LECTURE 13: CUSPIDAL REPRESENTATIONS
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1 The case of finite fields

For motivation, we will consider the analogous situation for the case of finite fields - many features of the representation theory of groups over local fields is similar to this easier case. We want to answer:

Problem 1. Describe all complex irreducible representations of $G = \mathrm{GL}_2(k)$ where $k = \mathbf{F}_q$ is a finite field.

Note that this finite group G has $q^2 - 1$ conjugacy classes (as one may verify by studying rational canonical form, etc.), so there will be $q^2 - 1$ different irreducible representations.

We write $B = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, $N = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, and $T = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$ for the k-points of the upper triangular Borel subgroup, its unipotent radical, and the diagonal split maximal torus of GL_2, respectively.

We may define *principal series* for G analogously to the local field case. This rests on the following:

Proposition 1.1. For π a complex irreducible representation of G, the following are equivalent:

(i) $\pi|_N$ contains the trivial representation of N.

(ii) There is a G-embedding $\pi \hookrightarrow \mathrm{Ind}_B^G \chi$ for some character $\chi \colon B/N = T \to \mathbf{C}^\times$.

Definition 1.2. If π satisfies the above equivalent conditions, we say that it is a *principal series representation*.

Remark 1.3. The condition (ii) is the better definition, since it agrees with the definition in the local field case. Moreover, since all Borel subgroups of GL_2 are conjugate, this manifestly does not depend on the choice of B.

Proof. We have $\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \chi) = \mathrm{Hom}_B(\pi, \chi)$ by Frobenius reciprocity. Thus, it is clear that (ii) implies (i).

Conversely, suppose that (i) holds. We need to construct a nonzero B-map $\pi \to \chi$ for some character χ of B/N. Let $(\sigma, V) \subseteq \mathrm{Res}_B^G \pi$ be an irreducible representation of B containing the trivial representation of N. [1] Thus, $V^N \neq 0$. Since N is normal in B, this is B-stable, so $V^N = V$, and (σ, V) is thus an irreducible representation of $B/N = T$. Thus, (σ, V) is a character χ of T, and we get a map from π to χ. □

Since $T = F^\times \times F^\times$, a character χ of T is of the form $\chi = \chi_1 \otimes \chi_2$. We may define $\chi^w \colon t \mapsto \chi(wtw^{-1})$ for $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and this is equal to $\chi_2 \otimes \chi_1$.

We have:

[1] Really, it's better to let (σ, V) be a quotient of $\mathrm{Res}_B^G \pi$ rather than a sub-representation, but in the finite field case this doesn't matter because complex representations of finite groups are always semisimple.

Proposition 1.4. (i) $\operatorname{Hom}_G(\operatorname{Ind}_B^G \chi, \operatorname{Ind}_B^G \psi) = 0$ unless $\chi = \psi$ or $\chi = \psi^w$.

(ii) $\operatorname{Hom}_G(\operatorname{Ind}_B^G \chi, \operatorname{Ind}_B^G \chi)$ is one-dimensional unless $\chi = \chi^w$, in which case it is two-dimensional. In other words, when $\chi \neq \chi^w$, $\operatorname{Ind}_B^G \chi$ is irreducible, and when $\chi = \chi^w$, it is the direct sum of two irreducible representations.

Proof. By Frobenius reciprocity, we have:

$$\operatorname{Hom}_G(\operatorname{Ind}_B^G \chi, \operatorname{Ind}_B^G \psi) = \operatorname{Hom}_B(\operatorname{Res}_B^G \operatorname{Ind}_B^G \chi, \psi)$$

and by Mackey theory,

$$\operatorname{Res}_B^G \operatorname{Ind}_B^G \chi = \bigoplus_{y \in B \backslash G / B} \operatorname{Ind}_{B \cap y^{-1} B y}^{B} \left(\operatorname{Res}_{B \cap y^{-1} B y}^{y^{-1} B y} (\chi^y) \right)$$

Here, $G = B \coprod BwB$, so we may take $y = 1, w$, and the result follows. □

Corollary 1.5. There are $\frac{1}{2}(q^2 + q) - 1$ irreducible principal series representations.

