LECTURE 11: ADMISSIBLE REPRESENTATIONS AND SUPERCUSPIDALS I

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Let L is a global field, G a reductive group over L, and π an irreducible cuspidal automorphic representation of $G(\mathbf{A}_L)$. Recall that this implies that π is an admissible representation and that Flath's theorem says we may decompose π as $\pi = \otimes' \pi_v$ with π_v an irreducible admissible representation of $G(L_v)$ (at least for non-archimedean v), and v running through the places of L. Some properties of π may be read off from the corresponding properties of π_v: for example, the "levels" correspond, and the Hecke eigenvalue a_p of π corresponds to the Hecke eigenvalue of π_p (e.g. when $L = \mathbf{Q}$ and $G = \mathrm{GL}_2$).

Today, we will restrict our attention to the case of a non-archimedean local field F, and the group $G = \mathrm{GL}_2(F)$. Recall that an admissible representation π of G is a complex vector space π with a G-action such that:

- $$\pi = \bigcup_{\substack{K \subseteq G \\ K \text{ open compact}}} \pi^K$$

  where $\pi^K = \{v \in \pi \mid \pi(k) \cdot v = v \ \forall k \in K\}$. A representation satisfying this condition is called *smooth*.

- $\dim_{\mathbf{C}} \pi^K < \infty$ for any open compact subgroup $K \subseteq G$.

Inside the group $G = \mathrm{GL}_2(F)$, recall that there is a particular subgroup, called the *mirabolic subgroup*:

$$M = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix} \right) \mid a \in F^\times, b \in F \right\}$$

This has a unipotent subgroup

$$U = \left\{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right) \mid b \in F \right\}$$

We fix the identification $\mathbf{G}_a(F) \xrightarrow{\sim} U$ sending b to $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right)$, and think of U as $\mathbf{G}_a(F)$.

Fix a non-trivial additive character $\psi \colon F \to \mathbf{C}^\times$. Via the above identification, we think of this as a character on U. If π is an irreducible admissible representation of G, we define:

Definition 1. A *Whittaker model for* π is a G-embedding $\iota \colon \pi \hookrightarrow \mathrm{Ind}_U^G \psi$.

Recall that $C(G)^\infty$ is the complex vector space of $\mathbf{C}$-valued functions on G that are invariant by right translation by some open compact subgroup $K \subseteq G$. G acts on this space by right translation. We may think of $\mathrm{Ind}_U^G \psi$ as:

$$\mathrm{Ind}_U^G \psi = \{ f \in C(G)^\infty \mid f(ug) = \psi(u) f(g) \ \forall u \in U, g \in G \}$$

Likewise, we have:

$$\mathrm{Ind}_U^M \psi = \{ f \in C(M)^\infty \mid f(um) = \psi(u) f(m) \ \forall u \in U, m \in M \}$$

We define:

Definition 2. A *Kirillov model* for π is an M-embedding $\pi \hookrightarrow \operatorname{Ind}_U^M \psi$.

Remark 3. Since π might not be irreducible as an M-module, it is no longer obvious that a non-zero M-homomorphism from π to $\operatorname{Ind}_U^M \psi$ is an embedding. Nevertheless, this is true, as we will show later.

For $f \in \operatorname{Ind}_U^M \psi$¡ we may identify f as a function on $F^\times$ via:

$$f(x) := f\left(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$$

We may compute the M-action on such f by:

$$\left(\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \cdot f\right)(x) = f\left(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)\right) = f\left(\left(\begin{smallmatrix} ax & bx \\ 0 & 1 \end{smallmatrix}\right)\right) = f\left(\left(\begin{smallmatrix} 1 & bx \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} ax & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = \psi(bx)f(ax)$$

Let $K(F^\times)$ be the space of functions on $F^\times$ that are left translation-invariant by some $1 + \mathfrak{p}^n \subseteq F^\times$ and are supported on a compact subset of F. Here, $\mathfrak{p}$ is the maximal ideal of $\mathscr{O}_F$.

Now, we have:

Lemma 4. The map given by restricting $f \in \operatorname{Ind}_U^M \subseteq C(M)^\infty$ to $\left(\begin{smallmatrix} F^\times & 0 \\ 0 & 1 \end{smallmatrix}\right)$ defines an isomorphism $\operatorname{Ind}_U^M \psi \xrightarrow{\sim} K(F^\times)$.

