LECTURE 16: UNITARY REPRESENTATIONS
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Let F be a local field with valuation ring $\mathscr{O}_F$, and G_F the group $\mathrm{GL}_2(F)$. If (π, V) is an admissible representation of G_F, we define:

Definition 1. (π, V) is *pre-unitary* if it admits a positive definite G_F-invariant Hermitian form.

We say pre-unitary rather than unitary, since V may not be complete. However, given a pre-unitary representation (π, V), we may complete V to a Hilbert space $\widehat{V}$ and extend the action of π to $\widehat{\pi}$. This preserves the notion of irreducibility in an appropriate sense:

Lemma 2. If (π, V) is a pre-unitary admissible representation, then (π, V) is algebraically irreducible if and only if $(\widehat{\pi}, \widehat{V})$ is topologically irreducible: i.e. it has no non-trivial invariant *closed* subspaces.

Proof. If we let ρ be a (finite dimensional) irreducible representation of $\mathrm{GL}_2(\mathscr{O}_F)$, then we write $V(\rho)$ for the set of all vectors $v \in V$ which transform under $\mathrm{GL}_2(\mathscr{O}_F)$ according to ρ, i.e. the span of all of the copies of ρ inside $\mathrm{Res}^{G_F}_{\mathrm{GL}_2(\mathscr{O}_F)} V$. We may define $\widehat{V}(\rho)$ similarly. We will show that $V(\rho) = \widehat{V}(\rho)$. By admissibility of V, $V = \bigoplus_\rho V(\rho)$, and $V(\rho)$ are mutually orthogonal and finite-dimensional. This implies that:

$$\widehat{V} = \widehat{\bigoplus}_\rho V(\rho),$$

the Hilbert direct sum of the $V(\rho)$. Thus, $\widehat{V}(\rho) = V(\rho)$.

Hence V is the set of all $\mathrm{GL}_2(\mathscr{O}_F)$-finite vectors in $\widehat{V}$. Because $\mathrm{GL}_2(\mathscr{O}_F)$ is compact, $\mathrm{GL}_2(\mathscr{O}_F)$-finite vectors are dense in every closed invariant subspace of $\widehat{V}$.

Therefore $V \cap W'$ is dense in any G_F-invariant closed subspace W' of $\widehat{V}$. Thus, any closed invariant subspace $W' \subseteq \widehat{V}$ is the closure of $V \cap W'$. If V is algebraically irreducible, this means that $V \cap W' = 0$ or $V \cap W' = V$, so $W' = 0$ or $W' = \widehat{V}$. Therefore if (π, V) is algebraically irreducible, then $(\widehat{\pi}, \widehat{V})$ is topologically irreducible.

On the other hand, we need to show that if $V_0 \subseteq V$ is a nontrivial proper invariant subspace, then $\overline{V_0} \subsetneq \widehat{V}$. For V_0 an invariant subspace of V, we see that if we let $V_0(\rho) = V(\rho) \cap V_0$, then $V_0 = \bigoplus_\rho V_0(\rho)$. Since we also have $\widehat{V} = \widehat{\bigoplus}_\rho \widehat{V}(\rho)$ (recall $\widehat{V}(\rho) = V(\rho)$), we see that the closure $\overline{V_0}$ of V_0 in $\widehat{V}$ must be $\overline{V_0} = \widehat{\bigoplus}_\rho V_0(\rho)$. Thus $\overline{V_0}$ is invariant and is nontrivial exactly when V_0 is, so we see that if $\widehat{V}$ is topologically irreducible, $\overline{V_0} = 0$ or $\overline{V_0} = \widehat{V}$, so $V_0 = 0$ or $V_0 = V$. Therefore (π, V) is algebraically irreducible if $(\widehat{\pi}, \widehat{V})$ is topologically irreducible.

□

Remark 3. It is not *a priori* obvious that any topologically irreducible Hilbert space representation $\widehat{V}$ of G_F arises in this way from an irreducible admissible representation of G_F. In other words, why should the space of smooth vectors (i.e. the vectors which are fixed by an open subgroup of G_F) of $\widehat{V}$ be non-zero?

Theorem 4. Let (π, V) be an infinite-dimensional[1] irreducible admissible representation of G_F. It is pre-unitary *exactly* in these cases:

(1) π is super-cuspidal and the central character ω_π satisfies $|\omega_\pi(t)| = 1$ for all t.

(2) $\pi = \pi_{\mu_1,\mu_2}$ is a non-special principal series with μ_1, μ_2 unitary characters.

