LECTURE 1: INTRODUCTION - CONVERSE THEOREM AND REDUCTION THEORY
LECTURE BY BRIAN CONRAD
STANFORD NUMBER THEORY LEARNING SEMINAR
OCTOBER 4, 2017
NOTES BY DAN DORE

We'll start by discussing the classical Converse Theorem, which is discussed in [3, §4.3]. We'll only discuss the cusp form case, but [3] does the general modular form case. Eventually, we'll want to switch the perspective from modular forms to automorphic representations, which is far more robust.

First, we need some notation. For $N \geq 1$, $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$, $k \geq 1$, and $f \in S_k(N, \chi)$ a weight k cusp form on $\Gamma_1(N)$ with character χ. We think of χ as being a map $\chi : (\mathbf{Z}/N\mathbf{Z})^\times = \Gamma_0(N)/\Gamma_1(N) \to \mathbf{C}^\times$, i.e. f is invariant for $\Gamma_1(N)$ and invariant up to χ on $\Gamma_0(N)$. The isomorphism between $\Gamma_0(N)/\Gamma_1(N)$ and $(\mathbf{Z}/N\mathbf{Z})^\times$ is given by mapping $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to a/d. We can write the Fourier expansion of f as $\sum_{n\geq 1} a_n q^n$ with $q = e^{2\pi i z}$ and $a_n = O(n^{k/2})$.

To talk about invariance properties of functions on the upper half-plane we will use the following notation. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(R)^+$ and f a function on the upper half-plane. We define:

Definition 1.
$$(f|_k g)(z) = (\det g)^{k/2} (cz + d)^{-k} f(g \cdot z)$$

Here, g acts on $\mathbf{H}$ by fractional linear transformations.

Let $W_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$; then $f|_k W_N = \left(\sqrt{N}z\right)^{-k} f\left(\frac{-1}{Nz}\right)$.

Define $\Lambda_N(s, f) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s) L(s, f)$, with $L(s, f) = \sum_{n\geq 1} \frac{a_n}{n^s}$ for $\mathrm{Re}(s) > \frac{k}{2} + 1$ the L-function of f. We can also write this as a variant on a Mellin transformation:

$$\Lambda_N(s, f) = \int_0^\infty f\left(\frac{it}{\sqrt{N}}\right) t^s \frac{dt}{t}$$

Then we have the fundamental theorem:

Theorem 2 (Hecke). (1) $\Lambda_N(s, f)$ is *entire* and bounded in vertical strips (we'll abbreviate this as "BVS").

(2) $\Lambda_N(s, f) = i^k \Lambda_N(k - s, f|_k W_N)$

The proof of part (i) uses the Phragmén-Lindelöf theorem, and is reminiscent of Riemann's proof of analytic continuation of the zeta function via the functional equation.

Note that if $N = 1$, $\chi = 1$, so $f|_k W_N = f$, and thus the second part of the theorem says that $\Lambda_1(s, f) = i^k \Lambda_1(k - s, f)$. In this case, Hecke gives a converse: if f is a function given by a Fourier series $f = \sum_n a_n q^n$ with $a_n = O(n^\alpha)$, then there is a criterion on $L(s, f)$ which implies that $f \in S_k(1)$.

How do we generalize this converse beyond level 1? The first issue is that we have two possibly different Dirichlet series coming from f and $f|_k W_N$, so there isn't a clear L-function criterion to

guess. However, we can remedy this by looking at more general L-functions by twisting by various Dirichlet characters. Weil's theorem will give a criterion for a Dirichlet series to be the L-series of a modular form in terms of the analytic properties of the L-series as well as properties of a big family of twists.

Weil was inspired by a stronger version of Hecke's theorem, allowing these "twists". Let $\psi : (\mathbf{Z}/m\mathbf{Z})^\times \to \mathbf{C}^\times$ be a *primitive* Dirichlet character, meaning that it is not just the composition of a character for a divisor d of m with the projection from $(\mathbf{Z}/m\mathbf{Z})^\times$ to $(\mathbf{Z}/d\mathbf{Z})^\times$ (non-primitive characters may have L-series with extra zeros where n is coprime to d but not to m).

With $f \in S_k(N, \chi)$ still, we define:

$$f_\psi = \sum_{n \geq 1} \psi(n) a_n q^n \in S_k(Nm^2, \chi\psi^2)$$

Here, as usual, we extend ψ to a function on $\mathbf{Z}$ by setting it to 0 if $\gcd(n, m) \neq 1$. The fact that this is a cusp form with level Nm^2 is not obvious and requires proof, which is covered in §4.3 of Miyake's book. In addition, we define:

$$\Lambda_N(s, f, \psi) = \Lambda_N(s, f_\psi) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s) L(s, f_\psi)$$

with the L-function $L(s, f_\psi) = \sum_{n \geq 1} \frac{\psi(n) a_n}{n^s}$. Now, we have the stronger version of Hecke's theorem:

Theorem 3. If $(m, N) = 1$ then:

(1') $\Lambda_N(s, f, \psi)$ is entire and BVS.

