LECTURE 10: ADMISSIBLE REPRESENTATIONS AND SUPERCUSPIDALS I

LECTURE BY CHENG-CHIANG TSAI
STANFORD NUMBER THEORY LEARNING SEMINAR
DECEMBER 13, 2017
NOTES BY DAN DORE AND CHENG-CHIANG TSAI

We will consider a connected reductive group G over a global field L, with adele ring $\mathbf{A} = \mathbf{A}_L$, archimedean part $\mathbf{A}_\infty = \prod_{v \mid \infty} L_v$, and non-archimedean part $\mathbf{A}^\infty = \prod'_{v \nmid \infty} L_v$. Let $Z = Z(G)$ be the center and fix some character $\chi\colon Z(L)\backslash Z(\mathbf{A}) \to \mathbf{C}^\times$.

Fix a choice of a maximal compact subgroup $K_\infty \subseteq G(\mathbf{A})$, and let $Z(\mathfrak{g}_\infty)$ be the center of the universal enveloping algebra c. Recall an automorphic form is a function $f\colon G(\mathbf{A}) \to \mathbf{C}$ such that:

(i) $f(\gamma x) = f(x)$ for all $\gamma \in G(L)$.

(ii) $f(zx) = \chi(z) f(x)$ for all $z \in Z(\mathbf{A})$.

(iii) There exists an open compact subgroup $K^\infty \subseteq G(\mathbf{A}^\infty)$ such that $f(xg) = f(x)$ for all $g \in K^\infty$.

(iv-1) For all $x \in G(\mathbf{A}_\infty), y \in G(\mathbf{A}^\infty)$, the function $f(xy)$ is smooth in x, and f is right $Z(\mathfrak{g}_\infty)$-finite (i.e. $\dim_{\mathbf{C}} f \cdot Z(\mathfrak{g}_\infty)$ is finite, where we have $f \cdot v(x) = \frac{d}{dt} f(x e^{tv})$ for $v \in \mathfrak{g}_\infty$ and e^{tv} the exponential map $\mathfrak{g}_\infty \to G(\mathbf{A}_\infty)$).

(iv-2) f is right K_∞-finite (i.e. $\dim_{\mathbf{C}} f \cdot K_\infty$ is finite).

(v) f is "slowly decreasing".

Now, condition (iv-2) implies that $f \cdot K_\infty$ is a finite-dimensional complex representation of the compact Lie group K_∞ (one should check that this action is continuous in this finite-dimensional case). We may decompose this representation as $f \cdot K_\infty = \oplus_{\rho \in \mathrm{Irr}(K_\infty)} \rho^{e(\rho)}$ where this sum has finitely many nonzero terms, and $\mathrm{Irr}(K_\infty)$ is the set of isomorphism classes of irreducible complex representations of K_∞. One may restrict to those f that only one isomorphism class of ρ appears in the sum; such a projection is given by $f \mapsto (x \mapsto \frac{1}{|K_\infty|} \int_{K_\infty} f(xk) \bar{\pi}_\rho(k) dk)$, where $\bar{\pi}_\rho(k)$ is the complex conjugate of the character of ρ. Likewise, $f \cdot Z(\mathfrak{g}_\infty) \simeq Z(\mathfrak{g}_\infty)/J$ for some ideal $J \lhd Z(\mathfrak{g}_\infty) \simeq \mathbf{C}[t_1, \ldots, t_n]$ has finite codimension.

We write $\mathscr{A}(G, K^\infty, \rho, J)$ for the space of such f which are right invariant by K^∞, live in the ρ-isotypic component, and are (right)-annihilated by J. Likewise, we write $\mathscr{A}_0(G, K^\infty, \rho, J)$ for the cuspidal ones.

Now, we have the following big theorems:

Theorem 1 (Harish-Chandra)**.** When L is a number field, $\dim_{\mathbf{C}} \mathscr{A}(G, K^\infty, \rho, J) < \infty$.

Theorem 2 (Harder)**.** When L is a global function field, $\dim_{\mathbf{C}} \mathscr{A}_0(G, K^\infty) < \infty$.