As an important example, we have the *Steinberg representation*, defined by $\operatorname{Ind}_B^G 1_B = 1_G \oplus \operatorname{St}_G$. We define:

Definition 1.6. The irreducible representations of G which are not of one of the above form are called cuspidal. Equivalently, these are exactly the irreducible representations which do not contain the trivial representation of N.

Remark 1.7. Where does the term cuspidal come from? In the context of modular forms, this corresponds to saying that the automorphic representation coming from f vanishes at the cusp; equivalently, the 0th Fourier coefficient vanishes, which corresponds to saying that the integral of f along a length-one horizontal strip is 0. In general, the right condition should be that π contains a copy of the trivial representation of the unipotent radical of every parabolic.

By the above count, there are $\frac{1}{2}(q^2 - q)$ cuspidal representations. We will now give a construction of some cupsidal representations which will turn out to account for all of them.

Let ℓ/k be a quadratic extension, and let $\theta\colon \ell^\times \to \mathbf{C}^\times$ be a character such that $\theta^q \neq \theta$: such characters are called *regular characters*, and this is a "genericity" condition saying that θ is not the norm of a character of $\ell^\times$.

By choosing a k-basis for ℓ, we obtain an embedding $\ell^\times \longrightarrow \operatorname{GL}_2(k)$ given by sending x to the k-linear map given by multiplying by x. Note that this map is well-defined up to changing our basis for ℓ/k, which corresponds to conjugation by an element of $\operatorname{GL}_2(k)$. Call the image of $\ell^\times$ under this map E.

Thus, we have a character $\theta\colon E \to \mathbf{C}^\times$. Fix a nontrivial character ψ of $N \simeq F$. We define a character $ZN \simeq Z \times N \to \mathbf{C}^\times$ defined by $zn \mapsto \theta(z)\psi(n)$. Now we may consider the virtual representation $\pi_\theta = \operatorname{Ind}_{ZN}^G \theta\psi - \operatorname{Ind}_E^G \theta$. Note that this is independent of our choice of ψ because B-conjugation on N acts transitively on the set of nontrivial characters.

One may ask where all of this comes from. The idea is that one would like to induce from a nontrivial character of N to get a cuspidal representation (and not a principal series). On the other hand, we want to induct from as large a group as possible in order to have the best hope of getting an irreducible representation. We know that Z will have to act via a character by Schur's Lemma, so we throw in the center as well, acting by some character (here, θ). Then we hope that the result is irreducible. In fact, it is not, but it becomes so if we subtract off $\operatorname{Ind}_E^G \theta$.

Proposition 1.8. (i) π_θ is an irreducible cuspidal representation.

(ii) $\pi_{\theta_1} \simeq \pi_{\theta_2}$ iff $\theta_1 = \theta_2$ or $\theta_1 = \theta_2^q$.

This immediately yields, by an easy count, the following:

Corollary 1.9. The π_θ yield *all* of the irreducible cupsidal representations.

Now, we need to prove the proposition:

Proof. We compute the character χ_θ of π_θ. We get

$$\chi_\theta(z) = (q-1)\theta(z), z \in Z$$

$$\chi_\theta(zn) = -\theta(z), z \in Z, n \in N$$

$$\chi_\theta(y) = -(\theta(y) + \theta^q(y)), y \in E\backslash Z$$

and $\chi_\theta(g) = 0$ if g is not conjugate to an element of $ZN \cup E$. One then checks that $\langle \chi_\theta, \chi_\theta \rangle = 1$, so since χ_θ is positive-dimensional, it is irreducible. This character table also shows that $\pi_{\theta_1} \simeq \pi_{\theta_2}$ iff $\theta_1 = \theta_2$ or θ_2^q. Finally, we have

$$\frac{1}{|N|} \sum_{n \in N} \chi_\theta(n) = 0$$

which shows that the projection onto the N-invariant subspace of π_θ is 0; i.e., π_θ doesn't contain a copy of the trivial character of N, so it is cuspidal. □

2 The case of local fields

Now, we return to the local field case: let F be a non-archimedean local field and let $G = \mathrm{GL}_2(F)$. We also let B, N, T be the F-points of the upper triangular Borel subgroup, its unipotent radical, and the diagonal maximal torus, respectively. We want to discuss cuspidal representations of G. Note: all representations considered in the sequel will be assumed to be smooth.

