LECTURE 15: SPHERICAL AND UNITARY REPRESENTATIONS
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1 Spherical Representations

First, we fix some notation. Let F be a local field with ring of integers $\mathscr{O}_F$ and maximal ideal $\mathfrak{m} = (\varpi)$, and let (π, V) be an irreducible smooth representation of $\mathrm{GL}_2(F)$. We have the following definition:

Definition 1.1. An irreducible smooth representation (V, π) is called *spherical* if $V^{\mathrm{GL}_2(\mathscr{O}_F)} \neq \{0\}$.

Our goal will be to classify all such representations. First, we will review Hecke algebras:

1.1 Hecke Algebras

Let G be any locally profinite group. We want to introduce an algebraic gadget $\mathcal{H}_G$ to study the representation theory of G in terms of modules over $\mathcal{H}_G$. For example, in the case of a finite group G it is useful to understand complex representations of G as modules over the group algebra $\mathbf{C}[G]$. However, in the case of locally profinite groups we are primarily interested in smooth representations of G, so we need to slightly change the definition of group algebra $\mathbf{C}[G]$ to be able to find an analogue of the smoothness condition on the algebraic side.

We start with the space $C_c^\infty(G)$ of all locally constant compactly supported functions on G. We define a structure of a (non-unital) algebra on the space $C_c^\infty(G)$ via the operation of convolution:

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g)\, dx$$

Remark 1.2. Here we implicitly fixed a left Haar measure. The convolution operation does depend on this choice. However, different left Haar measures will give isomorphic Hecke algebras. So it will not cause any serious problems.

Remark 1.3. The space $C_c^\infty(G)$ has a natural left action λ of the group G. Namely, $\lambda(g)(\phi) = (x \mapsto \phi(g^{-1}x))$. Also, it has a natural left action ρ defined in the similar way: $\rho(g)(\phi) = (x \mapsto \phi(xg))$. In what follows we will say that an element $\phi \in C_c^\infty(G)$ is left/right invariant meaning invariance under the actions of left/right translation from above.

Proposition 1.4. Let G be a locally profinite group, then the following is true:

(1) For any function $\phi \in C_c^\infty(G)$ there is an open and compact subgroup K s.t. f is K bi-invariant.

(2) For any two functions $\phi_1, \phi_2 \in C_c^\infty(G)$ their convolution $\phi_1 * \phi_2$ is also an element of $\mathcal{H}_G$.

(3) For any three functions $\phi_1, \phi_2, \phi_3 \in C_c^\infty(G)$ we have an equality

$$(\phi_1 * \phi_2) * \phi_3 = \phi_1 * (\phi_2 * \phi_3).$$

In other words, the convolution operation defines a structure of a (non-unital) associative algebra on $C_c^\infty(G)$.

Proof. (1) We know that $\phi \in C_c^\infty(G)$ is locally constant. Thus, for every point $g \in G$ we can choose an open subset U_g s.t. U_g contains g and $f|_{U_g}$ is a constant function. Let us consider $U_g' := U_g g^{-1}$, it is an open neighborhood of 1_G, so using the fact that G is a locally profinite group we can find an open compact subgroup $K_g \subset U_g'$, then $\{K_g g\}_{g\in G}$ is a covering of G and $f|_{K_g g}$ is constant for each $g \in G$. Since $\mathrm{Supp}(\phi)$ is compact, we can choose a finite number of elements $g_1, \dots, g_n$ s.t. $\mathrm{Supp}(\phi) \subset K_{g_1}g_1 \cup K_{g_2}g_2 \cup \cdots \cup K_{g_n}g_n$. Without loss of generality we can assume that $K_{g_i}g_i \subset \mathrm{Supp}(\phi)$ for each of these g_i. Let us choose $K' := \cap_{i=1}^n K_{g_i}$ to be the intersection of all these K_{g_i}. We claim that ϕ is K' left-invariant.

First, note that left multiplication by K' preserves each of $K_{g_i}g_i$. Therefore, left multiplication by K preserves their union $\cup_{i=1}^n K_{g_i}g_i = \mathrm{Supp}(\phi)$. Therefore,

$$(\lambda(k)\phi)(x) = \phi(k^{-1}x) = 0 = \phi(x)$$

for all $x \notin \mathrm{Supp}(\phi)$ and $k \in K'$.

On the other hand, if $x \in \mathrm{Supp}(\phi)$ we can choose g_i and $k_i \in K_{g_i}$ s.t. $x = k_i g_i$. Then

$$(\lambda(k)\phi)(x) = (\lambda(k)\phi)(k_i g_i) = \phi(k^{-1}k_i g_i) = \phi(g_i) = \phi(k_i g_i),$$

where the last two equalities follow from the fact that $\phi|_{K_{g_i}g_i}$ is constant and that left multiplication by K' preserves $K_{g_i}g_i$.

All in all, we have shown that ϕ is K' left-invariant for some open compact subgroup of G. The same argument (with gK_g in place of $K_g g$) shows that there is an open compact subgroup K" s.t. ϕ is K" right-invariant. But then $K := K' \cap K$" is also an open compact subgroup of G and ϕ is K bi-invariant!

(2) Consider an open and compact subgroup K_1 (resp. K_2) such that ϕ_1 (resp. ϕ_2) is K_1 (resp. K_2) bi-invariant. Then taking the intersection $K := K_1 \cap K_2$ we can assume that both ϕ_1 and ϕ_2 are K bi-invariant for an open compact subgroup K. In this case each of ϕ_i can be written as a linear combination of characteristic functions of double cosets Kg_iK for some $g_i \in G$. Therefore it suffices to show that $\mathbf{1}_{Kg_1K} * \mathbf{1}_{Kg_2K}$ is locally constant and compactly supported. Indeed,

$$\mathbf{1}_{Kg_1K} * \mathbf{1}_{Kg_2K}(g) = \int_G \mathbf{1}_{Kg_1K}(x)\mathbf{1}_{Kg_2K}(x^{-1}g)\,dx = \int_{Kg_1K} \mathbf{1}_{Kg_2K}(x^{-1}g)\,dx = \mu(Kg_1K \cap gKg_2^{-1}K)$$

