LECTURE 12: PRINCIPAL SERIES
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Last time, we discussed *supercuspidal representations* of $G_F := \mathrm{GL}_2(F)$ for F a non-archimedean local field. These are irreducible admissible representations π such that the "matrix coefficient" function $g \mapsto \langle g\phi, \psi \rangle$ is compactly supported mod the center of G for any $\phi \in \pi, \psi \in \check{\pi}$.

Fix once and for all a nontrivial additive character $\psi : F \to \mathbf{C}^\times$. Any irreducible admissible representation π admits a *Kirillov model*: an embedding $\pi \hookrightarrow S(F)$, the space of "Schwarz functions" on F:

$$\left\{ f\colon F^\times \to \mathbf{C} \mid f \text{ is locally constant with compact support on } F \right\}$$

such that the mirabolic subgroup acts by

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f(x) = \psi(bx) f(ax)$$

Now, we will discuss *principal series* representations:

Definition 1. Let $\mu_1, \mu_2\colon F^\times \to \mathbf{C}^\times$ be multiplicative characters. Then the principal series representation ρ_{μ_1,μ_2} associated to these is the space of locally constant functions $\varphi\colon G_F \to \mathbf{C}$ such that:

$$\varphi\big(\left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right) \cdot g\big) = \mu_1(a)\mu_2(b)|a/b|^{1/2}\varphi(g)$$

where G_F acts by right translation.

A more conceptual way to define these is to consider the Borel subgroup $B = \left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\right\}$ with unipotent radical $U = \left\{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\right\}$. Then $B/U \simeq \left\{\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)\right\} \simeq F^\times \times F^\times$. So we may take characters χ_1, χ_2 of $F^\times$ and consider the representation $\mathrm{Ind}_B^{G_F}(\chi_1 \otimes \chi_2)$. This is essentially ρ_{μ_1,μ_2} up to the scaling factor $|a/b|^{1/2}$.

The scaling factor is there in order to make the following theorem hold:

Theorem 2. The contragredient representation of ρ_{μ_1,μ_2} is $\rho_{\mu_1^{-1},\mu_2^{-1}}$.

The principal series representations, along with the supercuspidals, account for all irreducible admissible representations:

Theorem 3. If π is an irreducible admissible representation of $G_F = \mathrm{GL}_2(F)$ and not supercuspidal, then π is a subrepresentation of some ρ_{μ_1,μ_2}.

Theorem 4. Let $\mu := \mu_1/\mu_2$. Then we have the following cases:

- When $\mu(x) \neq |x|, |x|^{-1}$, $\pi_{\mu_1,\mu_2} := \rho_{\mu_1,\mu_2}$ is irreducible.

- If $\mu(x) = |x|$, ρ_{μ_1,μ_2} contains a codimension one representation which is irreducible:

$$\pi_{\mu_1,\mu_2} := \left\{ \varphi \mid \int \varphi\big(w^{-1} \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\big)\, dx = 0 \right\}$$

where $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

- If $\mu(x) = |x|^{-1}$, then there is a one-dimensional invariant subspace V of ρ_{μ_1,μ_2} such that $\pi_{\mu_1,\mu_2} := \rho_{\mu_1,\mu_2}/V$ is irreducible: $V = \left\{ \varphi \mid \varphi\big(w^{-1} \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\big) \text{ is constant} \right\}$

Theorem 5. The principal series π_{μ_1,μ_2} are not supercuspidal, and $\pi_{\mu_1,\mu_2} \simeq \pi_{\lambda_1,\lambda_2}$ iff $(\mu_1, \mu_2) = (\lambda_1, \lambda_2)$ or $(\mu_1, \mu_2) = (\lambda_2, \lambda_1)$.

Thus, together with Theorem 3, this shows that the set of isomorphism classes of irreducible representations is exactly the set of supercuspidal representations along with π_{μ_1,μ_2} for every unordered pair $\{\mu_1, \mu_2\}$ of characters of $F^\times$.