Example 3. Let $L = \mathbf{Q}$, $G = \mathrm{GL}_2$, $\chi = 1$, $K_\infty = \mathrm{O}_2(\mathbf{R})$, $K^\infty = K_0(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{Z}) \mid c \equiv 0 \mod N\}$. Fix a weight $k \geq 1$, and let $\rho(r(\theta)) = e^{-ik\theta}, \rho\Big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\Big) = \mathrm{id}$, where $r(\theta)$ parametrizes $\mathrm{SO}_2(\mathbf{R}) \simeq S^1$. Finally, let $J = (\Delta - (-\frac{k}{2}(\frac{k}{2} - 1)))$, where Δ is the Casimir operator in $Z(\mathfrak{g}_\infty) \simeq \mathbf{C}[\Delta, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)]$.

Then $\mathscr{A}(G, K^\infty, \rho, J) \simeq M_k(\Gamma_0(N))$ and $\mathscr{A}_0(G, K^\infty, \rho, J) \simeq S_k(\Gamma_0(N))$, as we saw in the lectures on the relationship of the classical theory to the adelic theory.

We may write $\mathscr{A}(G)$ (resp. $\mathscr{A}_0(G)$) to be the union of $\mathscr{A}(G, K^\infty, \rho, J)$ (resp. $\mathscr{A}_0(G, K^\infty, \rho, J)$) over all possible choices of K^∞, ρ, J: these are respectively the spaces of automorphic forms and cusp forms.

Recall the following definition:

Definition 4. A $(\mathfrak{g}_\infty, K_\infty)$-module is a $\mathbf{C}$-vector space with an action of $U(\mathfrak{g}_\infty)$ and K_∞ with compatibilities:

(i) The K_∞-action is smooth and can be differentiated to the $U(\mathrm{Lie}\ K_\infty)$-action, when we regard this as a subspace of $U(\mathfrak{g}_\infty)$.

(ii) Exchanging the K_∞ and $\mathfrak{g}_\infty$ action amounts to the adjoint action of K_∞ on $\mathfrak{g}_\infty$.

This allows us to define:

Definition 5. An *admissible* representation π of $G(\mathbf{A})$ is a representation of $G(\mathbf{A}^\infty)$ which is simultaneously a $(\mathfrak{g}_\infty, K_\infty)$-module such that the finite-dimensionality holds: i.e. the subspace $\pi^{K^\infty,(\rho,J)}$ consisting of the right K^∞-invariant (ρ, J)-isotypic vectors is finite-dimensional for all K^∞, ρ, J as above, and $\pi = \cup \pi^{K^\infty,(\rho,J)}$.

Remark 6. Note that this isn't actually a representation of $G(\mathbf{A})$.

We've seen via Theorems 1 and 2 that the spaces $\mathscr{A}(G)$ over a number field and $\mathscr{A}_0(G)$ over any global field are admissible representations. We remark that when L is a number field, $\mathscr{A}(G)$ is not a representation of $G(\mathbf{A})$ unless $G(\mathbf{A}_\infty)$ is compact, for the K_∞-finite condition cannot hope to be preserved under a reasonable $G(\mathbf{A}_\infty)$-action.

Now we have the following big theorem which allows us to reduce certain questions about adelic representations to their local versions.

Theorem 7 (Flath)**.** An irreducible admissible representation of $G(\mathbf{A})$ can be written as $\pi \simeq \otimes'_v \pi_v$ for unique (up to isomorphism) irreducible admissible representations π_v of $G(L_v)$.

We still need to say what the right hand side of this theorem means. To do this, first fix a non-archimedean local field F. We define:

Definition 8. Let π be a representation of $G(F)$. For any compact open subgroup K, write π^K for the K-fixed vectors in π. We say:

(1) π is *smooth* if $\pi = \cup_K \pi^K$ where K ranges over the set of compact open subgroups of $G(F)$.

(2) π is *admissible* if it is smooth and if $\dim_{\mathbf{C}} \pi^K$ is finite for all such K.