This equation implies that $\mathrm{Supp}(1_{Kg_1K} * \mathbf{1}_{Kg_2K})$ is a subset of the compact set Kg_1Kg_2K, so this function is compactly supported. Also, this function is K bi-invariant since

$$\begin{aligned}\mathbf{1}_{Kg_1K} * \mathbf{1}_{Kg_2K}(k_1gk_2) = \mu(Kg_1K \cap k_1gk_2Kg_2^{-1}K) &= \\ \mu(Kg_1K \cap k_1gKg_2^{-1}K) &= \\ \mu(Kg_1K \cap gKg_2^{-1}K) &= \\ \mathbf{1}_{Kg_1K} * \mathbf{1}_{Kg_2K}(g).\end{aligned}$$

(3) The main tool to prove associativity is Fubini's Theorem. We need to manipulate with integrals using Fubini's Theorem several times to prove associativity. The manipulations below do the job.

$$(\phi_1 * (\phi_2 * \phi_3))(g) = \int_G \phi_1(x)(\phi_2 * \phi_3)(x^{-1}g)\, dx =$$
$$\int_G \phi_1(x) \left(\int_G \phi_2(y)\phi_3(y^{-1}x^{-1}g)\, dy \right) dx =$$
$$\int_{G\times G} \phi_1(x)\phi_2(y)\phi_3(y^{-1}x^{-1}g)\, dy\, dx =$$
$$\int_{G\times G} \phi_1(x)\phi_2(x^{-1}y)\phi_3(y^{-1}xx^{-1}g)\, dy\, dx =$$
$$\int_{G\times G} \phi_1(x)\phi_2(x^{-1}y)\phi_3(y^{-1}g)\, dy\, dx =$$
$$\int_G \left(\int_G \phi_1(x)\phi_2(x^{-1}y)\, dx \right) \phi_3(y^{-1}g)\, dy =$$
$$\int_G (\phi_1 * \phi_2)(y)\phi_3(y^{-1}g)\, dy = ((\phi_1 * \phi_2) * \phi_3)(g)$$

□

Definition 1.5. The *Hecke algebra* $\mathcal{H}_G$ is the associative algebra $(C_c^\infty(G), *)$.

Example 1.6. If G is a discrete group, then $\mathcal{H}_G \cong \mathbf{C}[G]$. So the notion of Hecke algebra is a "generalization" of the notion of group algebra that keeps track of topology on G.

The whole point of this definition is that every smooth representation of G can be "extended" to a left module over $\mathcal{H}_G$. Namely, given a smooth representation (V, π) and an element $\phi \in \mathcal{H}_G$ we can *define* an action of ϕ on V by the formula

$$\widetilde{\pi}(\phi)v := \int_G \phi(g)\pi(g)v\, dg$$

Remark 1.7. One needs to explain why this integral makes sense at all. The issue is that V is usually of infinite dimension and we even don't specify any topology on it. So it is not clear why this construction is well-defined. Actually we need to use the smoothness assumption to guarantee that this integral exists. For the detailed discussion of it, you can look at [1][3.2 and 4.1]. The key point is that the function $\Psi(g) = \phi(g)\pi(g)v$ lies in the space of locally constant compactly supported functions on G with values in V and one can really integrate such functions.

Proposition 1.8. Let (V, π) be a smooth representation of a locally profinite group G, then $(V, \widetilde{\pi})$ is a left $\mathcal{H}_G$-module.

Proof. The only thing we really need to prove here is that $\widetilde{\pi}(\phi_1)\widetilde{\pi}(\phi_2)v = \widetilde{\pi}(\phi_1 * \phi_2)v$ for any $\phi_1, \phi_2 \in \mathcal{H}_G$ and $v \in V$. The same idea as one in the proof of associativity of convolution works here. Namely, we just manipulate with integrals using Fubuni's theorem:

$$
\begin{aligned}
\widetilde{\pi}(\phi_1 * \phi_2)(g)v = \int_G (\phi_1 * \phi_2)(g)\pi(g)v \, dg = \\
\int_G \left(\int_G \phi_1(x)\phi_2(x^{-1}g) \, dx \right) \pi(g)v \, dg = \\
\int_{G\times G} \phi_1(x)\phi_2(x^{-1}g)\pi(g)v \, dx \, dg = \\
\int_{G\times G} \phi_1(x)\phi_2(g)\pi(xg)v \, dx \, dg = \\
\int_{G\times G} \phi_1(x)\pi(x)\phi_2(g)\pi(g)v \, dg \, dx = \\
\int_G \phi_1(x)\pi(x) \left(\int_G \phi_2(g)\pi(g)v \, dg \right) \, dx = \\
\int_G \phi_1(x)\pi(x)\widetilde{\pi}(\phi_2)v \, dx = \\
\widetilde{\pi}(\phi_1)\widetilde{\pi}(\phi_2)v.
\end{aligned}
$$

□

Now we want to describe all left $\mathcal{H}_G$-modules that come from smooth representations of the group G. In order to do this, we need to invoke more structure on the Hecke algebra $\mathcal{H}_G$. Usually $\mathcal{H}_G$ is not unital (unless it is discrete), however, there are lots of idempotents: let K be any compact open subgroup of G. We define:

$$
e_K \in \mathcal{H}_G \colon x \mapsto \begin{cases} 0 & x \notin K \\ \frac{1}{\mu(K)} & x \in K \end{cases}
$$

We have the following properties:

Proposition 1.9. (1) $e_K * e_K = e_K$

(2) $f \in \mathcal{H}_G$ is left invariant by K (i.e. $f(kx) = f(x)$ for all $x \in G$ and all $k \in K$) iff $f = e_K * f$

(3) $f \in \mathcal{H}_G$ is right invariant by K (i.e. $f(xk) = f(x)$ for all $x \in G$ and all $k \in K$) iff $f = f * e_K$.

This proposition means that the Hecke algebra $\mathcal{H}_G$ has more structure than just being an associative algebra, it is an "idempotented algebra".

Definition 1.10. An *idempotented algebra* over a field k is a pair (H, E), where H is an associative k-algebra and E is a set of idempotents in H satisfying the conditions:

(1) For all $e_1, e_2 \in E$ there is $f \in E$ such that $e_1 f = f = f e_1$ and $e_2 f = f = f e_2$.

(2) For all $h \in H$ there is $e \in E$ such that $eh = h = eh$.

Denote the set of all compact open subgroups of G by $\mathcal{K}_G$.

Proposition 1.11. Let G be a locally profinite group, then a pair $(\mathcal{H}_G, \{e_K\}_{K\in\mathcal{K}_G})$ is an idempotented algebra.