Given a (smooth) G-representation V, we may consider V_N, the maximal quotient of V on which N acts trivially. Letting $V(N)$ be the span of $nv - v$ for all $v \in V, n \in N$, we have $V_N = V/V(N)$. Now:

Definition 2.1. V_N is called the *Jacquet module* of V.

We have:

Proposition 2.2. The functor $V \rightsquigarrow V_N$ is an exact functor from the category of smooth N-representations to the category of complex vector spaces.

Lemma 2.3. For $v \in V$, we have $v \in V(N)$ iff there exists a compact open subgroup $N_0 \subseteq N$ such that $\int_{N_0} \pi(n) \cdot v \, dn = 0$.

Proof. See [BH, Ch. 3, §8.1, Lemma, (2)]. □

Now, we may define cuspidal representations:

Definition 2.4. Let π be an irreducible smooth G-representation. We say that π is *cuspidal* if $V_N = 0$.

Proposition 2.5. V is cuspidal iff there is no G-embedding $V \hookrightarrow \operatorname{Ind}_B^G \chi$.

Proof. This is very similar to the proof in the finite field case. The only additional step is to show that if $V_N \neq 0$, then it contains an irreducible (nontrivial) $T = B/N$-quotient. For this, we first note that V_N is a finitely-generated $\mathbf{C}[T]$-module. Indeed, $G = BK$, where $K = \mathrm{GL}_2(\mathcal{O}_F)$. Since V is irreducible over G, any $0 \neq v \in V$ generates V as a $\mathbf{C}[G]$-module. But $v \in V^{K'}$ for some compact open $K' \subset K$, so if $k_1, \ldots, k_r$ form a set of coset representatives for K/K', then the $k_i v$ generate V, hence also V_N, over B.

Now let $\{v_1, \ldots, v_r\}$ be a minimal set of $\mathbf{C}[T]$ generators for V_N. By Zorn's Lemma, there is a maximal T-subspace W of V_N containing $v_1, \ldots, v_{r-1}$ but not v_r. Then V_N/W is an irreducible nontrivial T-quotient of V_N. □

Definition 2.6. Let (π, V) be a G-representation, $\check{v} \in \check{V}$, $v \in V$. Then we define the matrix coefficient $\gamma_{\check{v}\otimes v}\colon G \to \mathbf{C}$ by $g \mapsto \langle \check{v}, g \cdot v\rangle$. Then we define the space $C(\pi)$ to be the $\mathbf{C}$-linear span of these coefficients. We say that π is *γ-cuspidal* if f is compactly supported mod Z for all $f \in C(\pi)$. [2]

Remark 2.7. If V is irreducible, then Z acts via a central character, hence the support of any matrix coefficient is invariant under Z-translation, so the best one can hope for is that the support is compact mod Z (rather than just being compact).

Proposition 2.8. (i) If π is irreducible, smooth, and γ-cuspidal, then it is admissible.

(ii) If π is irreducible and admissible and there exists some $f \neq 0$ in $C(\pi)$ such that f is compactly supported mod Z, then this is true for all $g \in C(\pi)$.