Remark 9. Note that if π is *any* representation of $G(F)$, we may form a sub-representation $\pi_{\mathrm{smooth}} = \cup_K \pi^K$, and this is a smooth representation of $G(F)$.

Theorem 10. [1] An irreducible smooth (finite-length) representation is admissible.

[1] This theorem is likely due to Bernstein, but Cheng-Chiang is not sure.

We won't prove this theorem (and hopefully we won't need to use it) in this seminar, but references can be found in [1] and (at least in the GL_2 case) [3, §10]. The proof is difficult and involves an analysis of the 'supercuspidal' representations, which we will discuss later.

Now, let us fix a Haar measure μ on $G(F)$. For any $K \subseteq G(F)$ an open compact subgroup, we define:

$$\mathcal{H}(G(F), K) = \{f \in C_c(G(F)) \mid f(g_1 x g_2) = f(x)\ \forall g_1, g_2 \in K\}$$

Here, $C_c(G(F))$ is the space of compactly supported functions on $G(F)$. Convolution makes this into an algebra with unit $e_K = \mu(K)^{-1}\mathbb{1}_K$ (where $\mathbb{1}$ denotes the characteristic function).

Now, for all $f \in \mathcal{H}(G(F), K)$, we may write $f = \sum_{i \in I} c_i \mathbb{1}_{g_i K}$ for a finite set I, $c_i \in \mathbf{C}$, $g_i \in G(F)$. If $v \in \pi$, by smoothness there is some K_1 such that $v \in \pi^{K_1}$, so by shrinking K if necessary (noting that $\mathcal{H}(G(F), K) \longrightarrow \mathcal{H}(G(F), K')$ for any $K' \subseteq K$), we may define:

$$f \cdot v = \frac{1}{\mu(K)} \sum_{i \in I} c_i \pi(g_i) \cdot v = \int_{g \in G(F)} f(g)\pi(g) \cdot v \, d\mu(g)$$

Now, if $K_1 \subseteq K_2$, the natural inclusion $\mathcal{H}(G(F), K_2) \longrightarrow \mathcal{H}(G(F), K_1)$ is not a map of algebras, because it does not send the unit e_{K_2} to e_{K_1}. However, e_{K_2} is an idempotent in $\mathcal{H}(G(F), K_1)$.

This allows us to define:

Definition 11. The *full Hecke algebra* of $G(F)$ is the algebra of compactly supported smooth functions, which we may write as: $\mathcal{H}(G(F)) = C_c^\infty(G(F)) = \cup_K \mathcal{H}(G(F), K)$.

Note that this is not actually a (unital) algebra, because the inclusions in the above union do not preserve the units. Regardless, if π is a smooth representation, $\pi = \cup \pi^K$, so π is naturally a $\mathcal{H}(G(F))$-module. Furthermore, if e_K is the idempotent corresponding to K (i.e. the image of the unit $e_K \in \mathcal{H}(G(F), K)$ under the inclusion into $\mathcal{H}(G(F))$), we have $e_K \cdot \pi = \pi^K$.

Definition 12. A $\mathcal{H}(G(F))$-module V is smooth if $V = \cup_K \, e_K \cdot V$ where K ranges over the compact open subgroups of $G(F)$.

Lemma 13. There is a natural equivalence of categories between smooth representations of $G(F)$ and smooth $\mathcal{H}(G(F))$-modules.

We described one side of this equivalence, i.e. the action of $\mathcal{H}(G(F))$ on any $G(F)$-module π. Conversely, if one begin with a smooth $\mathcal{H}(G(F))$-module, one may define $\pi(g)v := \frac{1}{\mu(K)}\mathbb{1}_{gK} \cdot v$ for some small enough compact open subgroup K.

Lemma 14. Let V be an irreducible smooth $\mathcal{H}(G(F))$-module, and suppose that for some K, $V^K := e_K \cdot V \neq 0$. Then V^K is an irreducible $\mathcal{H}(G(F), K)$-module.

Proof. We have $\mathcal{H}(G(F), K) = e_K * \mathcal{H}(G(F)) * e_K$, i.e. any function may be made left and right K-invariant by convolution on both sides by e_K, so we see easily that V^K is a $\mathcal{H}(G(F), K)$-module.