Proof. We need to check two axioms of an idempotented algebra. Firstly, given two open compact subgroups K_1, K_2 their intersection $K := K_1 \cap K_2$ is also open compact subgroup. And it is easy to see that $e_K * e_{K_1} = e_{K_1} = e_{K_1} * e_K$ and the same for e_{K_2}. Thus the first condition is verified.

Now let's verify the second condition. Given any $h \in \mathcal{H}_G$ Proposition 1.4 guarantees that there is an open compact subgroup K such that h is K bi-invariant. Then Proposition 1.9 implies that $e_K * h = h = h * e_K$. □

Remark 1.12. In what follows we will slightly abuse notations and denote the idempotented algebra $(\mathcal{H}_G, \{e_K\}_{K\in\mathcal{K}_G})$ just by $\mathcal{H}_G$.

Finally, we can formulate conditions that will characterize all left $\mathcal{H}_G$-modules that come from smooth representations.

Definition 1.13. Let (H, E) be an idempotented algebra and let M be a left H-module. We say that it is *smooth* if for all $v \in V$, there exists some idempotent $e \in E$ such that $em = m$.

Remark 1.14. In what follows we will always denote left multiplication on a $\mathcal{H}_G$-module M by $*$. In particular, we will write $\phi * m$ meaning a product ϕm with respect to corresponding left $\mathcal{H}_G$-module structure on M.

Theorem 1.15. There is an equivalence of categories between the category of smooth representations of G and the category of smooth representations of $\mathcal{H}_G$. This equivalence sends a smooth representation (V, π) to a left $\mathcal{H}_G$-module $(V, \widetilde{\pi})$.

Proof. : If (V, π) is a representation of G, we define a left $\mathcal{H}_G$-module $(V, \widetilde{\pi})$ on the same vector space by:

$$\widetilde{\pi}(\phi) \colon v \mapsto \int_G \phi(g)\pi(g)v \, dg$$

Proposition 1.8 guarantees that $(V, \widetilde{\pi})$ is a left $\mathcal{H}_G$-module. But we also need to check that this module is smooth. In other words, for any element $v \in V$ we need to produce an open compact subgroup K such that $\widetilde{\pi}(e_K)v = v$. Since V is a smooth representation of G we can find an open compact subgroup K such that $v \in V^K$. We claim that this implies that v is an e_K-invariant vector. Indeed,

$$\widetilde{\pi}(e_K)v = \int_G e_K(g)\pi(g)v \, dg = \frac{1}{\mu(K)} \int_K \pi(g)v \, dg = \frac{1}{\mu(K)} \int_K v \, dg = v.$$

We also need to define a functor on morphisms. In order to do this, we need to check if $f : (V, \pi) \to (V', \pi')$ is a morphism of G-representations, then

$$\widetilde{\pi'}(\phi) \circ f = f \circ \widetilde{\pi}(\phi) \text{ for any } \phi \in \mathcal{H}_G.$$

This straightforward computation is left to the reader. It implies that correspondence $(V, \pi) \mapsto (V, \widetilde{\pi})$ is functorial.

Conversely, if M is a left $\mathcal{H}_G$-module, we define the structure of a G-representation on M as follows: if $m \in M$ is such that $m = e_K * m$, we define $g \cdot m = (\lambda(g)e_K) * m$, where $\lambda(g)$ is the action of left multiplication by g^{-1} on $\mathcal{H}_G$ i.e. $\lambda(g)\colon f \mapsto (x \mapsto f(g^{-1}x))$. We need to check that this gives a well-defined functor from the category of smooth representations of $\mathcal{H}_G$ (i.e. that this definition does not depend on the choice of K with $m = e_K * m$), and that these two functors are mutually inverse.

Firstly, let us check that it is well-defined. In order to do this, we can define $g \cdot m$ more generally. Write m as a finite linear combination $m = \sum_{i=1}^{N} \phi_i * m_i$ for some $\phi_i \in \mathcal{H}_G$ and $m_i \in M$ and define $g \cdot m = \sum_{i=1}^{n} \lambda(g)\phi_i * m_i$. Let us check that this action is well-defined, it suffices to show that whenever $0 = \sum_{i=1}^{n} \phi_i * m_i$ we have $0 = \sum_{i=1}^{n} \lambda(g)\phi_i * m_i$. Denote the sum $\sum_{i=1}^{n} \lambda(g)\phi_i * m_i$ by n. Since M is a smooth representation of $\mathcal{H}_G$ there exists an open compact subgroup K such that $e_K * n = n$. Then we have:[1]

$$\begin{aligned} n = e_K * n = e_K * \left(\sum_{i=1}^{n} \lambda(g)\phi_i * m_i\right) &= \sum_{i=1}^{n} (e_K * (\lambda(g)\phi_i)) * m_i \\ = \sum_{i=1}^{n} ((\rho(g^{-1})e_K) * \phi_i)) * m_i &= \rho(g^{-1})e_K \sum_{i=1}^{n} \phi_i * m_i = 0. \end{aligned}$$

Secondly, we need to show that this representation is smooth. Indeed, for any $m \in M$ there is an open compact subgroup K such that $e_K * m = m$. Then it is clear that for all $k \in K$

$$k \cdot m = \lambda(k)e_K * m = e_K * m = m.$$

Therefore, $m \in V^K$ and this means that the representation is smooth.

Finally, we show that this construction is functorial. Namely, given a morphism $f : M \to M'$ between two left smooth $\mathcal{H}_G$-modules, we want to show that $f(g \cdot m) = g \cdot (fm)$ for any $m \in M$ and $g \in G$. This will imply that f is a morphism of corresponding smooth G-representations. Pick any $m \in M$ and choose K, K' two open compact subgroups such that $e_K * m = m$ and $e_{K'} * (f(m)) = f(m)$. We may and do assume that $K = K'$ by passing to the intersection $K \cap K'$. In this case we have:

$$f(g \cdot m) = f((\lambda(g)e_K) * m) = (\lambda(g)e_K) * f(m) = g \cdot f(m).$$

All in all, we have defined functors in both directions. The fact that they are mutually inverse is left to the reader as an exercise.

□

1.2 The Hecke Algebra of a pair (G, K)

Our goal now is to understand irreducible smooth representations of G with a non-zero K-fixed vector. Namely, under the of categories from Theorem 1.15, we want to identify the subspaces V^K with something on the Hecke algebra side.