Now, we prove Theorem 3:

Proof. Suppose π is not supercuspidal: then its Kirillov model $K(\pi)$ is not equal to the space of Schwarz functions on $F^\times$, i.e. $S(F^\times) \subsetneq K(\pi)$. Defining $V = K(\pi)/S(F^\times)$, we obtain a finite-dimensional ([1, §1.2, Th. 1]) non-trivial vector space. We get a smooth representation of the upper triangular Borel subgroup $B = \left\{ \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \right\}$. Indeed, $B = ZM$, where M is the mirabolic subgroup, and M preserves $S(F^\times)$, while Z acts via the central character of π. The unipotent radical $U = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ acts trivially, since for any $f \in K(\pi)$, $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) f - f \colon x \mapsto f(x)\big(\psi(bx) - 1\big)$ is in $S(F^\times)$.

Thus, V gives us a finite-dimensional smooth representation of $F^\times \times F^\times \simeq B/U$. The space V admits a quotient which is an irreducible representation of the commutative group B/U, which must be one-dimensional by Schur's lemma. Such a one-dimensional representation must be of the form $\chi_1 \otimes \chi_2$ for χ_1, χ_2 characters of $F^\times$. We may write $\chi_1 = \mu_1 \cdot |\cdot|^{1/2}, \chi_2 = \mu_2 \cdot |\cdot|^{-1/2}$.

Thus, we have a non-zero linear map of B-representations $L \colon K(\pi)/S(F^\times) \to (\chi_1 \otimes \chi_2)$. In other words, we have:

$$L\big(\left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right) \cdot f\big) = \mu_1(a)\mu_2(b) \left|\frac{a}{b}\right|^{1/2} L(f)$$

By Frobenius reciprocity, we see that $\mathrm{Hom}_{G_F}(\pi, \mathrm{Ind}_B^{G_F}(\chi_1 \otimes \chi_2)) \neq 0$. Since $\mathrm{Ind}_B^{G_F}(\chi_1 \otimes \chi_2) = \rho_{\mu_1,\mu_2}$, we see that π embeds into the principal series ρ_{μ_1,μ_2}. □

Next, we prove Theorem 2:

Proof. Consider a locally compact group G (such as G_F) with a subgroup B. Then we want to have the "Fubini" theorem:

$$\int_G \varphi(g)\, dg = \int_{B\backslash G} \left(\int_B \varphi(bg)\, db \right) d\overline{g}$$

Here, dg, db are left-invariant Haar measures. This all works when B is unimodular, but is more complicated in general, since the function inside might not be left-invariant by B.

Defining $f(g) = \int_B \varphi(bg)\, db$, so we want:

$$\int_G \varphi(g)\, dg = \int_{B\backslash G} f(g)\, d\overline{g}$$

We have $f(b'g) = \int_B \varphi(bb'g)\, db$. Letting $\rho = bb'$, this is:

$$f(b'g) = \int_B \varphi(\rho g)\, d(\rho b'^{-1})$$

When B is unimodular, $d(\rho b'^{-1}) = d\rho$ so $f(b'g) = f(g)$, but this is not always true. In general, we have a character Δ_B (called the modulus character) so that $d(\rho b'^{-1}) = \Delta_B(b'^{-1})d\rho$. Thus, $f(b'g) = \Delta_B(b'^{-1})f(g)$.

Note that $G(F)$ is unimodular when G is a reductive group over F, since we have $G = Z \cdot \mathscr{D}G$ with Z the center and $\mathscr{D}G$ the derived subgroup. But the modulus character always kills the center Z (since right translation by elements of Z is the same thing as left translation). Further, the modulus character is induced by an algebraic character on $\mathscr{D}G$ (the absolute value of the determinant of the adjoint representation of G), and there are no nontrivial such characters, since $\mathscr{D}G$ equals its own derived group. So G is unimodular. However, Borel subgroups are not reductive, so B may - and in fact does - have a nontrivial modulus character, namely,

$$\Delta_B\big(\left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right)\big) = |b/a|$$

We may define an integral $\int_{B\backslash G} f(\overline{g})\, d\overline{g}$ for functions f such that $f(b'g) = \Delta_B(b'^{-1})f(g)$ for $b \in B$.

Now, let's return to the setting of $G_F = \mathrm{GL}_2(F)$. We have the Iwasawa decomposition[1] $G = KB$ with $K = \mathrm{GL}_2(\mathscr{O}_F)$ a closed compact subgroup, and one can use this to show that integration over $B\backslash G$ is the same as integration over K.