Proof. First, we see that this map lands inside $K(F^\times)$:

(i) We have $\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot f\right)(x) = f(ax)$. Since f is right translation-invariant by an open compact subgroup of M, if we take a sufficiently close to 1, this shows $f(x) = f(ax)$.

(i) Similarly, $f(x) = \left(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \cdot f\right)(x) = \psi(bx)f(x) = f(x)$ for $b \in \mathfrak{p}^n$ for some large enough n, i.e. $\psi(bx) = 1$ for all $x \in \operatorname{supp}(f)$, $b \in \mathfrak{p}^n$. Suppose ψ is non-trivial on some $\mathfrak{p}^{-m}$. Then we have $\operatorname{supp}(f) \subset \mathfrak{p}^{-m-n+1}$ is compact.

Next, if $f \in K(F^\times)$, we may extend f to an element of $\operatorname{Ind}_U^M(\psi)$ by defining $f\left(\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)\right) = \psi(b)f(a)$. The arguments above run in reverse show that $f \in C(M)^\infty$ (because any element of M can be written as a product $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$) is stable by right translation by an open compact subgroup of M. The same calculations let us easily check that these operations are inverse to each other. □

Now, Frobenius reciprocity gives:

$$\operatorname{Hom}_G(\pi, \operatorname{Ind}_U^G \psi) = \operatorname{Hom}_G(\pi, \operatorname{Ind}_M^G \operatorname{Ind}_U^M \psi) = \operatorname{Hom}_M(\operatorname{Res}_M^G \pi, \operatorname{Ind}_U^M \psi)$$

Thus, if π has a Kirillov model, then π has a Whittaker model (since maps from the irreducible G-representation π to $\operatorname{Ind}_U^G \psi$ must be embeddings).

We have:

Theorem 5. Let π be an irreducible admissible representation of GL_2 that is not one-dimensional. Then π has a Kirrilov model (for any fixed choice of ψ).

Proof. See [3, §1.2-1.6], [1, §4.4], or [2, §3.6]. □

In addition, the Kirrilov model for π is essentially unique:

Lemma 6.

$$\dim \operatorname{Hom}_M(\operatorname{Res}^G_M \pi, \operatorname{Ind}^M_U \psi) = 1$$

Proof. Suppose we have a Kirrilov model $j\colon \pi \hookrightarrow \operatorname{Ind}^M_U \psi \simeq K(F^\times)$. Consider the linear functional on π given by $L_j\colon f \mapsto (j(f))(1)$. Since for $a \in F^\times$ we have $j(f)(a) = \left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot j(f)\right)(1) = j\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot f\right)(1)$, the functional $L_j\colon \pi \to \mathbf{C}$ determines $j\colon \pi \hookrightarrow K(F^\times)$.

Now, note that if $f \in \ker L_j$, then $(j(f))(1) = 0$, so $(j(f))(a) = 0$ for all $a \in F^\times$ sufficiently close to 1, i.e. when $v(a-1) \geq n_0$ (where v is the valuation on F) for some n_0 sufficiently large.

Suppose ψ is non-trivial on $\mathfrak{p}^{-m}$ for some m, so that $\int_{\mathfrak{p}^{-m}} \psi(b)db = 0$. Now, we have, for all $x \in F^\times$ and for all $n \geq m + n_0$

$$\begin{aligned} \int_{\mathfrak{p}^{-n}} \psi(b)^{-1} j\left(\pi\left(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\right) \cdot f\right)(x)\, db &= \int_{\mathfrak{p}^{-n}} \psi(b)^{-1}\psi(bx)(j(f))(x)\, db \\ &= \int_{\mathfrak{p}^{-n}} \psi(b(x-1))(j(f))(x)\, db = 0 \end{aligned}$$

Now, since j is an embedding of π into a space of functions on $F^\times$, this says that:

$$I_n(f) := \int_{\mathfrak{p}^{-n}} \psi(b)^{-1} \pi\left(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\right) \cdot f\, db = 0$$

whenever $n \geq n_0$.