We define:

[1] Does the identity $e_K * (\lambda(g)\phi_i) = (\rho(g^{-1})e_K) * \phi_i$ require that the Haar measure is unimodular?

Definition 1.16. For G a locally profinite group with a compact open subgroup $K \subseteq G$, we define the Hecke algebra $\mathcal{H}_{G,K}$ by:

$$\mathcal{H}_{G,K} := e_K * \mathcal{H}_G * e_K$$

The Proposition 1.9 implies that this consists of the set of locally constant compactly supported functions on G which are invariant under left and right multiplication by K.

Proposition 1.17. $\mathcal{H}_{G,K}$ is an associative algebra with unit.

Proof. It is an immediate consequence of Proposition 1.9 □

The analogous theorem to Theorem 1.15 is:

Theorem 1.18. For any locally profinite group G and compact open subgroup $K \subseteq G$, there is a bijection between the set of isomorphism classes of irreducible smooth representations (π, V) of G with $V^K \neq 0$ and the set of isomorphism classes of irreducible smooth representations of $\mathcal{H}_{G,K}$. [2]

Remark 1.19. It is not true that the category of smooth representations (π, V) of G with $V^K \neq 0$ is equivalent to some category of smooth representations of $\mathcal{H}_{G,K}$: the problem is that we could take the direct sum of (π, V) with $V^K \neq 0$ with (π', V') with $(V')^K = 0$. Thus, we need some sort of irreducibility condition, or at least we need to require that $\mathbf{C}[G] \cdot V^K = V$.

Proof. (sketch) Given a representation (π, V), we may pass to the $\mathcal{H}_G$-representation $(\widetilde{\pi}, V)$. Then $V^K = e_K * V$, which is a $\mathcal{H}_{G,K}$-module since $e_K * \mathcal{H}_G * e_K * (e_K * V) = e_K * \mathcal{H}_G * V = e_K * V$.

How do we go back? We send a smooth representation M of $\mathcal{H}_{G,K}$ to $\mathcal{H}_G \otimes_{\mathcal{H}_{G,K}} M/X$, where X is the maximal G-subspace such that $X^K = 0$. □

Remark 1.20. Another way of saying this theorem is that irreducible smooth G-representations (π, V) with $V^K \neq 0$ are equivalent to irreducible smooth $\mathcal{H}_G$-representations $(\widetilde{\pi}, V)$ with $V^K \neq 0$ and that these in turn are equivalent to irreducible smooth representations of $\mathcal{H}_{G,K}$.

1.3 The Spherical Hecke Algebra of $\mathrm{GL}_n(F)$

Now, we will restrict to the special case that $G = \mathrm{GL}_n(F)$ and $K = \mathrm{GL}_n(\mathscr{O}_F)$. Then we define:

Definition 1.21. The *spherical Hecke algebra of* $\mathrm{GL}_n(F)$ is $\mathcal{H}^0 := \mathcal{H}_{G,K}$ with G, K as defined above.

Remark 1.22. We normalize left Haar measure μ on $\mathrm{GL}_n(F)$ such that $\mu(\mathrm{GL}_n(\mathscr{O}_F)) = 1$.

We will also denote $\mathcal{H}_G$ by just $\mathcal{H}$. We have the following theorem:

Theorem 1.23. $\mathcal{H}^0$ is commutative.

[2] We went back and forth a lot about what the best statement of this result is: can this be beefed up to an equivalence of categories?

Proof. Let i be the transpose map $G \to G$. This is an (anti-) involution, i.e. $i(xy) = i(y)i(x)$, that fixes the subgroup K. This (anti)-involution induces an (anti-)involution of the spherical Hecke algebras $i^* : \mathcal{H}^0 \to \mathcal{H}^0$ defined by the rule:

$$i^*(\phi)(g) = \phi(i(g)).$$

The key is that we may identify the double coset space $K\backslash G/K$ as:

$$K\backslash G/K = \left\{ \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_n} \end{pmatrix} \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \right\} (*).$$

Thus each double coset is represented by a diagonal matrix. As a result, each double coset is fixed by the involution i. This implies that characteristic functions of any double coset is fixed by the induced involution i^*. But any element of the spherical Hecke Algebra $\mathcal{H}^0$ is a finite $\mathbf{C}$-linear combination of characteristic functions of some double cosets. Therefore, i^* is the identity morphism! In particular, we have

$$xy = i^*(xy) = i^*(y)i^*(x) = yx.$$

So, we are done.

□

This has the following striking corollary:

Corollary 1.24. For any spherical irreducible *admissible* representation (V, π), we have $\dim V^K = 1$.

Proof. Theorem 1.18 states that V^K is a simple $\mathcal{H}^0$-module. However, $\mathcal{H}^0$ is a commutative $\mathbf{C}$-algebra (Theorem 1.23) and V^K is a *finite-dimensional* $\mathbf{C}$-vector space (admissibility condition) with a structure of a simple $\mathcal{H}^0$-module. Then V must be one-dimensional due to Schur's Lemma. □

Remark 1.25. Actually, it turns out that we don't need an admissibility condition, it is enough to assume that (V, π) is smooth. We will see this later.

Although Theorem 1.23 is pretty nice, we will really need to know more about $\mathcal{H}^0$ in order to classify all spherical representations of $\mathrm{GL}_2(F)$.

Before giving a more concrete description of $\mathcal{H}^0$ we need to introduce some definitions from combinatorics.

Definition 1.26. An ordered set of n elements $\lambda = (\lambda_1, \ldots, \lambda_n)$ is called *a partition of order* n, if $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n$. We say that partition λ is a partition of k, if $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_n = k$.

Definition 1.27. We say that a partition λ (of order n) of an integer k is *greater or equal* to a partition μ (of order n) of integer k, if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i \text{ for all } 1 \leq i \leq n.$$

In this case write $\lambda \preccurlyeq \mu$.

Definition 1.28. We say that a partition λ (of order n) is *non-negative*, if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. In particular, $|\lambda| \geq 0$ for any non-negative partition.