Now, we may define a pairing between ρ_{μ_1,μ_2} and $\rho_{\mu_1^{-1},\mu_2^{-1}}$ by:

$$\langle \varphi, \varphi' \rangle = \int_{B_F\backslash G_F} \varphi\varphi'\, dg$$

for $\varphi \in \rho_{\mu_1,\mu_2}, \varphi' \in \rho_{\mu_1^{-1},\mu_2^{-1}}$. To check that this integral makes sense, it suffices to verify that $(\varphi\varphi')$ transforms under left multiplication by B_F via the modulus character as above:

$$\begin{aligned}
\varphi\varphi'\big(\left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right) \cdot g\big) &= \varphi\big(\left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right) \cdot g\big) \cdot \varphi'\big(\left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right) \cdot g\big) \\
&= \Big(\mu_1(a)\mu_2(b)|a/b|^{1/2}\varphi(g)\Big) \cdot \Big(\mu_1^{-1}(a)\mu_2^{-1}(b)|a/b|^{1/2}\varphi'(g)\Big) \\
&= |a/b|(\varphi\varphi')(g) \\
&= \Delta_B\Big(\left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right)^{-1}\Big)(\varphi\varphi')(g)
\end{aligned}$$

It remains to check that this pairing is nondegenerate. Restricting to $K := \mathrm{GL}_2(\mathscr{O}_F)$, we see that for $a, b \in \mathscr{O}_F^\times$,

$$\overline{\varphi}\big(\left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right) \cdot g\big) = |a/b|^{1/2}\overline{\mu_1(a)\mu_2(b)}\overline{\varphi}(g) = \mu_1^{-1}(a)\mu_2^{-1}(b)\overline{\varphi}(g)$$

[1] as can be seen by hand for the case $G = \mathrm{GL}_2$ by identifying $G/B \simeq \mathbf{P}^1$ as $\mathscr{O}_F$-schemes and seeing directly that $G(\mathscr{O}_F) \to G/B(\mathscr{O}_F) \simeq \mathbf{P}^1(\mathscr{O}_F)$ is surjective; a general proof is given at [2]

Thus, by choosing (as we may) φ' such that $\varphi'|_K = \overline{\varphi}|_K$, we get:

$$\int_{\mathrm{GL}_2(\mathscr{O}_F)} \varphi\overline{\varphi}\, dg \neq 0$$

whenever $\varphi \neq 0$. Thus, the pairing is non-degenerate and therefore defines an isomorphism from $\rho_{\mu_1^{-1},\mu_2^{-1}}$ to $\rho_{\mu_1,\mu_2}^{\vee}$. □

Next, we prove Theorem 4. We do this by constructing something like a Kirillov model for ρ_{μ_1,μ_2}. The idea is to first replace the function $\phi : G_F \to \mathbf{C}$ with the function $\Phi : F \to \mathbf{C}$ defined by $\Phi(x) := \phi(w^{-1} \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right))$, where $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The function Φ determines ϕ (see below).

We then replace Φ with its Fourier transform:

$$\widehat{\varphi}(x) = \mu_2(x)|x|^{1/2} \int_F \varphi(y)\overline{\psi(xy)}\, dy$$

Then:

$$\left(\widehat{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \varphi(x)}\right) = \psi(bx)\widehat{\varphi}(ax)$$

so this behaves like a Kirillov model should. Unfortunately, this doesn't quite work as stated because of convergence issues, but it is the right idea.

Let $\varphi\colon \mathrm{GL}_2(F) \to \mathbf{C}$ be an element of ρ_{μ_1,μ_2}, so:

$$\varphi\big(\left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right) \cdot g\big) = \mu_1(a)\mu_2(b) \left|\frac{a}{b}\right|^{1/2} \varphi(g)$$

We may check that Φ determines φ, via the identity, for $c \neq 0$:

$$g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{pmatrix} c^{-1}\det g & * \\ 0 & c \end{pmatrix} w^{-1} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix} \tag{1}$$

which implies that

$$\begin{aligned}\varphi(g) &= \mu_1(c^{-1}\det g)\mu_2(c)|\det g|^{1/2}|c|^{-1}\varphi\left(w^{-1}\begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}\right) \\ &= \mu_1(c^{-1}\det g)\mu_2(c)|\det g|^{1/2}|c|^{-1}\Phi(d/c)\end{aligned}$$