Proof. We will sketch the arguments. For more details, see [BH, Ch. 3, §10.1, Prop.].
(i) Let K be a compact open in G. We want to show that V^K is finite-dimensional. Fix $0 \neq v \in V^K$. (If $V^K = 0$, then there is nothing to show.) Consider the map $(\check{V})^K \to C(\pi)$ defined by $\check{v} \mapsto \langle \check{v}, v\rangle$. Irreducibility of V implies that this map is injective. The image is contained in the set of functions whose support is a finite union of cosets $ZKgK$. This is a space of countable dimension. But $(\check{V})^K = (V^K)^*$, so if V^K is infinite-dimensional, then this space is of uncountable dimension; therefore V^K must be finite-dimensional.
(ii) $V, \check{V}$ are irreducible G-spaces, hence for K a sufficiently small compact open, $V^K, \check{V}^K$ are irreducible finite-dimensional $\mathcal{H}(G,K)$-spaces. The Jacobson density theorem then implies that $\check{V} \otimes V$ is an irreducible $\mathcal{H}(G,K) \otimes \mathcal{H}(G,K) = \mathcal{H}(G \times G, K \times K)$-space. Since this holds for all small K, it follows that $\check{V} \otimes V$ is an irreducible $(G \times G)$-space. The surjective $(G \times G)$-map $\check{V} \otimes V \to C(\pi)$ is therefore an isomorphism, so every matrix coefficient is in the $(G \times G)$-span of any fixed nonzero one. Therefore, if any nonzero matrix coefficient is compactly supported mod Z, then so is every other one. □

Proposition 2.9. Let π be an irreducible smooth G-representation. Then π is cuspidal iff it is γ-cuspidal.

[2] This is what was defined as supercuspidal in previous talks.

Corollary 2.10. Every irreducible smooth representation of G is admissible.

We will prove Proposition 2.9:

Proof. Assume that (π, V) is cuspidal, so $V_N = 0$ or $V = V(N)$. Let $f = \gamma_{\check{v}\otimes v}$, and let K' be a compact open normal subgroup of $K = \mathrm{GL}_2(\mathscr{O}_F)$ which fixes v and $\check{v}$. Let $k_1, \ldots, k_r$ be coset representatives for K/K'. The support of f is a union of cosets $ZK'gK'$.

Let $t = \left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)$ for ϖ a uniformizer of F. Then $\{t^n \mid n \geq 0\}$ forms a set of coset representatives for $ZK\backslash G/K$. Thus, we have:

$$\mathrm{supp}(f) = \bigcup_{1\leq i,j\leq r} ZK'\left(\mathrm{supp} f_{ij} \cap \{t^n \mid n \geq 0\}\right) K'$$

With $f_{ij} \colon g \mapsto f\left(k_i^{-1} g k_j\right)$. Now, the lemma below shows that $\mathrm{supp} f_{ij} \cap \{t^n \mid n \geq 0\}$ is a *finite* set, which completes the proof. $\square$

Lemma 2.11. If $f \in C(\pi)$ for a cuspidal representation π, $f(t^m) = 0$ for $m \gg 0$.

Proof. We compute $\langle \check{v}, t^m \cdot v\rangle$. Since $V = V(N)$, if $v \in V$, then $\int_{N_0} n \cdot v \, dn = 0$ for some compact open subgroup $N_0 \subseteq N$ by Lemma 2.3. Let $N_1 \subset N$ be a compact open subgroup fixing $\check{v}$. Then for some positive constants c, c':

$$\begin{aligned}\langle \check{v}, t^m v\rangle &= c \cdot \int_{N_1} \langle x^{-1}\check{v}, t^m v\rangle \, dx \\ &= c \cdot \int_{N_1} \langle t^{-m}\check{v}, t^{-m} x t^m v\rangle \, dx \\ &= c' \langle t^{-m}\check{v}, \int_{t^{-m}N_1 t^m} n v dn\rangle\end{aligned}$$

Now, $t^{-m}N_1 t^m \supseteq N_0$ for $m \gg 0$, so the integral is 0, since the integral over the subgroup N_0 is 0. $\square$

We still have to prove the converse; that is, γ-cuspidal $\Longrightarrow$ cuspidal. So suppose that (π, V) is irreducible γ-cuspidal. In particular, it is admissible by Prop. 2.8(i). Then $(\check{\pi}, \check{V})$ is also irreducible admissible. Let $K_n := 1 + \mathfrak{p}^n M_2(\mathcal{O}_F)$, where $\mathfrak{p} = \varpi\mathcal{O}_F$ is the maximal ideal of $\mathcal{O}_F$. Fix $v \in V$, and choose $n \geq 1$ such that K_n fixes v.