If $W \subseteq V^K$ is a non-trivial proper $\mathcal{H}(G(F), K)$-submodule, then $\mathcal{H}(G(F)) \cdot W$ is a non-trivial proper $\mathcal{H}(G(F))$-submodule of V: as $W \subseteq V^K$, we have $\mathcal{H}(G(F)) \cdot W = \mathcal{H}(G(F))e_K W$. Now $e_K e_K w = e_K w = w \in \mathcal{H}(G(F)) \cdot W$ for any nonzero w, so this is nonzero. If we had $V = \mathcal{H}(G(F)) \cdot W = \mathcal{H}(G(F)) \cdot e_K \cdot W$, we would have $V^K = e_K V = e_K \mathcal{H}(G(F)) e_k \cdot W = \mathcal{H}(G(F), K)W = W$, contrary to hypothesis. □

Lemma 15. The isomorphism type of V^K as a $\mathcal{H}(G(F), K)$-module determines the isomorphism type of V as a $\mathcal{H}(G(F))$-module.

Proof. Assume that V_1, V_2 are irreducible smooth $\mathcal{H}(G)$-modules with V_1^K, V_2^K non-trivial such that we have an isomorphism $\phi\colon V_1^K \to V_2^K$ of $\mathcal{H}(G(F), K)$-modules. Fix any non-zero $w \in V_1^K$. As V_1 is irreducible, $V_1 = \mathcal{H}(G) \cdot w$. Thus, we may try to extend ϕ by setting $\phi(f \cdot w) = f \cdot \phi(w)$. In order to show that this is an isomorphism, it suffices to show that it is well-defined (since both V_i are irreducible and $\phi(w) \neq 0$). It suffices to show that if $f \cdot w = 0$, then $f \cdot \phi(w) = 0$ for any $f \in \mathcal{H}(G)$. Suppose that $f \cdot w = 0$. Then since $w \in V^K$, for any $f' \in \mathcal{H}(G)$, we have $(e_K * f' * f) \cdot w = (e_K * f' * f * e_K) \cdot w = 0$. But $e_K * f' * f * e_K \in \mathcal{H}(G, K)$, so by assumption we have:

$$0 = \phi\big((e_K * f' * f * e_K) \cdot w\big) = (e_K * f' * f * e_K) \cdot \phi(w) = (e_K * f') \cdot (f \cdot \phi(w))$$

Thus, letting $v = f \cdot \phi(w) \in V_2$, we see that v is annihilated by $(e_K * f')$ for any $f' \in \mathcal{H}(G)$. However, since V_2 is irreducible, if $v \neq 0$, $V_2 = \mathcal{H}(G) \cdot v$, so letting w' be a non-trivial vector in V_2^K, there is some $f' \in \mathcal{H}(G)$ such that $f' \cdot v = w'$. But then we have

$$0 = (e_K * f') \cdot v = e_K \cdot (f' \cdot v) = e_K \cdot w' = w' \neq 0$$

This is the desired contradiction. □

Lemma 16 (Schur's Lemma)**.** Suppose V is an irreducible smooth $\mathcal{H}(G(F))$-module and $\phi\colon V \to V$ is a $\mathcal{H}(G(F))$-morphism. Then ϕ is a scalar.

Proof. By irreducibility of V, $\mathrm{End}_{\mathcal{H}(G(F))}(V)$ is a division algebra (any nonzero endomorphism has a kernel which is a proper invariant subspace and therefore zero, and image which is a nonzero invariant subspace and therefore all of V). Since $\mathbf{C}$ is algebraically closed, any division algebra strictly containing $\mathbf{C}$ contains the field $\mathbf{C}(x)$, and therefore has countable dimension. Now, suppose ϕ is not a scalar. Then the sub-division algebra $\mathbf{C}(\phi)$ generated by ϕ in $\mathrm{End}_{\mathcal{H}(G(F))}(V)$ is larger than $\mathbf{C}$ and therefore we must have $\mathbf{C}(\phi) \simeq \mathbf{C}(x)$ by the above argument. Now, note that $\dim_{\mathbf{C}} \mathcal{H}(G)$ is countable because the topology of $G(F)$ is *separable*, i.e. there is a countable base of open sets, and we've seen that $\mathcal{H}(G(F), K)$ is finite-dimensional for any compact open subgroup K. Thus (since $V = \mathcal{H}(G) \cdot v$ for any non-zero $v \in V$), $\dim_{\mathbf{C}} V$ is countable. Therefore, $\dim_{\mathbf{C}} \mathrm{End}_{\mathcal{H}(G(F))}(V)$ is countable as well, so it cannot contain $\mathbf{C}(x)$. □