We have the following useful identity:

$$\Phi(x) = \mu(-1)\varphi(e)\mu^{-1}(x)|x|^{-1}$$

for $|x| \gg 0$, and with $\mu = \mu_1/\mu_2$. This comes from

$$w^{-1} \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = \begin{pmatrix} -1 & -x^{-1} \\ 0 & -1 \end{pmatrix} \left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & x \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ -x^{-1} & 1 \end{smallmatrix}\right)$$

for $x \neq 0$, which implies that

$$\Phi(x) = \varphi\big(w^{-1} \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\big) = \mu(-1)\mu_1(x)^{-1}\mu_2(x)\varphi\Big(\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)\Big)$$

and the last term equals $\varphi(e)$ for $|x| \gg 0$.

Note: the existence of such an identity comes from the Bruhat decomposition $G = B \sqcup BwB$. One then only has to find out what it is.

We want to define the Fourier transform of Φ by:

$$\widehat{\Phi}(x) = \mu_2(x)|x|^{1/2} \int_F \Phi(y)\overline{\psi(xy)}\, dy$$

However, we have a problem: the above integral is not necessarily absolutely convergent. For example, this will be the case if $\mu = 1$. We remedy this by first integrating over circles (which introduces cancellation), and then summing over all of the circles.

Suppose we have a locally constant function $\Phi\colon F \to \mathbf{C}$ such that $\Phi(x) = C \cdot \mu^{-1}(x)|x|^{-1}$ for $|x| \gg 0$.

Then we can define $\widehat{\Phi}(x)$ as:

$$\widehat{\Phi}(x) = \sum_{n \in \mathbf{Z}} \int_{v(y)=n} \Phi(y)\overline{\psi(xy)}\, dy$$

We may think of this as providing a value for the non-convergent integral $\int_F \Phi(y)\overline{\psi(xy)}\, dy$ discussed above.

We have:

Proposition 6. The series defining $\widehat{\Phi}(x)$ above converges uniformly on compact subsets of $F^\times$.

The image under $\Phi \mapsto \widehat{\Phi}$ of the locally constant functions Φ with $\Phi(x) = C \cdot \mu^{-1}(x)|x|^{-1}$ for $|x| \gg 0$ is the set of locally constant functions G on $F^\times$ which are compactly supported on F such that in a neighborhood of 0:

$$G(x) = \begin{cases} a\mu(x) + b & \mu(x) \neq 1, |x|^{-1} \\ av(x) + b & \mu(x) \equiv 1 \\ b & \mu(x) \equiv |x|^{-1} \end{cases}$$

Here $v(x)$ denotes the valuation of x, normalized so that $v(\varpi) = 1$, where ϖ is a uniformizer of F. So $v(\varpi^n x) = n$ for $x \in \mathcal{O}_F^\times$. The idea of the proof is to note that the space of Φ satisfying our condition at infinity is the direct sum of $S(F^\times)$ and the span of the function

$$f(x) := \begin{cases} \mu^{-1}(x)|x|^{-1} & |x| \geq 1 \\ 0 & |x| < 1 \end{cases}$$

The case in which $\Phi \in S(F^\times)$ is quite simple, so the crux of the proof is in the case $\Phi = f$, which is a somewhat involved calculation, cf. [1, §1.9, Lemma 9]. This characterization of the space of Fourier transforms shows:

Proposition 7. The dimension of $K(\rho_{\mu_1,\mu_2})/S(F^\times)$ is 2 if $\mu(x) \neq |x|^{-1}$ and 1 otherwise.

We also have:

Proposition 8. $\Phi \mapsto \widehat{\Phi}$ is injective except in the case when $\mu(x) = |x|^{-1}$, in which case the kernel is the span of the constant function $\Phi(x) = 1$.

The intuitive idea behind this is that Fourier inversion should show that the Fourier transform as a function on F determines Φ (by Fourier inversion). Our Fourier transforms, however, are only defined on $F^\times$, so the kernel should consist of those functions with Fourier transform a multiple of the Dirac distribution at 0. This consists of the constant functions. For a few more details, see [1, §1.9].

Thus, since we may recover φ from Φ, φ is determined by $\widehat{\Phi}$, up to a constant when $\mu(x) = |x|^{-1}$.