For any $\check{v} \in \check{V}^{K_n}$, we have $\langle \check{v}, t^m v\rangle = 0$ for m sufficiently large, by our γ-cuspidality assumption. Since $\check{V}^{K_n}$ is finite-dimensional, this implies that for suitable c (independent of $\check{v}$), $\langle \check{v}, t^m v\rangle = 0$ for all $m \geq c$. It follows that $\pi(e_{K_n})\pi(t^m)v = 0$ for all $m \geq c$, where $e_{K_n} \in \mathcal{H}(G)$ is the idempotent in the Hecke algebra projecting onto the K_n-fixed functions. Indeed, this is because for any $\check{v} \in \check{V}$, we have $\langle \check{v}, \pi(e_{K_n})\pi(t^m)v\rangle = \langle \pi(e_{K_n})\check{v}, \pi(t^m)v\rangle = 0$, since $\pi(e_{K_n})\check{v} \in \check{V}^{K_n}$.

Now let

$$N_j := \begin{pmatrix} 1 & \mathfrak{p}^j \\ 0 & 1\end{pmatrix}, N'_j := \begin{pmatrix} 1 & 0 \\ \mathfrak{p}^j & 1\end{pmatrix}, T_n := K_n \cap T$$

Then $K_n = N_n T_n N'_n$. Let $K_n^{(m)} := t^{-m} K_n t^m = N_{n-m} T_n N'_{n+m}$. Then for $m \geq c$,

$$0 = \pi(e_{K_n})\pi(t^m)v = \pi(t^m)\pi(e_{K_n^{(m)}})v = \pi(t^m) \sum_{x \in N_{n-m}/N_n} \pi(x)\pi(e_{K_n^{(m)}\cap K_n})v$$

where the last equality uses the isomorphism $N_{n-m}/N_n \xrightarrow{\sim} K_n^{(m)}/K_n^{(m)} \cap K_n$. But $K_n^{(m)} \cap K_n$ fixes v, so we get

$$\int_{N_{n-m}} \pi(x)v dx = 0$$

It follows that $v \in V(N)$ (Lemma 2.3). Since $v \in V$ was arbitrary, we deduce that $V(N) = V$, i.e., V is cuspidal. □

Now we finally give some examples of cuspidal representations of $\mathrm{GL}_2(F)$. Let λ be a cuspidal representation of $\mathrm{GL}_2(k)$, where k is the residue field of F. We inflate this to a representation of $K := \mathrm{GL}_2(\mathscr{O}_F)$. Now we have:

$$Z \cap K = \left\{ \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix} \right) \mid \lambda \in \mathscr{O}_F^\times \right\}$$

and we extend the character of $\lambda|_{Z\cap K}$ to a character θ of Z. This amounts to arbitrarily choosing a value for $\theta(\varpi)$. Now $\rho := \theta\lambda$ is an irreducible representation of ZK on some space W. Our first guess for constructing a G-representation might be to consider $\mathrm{Ind}_{ZK}^G \rho$. It is, however, more natural to consider a certain subrepresentation (which may agree with the induction). For a closed subgroup $H \subseteq G$, and a representation (V, π) of H, $\mathrm{Ind}_H^G \pi$ may be identified with the set of functions $f\colon G \to V$ such that $f(hg) = \pi(h) \cdot f(g)$ for all $h \in H, g \in G$, together with a smoothness condition. We may define a subrepresentation by only considering those functions as above that are also compactly supported mod H. We call this the *compact induction* of V, denoted $\mathrm{c} - \mathrm{Ind}_H^G V$.

The compact induction satisfies a dual form of Frobenius reciprocity:

Proposition 2.12. Let $K' \subset G$ be a closed subgroup of the locally profinite group G. If V is a K'-representation and T is a G-representation, then:

$$\mathrm{Hom}_G(\mathrm{c} - \mathrm{Ind}_{K'}^G V, T) = \mathrm{Hom}_{K'}(V, T)$$

We have:

Theorem 2.13. For ρ defined as above and $K' := ZK$, $\mathrm{c} - \mathrm{Ind}_{K'}^G \rho$ is an irreducible cuspidal representation of G.

