathcal{F} \to g'_* g'^* \mathcal{F}$ *is an isomorphism and* $R^q g'_* (g'^*\mathcal{F}) = 0$ *for* $q > 0$*. In particular,* g *is strongly acyclic if and only if for all such* $(X', \mathcal{F})$*,* $\mathcal{F} \to g'_* g'^* \mathcal{F}$ *is an isomorphism and* $R^q g'_* (g'^*\mathcal{F}) = 0$ *for* $q > 0$*.*

Proof. The last statement regarding strong acyclicity follows from the prior statement by writing a torsion sheaf as a limit of n-torsion

sheaves. So, it suffices to prove the first statement. The proof is a spectral sequence computation. Start with the spectral sequence $H^q(X', R^p g'_* g'^* \mathcal{F})$ abutting to $R^{p+q} g'_*(g'^* \mathcal{F})$:
(B.2)

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \iddots \\
0 \arrow[r] & H^2(X', R^0 g'_*(g')^*\mathcal{F}) \arrow[r] \arrow[u] & H^2(X', R^1 g'_*(g')^*\mathcal{F}) \arrow[r] \arrow[u] & H^2(X', R^2 g'_*(g')^*\mathcal{F}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & H^1(X', R^0 g'_*(g')^*\mathcal{F}) \arrow[r] \arrow[u] & H^1(X', R^1 g'_*(g')^*\mathcal{F}) \arrow[r] \arrow[u] & H^1(X', R^2 g'_*(g')^*\mathcal{F}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & H^0(X', R^0 g'_*(g')^*\mathcal{F}) \arrow[r] \arrow[u] & H^0(X', R^1 g'_*(g')^*\mathcal{F}) \arrow[r] \arrow[u] & H^0(X', R^2 g'_*(g')^*\mathcal{F}) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \cdots
\end{tikzcd}$$

We first demonstrate the desired equivalence in the case $p + q = 0$. If $p + q = 0$, the spectral sequence has already converged to $g'_* g'^* \mathcal{F}$. Thus, $\mathcal{F} \simeq g'_* g'^* \mathcal{F}$ implies $\phi^q_{X',\mathcal{F}}$ is a bijection. Conversely, if $\phi^q_{X',\mathcal{F}}$ is a bijection for all pairs $(X', \mathcal{F})$, it follows that the map $\mathcal{F} \simeq g'_* g'^* \mathcal{F}$ is a bijection on stalks, since any strictly henselian local ring can be written as a limit of finite type étale covers, and limits commute with cohomology (as g is qcqs).

To complete one implication, we suppose $R^q g'_* (g'^* \mathcal{F}) = 0$ for $q > 0$ and we show $\phi^q_{X',\mathcal{F}}$ is a bijection for $q > 0$. If $R^q g'_* (g'^* \mathcal{F}) = 0$ for $q > 0$, the spectral sequence has already converged at the E_2 page, so

$$H^q(X', \mathcal{F}) \simeq H^q(X', R^0 g'_* g'^* \mathcal{F}) \simeq H^q(Y', g'^* \mathcal{F}).$$

Here, the first map is induced by the isomorphism of the previous paragraph and the second map is coming from the spectral sequence. The composition is precisely $\phi^q_{X',\mathcal{F}}$, and so $\phi^q_{X',\mathcal{F}}$ is an isomorphism.

Conversely, suppose $\phi^q_{X',\mathcal{F}}$ is an isomorphism for all pairs $(X', \mathcal{F})$. Now, we inductively assume $R^m g'_* (g')^* \mathcal{F} = 0$ for all $m < q - 1$ and all pairs $(X', \mathcal{F})$. To complete the proof, it suffices to show $R^{q-1} g'_* (g'^* \mathcal{F}) = 0$. To accomplish this, we will show that boundary map

$$\partial : H^0(X', R^{q-1} g'_*(g'^* \mathcal{F})) \to H^q(X', R^0 g'_*(g'^* \mathcal{F}))$$

has 0 image and 0 kernel. Note that this will complete the proof because we are showing this holds for all pairs $(X', \mathcal{F})$, and so we can check for any fixed $(X', \mathcal{F})$ that all stalks of $R^{q-1} g'_* (g')^* \mathcal{F}$ are 0 by writing the stalk as a limit of the pullback of $\mathcal{F}$ to finite-type étale covers of X'.