Thus, the kernel of L_j is contained in the space of f such that $I_n(f) = 0$ for all sufficiently large n. We'll see that this is an equality: if $j(f)(1) \neq 0$, then we have:

$$\begin{aligned} j(I_n(f))(1) &= \int_{\mathfrak{p}^{-n}} \psi(b)^{-1} j\left(\pi\left(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\right) f\right)(1)\, db \\ &= \int_{\mathfrak{p}^{-n}} \psi(0)(j(f))(1)\, db \\ &= j(f)(1) \cdot \operatorname{vol}(\mathfrak{p}^{-n}) \\ &\neq 0 \end{aligned}$$

Thus, the kernel of I_n is contained in the kernel of L_j, so they are equal. This shows that the kernel of L_j does not depend on j, so different choices of j can only change L_j by a constant multiple. This shows an *embedding* $j\colon \operatorname{Res}^G_M \pi \hookrightarrow \operatorname{Ind}^M_U \psi$ is unique up to a constant. But it is not a priori clear that a non-trivial homomorphism j' from $\operatorname{Res}^G_M \pi$ to $\operatorname{Ind}^M_U \psi$ is injective. Choose some such j', and let j be an embedding of $\operatorname{Res}^G_M \pi$ into $\operatorname{Ind}^M_U \psi$. Since j' is non-trivial, we see that $L_{j'} \neq 0$. Now, if $f \in \ker L_j = \{f \mid I_n(f) = 0\ \forall n \gg 0\}$, the argument above shows that $L_{j'}(f) = 0$, so $\ker L_j \subseteq \ker L_{j'}$. Since these are both codimension-one subspaces of π, we must have $\ker L_j = \ker L_{j'}$ and thus $L_{j'} = cL_j$ for some $c \neq 0$. As before, this implies that $j' = cj$ for some $c \neq 0$ and we see that j' is an embedding after all. $\square$

Let us now fix an irreducible admissible representation π of dimension greater than 1, and fix a Kirillov model $j\colon \pi \hookrightarrow \operatorname{Ind}^M_U \psi$. We will drop j from the notation and regard π as a subspace of $\operatorname{Ind}^M_U \psi$. Define $S(F^\times) \subseteq K(F^\times)$ to be the subspace of functions that are compactly supported in $F^\times$, i.e. the elements of $K(F^\times)$ that vanish in a neighborhood of 0. We call these *Schwarz functions* on $F^\times$.

Lemma 7. $S(F^\times) \subseteq \pi$.

Proof. (sketch) First, we verify that $S(F^\times)$ is an irreducible representation of M. This can be done as follows: Suppose ψ is trivial on $\mathfrak{p}^{-m+1}$ but not on $\mathfrak{p}^{-m}$. Let $f \in S(F^\times)$ be non-trivial, and $\mathrm{supp}(f) \subset a + \mathfrak{p}^n$ for some $a \in F$. Then we have

$$\frac{1}{|\mathfrak{p}^{-m-n}|} \int_{\mathfrak{p}^{-m-n}} \psi(ba)^{-1}\psi(bx)f(x)db = \begin{cases} f(x) & \text{if } x \in a + \mathfrak{p}^{n+1} \\ 0 & \text{else} \end{cases}$$

In other words, $f \mapsto \int_{\mathfrak{p}^{-m-n}} \psi(ba)^{-1}(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \cdot f)db$ restricts f to $a + \mathfrak{p}^{n+1} \subset \mathrm{supp}(f)$. By using this, one can construct from any non-zero function in $S(F^\times)$ any characteristic on $F^\times$ and thus any function in $S(F^\times)$.

Then, we have $\pi \cap S(F^\times) \neq 0$, because $\pi\Big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\Big) \cdot f - f \in S(F^\times)$ for any $b \in F$. We claim that we may choose b, f such that this is non-zero. This is because:

$$\left(\pi\Big(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\Big) \cdot f - f\right)(x) = \big(\psi(bx) - 1\big)\, f(x)$$

Since ψ is non-trivial, this is not always equal to 0. Since $S(F^\times)$ is irreducible, this means that $S(F^\times) \subseteq \pi$. □

This allows us to state the next lemma:

Lemma 8. Let $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then we have:

$$\pi = S(F^\times) + \pi(w)S(F^\times)$$

Proof. Since π is an irreducible G-representation, π is spanned by $\pi(g) \cdot S(F^\times)$ for $g \in G$. Now, we may use the Bruhat decomposition to write $G = B \sqcup UwB$ for $B = M \cdot Z$ the upper triangular Borel subgroup of G (Z is the center of $G = \mathrm{GL}_2(F)$, i.e. the scalar matrices). Thus, π is generated by $S(F^\times)$ and $\pi(uw)S(F^\times)$ for $u \in U$.