Suppose $\mathscr{G}$ is a connected reductive group scheme over $\mathscr{O}_F$, and let $K = \mathscr{G}(\mathscr{O}_F)$. This is a compact subgroup, which is in fact maximal[2]. We call $\mathcal{H}(G(F), K) = \mathcal{H}(G(F), \mathscr{G}(\mathscr{O}_F))$ the *spherical Hecke algebra* (with respect to the choice of integral structure $\mathscr{G}$).

Lemma 17. The spherical Hecke algebra $\mathcal{H}(G(F), K)$ with $K = \mathscr{G}(\mathscr{O}_F)$ is commutative.

Proof. We'll just give a proof in the case that $G = \mathrm{GL}_n$. (See Remark 18 below for general groups.) Fix a choice of uniformizer $\varpi_F \in F$. Then we have a set of representatives of $K\backslash G(F)/K$ given

[2]This uses some Bruhat-Tits theory, and Cheng-Chiang is not aware of a good reference without introducing Bruhat-Tits theory first.

by:

$$\beta = \left\{ \begin{pmatrix} \varpi_F^{a_1} & & & \\ & \varpi_F^{a_2} & & \\ & & \ddots & \\ & & & \varpi_F^{a_n} \end{pmatrix} \mid a_1 \geq a_2 \geq \cdots \geq a_n \in \mathbf{Z} \right\}$$

This fact is essentially just the existence of Smith normal form, i.e. via the structure theory of modules over a PID.

Thus, we have a $\mathbf{C}$-basis for $\mathcal{H}(G(F), K)$ given by the functions $\{\mathbb{1}_{KgK}, g \in \beta\}$. Now, consider $\iota\colon \mathrm{GL}_n(F) \to \mathrm{GL}_n(f)$ given by $\iota(g) = {}^t g$. Then we have:

- $\iota(g_1 g_2) = \iota(g_2)\iota(g_1)$, i.e. ι is an (anti)-involution. [3]
- $\iota(K) = K$.
- $\iota(g) = g$ for any $g \in \beta$.

This shows that ι induces an anti-involution on $\mathcal{H}(G(F), K)$ which acts trivially on the functions $\mathbb{1}_{KgK}$ for $g \in \beta$, so therefore it acts trivially on $\mathcal{H}(G(F), K)$. Since this trivial map interchanges the order of multiplication on $\mathcal{H}(G(F), K)$, this algebra must be commutative. □

Remark 18. In general for G a reductive group scheme over $\mathscr{O}_F$, the transpose should be replaced by an automorphism of the group G that stabilizes the centralizer T of a fixed maximal split torus (T is then a maximal torus, as G is quasi-split.), and acts as $t \mapsto t^{-1}$ on that torus. The required automorphism of T stabilizes the set of roots, and one lifts to G in a way that stabilizes $K := G(\mathscr{O}_F)$. When G has no factor of type $\mathbf{A}$, $\mathbf{D}$ and $\mathbf{E}_6$, this automorphism is given by the longest element of the Weyl group.

For general group G, see [4, Theorem 4.1]

Now, we define:

Definition 19. We say that an irreducible admissible representation is *unramified* if $\pi^{\mathscr{G}(\mathscr{O}_F)} \neq 0$.

This gives a corollary to Lemma 17.