Lemma 1.29. Suppose that λ and μ are partitions (of length n) of integers k and l, respectively. We will denote by $\lambda + \mu$ the partition $\{\lambda_1 + \mu_1, \ldots, \lambda_n + \mu_n\}$ of $k + l$. Let

$$g = \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_n} \end{pmatrix},\ h = \begin{pmatrix} \varpi^{\mu_1} & & \\ & \ddots & \\ & & \varpi^{\mu_n} \end{pmatrix},$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Suppose that $(KgK)(KhK)$ contains the double coset

$$K \begin{pmatrix} \varpi^{\nu_1} & & \\ & \ddots & \\ & & \varpi^{\nu_n} \end{pmatrix} K, \quad \nu_1 \geq \nu_2 \geq \cdots \geq \nu_n.$$

Then ν is a partition of $k + l$ and $\nu \preccurlyeq \lambda + \mu$.

Proof. [2, Prop. 36] □

Theorem 1.30 (Satake isomorphism)**.** The spherical Hecke algebra $\mathcal{H}^0$ is canonically isomorphic

$$\mathcal{H}^0 \simeq \mathbf{C}[T_1, \ldots, T_n, T_n^{-1}]$$

Proof. We present here a full combinatorial proof for GL_n that is taken from [2], and a proof that works for more general reductive groups is in [3].

The p-adic Iwasawa decomposition for $\mathrm{GL}_n(F)$ states that any $\mathrm{GL}_n(\mathscr{O}_F)$ double coset is uniquely represented by a matrix of the form

$$\left\{ \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_n} \end{pmatrix} \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \right\}.$$

In other words, we have the following decomposition:

$$K\backslash G/K = \left\{ \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_n} \end{pmatrix} \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \right\}$$

Now, we define θ_i to be the characteristic function of the double coset represented by a matrix

$$A_k := \left\{ \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_n} \end{pmatrix} \mid \lambda_1 = \cdots = \lambda_i = 1, \lambda_{i+1} = \cdots = \lambda_n = 0 \right\}$$

.

More generally, for any partition $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ we define A_λ and θ_λ to be the matrix

$$\begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_n} \end{pmatrix}$$

and the characteristic function of the double coset

$$KA_\lambda K = K \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_n} \end{pmatrix} K.$$

We divide the proof of Theorem into several lemmas.

Lemma 1.31. The set of functions θ_λ form a $\mathbf{C}$-basis of $\mathcal{H}^0$ where λ run through all partitions of length n.

Proof. Any function $\phi \in \mathcal{H}^0$ is K bi-invariant, so its restriction to each K-double coset is constant. Moreover, the function $\phi \in \mathcal{H}^0$ is compactly supported, so we conclude that it can be written as a finite linear combination of characteristic functions of K-double cosets. But these functions are by the very definition equal to θ_λ for some λ. □

Lemma 1.32. The convolution $\theta_{(-1,-1,\ldots,-1)} * \theta_\lambda = \theta_{(\lambda_1-1,\ldots,\lambda_n-1)}$.

Proof. Let us directly compute this convolution

$$(\theta_{(-1,-1,\ldots,-1)} * \theta_\lambda)(g) = \int_G \theta_{(-1,-1,\ldots,-1)}(x)\theta_\lambda(x^{-1}g)dx =$$
$$\int_{\begin{pmatrix} \varpi^{-1} & & \\ & \ddots & \\ & & \varpi^{-1} \end{pmatrix} K} \theta_\lambda(x^{-1}g)dx = \theta_{(\lambda_1-1,\ldots,\lambda_n-1)}(g).$$

In this computation we used the fact that $\begin{pmatrix} \varpi^{-1} & & \\ & \ddots & \\ & & \varpi^{-1} \end{pmatrix}$ is a central element of G. Namely, we used that

$$K \begin{pmatrix} \varpi^{-1} & & \\ & \ddots & \\ & & \varpi^{-1} \end{pmatrix} K = \begin{pmatrix} \varpi^{-1} & & \\ & \ddots & \\ & & \varpi^{-1} \end{pmatrix} K.$$

□

Corollary 1.33. The element $\theta_n \in \mathcal{H}^0$ is invertible and its inverse is equal to $\theta_{(-1,\ldots,-1)}$.

Proof. Immediate from the definition of θ_n and Lemma 1.32 □

Now we can define a morphism $f : \mathbf{C}[T_1, \ldots, T_n, T_n^{-1}] \to \mathcal{H}^0$:

Lemma 1.34. There is a unique well-defined morphism $f\colon \mathbf{C}[T_1, \dots, T_n, T_n^{-1}] \to \mathcal{H}^0$ such that $f(T_i) = \theta_i$.

Proof. Since $\mathcal{H}^0$ is commutative (Theorem 1.23!) we can define $f' : \mathbf{C}[T_1, \dots, T_n] \to \mathcal{H}^0$ by mapping T_i to θ_i. In order to check that this map extends to a morphism $f : \mathbf{C}[T_1, \dots, T_n, T_n^{-1}] \to \mathcal{H}^0$ we only need to verify that θ_n is invertible. But it is the statement of Lemma 1.33, thus f' indeed extends to $f : \mathbf{C}[T_1, \dots, T_n, T_n^{-1}] \to \mathcal{H}^0$. □

We need to check injectivity and surjectivity of f. This will be done separately and the proofs will essentially rely on the Lemma 1.29.

Lemma 1.35. The morphism $f : \mathbf{C}[T_1, \dots, T_n, T_n^{-1}] \to \mathcal{H}^0$ is surjective.

Proof. Lemma 1.31 states that all functions of the form θ_λ generate $\mathcal{H}^0$ as a $\mathbf{C}$-vector space. Therefore it suffices to show that for any partition λ the function θ_λ is inside the image of f. Lemma 1.32 states that $\theta_n^r * \theta_{(\lambda_1,\dots,\lambda_n)} = \theta_{\lambda_1+r,\dots,\lambda_n+r}$ for any integral number r. In particular, we can apply it to $r = -\lambda_n$ to reduce to the case $\lambda_n = 0$. Thus, we may and do assume that $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k, 0, \cdots, 0)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. In particular, λ is non-negative, so $|\lambda|$ is also non-negative.

Therefore, it suffices to show that for any non-negative partition λ the element θ_λ is inside image of f. We prove this claim by induction on the absolute value $|\lambda|$.

If $|\lambda| = 0$, then $\theta_\lambda = 1$ and it lies inside image of $\mathcal{H}^0$, so we are done in that case.

Now we assume that the statement is proven for any partition ν s.t. $|\nu| < |\lambda|$ and we want to prove it for λ (also assuming that $|\lambda| > 0$).

Again, let us assume that $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k, 0, \cdots, 0)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ (such k exists since $|\lambda| > 0$).