First, we show $\ker \partial = 0$. Label by $\psi^r_{X',\mathcal{F}}$ the natural map

$$H^r(X', R^0 g'_*(g'^*\mathcal{F})) \xrightarrow{\psi^r_{X',\mathcal{F}}} H^r(Y', g'^*\mathcal{F}).$$

Recall that $\phi^r_{X',\mathcal{F}}$ is given as the composition

$$H^r(X', \mathcal{F}) \simeq H^r(X', R^0 g'_*(g'^*\mathcal{F})) \xrightarrow{\psi^r_{X',\mathcal{F}}} H^r(Y', g'^*\mathcal{F}).$$

The first first map is an isomorphism by the $q = 0$ case, which we are inductively assuming. Since we are assuming $\phi^m_{X',\mathcal{F}}$ is an isomorphism for $m < q-1$, it follows that $\psi^m_{X',\mathcal{F}}$ is an isomorphism for $m < q-1$. However, we also know that the spectral sequence abuts to $H^r(Y', g'^*\mathcal{F})$. Taking $r = q-1$, and recalling our inductive assumption that $R^m g'_*(g'^*\mathcal{F}) = 0$ for $m < q-1$, we obtain the exact sequence

$$0 \longrightarrow H^{q-1}(X', R^0 g'_*(g'^*\mathcal{F})) \xrightarrow{\psi^{q-1}_{X',\mathcal{F}}} H^{q-1}(Y', g'^*\mathcal{F}) \longrightarrow \ker\delta \longrightarrow 0.$$

Since $\psi^{q-1}_{X',\mathcal{F}}$ is an isomorphism as described above, we obtain that $\ker\delta = 0$.

To complete the proof, we show $\operatorname{im}\delta = 0$. For this, we only need observe that the diagram

$$\begin{tikzcd} H^q(X', R^0 g'_*(g'^*\mathcal{F})) \arrow[rr, "h"] \arrow[dr, "\phi^q_{X',\mathcal{F}}"'] & & \operatorname{coker}\delta \arrow[dl] \\ & H^q(Y', (g')^*\mathcal{F}) & \end{tikzcd}$$

commutes. Since ϕ is an isomorphism, and h is a surjection, h must in fact be an isomorphism. Therefore, $\operatorname{coker}\delta = H^q(X', R^0 g'_*(g'^*\mathcal{F}))$. Then, $\psi^q_{X',\mathcal{F}}$ is a bijection, so $\operatorname{im}\delta = 0$, completing the proof. □

We now prove another lemma using spectral sequences, which verifies that if a map is strongly acyclic for a certain sheaf, it is also strongly acyclic for the pushforward of that sheaf along a finite map. This will be used in proving Proposition B.2.4.

Lemma B.2.3. *Suppose* $g : Y \to X$ *is acyclic. Then for any finite morphism* $f : X' \to X$, g *is strongly acyclic for* $(X, f_*\mathbf{Z}/(n))$.

Proof. We will verify $g_* g^* f_* \mathbf{Z}/(n)$ and $R^i g_*(g^* f_* \mathbf{Z}/(n)) = 0$ for $i > 0$. First, we check $g_* g^* f_* \mathbf{Z}/(n)$. Indeed, define Y', f', g' so that the square

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & X' \arrow[d, "f"] \\ Y \arrow[r, "g"] & X \end{tikzcd} \tag{B.3}$$

is Cartesian. By our assumption that g is acyclic combined with proper base change, we have

$$\begin{aligned} f_* \mathbf{Z}/(n) &\simeq f_* g'_* g'^* \mathbf{Z}/(n) \\ &\simeq g_* f'_* g'^* \mathbf{Z}/(n) \\ &\simeq g_* g^* f_* \mathbf{Z}/(n), \end{aligned}$$

as desired.

It only remains to show $R^i g_*(g^* f_* \mathbf{Z}/(n)) = 0$ for $i > 0$. Again, by proper base change, we have $g^* f_* \mathbf{Z}/(n) \simeq f'_* g'^* \mathbf{Z}/(n)$. So, we will show $R^i g_*(f'_* g'^* \mathbf{Z}/(n)) = 0$ for $i > 0$. In turn, since $g'^* \mathbf{Z}/(n) \simeq \mathbf{Z}/(n)$, we will show $R^i g_*(f'_* \mathbf{Z}/(n)) = 0$. Because we are assuming g is acyclic, we know $R^i g'_*(\mathbf{Z}/(n)) \simeq R^i g'_*(g'^* \mathbf{Z}/(n)) \simeq 0$. Also, because f is finite, f_* is exact by Lemma 2.4.2, so $R^i f_* = 0$ for $i > 0$. By the composition of functors spectral sequence, we obtain

$$R^i (f \circ g')_* (\mathbf{Z}/(n)) = 0$$

for $i > 0$. Since $f \circ g' = g \circ f'$, we also obtain $R^i (g \circ f')_* (\mathbf{Z}/(n)) = 0$. Further, $R^i f'_* (\mathbf{Z}/(n)) \simeq 0$ as f' is finite. From the composition of functors spectral sequence, we see the spectral sequence for $g \circ f'$ has already converged on the E_2 page, with the only potentially nonzero terms being $R^i g_*(f'_* \mathbf{Z}/(n))$. Since we know this spectral sequence converges to 0, it follows that $R^i g_*(f'_* \mathbf{Z}/(n))$, as we wanted to show. □