(2') $\Lambda_N(s, f, \psi) = i^k C_\psi \Lambda_N(k - s, f|_k W_N, \overline{\psi})$

Here, we define the constant C_ψ by:

$$C_\psi = \frac{\chi(m)\psi(N)}{m} W(\psi)^2, \quad \text{where } W(\psi) = \sum_{a \in (\mathbf{Z}/m\mathbf{Z})^\times} \psi(a) e^{2\pi i \frac{a}{m}}$$

$$= \chi(m)\psi(-N) \frac{W(\psi)}{W(\overline{\psi})}$$

The W's are just Gaußsums, and the identity above comes from the classical fact that $W(\psi)W(\overline{\psi}) = \psi(-1)m$.

Now we can state Weil's converse theorem:

Theorem 4 (Weil's Converse Theorem). Consider $f = \sum_{n \geq 1} a_n q^n, g = \sum_{n \geq 1} b_n q^n$ where $a_n, b_n = O(n^\alpha)$. Fix $N \geq 1, k \geq 1, \chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ such that $\chi(-1) = (-1)^k$ (this last condition is necessary for $S_k(N, \chi)$ to be non-zero). Then $f \in S_k(N, \chi)$ with $g = f|_k W_N$ if:

(i) $\Lambda_N(s, f), \Lambda_N(s, g)$ are entire, BVS, and $\Lambda_N(s, f) = i^k \Lambda_N(k - s, g)$.

2

(ii) For all primes $p \nmid N$ and primitive Dirichlet characters $\psi : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{C}^\times$, conditions (1')
and (2') from the stronger version of Hecke's theorem hold.

Miyake states a slightly more optimal version of this theorem where we can replace condition
(ii) by the analogous condition with a set of primes of positive density.

Weil's proof is a lot of gritty group theory with $\mathrm{SL}_2(\mathbf{Z})$, which is the sort of thing you read once
and never again. Good for Weil. We'll study this theorem via representation theory.

Remark 5. If $f = \sum_n a_n q^n \in S_k(N, \chi)$ is nonzero then f is an eigenform for all Hecke operators
$T(n)$ with $n \geq 1$ if and only if $a_1 \neq 0$ and $L(s, f) = \sum_n \frac{a_n}{n^s}$ is given by an Euler product:

$$L(s, f) = a_1 \prod_p \left(1 - \lambda_p p^{-s} + \chi(p) p^{k-1-2s} \right)^{-1}$$

Then, $f|_{T_p} = \lambda_p f$ for all primes p, and $a_p = \lambda_p a_1$.

Our major goal in this seminar will be to generalize the special case of Weil's theorem where f
is a "new" eigenform to the setting of automorphic representations of $\mathrm{GL}_2(\mathbf{A}_k)$ for any global field
k (with $\mathbf{A}_k$ the adele ring of k). What does this mean? We'll discuss a way to assign L-functions
to representation-theoretic data, and then we'll show that if the L function satisfies a natural list
of necessary conditions, then in fact the original representation-theoretic data "comes from" an
automorphic representation.

Remark 6. Even in the case $k = \mathbf{Q}$, we'll prove a statement that is stronger than Weil's theorem
because we'll incorporate Maass forms.

How do we relate cusp forms to $\mathrm{GL}_2(\mathbf{A_Q})$? First, there are various ways of stuffing $S_k(N, \chi)$
into function spaces related to coset spaces for SL_2b. In particular, we can define an injection
$S_k(N, \chi) \hookrightarrow L^2\left(\Gamma_1(N)\backslash\mathrm{SL}_2(\mathbf{R})\right)_{\mathrm{cusp}}$ by sending f to $(\phi_f : g \mapsto (f|_k g)(i))$. The subscript
"cusp" is referring to a condition we'll apply to elements of the Hilbert space to capture the idea
of vanishing at the cusps. Here, L^2 is with respect to a Haar measure (and in this case, the left
and right Haar measures agree; if this is true for a group G, we say the group is *unimodular*). To
translate the Hecke operators to the L^2-side, it is more convenient to use GL_2 rather than SL_2 and to
"adelize". An advantage of evaluating at i is that the stabilizer in $\mathrm{SL}_2(\mathbf{R})$ of i is $\mathrm{SO}(2)$, and the L^2
inner product in the double coset space $\Gamma_1(N)\backslash\mathrm{SL}_2(\mathbf{R})/\mathrm{SO}(2)$ corresponds to the Petersson inner
product for cusp forms (given by an explicit integral on the upper half-plane).