Under these assumptions we introduce a non-negative partition $\nu = (\lambda_1 - 1, \lambda_2 - 1, \cdots, \lambda_k - 1, \cdots, 0, \cdots, 0)$ (ν is a non-negative partition due to the assumptions on k) and consider the convolution $\theta_k * \theta_\nu$. Lemma 1.31 ensures that we can write $\theta_k * \theta_\nu$ as a linear combination $\sum_\mu a_\mu \theta_\mu$ where each μ is a partition. The essential idea of the proof is to understand explicitly for which μ the corresponding coefficient a_μ is non-zero.

Let us try to compute $\theta_k * \theta_\nu$:

$$\begin{aligned}
\theta_k * \theta_\nu(g) &= \int_G \theta_k(x)\theta_\mu(x^{-1}g)dx \\
&= \int_{KA_kK} \theta_{(\lambda_1-1,\lambda_2-1,\cdots,\lambda_k-1,0,\cdots,0)}(x^{-1}g)dx \\
&= \begin{cases} 0, g \notin KA_kKA_\mu K \\ \text{not } 0, g \in KA_kKA_\mu K \end{cases}
\end{aligned}$$

Now we need to remember that $a_\nu \neq 0$ if and only if $(\theta_k * \theta_\nu)(A_\nu) \neq 0$. The computation above shows that the latter condition is equivalent to the combinatorial condition $A_\nu \in KA_kKA_\mu K$ which in turn is equivalent to the condition $KA_\nu K \subset KA_kKA_\mu K$. And now we can use Lemma 1.29 to conclude that if the condition holds, then $\nu \preccurlyeq (1_1, 1_2, \cdots, 1_k, 0, \cdots, 0) + (\lambda_1 - 1, \lambda_2 - 1, \cdots, \lambda_k - 1, 0, \cdots, 0) = (\lambda_1, \lambda_2, \cdots, \lambda_k, 0, \cdots, 0) = \lambda$. Thus, we see that if $a_\nu \neq 0$ then $|\nu| < |\lambda|$ or $\nu = \lambda$.

Moreover, it is clear that the double coset $KA_\lambda K$ occurs among double cosets inside $KA_kKA_\mu K$, so $a_\lambda \neq 0$! Hence, we conclude that

$$\theta_k * \theta_\nu = a_\lambda \theta_\lambda + \sum_{\mu, |\mu| < |\lambda|} a_\mu \theta_\mu.$$

By induction each of θ_μ and θ_ν are inside the image of f! Also $\theta_k = f(T_k)$ is inside the image of f, therefore each of $a_\mu \theta_\mu$ and $\theta_k * \theta_\nu$ are inside the image of f as f is a homomorphism of $\mathbf{C}$-algebras! Thus, we see that $a_\lambda \theta_\lambda$ is inside the image of f and since a_λ is non-zero, we conclude that θ_λ is inside this image!

So, the morphism f is indeed surjective. □

The last step is injectivity of f.

Lemma 1.36. The morphism $f : \mathbf{C}[T_1, \dots, T_n, T_n^{-1}] \to \mathcal{H}^0$ is injective.

Proof. First of all, we need to note that $\mathcal{H}^0$ is actually a graded ring and f is a morphism of graded $\mathbf{C}$-algebras (with respect to some grading on $\mathbf{C}[T_1, \dots, T_n, T_n^{-1}]$).

Indeed, Lemma 1.31 says that $\mathcal{H}^0$ is generated by θ_λ as $\mathbf{C}$-vector space for different partitions λ. This means that $\mathcal{H}^0 := \bigoplus_{|\lambda|=i} \mathbf{C}\theta_\lambda$. Let us define homogeneous elements of degree i as linear combinations $h = \sum_{|\lambda|=i} a_\lambda \theta_\lambda$ of θ_λ for $|\lambda| = i$. In order to conclude that this defines grading on $\mathcal{H}^0$ we need to show that $\deg(h_1 * h_2) = \deg(h_1)\deg(h_2)$ for any homogeneous elements $h_1, h_2 \in \mathcal{H}^0$. This statement essentially boils down to the following claim

$$\deg(\theta_\lambda * \theta_\mu) = |\lambda| + |\mu|.$$

Let us use the same strategy as in the proof Lemma 1.35, we know that $\theta_\lambda + \theta_\mu$ can be written as $\sum_\nu a_\nu \theta_\nu$ and we want to show that $a_\nu \neq 0$ implies that $|\nu| = |\lambda| + |\mu|$. Ok, let us start with computing $\theta_\lambda * \theta_\mu$:

$$\begin{aligned} \theta_\lambda * \theta_\mu(g) &= \int_G \theta_\lambda(x)\theta_\mu(x^{-1}g)dx \\ &= \int_{KA_\lambda K} \theta_\mu(x^{-1}g)dx \\ &= \begin{cases} 0, g \notin KA_\lambda KA_\mu K \\ \text{not } 0, g \in KA_\lambda KA_\mu K \end{cases} \end{aligned}$$

So, the same argument as in the proof of Lemma 1.35 shows that $a_\nu \neq 0$ if and only if $KA_\nu K \subset KA_\lambda KA_\mu K$. But Lemma 1.29 shows that this condition implies that ν is a partition of $|\lambda| + |\mu|$ or, in other words, $|\nu| = |\lambda| + |\mu|$. Hence, $\theta_\lambda * \theta_\mu = \sum_{\nu, |\nu| = |\lambda| + |\mu|} a_\nu \theta_\nu$ and this means that $\theta_\lambda * \theta_\mu$ is a homogeneous element and, moreover, its degree is precisely $|\lambda| + |\nu|$. Therefore, $\mathcal{H}^0$ is actually a graded algebra.

Moreover, let us define a grading on $\mathbf{C}[T_1, \dots, T_n, T_n^{-1}]$ such that $\deg(T_i^n) = ni$. It is straightforward to see that this extends to a grading on $\mathbf{C}[T_1, \dots, T_n, T_n^{-1}]$ and that f is a graded morphism with respect to these gradings on $\mathbf{C}[T_1, \dots, T_n, T_n^{-1}]$ and $\mathcal{H}^0$.