(3) $\pi = \pi_{\mu_1,\mu_2}$ is a non-special principal series with $\mu_2 = \overline{\mu_1}^{-1}$ *and* $\mu := \mu_1\mu_2^{-1} = |x|^\sigma$ for $0 < \sigma < 1$.

(4) $\pi = \pi_{\mu_1,\mu_2}$ is a special principal series such that $\mu_1(x) = |x|^{1/2}\chi(x), \mu_2(x) = |x|^{-1/2}\chi(x)$ for χ a unitary character.

Furthermore, on the Kirillov model $\mathcal{K}(\pi)$ of (π, V), the invariant scalar product takes the form

$$(\xi, \eta) = \int \xi(x)\overline{\eta(x)}d^\times x.$$

The rest of these notes concern the proof of this theorem.

1 Super-cuspidal case

First, we will consider case (1), where π is super-cuspidal. The condition that $|\omega_\pi(t)| = 1$ is necessary because for any $\xi, \eta \in V$, we have by G_F-invariance of the Hermitian pairing $(\cdot, \cdot)$ on V:

$$(\xi, \eta) = \Big(\pi\big(\left(\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right)\big)\xi, \pi\big(\left(\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right)\big)\eta\Big) = \omega_\pi(t)\,\overline{\omega_\pi(t)}\,(\xi, \eta)\,.$$

Thus, by non-degeneracy, $|\omega_\pi(t)| = 1$ for all t.

Now, assume that this condition holds. Fix $\zeta_0 \in \check{V}$ a non-zero vector. For any $\xi, \eta \in V$, we consider the following function on G_F:

$$f_{\xi,\eta} \colon g \mapsto \langle\pi(g)\xi, \zeta_0\rangle\overline{\langle\pi(g)\eta, \zeta_0\rangle}.$$

Here, $\langle\cdot, \cdot\rangle$ is the canonical pairing on $V \times \check{V}$. Then $f_{\xi,\eta}$ is a function from G_F to $\mathbf{C}$ which is compactly supported modulo the center Z_F of G_F - this follows from the definition of super-cuspidality, because $f_{\xi,\eta}$ is a product of matrix coefficients. (See Theorem 3 in [1].)

Now, it makes sense to integrate over G_F to obtain a Hermitian pairing:

$$(\xi, \eta) := \int_{G_F/Z_F} \langle\pi(g)\xi, \zeta_0\rangle\overline{\langle\pi(g)\eta, \zeta_0\rangle}\,dg.$$

Note that the integrand is a Z_F-invariant function because the central character is unitary by assumption. It is not hard to see that this is G_F-invariant by unimodularity of G_F.

Now, we must check that this is positive-definite. Assume that $(\xi, \xi) = 0$. This means that for all g, $\pi(g)\xi$ is orthogonal to ζ_0, or equivalently (by G_F-invariance of $\langle\cdot, \cdot\rangle$) that ξ is orthogonal to $\check{\pi}(g)\zeta_0$. Since $\check{\pi}$ is irreducible, this implies that $\check{\pi}(G) \cdot \zeta_0$ generates $\check{V}$ and therefore that $\xi = 0$.

[1] so it does not factor through the determinant

Next, we will describe this pairing on the Kirillov model. We claim that it is given by:

$$(\xi, \eta) = \int \xi(x)\overline{\eta(x)}\, d^\times x.$$

It suffices to show that this gives an invariant inner product on $\mathcal{K}(\pi)$, since these are unique up to a scalar: an invariant inner product on (π, V) is determined by the associated isomorphism $\overline{\pi} \xrightarrow{\sim} \check{\pi}$, so two distinct invariant inner products differ by an automorphism of the irreducible G_F-module $\check{V}$. This must be a scalar by Schur's lemma.

The pairing defined above is clearly a positive-definite Hermitian inner product, so it suffices to show that the pairing is G_F-invariant. It actually suffices to check this invariance under the family of operators $F_{a,b}\colon [x \mapsto \xi(x)] \mapsto [x \mapsto \psi_F(bx)\xi(ax)]$:

$$\begin{aligned}\int \psi_F(bx)\xi(ax)\overline{\psi_F(bx)\eta(ax)}d^\times x &= \int \psi_F(bx)\xi(x)\overline{\psi_F(bx)\eta(x)}d^\times x \\ &= \int \xi(x)\overline{\eta(x)}d^\times x.\end{aligned}$$

since we assume ψ_F unitary.