Why should we expect "adelizing" to be useful? First off, we'll see that the above coset
space $\Gamma_1(N)\backslash\mathrm{SL}_2(\mathbf{R})$ is isomorphic to $K_1(N)\backslash\mathrm{SL}_2(\mathbf{A_Q})/\mathrm{SL}_2(\mathbf{Q})$, with $K_1(N)$ the compact open
subgroup $K_1(N) = \{\gamma \in \mathrm{SL}_2(\widehat{\mathbf{Z}}) \mid \gamma \cong \left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \mod N\}$.

In order to relate these coset spaces, we need the *strong approximation theorem*:

Theorem 7 (Strong approximation theorem for SL_2[1]). If k is a global field, then for any finite
non-empty set of places S, $\mathrm{SL}_2(k) \subseteq \mathrm{SL}_2(\mathbf{A}_k^S)$ is dense.

---

[1]Or, more generally, for any split simply connected semisimple group

This is a corollary of the classical strong approximation theorem for $\mathbf{G}_a$, which states that $k \subseteq \mathbf{A}_k^S$ is dense, using the fact that the upper and lower unipotent subgroups $U^\pm \simeq \mathbf{G}_a$ generate SL_2. Then, using structure theory for semisimple groups, we can deduce the general case from the SL_2 case. In addition, we can remove the hypothesis that the group is split with more effort. See [1].

A special case of this is that $\mathrm{SL}_2(\mathbf{Q})$ is dense inside $\mathrm{SL}_2(\mathbf{A}_f)$ ($\mathbf{A}_f = \mathbf{A}_{\mathbf{Q}}^\infty$ is the ring of finite adeles). $\mathrm{SL}_2(\widehat{\mathbf{Z}})$ is a compact open subgroup of $\mathrm{SL}_2(\mathbf{A}_f)$, and it is a classical fact that $\mathrm{SL}_2(\mathbf{Z})$ is dense inside this.

Now, we can use this to show that we have a homeomorphism

$$\Gamma_1(N)\backslash\mathrm{SL}_2(\mathbf{R}) \xrightarrow{\sim} K_1(N)\backslash\left(\mathrm{SL}_2(\mathbf{R}) \times \mathrm{SL}_2(\widehat{\mathbf{Z}})\right)/\mathrm{SL}_2(\mathbf{Z})$$

with the $\mathrm{SL}_2(\mathbf{Z})$ factor embedded diagonally. Note that $\Gamma_1(N) = \mathrm{SL}_2(\mathbf{Z}) \cap K_1(N)$. Additionally, strong approximation over $\mathbf{Q}$ shows us that this is also equal to:

$$K_1(N)\backslash\mathrm{SL}_2\left(\mathbf{A}_{\mathbf{Q}} = \mathbf{R} \times \mathbf{A}_f\right)/\mathrm{SL}_2(\mathbf{Q})$$

Later, we'll see that $S_k(N, \chi)$ embeds into $L^2\left(\mathrm{GL}_2(\mathbf{Q})\backslash\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}), \widetilde{\chi}\right)_{\mathrm{cusp}}$ in a way such that the Hecke theory on the left side will go over to the right regular representation of $\mathrm{GL}_2(\mathbf{A}_f)$ on the L^2 space. If Z is the maximal central torus of GL_2, we'll see that the L^2 functions we'll consider are defined on $\mathrm{GL}_2(\mathbf{Q})\backslash\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})/Z(\mathbf{A})$ up to multiplication by $\widetilde{\chi}$; this double coset space has finite volume, so the analysis is well-behaved.

Our goal is to understand the irreducible Hilbert space representations π of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ that occur inside this "cuspidal" part of the L^2 space. This will be the representation-theoretic incarnation of the "new" cuspidal eigenforms. All of this makes sense for more general reductive groups G over global fields k.

The idea will be that we can decompose π into a "restricted tensor product" $\pi = \widehat{\otimes}_v \pi_v$. The π_v will be irreducible "admissible" Hilbert space representations of $G(k_v)$, and the "restricted" condition will be that all but finitely many π_v are "spherical". In the GL_1 case, this is like looking at products of characters of $k_v^\times$ such that all but finitely many are unramified. In order to get an idèle class character, we need this product to vanish on $k^\times$, and we'll come up with an analogous condition in general.

We'll classify the π_v (this is essentially functional analysis!), define L-functions and "ϵ-factors" $L(s, \pi_v), \epsilon(\pi_v, s)$ for the local components, and define $L(s, \pi) = \prod_v L(s, \pi_v)$, $\epsilon(\pi, s) = \prod_v \epsilon(\pi_v, s)$.