We conclude that the kernel $\ker f$ must be a graded ideal. Thus to see that f is injective it suffices to show that its kernel doesn't contain any non-zero homogeneous elements. Suppose on the contrary that there is a non-zero homogeneous polynomial G of degree k s.t. $f(G) = 0$. Since G is homogeneous of degree k we can write it as a finite sum of the following form(exercise!)

$$G = \sum_{\lambda, |\lambda|=k} a_\lambda T_1^{\lambda_1-\lambda_2} T_2^{\lambda_2-\lambda_3} \dots T_{n-1}^{\lambda_{n-1}-\lambda_n} T_n^{\lambda_n}(*), \tag{1}$$

If $f(G) = 0$, then we see that

$$f(G) = \sum_{\lambda, |\lambda|=k} a_\lambda \theta_1^{\lambda_1-\lambda_2} * \theta_2^{\lambda_2-\lambda_3} * \dots * \theta_{n-1}^{\lambda_{n-1}-\lambda_n} * \theta_n^{\lambda_n}$$

Now note that the iterated argument from the beginning of the proof of this Lemma shows that when we expand each term $\theta_1^{\lambda_1-\lambda_2} * \theta_2^{\lambda_2-\lambda_3} * \dots * \theta_{n-1}^{\lambda_{n-1}-\lambda_n} * \theta_n^{\lambda_n}$ we get a sum $\sum_{\mu, \mu \preccurlyeq \lambda} b_\nu \theta_\mu$, where $b_\mu \in \mathbf{C}$ and $b_\lambda \neq 0$. In particular, we have

$$f(G) = \sum_{\lambda, |\lambda|=k} a_\lambda \theta_1^{\lambda_1-\lambda_2} * \theta_2^{\lambda_2-\lambda_3} * \dots * \theta_{n-1}^{\lambda_{n-1}-\lambda_n} * \theta_n^{\lambda_n} \tag{2}$$

$$\sum_{\lambda, |\lambda|=k} a_\lambda \Big(\sum_{\mu, |\mu| \preccurlyeq |\lambda|} b_\mu \theta_\mu \Big). \tag{3}$$

Choose λ such that a_λ is a maximal (with respect to a partial order on partitions) non-zero coefficient in (1) (it exists since the sum in (1) is finite and G is a non-zero polynomial), then we see from (3) that

$$f(G) = a_\lambda b_\lambda \theta_\lambda + \text{something},$$

where "something" doesn't contain θ_λ in its decomposition and $a_\lambda b_\lambda \neq 0$ (because $a_\lambda \neq 0$ by the hypothesis and $b_\lambda \neq 0$ by the argument above). Hence, $f(G)$ cannot be a zero in $\mathcal{H}^0$ since the set of $\{\theta_\nu\}$ for all partitions μ form a $\mathbf{C}$-basis of $\mathcal{H}^0$. Contradiction! □

Lemmas 1.35 and 1.36 imply together that $f\colon \mathbf{C}[T_1, \dots, T_n, T_n^{-1}] \to \mathcal{H}^0$ is an isomorphism of $\mathbf{C}$-algebras.

□

Corollary 1.37. For any spherical irreducible *smooth* representation (V, π), we have $\dim_{\mathbf{C}} V^K = 1$.

Proof. Theorem 1.18 states that V^K is a simple $\mathcal{H}^0$-module. Since $\mathcal{H}^0$ is commutative, we conclude that V^K must be a quotient of $\mathcal{H}^0$ by its maximal ideal. In principle, it could be that this quotient is of infinite degree over $\mathbf{C}$, however in our situation the Satake Isomorphim (Theorem 1.30) guarantees that $\mathcal{H}^0$ is a commutative *finitely generated* $\mathbf{C}$-algebra. Therefore, the Hilbert Nullstellensatz implies that this quotient is isomorphism to $\mathbf{C}$ as a $\mathbf{C}$-algebra. In particular, $\dim_{\mathbf{C}} V^K = 1$. □

1.4 Classification of Spherical Representations of GL_2

We have the following classification of irreducible smooth spherical representations:

Theorem 1.38. If (V, π) is an irreducible admissible spherical representation of $\mathrm{GL}_2(F)$, then there are two options:

(1) $V \simeq \rho_{\chi_1,\chi_2}$, a non-special (i.e. $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$) principal series with χ_1, χ_2 unramified.

(2) $(V, \pi) \simeq (\mathbf{C}, \chi(\det(-)))$ with $\chi\colon F^\times \to \mathbf{C}^\times$ an unramified character.

We will use the Satake isomorphism to prove Theorem 1.38:

Proof. First, we need to check that the representations (1) and (2) in the statement of the theorem are actually spherical.

For case (2), this is easy: if $k \in K = \mathrm{GL}_2(\mathscr{O}_F)$, then $\det(k) \in \mathscr{O}_F^\times$. Since χ is assumed to be unramified, $\chi|_{\mathscr{O}_F^\times} \equiv 1$, and thus $V^K = V$.

For case (1), we have:

$$\rho_{\chi_1,\chi_2} = \iota_{B(F)}^{G(F)} \chi_1 \boxtimes \chi_2$$

where ι is a "twisted induction". We may identify:

$$\rho_{\chi_1,\chi_2} = \{f\colon G \to G \mid f\big(\left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right) \cdot g\big) = \chi_1(a)\chi_2(b)|a/b|^{1/2} f(g)\}$$

with G acting by right multiplication.

We have the Cartan decomposition $\mathrm{GL}_2(F) = B(F) \cdot \mathrm{GL}_2(\mathscr{O}_F)$. Thus, if $g \in \mathrm{GL}_2(F)$, we may write:

$$g = \left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right) k$$

So, f is right invariant by k iff the following formula holds:

$$f(g) = f\big(\left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right) \cdot k\big) = \chi_1(a)\chi_2(b)|a/b|^{1/2} f(e)$$

We need to show that such an f exists inside ρ_{χ_1,χ_2}. We *define* f by the formula:

$$f\big(\left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right) \cdot k\big) = \chi_1(a)\chi_2(b)|a/b|^{1/2}$$

We may check that f is well-defined iff $\chi_1(a)\chi_2(b) = 1$ for all $(a, b) \in \mathscr{O}_F^\times \times \mathscr{O}_F^\times$. This is the case exactly when χ_1, χ_2 are unramified, i.e. exactly when $\chi_1|_{\mathscr{O}_F^\times}, \chi_2|_{\mathscr{O}_F^\times} \equiv 1$. (we may take $a = 1$ or $b = 1$ to see that this is necessary).