This gives invariance under the mirabolic subgroup $H = \{\left(\begin{smallmatrix} a & b \\ 0 & 1\end{smallmatrix}\right)\}$ by the definition of the Kirillov model. Since we are in the supercuspidal case, $\mathcal{K}(\pi) = \mathscr{S}(F^\times)$. In fact the mirabolic subgroup acts irreducibly on $\mathscr{S}(F^\times)$. Since we know that a G_F-invariant (hence H-invariant) bilinear form exists by the previous discussion, we see that up to a nonzero constant, this G_F-invariant bilinear form must equal $(\xi, \eta) = \int \xi(x)\overline{\eta(x)}\, d^\times x$ as we desired to show.

This completes the case of super-cuspidal representations, so it suffices to consider the case of principal series.

2 Principal series case

Let's assume that (π, V) is a pre-unitary irreducible admissible representation. We have a complex semi-linear map J from V to $\check{V}$ defined by :

$$\langle \xi, J\eta \rangle = (\xi, \eta).$$

This gives an isomorphism from $\overline{\pi}$ to $\check{\pi}$.

Now, assume that $\pi = \pi_{\mu_1,\mu_2}$ is a non-special principal series representation. This is defined on the space $\mathcal{B}_{\mu_1,\mu_2}$ of locally constant function φ on G_F such that:

$$\varphi\big(\left(\begin{smallmatrix} a & * \\ 0 & b\end{smallmatrix}\right) \cdot g\big) = \mu_1(a)\mu_2(b)|a/b|^{1/2}\varphi(g).$$

Now, we have an isomorphism $\mathcal{B}_{\mu_1,\mu_2} \xrightarrow{\sim} \mathcal{B}_{\overline{\mu_1},\overline{\mu_2}} \simeq \overline{\pi}$ sending φ to $\overline{\varphi}$. We know from a previous lecture that $\check{\pi} \simeq \mathcal{B}_{\mu_2^{-1},\mu_1^{-1}}$.

Now, there are two cases, corresponding to cases (2) and (3) in the statement of the theorem. The first of these is the case that $\mu_1^{-1} = \overline{\mu_1}$, $\mu_2^{-1} = \overline{\mu_2}$, and this means that μ_1, μ_2 are unitary. The second is the case that $\mu_1^{-1} = \overline{\mu_2}$, $\mu_2^{-1} = \overline{\mu_1}$.

Since $\mathcal{B}_{\mu_1,\mu_2} \simeq \mathcal{B}_{\lambda_1,\lambda_2}$ iff $\{\mu_1, \mu_2\} = \{\lambda_1, \lambda_2\}$ as unordered pairs, these possibilities are necessary and sufficient for $\overline{\pi}$ and $\check{\pi}$ to be G_F-isomorphic. As we saw above, this is necessary for π to be pre-unitary, but it is not clearly sufficient: an isomorphism from $\overline{\pi}$ to $\check{\pi}$ induces a non-degenerate Hermitian invariant bilinear pairing on V, but it is not *a priori* clear that it should be positive definite.

2.1 Case (2)

In the first case, we define the bilinear pairing by:

$$(\varphi_1, \varphi_2) := \int_{B_F\backslash G_F} \varphi_1(g)\overline{\varphi_2(g)}\, dg = \int_{\mathrm{GL}_2(\mathscr{O}_F)} \varphi_1(m)\overline{\varphi_2(m)}\, dm.$$

The second equality follows from the Cartan decomposition $B_F \cdot \mathrm{GL}_2(\mathscr{O}_F) = G_F$.

The notes from Lecture 12 of this seminar ([5]) show that this integral defines a non-degenerate G_F-invariant bilinear pairing (this was how we identified the contragredient of π_{μ_1,μ_2} with $\pi_{\mu_1^{-1},\mu_2^{-1}}$). To show that this is Hermitian and positive-definite, we will compute in the Kirillov model $\mathcal{K}(\pi)$. To $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$, we associate $\xi_\varphi \in \mathcal{K}(\pi)$, defined by:

$$\xi_\varphi(x) := \mu_2(x)|x|^{1/2} \int \varphi\Big(w^{-1} \left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)\Big)\overline{\psi_F(xy)}\, dy.$$

Here, w is the Weyl group generator $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Now, we use the following identity, which comes from explicitly realizing Bruhat decomposition for GL_2:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{pmatrix} c^{-1}\det g & * \\ 0 & c \end{pmatrix} w^{-1} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix},$$

which is valid whenever $c \neq 0$. This implies that:

$$\int_{B_F\backslash G_F} \varphi_1(g)\overline{\varphi_2(g)}\, dg = \int \varphi_1\big(w^{-1} \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\big)\overline{\varphi_2\big(w^{-1} \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\big)}\, dx.$$