Let $X := \mathrm{c} - \mathrm{Ind}_{K'}^G \rho$. We have a K'-embedding $j : W \hookrightarrow X$. Via Frobenius reciprocity (Proposition 2.12), this embedding in $\mathrm{Hom}_{K'}(W, \mathrm{c} - \mathrm{Ind}_{K'}^G W)$ corresponds to the identity function in $\mathrm{Hom}_G(\mathrm{c} - \mathrm{Ind}_{K'}^G W, \mathrm{c} - \mathrm{Ind}_{K'}^G W)$. Its image is the set of functions supported in K'. Explicitly,

$$j(w)(g) := \begin{cases} \rho(g)w & g \in K' \\ 0 & g \notin K' \end{cases}$$

We will first construct a non-zero matrix coefficient of G on X which is compactly supported mod Z. Let $j\colon W \hookrightarrow X$ be the embedding discussed above, and $\check{j}\colon \check{W} \hookrightarrow \check{X} := \mathrm{c} - \mathrm{Ind}_{K'}^G \check{W}$ be the analogous such embedding for $\check{W}$.[3] We can compute:

$$\langle \check{f}, f \rangle = \int_{K' \backslash G} \left\langle \check{f}(g), f(g) \right\rangle \, d\overline{g}$$

[3] warning: $\check{j}$ is not the contragredient of j

Let $w \in W, \check{w} \in \check{W}$ be non-zero. Then $\langle \check{w}, g \cdot w \rangle = \langle j(\check{w}), g \cdot \check{j}(\cdot w) \rangle = 0$ unless $g \in K'$ because $j(w)$ and $\check{j}(\check{w})$ are supported in K'. Therefore, this is a nonzero matrix coefficient supported on K', which is compact mod Z. It therefore suffices to prove that X is irreducible over G.

Now Z acts on X via w_ρ, the central character of ρ. In other words, $z \cdot f(g) = f(gz) = f(zg) = \rho(z) \cdot f(g) = w_\rho(z) \cdot f(g)$ for $z \in Z \subseteq K'$. Since $Z \backslash K'$ is compact, X splits into a direct sum of K'-isotypic components. We want to show that:

Proposition 2.14. $j(W) = X^\rho$, the ρ-isotypic component.

This proposition follows from (and is in fact, equivalent to) the following lemma, which is the key point of the entire argument:

Lemma 2.15. $\mathrm{Hom}_{K'}(W, X)$ is one-dimensional.

Now, to prove the irreducibility of X, we let $0 \neq Y \subseteq X$ be a G-invariant subspace. Note that Y is K'-semisimple, for the same reason as X is. We have:

$$0 \neq \mathrm{Hom}_G(Y, X) \subseteq \mathrm{Hom}_G(Y, \mathrm{Ind}_{K'}^G \rho) = \mathrm{Hom}_{K'}(Y, \rho)$$

Thus, $Y^\rho \neq 0$, so $0 \neq Y^\rho \subseteq X^\rho = j(W)$, which is irreducible, so $Y^\rho = j(W)$. In particular, $j(W) \subset Y$. Since $j(W)$ generates X as a G-space, $Y = X$, so X is irreducible.

It only remains to prove Lemma 2.15.

For $g \in G$, let $(K')^g = g^{-1} K' g$, and ρ^g be the representation of $(K')^g$ with space W defined by $h \mapsto \rho(ghg^{-1})$.

Lemma 2.16. $\mathrm{Hom}_{(K')^g \cap K'}(\rho^g, \rho) = 0$ unless $g \in K'$.

Let us assume this lemma for the moment and see how to complete the proof of Lemma 2.15.