But we have:

$$\pi(uw)f - \pi(w)f = (\pi(u) - 1)(\pi(w)f) \in S(F^\times)$$

(as we may see by writing $u = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ and using the explicit description of the action of M on $K(F^\times)$). □

We will need the following general definition:

Definition 9. Let π be a smooth representation of G. The *contragredient representation* $\check{\pi}$ is the space of *smooth* functionals on π, i.e. we have:

$$\check{\pi} = \{\ell \colon \pi \to \mathbf{C} \mid \exists \text{ compact open } K \text{ such that } \ell(\pi(k)f) = \ell(f)\ \forall f \in \pi\}$$

In other words, $\check{\pi}$ is the subspace of smooth vectors in the linear dual of π.

If π is admissible, then for all K, $\pi = \pi^K \oplus \pi_K$, where π_K is the kernel of $e_K = \frac{1}{\mu(K)}\mathbb{1}_K \in \mathcal{H}(G, K)$, since e_K acts on π as projection onto π^K. This implies that the linear dual of π splits as $\pi^* = (\pi^K)^* \oplus (\pi_K)^*$, so $(\pi^*)^K = (\pi^K)^*$. Since $(\pi^*)^K = (\check{\pi})^K$, this shows that $(\check{\pi})^K$ is finite-dimensional, and thus $\check{\pi}$ is admissible. Moreover, the natural pairing $(\cdot,\cdot)\colon \pi\times\check{\pi}\to \mathbf{C}$ is non-degenerate, as it restricts to a non-degenerate pairing on $\pi^K\times(\check{\pi})^K$. Additionally, this shows that the natural map $\pi\to\check{\check{\pi}}$ is an isomorphism, since both sides are the sum of K-fixed subspaces, yet the map restricts to an isomorphism $\pi^K\cong\check{\check{\pi}}^K$ for each K.

Let us also remark that if π,π' are admissible representations, and there exists a non-degenerate pairing $(\cdot,\cdot)\colon \pi\times\pi'\to\mathbf{C}$ such that $(\pi(g)f,\pi'(g)f')=(f,f')$ for all $f\in\pi, f'\in\pi', g\in G$, then the natural map $\pi'\to\check{\pi}$ is an isomorphism. Non-degeneracy implies that the map is injective, and we can check surjectivity by passing to $(\pi')^K$, $(\check{\pi})^K$, and comparing dimensions.

Now, fix an irreducible admissible representation π and a Kirillov model $\pi\hookrightarrow K(F^\times)$. Recall that $\pi\big(\left(\begin{smallmatrix} c & 0\\ 0 & c\end{smallmatrix}\right)\big)$ acts by a constant by Schur's lemma. Thus, we may write:

$$\pi\big(\left(\begin{smallmatrix} c & 0\\ 0 & c\end{smallmatrix}\right)\big)=w_\pi(c)\,\mathrm{id}_\pi$$

We have:

Theorem 10. The contragredient representation $\check{\pi}$ of π can be realized in the same underlying abstract vector space as π with the action $\check{\pi}(g)=w_\pi(\det g)^{-1}\pi(g)$.

Proof. See [3, §1.6]. □

Then, a Kirillov model $\check{j}$ for $\check{\pi}$ is given by $\check{j}(f)(x)=w_\pi(x)^{-1}j(f)(x)$. This gives us an embedding $j\times\check{j}\colon \pi\times\check{\pi}\hookrightarrow K(F^\times)\times K(F^\times)$. This lets us define a pairing on $\pi\times\check{\pi}$ by:

$$(f,\check{f}) := \int_{F^\times} f_1(x)\check{f}(-x)\,d^*x+\int_{F^\times} f_2(x)(\check{\pi}\check{f})(-x)\,d^*x$$

Here, we choose $f_1,f_2\in S(F^\times)$ such that $f=f_1+\pi(w)f_2$.