Now, we want to show that ρ_{μ_1,μ_2} is irreducible when $\mu \neq |x|, |x|^{-1}$. We work with the Kirillov model for ρ defined above. Let $V \subsetneq \rho := \rho_{\mu_1,\mu_2}$ be a non-trivial invariant subspace. Given $0 \neq f \in V$, we have $0 \neq \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) f - f \in S(F^\times)$, since this function sends x to $(\psi(bx) - 1)f(x)$, which vanishes at 0. Since $S(F^\times)$ is irreducible as a representation of the mirabolic subgroup of G_F (as we saw in the previous lecture), it follows that $V \supset S(F^\times)$. Further, since $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) f - f \in S(F^\times) \subset V$ for all $f \in \pi$, it follows that the unipotent group $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ acts trivially on $V^\perp$, the orthogonal complement of V.

So let $h \in V^\perp$. Then we have:

$$h\Big(\left(\begin{smallmatrix} t' & * \\ 0 & t'' \end{smallmatrix}\right) w^{-1} \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\Big) = \mu_1^{-1}(t')\mu_2^{-1}(t'')|t'/t''|^{1/2} h(w^{-1})$$

Thus, h is determined by $h(w^{-1})$, since the big cell is dense in G_F, by (1). It follows that $\dim(V^\perp) \leq 1$, so $\operatorname{codim}(V) \leq 1$. In fact, we will see that if $\mu(x) \neq |x|$, then there can be no such $h \in \rho_{\mu_1^{-1},\mu_2^{-1}}$ with $h(w^{-1}) \neq 0$, hence $V = \pi$, so π is irreducible. On the other hand, if $\mu(x) = |x|$, then there is a codimension-one invariant subspace of π, namely the orthogonal complement of the one-dimensional space of constant functions living as an invariant subspace of $\rho_{\mu_1^{-1},\mu_2^{-1}}$.

Writing $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have

$$h(g) = \mu_1^{-1}(\det g)|\det g|^{1/2}\mu(c)|c|^{-1}h(w^{-1})$$

and this approaches $h\Big(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\Big)$ as $c \to 0$. This is:

$$h\Big(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\Big) = \mu_1^{-1}(a)\mu_2^{-1}(d)|a/d|^{1/2}h(1)$$

Since this holds for all $a, d \in F^\times$, and all $b \in F$, there are no such nonzero h possible unless $\mu(x) = |x|$. This proves Theorem 4.

Finally, we will prove Theorem 5, saying that $\pi_{\mu_1,\mu_2} \simeq \pi_{\lambda_1,\lambda_2}$ unless $(\lambda_1, \lambda_2) = (\mu_1, \mu_2)$ or (μ_2, μ_1).

Proof. We have:

$$\rho_{\mu_1^{-1},\mu_2^{-1}} = \check{\rho}_{\mu_1,\mu_2} = w^{-1}_{\rho_{\mu_1,\mu_2}} \otimes \rho_{\mu_1,\mu_2}$$

Here, $w_{\rho_{\mu_1,\mu_2}}$ is the central character of ρ_{μ_1,μ_2}. Now, we have an isomorphism $\rho_{\mu_1^{-1},\mu_2^{-1}} \xrightarrow{\sim} (\mu_1\mu_2)^{-1} \otimes \rho_{\mu_2,\mu_1}$ given by sending φ to $g \mapsto (\mu_1\mu_2)(\det g)\varphi(g)$. This shows that $\pi_{\mu_1,\mu_2} \simeq \pi_{\mu_2,\mu_1}$, and one can check that this even works in the case when $\mu(x) = |x|$ or $|x|^{-1}$.

To show that there are no other coincidences, we may look at the behavior of the functions in the (unique!) Kirillov model near 0.

Further, the fact that these representations are not supercuspidal may be seen from the fact that, as we have seen, $\pi/K(\pi)$ is nonzero in all of these cases, where $K(\pi)$ denotes the Kirillov model of π. □

References

[1] R. Godement, *Notes on Jacquet-Langlands' Theory*, The Institute for Advanced Study, 1970, available at http://math.stanford.edu/ conrad/conversesem/refs/godement-ias.pdf.

[2] https://mathoverflow.net/questions/255702/non-archimedean-qr-factorization-iwasawa-dec