Defining $\Phi_i(x) := \varphi_i\big(w^{-1} \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\big)$, the above identity may be rephrased as:

$$(\varphi_1, \varphi_2) = \int \Phi_1(x)\overline{\Phi_2(x)}\, dx.$$

We may use Fourier inversion to see:

$$(\varphi_1, \varphi_2) = \int \Phi_1(x)\overline{\int \widehat{\Phi_2(y)}\psi_F(xy)\, dy}\, dx = \int \widehat{\Phi_1}(y)\overline{\widehat{\Phi_2}(y)}\, dy.$$

We are able to do this because, as discussed in Section 1.9 of [1], Φ corresponding to $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ must be proportional to $\mu(x)^{-1}|x|^{-1}$ for $|x|$ large. Since $\mu(x) := \mu_1(x)\mu_2^{-1}(x)$ satisfies $|\mu(x)| = 1$, we see that Φ is square integrable on F, so $\widehat{\Phi}$ is its Fourier transform in the L^2 sense. Furthermore, we can determine the image of the map $\Phi \to \widehat{\Phi}$, from which we are able to deduce that the $\widehat{\Phi}$ are integrable and $\int \widehat{\Phi}(y)\psi_F(xy)\, dy = \Phi(x)$.

Therefore

$$\begin{aligned}(\varphi_1,\varphi_2) &= \int \widehat{\Phi_1}(y)\overline{\widehat{\Phi_2}(y)}\, dy \\ &= \int \mu_2(y)^{-1}|y|^{-1/2}\xi_{\varphi_1}(y)\mu_2(y)^{-1}|y|^{-1/2}\xi_{\varphi_2}(y)y \\ &= \int \xi_{\varphi_1}(x)\xi_{\varphi_2}(x)d^\times x,\end{aligned}$$

using the relation $d^\times x = |x|^{-1}dx$ and the fact that μ_2 is unitary.

In this form, it is easy to check that the pairing is positive definite and Hermitian. This settles case (2), where μ_1, μ_2 are unitary.

2.2 Case (3)

Now, we must settle case (3): $\pi = \pi_{\mu_1,\mu_2}$ with $\overline{\mu_1} = \mu_2^{-1}$ and $\overline{\mu_i} \neq \mu_i^{-1}$ (i.e. μ_i not unitary). Since we know what characters of $F^\times$ must look like, this means that $\mu = \mu_1\mu_2^{-1} = |x|^\sigma$ for $\sigma \neq 0$ (if $\sigma = 0$, the μ_i would be unitary). Without loss of generality, we may assume that $\sigma > 0$ since we may switch μ_1 and μ_2 if desired.

We will define an operator $\mathbf{A}\colon \mathcal{B}_{\mu_1,\mu_2} \to \mathcal{B}_{\overline{\mu_1}^{-1},\overline{\mu_2}^{-1}} = \mathcal{B}_{\mu_2,\mu_1}$ by:

$$(\mathbf{A}\varphi)(g) := \int \varphi\big(w \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \cdot g\big)\, dx.$$

For fixed g, the integrand $\varphi\big(w \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \cdot g\big)$ grows as $\mu^{-1}(x)|x|^{-1} = |x|^{-\sigma-1}$ for large $|x|$, so this integral converges (see p.1.28, [1]).

We will show that $\mathbf{A} \neq 0$ and that $\mathbf{A}$ is G_F-equivariant.

To see that $\mathbf{A}$ is nonzero, we use the function

$$f(g) = f(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) := |\det g|^{1/2}|c|^{-1}\mu^{-1}(c)\mu_1(\det g)$$

for $c \neq 0$ and $f(g) = 0$ otherwise. It can be checked that $f \in \mathcal{B}_{\mu_1,\mu_2}$, and furthermore, $(\mathbf{A}f)(1) = 1$. Hence $\mathbf{A} \neq 0$.

To see that $\mathbf{A}$ is G_F-equivariant, we note that for $h, g \in G_F$

$$\begin{aligned}(\pi_{\mu_2,\mu_1}(g)\mathbf{A}\varphi)(h) &= (\mathbf{A}\varphi)(hg) \\ &= \int \varphi\big(w \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \cdot hg\big)\, dx \\ &= \int (\pi_{\mu_1,\mu_2}\varphi)\big(w \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \cdot h\big)\, dx \\ &= (\mathbf{A}\pi_{\mu_1,\mu_2}\varphi)(h),\end{aligned}$$

so $\mathbf{A}$ is G_F-equivariant as we desired to show.