We may define a pairing, for $f,f'\in\pi$:

$$(f,f')=\int_{F^\times} f_1(x)w_\pi(-x)^{-1}f'(-x)\,d^*x+\int_{F^\times} f_2(x)w_\pi(-x)^{-1}(\pi(w)f')(-x)\,d^*x$$

Thus, the essential content of Theorem 10 says that this pairing is non-degenerate, well-defined (i.e. it does not depend on the choice of representation of f in terms of f_1,f_2), and that:

$$(\pi(g)f,\pi(g)f')=w_\pi(\det g)(f,f')$$

Now, we may define:

Definition 11. An irreducible admissible representation π of G is called *supercuspidal* if for all $f\in\pi$, $\check{f}\in\check{\pi}$, the "matrix coefficient" $g\mapsto(\pi(g)f,\widehat{f})$ is a compactly supported function on G mod the center Z of G.

Note that this definition makes sense for any reductive group G, since the definition of $\check{\pi}$ of the pairing between π and $\check{\pi}$ are both perfectly general.

We have the following theorem:

Theorem 12. π is supercuspidal iff $\pi=S(F^\times)$ in the Kirillov model.

Remark 13. This theorem does not yet show that any supercuspidal representations exist! Note that in general $S(F^\times)$ is only a M-subrepresentation of π.

Proof. First, suppose that $\pi = S(F^\times)$. Note that by the previous discussion, this implies that $\check{\pi} = S(F^\times)$ as well, since they have the same underlying space and essentially the same Kirillov model (up to a central character, which does not affect the condition of being compactly supported away from 0).

Then for all $f \in \pi, \check{f} \in \check{\pi}$, we have:

$$\left(\pi\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) f, \check{f}\right) = \int_{F^\times} f(ax) \cdot \check{f}(-x)\, d^*x$$

Since $f, \check{f} \in S(F^\times)$, we have $\check{f}(-x) = 0$ for x outside a compact subset of $F^\times$. Thus (since inversion and multiplication are continuous on $F^\times$), $\operatorname{Supp}(\check{f})^{-1} \operatorname{Supp}(f)$ is a compact subset of $F^\times$ and the integrand vanishes for a outside this set.

Recall that we have the Cartan decomposition: $G = \cup_{a \in F^\times} K \cdot \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot K \cdot Z$ for $K = \mathrm{GL}_2(\mathscr{O}_F)$. Fix $f \in \pi, \check{f} \in \check{\pi}$, and let $K_1 \subseteq K$ be such that K_1 fixes both f and $\check{f}$. Since $[K : K_1] < \infty$, the K-orbit of f is finite and similarly for $\check{f}$. Let $f_1, \ldots, f_s, \check{f}_1, \ldots, \check{f}_r$ be the elements of these orbits. We see that there is a compact subset $\Omega \subseteq F^\times$ such that

$$\left(\pi\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \cdot f_i, \check{f}_j\right) = 0$$

for all i, j and any $a \in F^\times - \Omega$.

Then, for any g, we may write:

$$\begin{aligned}(\pi(g)f, \check{f}) &= \left(\pi(k_1 \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) k_2 z) f, \check{f}\right) \\ &= w_\pi(z) \left(\pi \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \pi(k_2) f, \pi(k_1^{-1}) \check{f}\right) \\ &= w_\pi(z) \left(\pi\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \cdot f_i, \check{f}_j\right)\end{aligned}$$

Thus, this is 0 unless $g \in \cdot K \cdot \left(\begin{smallmatrix} \Omega & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot K \cdot Z$, and this set is compact mod Z. Therefore, π is supercuspidal.

Conversely, suppose that π is supercuspidal. Take any $\check{f} \in \check{\pi}$. For any $f \in S(F^\times) \subset \pi$ and $a \in F^\times$ we again have:

$$\left(\pi\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \cdot f, \check{f}\right) = \int_{F^\times} f(ax) \check{f}(-x) d^*x$$

and we know that as a function of a, this is compactly supported in $F^\times$. As $\check{f}$ is smooth, as a function on $F^\times$ it is locally constant under some $1 + \mathfrak{p}^n$. By taking $f = \mathbb{1}_{1+\mathfrak{p}^n}$, the above compact support property implies that $\check{f} \in S(F^\times)$. Thus, $\check{\pi} = S(F^\times)$ and therefore we know that $\pi = S(F^\times)$ by the relationship between the Kirillov models of π and $\check{\pi}$. □

References

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