Note that this argument shows that ρ_{χ_1,χ_2} has a K-fixed vector whenever χ_1, χ_2 are unramified, even when ρ_{χ_1,χ_2} is special. However, in the special case, ρ_{χ_1,χ_2} will not be irreducible. The rest of this proof will imply that the associated irreducible representation π_{χ_1,χ_2} is not spherical, or we could see this directly.

Now, we need to show that any irreducible smooth spherical representation is of one of the two forms stated in the theorem. We first compute the character of $\mathcal{H}^0$ that corresponds to a spherical principal series. This sends:

$$T_1 \mapsto \int_G \mathbb{1}_{K\left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)K} \varphi(g)\, dg,$$

$$T_2 \mapsto \int_G \mathbb{1}_{K\left(\begin{smallmatrix}\varpi & 0\\ 0 & \varpi\end{smallmatrix}\right)K}\varphi(g)\,dg$$

We may calculate (as has been done in Lecture 9):

$$K\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)K = \left(\begin{smallmatrix}1 & 0\\ 0 & \varpi\end{smallmatrix}\right)K \cup \cup_{\alpha\in\mathscr{O}_F/\mathfrak{m}}\left(\begin{smallmatrix}\varpi & \alpha\\ 0 & 1\end{smallmatrix}\right)K$$

Here, α runs over a set of representatives for the residue field $\mathscr{O}_F/\mathfrak{m}$. Thus, we have:

$$\begin{aligned}T_1 &\mapsto \int_G \mathbb{1}_{K\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)K}\varphi(g)\,dg\\ &= \int_{\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi\end{smallmatrix}\right)K}\varphi(g)\,dg + \sum_\alpha \int_{\left(\begin{smallmatrix}\varpi & \alpha\\ 0 & 1\end{smallmatrix}\right)K}\varphi(g)\,dg\\ &= \chi_2(\varpi)q^{1/2} + q\left(q^{-1/2}\chi_1(\varpi)\right)\\ &= q^{1/2}(\chi_1(\varpi)+\chi_2(\varpi))\end{aligned}$$

Next, we have:

$$\begin{aligned}T_2 &\mapsto \int_G \mathbb{1}_{\left(\begin{smallmatrix}\varpi & 0\\ 0 & \varpi\end{smallmatrix}\right)K}\varphi(g)\,dg\\ &= \int_{\left(\begin{smallmatrix}\varpi & 0\\ 0 & \varpi\end{smallmatrix}\right)K}\varphi(g)\,dg\\ &= \chi_1(\varpi)\chi_2(\varpi)\end{aligned}$$

Now we will compute the associated character of the Hecke Algebra $\mathcal{H}^0$ in the case (2). Namely, we have

$$\begin{aligned}T_1 &\mapsto \int_G \mathbb{1}_{K\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)K}\varphi(g)\,dg\\ &= \int_{\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi\end{smallmatrix}\right)K}\chi(\det(g))\,dg + \sum_\alpha \int_{\left(\begin{smallmatrix}\varpi & \alpha\\ 0 & 1\end{smallmatrix}\right)K}\chi(\det(g))\,dg\\ &= \chi(\varpi) + q\chi(\varpi)\\ &= (q+1)\chi(\varpi)\end{aligned}$$

Next, we have:

$$\begin{aligned}T_2 &\mapsto \int_G \mathbb{1}_{\left(\begin{smallmatrix}\varpi & 0\\ 0 & \varpi\end{smallmatrix}\right)K}\varphi(g)\,dg\\ &= \int_{\left(\begin{smallmatrix}\varpi & 0\\ 0 & \varpi\end{smallmatrix}\right)K}\chi(\det g)\,dg\\ &= \chi(\varpi)^2\end{aligned}$$

Now, assume that (V,π) is a spherical irreducible smooth representation of G. Theorem 1.18 says it is uniquely determined by an irreducible representation of $\mathcal{H}^0$. However, Corollary 1.37 implies that irreducible representation of $\mathcal{H}^0$ is just a character. Therefore, in order to prove that we found all spherical irreducible smooth representations of G it suffices to show that for any character of $\mathcal{H}^0 \cong \mathbf{C}[T_1, T_2, T_2^{-1}]$ there exists a representation of the form (1) or (2) with the same character.

Assume that the character is given by $T_1 \mapsto \alpha_1$ and $T_2 \mapsto \alpha_2$. We form the polynomial $X^2 - q^{-1/2}\alpha_1 X + \alpha_2$. This has two roots β_1, β_2.We define two unramified characters χ_1, χ_2 by defining $\chi_1(\varpi) = \beta_1$, $\chi_2(\varpi) = \beta_2$. Then $\chi_1(\varpi)\chi_2(\varpi) = \beta_1\beta_2 = \alpha_2$ and $q^{1/2}(\chi_1(\varpi) + \chi_2(\varpi)) = q^{1/2}(\beta_1 + \beta_2) = \alpha_1$, so the character of (V, π) agrees with the character of ρ_{χ_1,χ_2}.

We are done whenever ρ_{χ_1,χ_2} is non-special - this occurs exactly when $\beta_1/\beta_2 \neq q^{\pm 1}$, since $|\varpi| = q^{-1}$. In the last case, we may assume that $\beta_1 = q^{-1}\beta_2$, and check that this gives us the character in case (2).

Firstly, note that this gives us a system of equations:

$$\begin{aligned}
&\beta_1 = q^{-1}\beta_2 \\
&\beta_1\beta_2 = \alpha_2 \\
&\beta_1 + \beta_2 = q^{-1/2}\alpha_1.
\end{aligned}$$

After a bit of work, we can conclude that $\alpha_1 = (1 + q)\alpha_2^{1/2}$ (with an appropriate choice of a sign for a square root). Now, we consider an unramified character $\chi : F^\times \to \mathbf{C}^\times$ defined by $\chi(\varpi) = \alpha_2^{1/2}$. Then according to the calculations above we see that the irreducible spherical representation $(\mathbf{C}, \det(\chi(-)))$ corresponds to the character

$$T_1 \mapsto (q + 1)\alpha_2^{1/2} = \alpha_1$$

$$T_2 \mapsto (\alpha_2^{1/2})^2 = \alpha_2.$$

So, we constructed an irreducible spherical representation of the form (1) or (2) with a given $\mathcal{H}^0$ character. Then Theorem 1.18 guarantees that all irreducible smooth spherical representations of $\mathrm{GL}_2(F)$ are of the form (1) or (2). □

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