It follows that an invariant scalar product on $\mathcal{B}_{\mu_1,\mu_2}$ *must* look like:

$$(\varphi_1, \varphi_2) = c\langle \mathbf{A}\varphi_1, \overline{\varphi_2}\rangle$$

for some constant c, since any invariant Hermitian product on $\mathcal{B}_{\mu_1,\mu_2}$ will give an isomorphism between $\overline{\pi_{\mu_1,\mu_2}} = \pi_{\overline{\mu_1},\overline{\mu_2}}$ and its dual $\mathcal{B}_{\mu_2,\mu_1}$, and those isomorphisms are unique up to a scalar by irreducibility of $\mathcal{B}_{\mu_2,\mu_1}$.

We will show that when $\sigma < 1$, this actually defines a positive-definite Hermitian pairing.

We have $\mathbf{A}\varphi_1 \in \mathcal{B}_{\mu_2,\mu_1}$, and $\overline{\varphi_2} \in \mathcal{B}_{\overline{\mu_1},\overline{\mu_2}} = \mathcal{B}_{\mu_2^{-1},\mu_1^{-1}}$. For $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$, we define:

$$\Phi(x) = \varphi\big(w^{-1}\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\big), \qquad \Phi'(x) = (\mathbf{A}\varphi)\big(w^{-1}\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\big).$$

Now, again using the Bruhat decomposition identity

$$\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) = \begin{pmatrix}c^{-1}\det g & *\\ 0 & c\end{pmatrix} w^{-1}\begin{pmatrix}1 & d/c\\ 0 & 1\end{pmatrix},$$

we may see that for $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ with $c \neq 0$:

$$\varphi(g) = \mu_1(\det g)|\det g|^{1/2}\mu^{-1}(c)|c|^{-1}\Phi(d/c),$$

since $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ transforms under upper triangular matrices in a specified way.

Furthermore:

$$\begin{aligned}
\Phi'(x) &= \int \varphi(w\left(\begin{smallmatrix}1 & -y\\ 0 & 1\end{smallmatrix}\right) w^{-1}\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right))dy\\
&= \int \varphi(\left(\begin{smallmatrix}1 & x\\ y & 1+xy\end{smallmatrix}\right))dy\\
&= \int \mu^{-1}(y)|y|^{-1}\Phi(\frac{1+xy}{y})dy \quad \text{(by the Bruhat decomposition identity)}\\
&= \int \mu^{-1}(y)\Phi(x+y^{-1})d^\times y\\
&= \int \mu(y)\Phi(x+y)d^\times y,
\end{aligned}$$

where in the last equality we make a change of variables $y \mapsto y^{-1}$.

Now, if we define:

$$(\varphi_1, \varphi_2) = c\langle \mathbf{A}\varphi_1, \overline{\varphi_2}\rangle,$$

we may use the above identities to see that:

$$\begin{aligned}
(\varphi_1, \varphi_2) &= c\int_{B_F\backslash G_F} (\mathbf{A}\varphi_1)(g)\overline{\varphi_2(g)}\, dg\\
&= c\int (\mathbf{A}\varphi_1)\big(w^{-1}\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\big)\overline{\varphi_2\big(w^{-1}\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\big)}\, dx\\
&= c\int \Phi_1'(x)\overline{\Phi_2(x)}\, dx\\
&= c\int\int \Phi_1(x+y)\overline{\Phi_2(x)}|y|^\sigma\, d^\times y\, dx.
\end{aligned}$$

For this to be positive-definite, we need to find some $c(\sigma) > 0$ such that:

$$c(\sigma) \int\int \Phi(x+y)\overline{\Phi(x)}|y|^\sigma \, dx \, d^\times y \geqslant 0$$

for any Φ defined as:

$$\Phi(x) = \varphi\big(w^{-1} \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\big)$$

for $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$.

The space of such Φ is the space $\mathscr{F}(\mu)$ of locally constant functions on F which are proportional to $\mu(x)^{-1}|x|^{-1}$ for $|x| \gg 0$ (see Section 1.9 in [1]). Since the space of Schwarz functions $\mathscr{S}$(F) are 0 for $|x| \gg 0$, we have $\mathscr{S}(F) \subseteq \mathscr{F}(\mu)$.

Note that $c(\sigma)|y|^\sigma d^\times y = c(\sigma) y^{\sigma-1} dy$.