If π is "cuspidal automorphic", meaning it can be embedded in the space L^2_{cusp} discussed earlier, then $L(s, \pi)$ is entire, BVS, and $L(s, \pi) = \epsilon(s, \pi)L(1 - s, \check{\pi})$, where $\check{\pi}$ is the dual representation (i.e. the "contragredient" representation on the dual vector space). We'll also have a similar story for $L(s, \omega \otimes \pi)$ for $\omega : \mathbf{A}^\times/k^\times \to S^1$ an idèle class character. We can define these L-functions for representations π which do not come from L^2_{cusp}, so it makes sense to ask if the good analytic properties of the $L(s, \omega \otimes \pi)$ *imply* that $\pi \subseteq L^2_{\mathrm{cusp}}$. The answer, proved by Jacquet and Langlands, is yes, and this is what we'll show this year (for GL_2).

One point to stress here is that even though we're ultimately concerned with functions with very nice properties, in order to prove anything serious, we need to make heavy use of functional-analytic methods via the Hilbert space structure of L^2_{cusp}. This isn't algebra!

Our "Step 0" will be to develop basic information about $G(k) \backslash G(\mathbf{A})$ for connected reductive groups G over a global field k.

Aside: why do we care about converse theorems? A historically important example is the fact that they convinced Weil and some of his contemporaries that the Shimura-Taniyama conjecture had a good reason to be true. Everybody expected that L-functions of elliptic curves should have analytic continuations and functional equations, and at least quadratic twists give other elliptic curves, so this theorem is "evidence" that the L-functions of elliptic curves should be modular. Knowing a converse theorem for general groups could be helpful in proving instances of Langlands functoriality: in some cases, we could understand how the L-functions transform under functorial constructions such as taking symmetric squares, and perhaps we could transfer the good analytic properties through these transformations. Then, a converse theorem would show that an associated construction on the automorphic side (done via the local Langlands correspondence) would still be automorphic.

Remark 8. Even though we ultimately only care about GL_2 over global fields, dealing with other reductive groups will be helpful: often, we can only establish the properties we want for groups over certain small-degree number fields, so we'll need to take Weil restrictions of GL_2 over a bigger field.

For X an affine scheme of finite type over a global field k, we can topologize $X(\mathbf{A}_k)$ with the locally compact subspace topology given from a closed immersion $X \hookrightarrow \operatorname{Spec} k[x_1, \ldots, x_n]$.[2] Fortunately, this does not depend on the choice of closed embedding into affine space, and is functorial in X, so in particular it is a topological group when X is a k-group. In addition, $X(k) \subseteq X(\mathbf{A}_k)$ is discrete (since we can reduce this statement to the case of affine space).

In particular, to get the usual topology on $\mathbf{A}_k^{\times}$, we use the *closed* embedding $\mathbf{G}_m \hookrightarrow \operatorname{Spec} k[x, y]$ as $t \mapsto (t, t^{-1})$. This is *NOT* the subspace topology induced from $\mathbf{A}_k$. Likewise, $\mathrm{GL}_n(\mathbf{A}_k)$ does not have the subspace topology induced from $\mathrm{Mat}_n(\mathbf{A}_k)$. However, since $\mathrm{SL}_n \hookrightarrow \mathrm{Mat}_n$ *is* a closed immersion, so $\mathrm{SL}_n(\mathbf{A}_k)$ *does* have the subspace topology from $\mathrm{Mat}_n(\mathbf{A}_k)$. This allows a "cheat" to give a stupid definition of the topology on $\mathrm{GL}_n(\mathbf{A}_k)$ by the closed immersion $\mathrm{GL}_n \hookrightarrow \mathrm{SL}_{2n}$ given by $g \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & g^{-1} \end{smallmatrix} \right)$.

Remark 9. We can also make good sense of $X(\mathbf{A}_k)$ for a separated finite type k-scheme X: see [2]

Exercise 10. Show that for a Borel k-subgroup $B \subset \mathrm{PGL}_2$, the injective continuous map

$$B(\mathbf{A}_k)/B(k) \to \mathrm{PGL}_2(\mathbf{A}_k)/\mathrm{PGL}_2(k)$$

is *not* a topological embedding.

Next time, we'll discuss the solution to this exercise and how to fix the problem to get functorial coset spaces.

---

[2]the typical notation $\mathbf{A}_k^n$ here is problematic...

References

[1] Brian Conrad, *Strong Approximation in Algebraic Groups*, `http://math.stanford.edu/~conrad/248BPage/handouts/strongapprox.pdf`. Handout from Math 248B: Modular Curves, taught Winter 2011 by Brian Conrad.

[2] _____ , *Weil and Grothendieck approaches to adelic points*, Enseign. Math. (2) **58** (2012), no. 1-2, 61–97.

[3] Toshitsune Miyake, *Modular forms*, Springer Monographs in Mathematics, Springer-Verlag, Berlin. Translated from the 1976 Japanese original by Yoshitaka Maeda.