Using the previous two lemmas, we now prove the main result of this subsection, which is used in showing the equivalences $(1A) \iff (1B)$ and $(2A) \iff (2B)$ in Proposition B.1.2.

Proposition B.2.4. *Suppose* $g : Y \to X$ *is qcqs. Then,* g *is acyclic if and only if it is strongly acyclic.*

Proof. If g is strongly acyclic, it follows from [Mi, IV §4, Lemma 4.6] (which implies that the base change of a strongly acyclic map along a quasi-finite morphism is strongly acyclic) that it is also acyclic. It remains to show the converse. Indeed, suppose that g is acyclic. We must show that for any map $f : X' \to X$ and any torsion sheaf $\mathcal{F}$ on X' with torsion orders invertible on X, g is strongly acyclic for $(X', \mathcal{F})$. By writing $\mathcal{F}$ as a limit of n-torsion sheaves for varying n, It suffices to demonstrate the statement when $\mathcal{F}$ is an n-torsion sheaf for a particular n. Noting that strong acyclicity can be reformulated in terms of a bijection of universal δ-functors via Lemma B.2.2, we may apply Proposition 2.5.1 using Lemma 2.5.5(1) and Lemma 2.5.5(4). For this application, we needed to verify that for any finite morphism $h_\lambda : X'_\lambda \to X'$, g is strongly acyclic for $(X', (h_\lambda)_* \mathbf{Z}/(n))$. Indeed, this follows from Lemma B.2.3. □

B.3. **Proof of Proposition B.1.2.** Using the preparatory results developed in § B.2 we are now ready to prove Proposition B.1.2.

Proof of Proposition B.1.2. First, note that by Lemma B.2.1, we have equivalences $(1B) \iff (1C)$ and $(2B) \iff (2C)$. Next, by Proposition B.2.4, it follows that we have equivalences $(1A) \iff (1B)$ and $(2A) \iff (2B)$. It only remains to verify that $(1C) \iff (2C) \iff (3)$, and in the case that g is locally of finite type, these are all equivalent to (4).

The equivalence $(2C) \iff (3)$ is [Mi, IV §4, Lemma 4.8]. The equivalence $(1C) \iff (2C)$ is [Mi, IV §4, Corollary 4.13]. Under the additional assumption that g is locally of finite type, it follows that (4) is equivalent to $(2C)$ by [Mi, IV §, Corollary 4.9]. Note that the definition of locally acyclic used in [Mi] is that given in $(2C)$. □

Remark B.3.1. Since much of the content of the above proof was obtained by citing [Mi], we now give some indication of the ideas used in showing the equivalences $(1C) \iff (2C) \iff (3)(\iff (4)$ in case g is locally of finite type).

For $(2C) \implies (3)$, Milne shows that the two sheaves in (3) are isomorphic by verifying isomorphisms on stalks. To do so, Milne uses standard properties of strictly henselian rings to reduce to verifying that strongly acyclic maps are preserved under quasi-finite base change. He then verifies strongly acyclic maps are preserved under quasi-finite base change by reducing to showing that they are preserved under finite base change. He shows they are preserved under finite base change using the proper base change theorem applied to the resulting finite map.

For showing $(3) \implies (2C)$, Milne writes the strict henselization as a limit of étale covers. Milne then reduces to verifying strong acyclicity for particular étale covers of X, which can be verifies using (3).

The equivalence of (4) and (3) in the case g is locally of finite type essentially falls out of the proof that $(2C) \implies (3)$ as it was only necessary to consider closed points in this implication.

Finally, the equivalence of (1C) and (2C) is deduced from an analogous statement for strongly acyclic morphisms. To prove this analogous statement, Milne uses that the notion of being strongly acyclic is local (via Lemma B.2.2) to reduce to the affine case. Milne then verifies g is strongly acyclic for particular sheaves pushed forward from the geometric fibers of the map induced by g on étale stalks. Finally, Milne uses Proposition 2.5.1 to verify g is strongly acyclic for all torsion sheaves of invertible torsion orders on any finite type étale cover of X.

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