We note that $g(y) = c(\sigma)y^{\sigma-1}$ must be a positive-definite function on F, meaning that for any n-tuple of elements $\{y_i\}_{i=1}^n$ in F, the matrix $(g(y_i - y_j))$ must be positive Hermitian.

This is because, as stated in Proposition 4.1, Chapter 1, [3], a function f_0 on a locally compact abelian group H is positive definite, if for any $f \in C_c(H)$, we have $\int f_0(y) \int f(y-x)\overline{f(-x)} dx dy \geqslant 0$, where dy is the Haar measure on that group.

We see that this condition is exactly equivalent to our previous condition

$$c(\sigma) \int\int \Phi(x+y)\overline{\Phi(x)}|y|^\sigma \, dx \, d^\times y \geqslant 0.$$

We will see that it is possible to choose such a $c(\sigma)$ iff $\sigma < 1$.

Then since $c(\sigma)y^{\sigma-1}$ is a positive definite function, the distributional Fourier transform of $c(\sigma)|y|^\sigma \, d^\times y = c(\sigma)y^{\sigma-1}dy$ must be a positive measure on F, by Bochner's theorem. (For proof of this theorem see p.19, [2].) This Fourier transform is proportional to $|x|^{1-\sigma} \, d^\times x$. Since $|x|^{1-\sigma}$ is only a locally L^1 function near 0 when $\sigma < 1$, this condition is necessary for this to be a measure on F.

Now, for $0 < \sigma < 1$, the definition of the γ-factor implies that there is a constant $\gamma(\sigma)$ for any $\Phi \in \mathscr{S}(F)$ such that:

$$\int \widehat{\Phi}(y)|y|^\sigma \, d^\times y = \frac{1}{\gamma(\sigma)} \int \Phi(x)|x|^{1-\sigma} \, d^\times x.$$

This is because if we define $L_\Phi(\chi, s) = \int_{F^\times} \Phi(x)\chi(x)|x|^s d^\times x$, there is a factor $\gamma(\chi, s)$ depending only on χ and s such that $L_\Phi(-\chi, 1-s) = \gamma(\chi, s)L_{\widehat{\Phi}}(\chi, s)$. (See p.1.41 in [1], or discussion in previous lectures.)

Thus, if we choose $c(\sigma) = \gamma(\sigma)$ and define $(\varphi_1, \varphi_2) = c(\sigma)\langle \mathbf{A}\varphi_1, \overline{\varphi_2}\rangle$, then we have:

$$(\varphi_1, \varphi_2) = \gamma(\sigma) \int\int \Phi_1(x+y)\overline{\Phi_2(x)}|y|^\sigma \, dx \, d^\times y.$$

It remains to show that the inner product takes the desired form on the Kirillov model. Since the Fourier transform of $y \mapsto \Phi_1(x+y)$ is $z \mapsto \widehat{\Phi_1}(z)\psi_F(xz)$, we see that

$$\gamma(\sigma) \int \Phi_1(x+y)|y|^\sigma d^\times y = \int \widehat{\Phi_1}(z)\psi_F(xz)|z|^{1-\sigma} d^\times z.$$

We are able to perform the Fourier transform since we can describe the behavior of the Φ_i for $|x|$ large - $\Phi_i(x)$ must be proportional to $\mu(x)^{-1}|x|^{-1} = x^{-\sigma-1}$ (see p.1.31, [1]), and $\sigma > 0$.

Then recalling the definition of ξ_φ the corresponding element to φ in the Kirillov model, we see that

$$\begin{aligned}(\varphi_1, \varphi_2) &= \gamma(\sigma) \int\int \Phi_1(x+y)\overline{\Phi_2(x)}|y|^\sigma \, dx \, d^\times y \\ &= \int\int \widehat{\Phi}_1(z)\psi_F(xz)\overline{\Phi_2(x)}|z|^{1-\sigma} \, dx \, d^\times z \\ &= \int\int \widehat{\Phi}_1(z)\psi_F(xz)\overline{\widehat{\Phi}_2}(z)|z|^{1-\sigma} \, d^\times z \\ &= \int\int \xi_{\varphi_1}(z)\overline{\xi_{\varphi_2}}(z)|\mu_2(z)|^{-2}|z|^{-1}|z|^{1-\sigma} d^\times z \\ &= \int\int \xi_{\varphi_1}(z)\overline{\xi_{\varphi_2}}(z) d^\times z,\end{aligned}$$

as we desired to show. We are able to perform the Fourier transforms since we can describe the behavior of the Φ_i for $|x|$ large (see p.1.31, [1]).