Corollary 20. If π is an irreducible admissible representation which is moreover unramified, $\pi^{\mathscr{G}(\mathscr{O}_F)}$ is a nonzero irreducible $\mathcal{H}(G(F), \mathscr{G}(\mathscr{O}_F))$-module with finite $\mathbf{C}$-dimension and thus $\dim_{\mathbf{C}} \pi^{\mathscr{G}(\mathscr{O}_F)} = 1$.

Now, we return to the global setting. Let G be a connected reductive group over the global field L. Suppose we have, for each non-archimedean place v an irreducible admissible representation π_v. In addition, suppose that for each archimedean place v, we have an irreducible smooth $(\mathrm{Lie}(G(L_v)), K_v)$-module π_v, where $K_v \subseteq G(L_v)$ is some maximal compact subgroup. Now choose any model $\mathscr{G}$ over $\mathscr{O}_L$.[4] For almost all v, G is reductive over $\mathscr{O}_{L_v}$. Suppose further that for almost all such v, π_v is unramified with respect to $\mathscr{G}(\mathscr{O}_{L_v})$.

[3] An "involution" of an associative algebra is defined to be a linear map squaring to the identity which interchanges the order of multiplication, so the prefix 'anti' is unnecessary.

[4] we also have to fix a choice of $\mathscr{O}_L$ when L is a function field

Remark 21. The above condition on π is independent of the choices of $\mathscr{G}$ and $\mathscr{O}_L$ (any two different choices are isomorphic over almost all places).

Now, we may define:

Definition 22. Given a family π_v of irreducible admissible representations which are unramified at almost all places (in the sense described above), we "define" the restricted tensor product:

$$\bigotimes_v{}' \pi_v = \bigcup_{|S|<\infty, S_0 \subseteq S} \left(\bigotimes_{v \in S} \pi_v \right) \otimes \left(\bigotimes_{v \notin S} \pi_v^{\mathscr{G}(\mathscr{O}_{L_v})} \right)$$

Here, S_0 is the set of 'bad' places: i.e. all archimedean places, all places v such that $\mathscr{G}$ is non-reductive over $\mathscr{O}_{L_v}$, and all places where π_v is ramified with respect to $\mathscr{G}(\mathscr{O}_{L_v})$.

To make sense of this definition, note that the factors on the right-hand side are all one-dimensional by Corollary 20, so their tensor product "does nothing". More precisely, $\otimes_v' \pi_v$ is really an inverse limit:

$$\otimes_v' \pi_v = \varinjlim_{\substack{S \\ S_0 \subseteq S \\ |S|<\infty}} \otimes_{v \in S} \pi_v$$

The transition maps in this direct system are specified as follows: if $S' = S \sqcup T$, we regard $\otimes_{v \in S} \pi_v$ as $(\otimes_{v \in S} \pi_v) \otimes \left(\otimes_{v \in T} \pi_v^{\mathscr{G}(\mathscr{O}_{L_v})} \right) \subseteq (\otimes_{v \in S} \pi_v) \otimes (\otimes_{v \in T} \pi_v)$. We need to make choices of identifications, for each $v \notin S_0$, of $\pi_v^{\mathscr{G}(\mathscr{O}_{L_v})}$ with $\mathbf{C}$ at the outset, but the representations obtained by two different such choices differ by a unique isomorphism.

Now, we will prove Theorem 7, Flath's Theorem:

Theorem 23. Let G be a connected reductive group over a global field L. Let π be any irreducible admissible representation of $G(\mathbf{A})$. Then we have:

$$\pi \simeq \bigotimes_v{}' \pi_v$$

where π_v are irreducible admissible representations of $G(L_v)$ such that with respect to any choice of model $\mathscr{G}$ over $\mathscr{O}_L$, all but finitely many π_v are unramified.

Proof. (Step 1): First, we will pretend that there are only two places v_1, v_2, both of which are non-archimedean. We want to show that an irreducible admissible representation π of $G(F_1) \times G(F_2)$ (here, $F_i := F_{v_i}$) is of the form $\pi \simeq \pi_1 \otimes \pi_2$ where π_i is an irreducible admissible representation of $G(F_i)$.