Let $\varphi \in \mathrm{Hom}_{K'}(W, X)$. For all $k \in K', g \in G, w \in W$, we have, thinking of $X = \mathrm{c} - \mathrm{Ind}_{K'}^G \rho$ as a space of functions on G:

(i) $\varphi(w)(k \cdot g) = \rho(k) \cdot (\varphi(w)(g))$

(ii) $\varphi(\rho(k) \cdot w)(g) = (k \cdot (\varphi(w)))(g) = \varphi(w)(gk)$

Fix $g \in G$, and consider $\psi_g \colon W \to W$ defined by $\psi_g(w) := \varphi(w)(g^{-1})$. We claim that $\psi_g \in \mathrm{Hom}_{(K')^g \cap K'}(\rho^g, \rho)$. Indeed, this amounts to showing that for $k \in K' \cap (K')^{(g)}$,

$$\psi_g(\rho(gkg^{-1})w) = \rho(k)(\psi_g(w))$$

i.e.,

$$\phi(\rho(gkg^{-1})w)(g^{-1}) = \rho(k)(\phi(w)(g^{-1}))$$

Well, the left hand side equals (thanks to (2))

$$\phi(w)(g^{-1}(gkg^{-1})) = \phi(w)(kg^{-1}) = \rho(k)(\phi(w)(g^{-1}))$$

where the last equality is by (1). This proves the claim.

Lemma 2.16 therefore implies that $\psi_g = 0$ unless $g \in K'$, so $\varphi(w)$ is supported on K'. This implies that $\varphi \colon W \to j(W)$, so $\varphi = \lambda \cdot j$ for some $\lambda \in \mathbf{C}$ by Schur's Lemma.

It remains to prove Lemma 2.16. This is where we will finally use cuspidality of the residual representation.

Proof. Note that the statement of this lemma depends only on the coset of g in $K'\backslash G/K'$. Indeed, if $k \in K'$ and $\phi \in \mathrm{Hom}_{(K')^g \cap K'}(\rho^g, \rho)$, then $\psi : v \mapsto \phi(k^{-1}v)$ lies in $\mathrm{Hom}_{(K')^{kg} \cap K'}(\rho^{kg}, \rho)$, while $\psi' : v \mapsto k^{-1} \cdot \phi(v)$ lies in $\mathrm{Hom}_{(K')^{gk} \cap K'}(\rho^{gk}, \rho)$.

Thus, we may assume that $g = t^n$ for some $n > 0$ with $t = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)$ as before.

Now, $(K')^g \cap K'$ contains $N_0 := \begin{pmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{pmatrix} \subseteq g^{-1} \begin{pmatrix} 1 & \mathfrak{p} \\ 0 & 1 \end{pmatrix} g$ with $\mathfrak{p} = \varpi \mathscr{O}_F$ the maximal ideal. This shows that ρ^g is trivial on N_0, since ρ is trivial on $\begin{pmatrix} 1 & \mathfrak{p} \\ 0 & 1 \end{pmatrix}$, because it is inflated from a residual representation, and residually, $\begin{pmatrix} 1 & \mathfrak{p} \\ 0 & 1 \end{pmatrix} = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$. But the residual representation λ does not contain the trivial representation on the unipotent radical by the assumption that λ is cuspidal, so there can be no nontrivial N_0-hom from ρ^g to ρ, because the latter contains no copy of the trivial representation of N_0. □

Remark 2.17. By the version of Frobenius reciprocity for compact induction, $\mathrm{Hom}_{K'}(W, X) - \mathrm{End}_G(X, X)$. This is an algebra under composition. There is a convolution algebra of functions, the so-called spherical Hecke algebra, denoted $\mathscr{H}(G, \rho)$, such that we have a natural isomorphism $\mathrm{End}_G(X, X) \simeq \mathscr{H}(G, \rho)$. Further, the space of functions in $\mathscr{H}(G, \rho)$ supported on $K'gK'$ is canonically isomorphic to $\mathrm{Hom}_{(K')^g \cap K'}(\rho, \rho^g)$. Thus, G-endomorphisms of X are naturally related to such homs, and if such a hom exists, then we say that g *intertwines* ρ with itself. For more on this, cf. [BH, Ch. 3, §11.1-11.3].

References

[BH] Colin Bushnell, Guy Henniart, *The Local Langlands Conjecture For* GL(2), Springer-Verlag, Berlin, 2006.