3 Special case

Finally, we must settle the final case (4), where $\pi = \pi_{\mu_1,\mu_2}$ is special. Without loss of generality, we may assume that $\mu = \mu_1\mu_2^{-1} = |x|$. If π is pre-unitary, $\overline{\pi} \simeq \check{\pi}$. As above, this implies that $\mu_1\overline{\mu_2} = 1$ (we cannot have μ_1, μ_2 both unitary, since this is incompatible with the assumption that $\mu = |x|$). One may show that in this case we have $\mu_1(x) = |x|^{1/2}\chi(x)$ and $\mu_2(x) = |x|^{-1/2}\chi(x)$, for χ a unitary character.

The space of π is $\mathcal{B}^0_{\mu_1,\mu_2} \subseteq \mathcal{B}_{\mu_1,\mu_2}$ defined by the condition that:

$$\int \varphi\big(w^{-1} \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\big) \, dx = 0.$$

By invariance of $\mathcal{B}^0_{\mu_1,\mu_2}$ as a subspace, if $\varphi \in \mathcal{B}^0_{\mu_1,\mu_2}$, then for any $g \in G_F$, we have:

$$\int \varphi\big(w^{-1} \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \cdot g\big) \, dx = 0.$$

Thus, we cannot use the formula for $\mathbf{A}$ that we used in the previous cases. Instead, we use a limit:

$$(\varphi_1, \varphi_2) := \lim_{\sigma \to 1^-} \gamma(\sigma) \int\int \Phi_1(x+y)\overline{\Phi_2(x)}|y|^\sigma \, dx \, d^\times y = \lim_{\sigma \to 1^-} \int \widehat{\Phi_1}(z)\overline{\widehat{\Phi_2}(z)}|z|^{1-\sigma} \, d^\times z.$$

We must verify that the properties of Φ_1, Φ_2 make the Fourier inversion make sense, and make sure that the limit exists and has the desired properties. We can do this Fourier inversion because $\sigma = 1$ implies that for $|x|$ large, Φ_1, Φ_2 are proportional to $\mu(x)^{-1}|x|^{-1} = |x|^{-2}$, so they are in $L^1 \cap L^2$ and we can apply Parseval's formula.

If $\varphi \in \mathcal{B}^0_{\mu_1,\mu_2}$, then $\widehat{\Phi}(0) = 0$. Because $\widehat{\Phi}$ is in fact locally constant, and vanish outside some compact subset of F, that gives the desired convergence of the limit (see p.1.29, [1]).

Since the limit exists, we can check that:

$$(\varphi_1, \varphi_2) = \int \widehat{\Phi_1}(x)\overline{\widehat{\Phi_2}(x)}\, d^\times x = \int \xi_{\varphi_1}(x)\overline{\xi_{\varphi_2}(x)}\, d^\times x.$$

This shows $(\cdot, \cdot)$ is Hermitian and positive definite. Thus, we need to check invariance under B_F and under w. The former comes from the formula for $(\cdot, \cdot)$ on the Kirillov model. What remains is to check invariance under w.

We have (see p.1.52, [1]) that

$$\gamma(\sigma) = \frac{1 - q^{-\sigma}}{1 - q^{\sigma - 1}}.$$

For $\varphi \in \mathcal{B}^0_{\mu_1,\mu_2}$, we have, via a change of variables:

$$\lim_{\sigma \to 1} \gamma(\sigma) \int \Phi(x)|x - y|^{\sigma - 1}\, dx = \lim_{\sigma \to 1} \frac{1 - q^{-\sigma}}{1 - q^{\sigma - 1}} \int \Phi(x + y)|x|^\sigma\, d^\times x.$$

We note that $\int \Phi(x + y)|x|d^\times x = \int \Phi(x + y)dx = 0$ for all $\varphi \in \mathcal{B}^0_{\mu_1,\mu_2}$. Therefore

$$\lim_{\sigma \to 1} \frac{1 - q^{-\sigma}}{1 - q^{\sigma - 1}} \int \Phi(x + y)|x|^\sigma\, d^\times x = \lim_{\sigma \to 1} \frac{(1 - q^{-\sigma}) \int \Phi(x + y)(|x|^\sigma - |x|)\, d^\times x}{1 - q^{\sigma - 1}}.$$