Now, by admissibility, we have $\pi = \cup_{K_1, K_2} \pi^{K_1 \times K_2}$, where each $\pi^{K_1 \times K_2}$ is a finite $\mathbf{C}$-dimensional module over $\mathcal{H} := \mathcal{H}(G(F_1) \times G(F_2), K_1 \times K_2)$. This algebra $\mathcal{H}$ decomposes as $\mathcal{H} \simeq \mathcal{H}_1 \otimes \mathcal{H}_2$, with $\mathcal{H}_i = \mathcal{H}(G(F_i), K_i)$. We may apply a result from the theory of finite-dimensional representations of unital algebras. This implies that any irreducible $\mathcal{H}$-module V of finite dimension over $\mathbf{C}$ is always a tensor product $V \simeq V_1 \otimes V_2$, where V_i is an irreducible $\mathcal{H}_i$-module which is unique up to isomorphism. This is proved in [2, 3.4.1].

Therefore, by the equivalence of categories in Lemma 13, the $\mathcal{H}_i$-modules V_i correspond to irreducible admissible representations $\pi^{(K_i)}$ of $G(F_i)$ which are fixed by K_i, and we have $\pi^{K_1\times K_2} \simeq \pi^{(K_1)} \otimes \pi_2^{(K_2)}$ as $G(F_1)\times G(F_2)$-modules.

One may shrink K_1, K_2 to $K_1' \subseteq K_2, K_2' \subseteq K_2$ and apply the same argument to get inclusions $\pi^{(K_i)} \longrightarrow \pi_i^{(K_i')}$ and an isomorphism $\pi^{K_1'\times K_2'} \simeq \pi^{(K_1')} \otimes \pi^{(K_2)'}$ which is compatible with these inclusions. Passing to the unions, we get $\pi = \cup_{K_1,K_2}\pi^{K_1\times K_2} \simeq (\cup_{K_1}\pi_1^{K_1}) \otimes (\cup_{K_2}\pi_2^{K_2})$, which is the desired result.

(Step 2): Now, we can extend the first step to any finite number of finite places. To include also the archimedean places, suppose v is an archimedean place. Write $\mathfrak{g}_v := \mathrm{Lie}G(L_v)$. An admissible representation of $G(L_v)$ is a $(\mathfrak{g}_v, K_v)$-module π such that the space of $v \in \pi$ which are (ρ, J)-isotypic has finite dimension over $\mathbf{C}$ for any choice of $\rho \in \mathrm{Irr}(K_v)$, $J \subseteq Z(\mathfrak{g}_v)$ where J has finite $\mathbf{C}$-codimension. Let A_{K_v} be the algebra of finite measures on K. The Hecke algebra this time is $\mathcal{H}(\mathfrak{g}_v, K_v) := U(\mathfrak{g}_v) \otimes A_K$. The (ρ, J)-isotypic subspace of π can be realized as the image of an idempotent operator in, which then plays the role analogous to that of e_K before so that Step 1 continues to apply.

(Step 3): For the given π, for all $w \in \pi$, there exists a finite set of places S such that w is fixed by $G(\mathscr{O}_{L_v})$ for all $v \notin S$. The space π^S consisting of such w is an irreducible admissible representation of $\prod_{v\in S} G(L_v)$. Here the irreducibility follows from that of π using the same argument as in Lemma 14. We may apply the previous steps to show $\pi^S = \oplus_{v\in S}\pi_v$.

(Step 4): We have $\pi = \cup_S \pi^S$. Thus, it suffices to check that the construction of the π_v is compatible with change in S.

□

References

[1] J. Bernstein, *Representations of p-adic Groups,* Course notes from Harvard University, Fall 1992, `http://www.math.harvard.edu/~gaitsgde/Jerusalem_2010/GradStudentSeminar/p-adic.pdf`

[2] D. Bump, *Automorphic forms and representations,* Cambridge Univ. Press, Cambridge, 1998.

[3] Colin J. Bushnell and Guy Henniart, *The local Langlands conjecture for* GL(2), Springer-Verlag, Berlin, 2006.

[4] P. Cartier, *Representations of p-adic groups: a survey*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, 1979, pp. 111–155.