Now if we observe that $\frac{d}{d\sigma}|x|^\sigma = -v(x)|x|^\sigma \log q$ (where $v(x)$ is the valuation) and noting that the integral $\int \Phi(x + y)v(x)dx$ is absolutely convergent (to justify the derivation under the integral), we are able to apply L'Hospital's rule to see that

$$\lim_{\sigma \to 1} \frac{\int \Phi(x + y)(|x|^\sigma - |x|)\, d^\times x}{1 - q^{\sigma - 1}} = \lim_{\sigma \to 1} \frac{\int \Phi(x + y)(-v(x))|x|^\sigma \log q\, d^\times x}{-q^{\sigma - 1} \log q}.$$

Therefore

$$\lim_{\sigma \to 1} \gamma(\sigma) \int \Phi(x)|x - y|^{\sigma - 1}\, dx = (1 - q^{-1}) \int \Phi(x + y)v(x)dx.$$

Now, this allows us to show that:

$$\begin{aligned}
(\varphi_1, \varphi_2) &= \lim_{\sigma \to 1^-} \gamma(\sigma) \int \int \Phi_1(x + y)\Phi_2(x)|y|^\sigma\, dx\, d^\times y \\
&= \lim_{\sigma \to 1^-} \gamma(\sigma) \int \Phi_2(x) \int \Phi_1(x + y)|y|^\sigma\, d^\times y\, dx \\
&= \lim_{\sigma \to 1^-} \left(1 - \frac{1}{q}\right) \int \Phi_2(x) \int \Phi_1(x + y)v(y)\, dy\, dx \\
&= \left(1 - \frac{1}{q}\right) \int \int \Phi_1(x)\Phi_2(y)v(x - y)\, dx\, dy.
\end{aligned}$$

Now, we may check invariance under the Weyl group element w:

$$\begin{aligned}\left(1-\frac{1}{q}\right)^{-1}(\pi(w)\varphi_1,\pi(w)\varphi_2) &= \int\int \varphi_1\big(w^{-1}\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)w\big)\overline{\varphi_2\Big(w^{-1}\left(\begin{smallmatrix}1 & y\\ 0 & 1\end{smallmatrix}\right)\Big)w}v(x-y)\,dx\,dy\\ &= \int\int \varphi_1\Big(\left(\begin{smallmatrix}1 & 0\\ -x & 1\end{smallmatrix}\right)\Big)\overline{\varphi_2\Big(\left(\begin{smallmatrix}1 & 0\\ -y & 1\end{smallmatrix}\right)\Big)}v(x-y)\,dx\,dy\\ &= \int\int \Phi_1(-x^{-1})|x|^{-2}\overline{\Phi_2}\big(-y^{-1}\big)|y|^{-2}v(x-y)\,dx\,dy\\ &= \int\int \Phi_1(x)\overline{\Phi_2(y)}v(x^{-1}-y^{-1})\,dx\,dy\\ &= \int\int \Phi_1(x)\overline{\Phi_2(y)}\,\big(v(x-y)-v(xy)\big)\,dx\,dy\\ &= \left(1-\frac{1}{q}\right)^{-1}(\varphi_1,\varphi_2)-\int\int \Phi_1(x)\overline{\Phi_2(y)}v(xy)\,dx\,dy.\end{aligned}$$

Now, we want the second term to vanish. We have:

$$\int\int \Phi_1(x)\overline{\Phi_2(y)}v(xy)\,dx\,dy = \int \overline{\Phi_2(y)}v(y)\int \Phi_1(x)\,dx\,dy + \int \Phi_1(x)v(x)\int \overline{\Phi_2(y)}\,dy\,dx.$$

Both terms vanish by the defining condition that $\varphi_i \in \mathcal{B}^0_{\mu_1,\mu_2}$: this says exactly that

$$\int \Phi_i(x)\,dx = 0.$$

Therefore we have invariance under the Weyl element w and hence under G_F, which is what we desired to show. This completes the proof of the theorem for the case of special representations

References

[1] Godement, R. *Notes on Jacquet-Langlands' Theory*, The Institute for Advanced Study, 1970, available at http://math.stanford.edu/ conrad/conversesem/refs/godement-ias.pdf.

[2] Rudin, W. *Fourier analysis on groups.* New York : Interscience 1962.

[3] Van Den Berg, C., and Gunnar Forst. *Potential theory on locally compact abelian groups*. Vol. 87. Springer Science & Business Media, 2012.

[4] `http://math.stanford.edu/~conrad/conversesem/Notes/L11.pdf`

[5] `http://math.stanford.edu/~conrad/conversesem/Notes/L12.pdf`