IRREDUCIBLE COMPONENTS OVER ARCHIMEDIAN FIELDS

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In this set of notes, after recalling past notation in § 1, we state and prove the classification of irreducible admissible $\mathrm{GL}_2(\mathbf{R})$ representations in § 2. We then state and sketch the similar classification over of irreducible admissible $\mathrm{GL}_2(\mathbf{C})$ representations in § 3.

1. Review Of Past Talks and Notation

Recall the following notation. Let F be either $\mathbf{R}$ or $\mathbf{C}$. Let $G_F = \mathrm{GL}_2(F)$ and K be a maximal compact, so $K = O(2, \mathbf{R})$ in the case $F = \mathbf{R}$ and $U(2)$ in the case $F = \mathbf{C}$. Let $\mathfrak{g}$ denote the lie algebra of G_F, viewed as a Lie algebra over the reals and let $\mathfrak{g}_{\mathbf{C}} := \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$ denote its complexification.

Let $\mu_1, \mu_2 : F^\times \to \mathbf{C}^\times$ denote two quasi-characters of F, which are by definition to be continuous maps, $F^\times \to \mathbf{C}^\times$. We next want to recall the definition of the sets of functions $\mathscr{B}(\mu_1, \mu_2)$. First, we recall what it means to be K-finite.

Definition 1.1. Let V be a K representation and $v \in V$. Then v is K**-finite** if there is a finite dimensional K-subrepresentation W with $v \in W \subset V$ so that the restriction of the representation to W, corresponding to the map $K \to \mathrm{GL}(W)$ is continuous.

We recall the definition of $\mathscr{B}(\mu_1, \mu_2)$.

Definition 1.2. Define $\mathscr{B}(\mu_1, \mu_2)$ as the set of those smooth functions $f : G_F \to \mathbf{C}$ so that f is K-finite via the right action of K on f and for all $g \in G_F, a_1 \in F^\times, a_2 \in F^\times, x \in F$,

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} f(g) \tag{1.1}$$

We next recall the definition of a certain type of representation of $(\mathfrak{g}, K)$, also known as a Harish-Chandra module.

Definition 1.3. A $(\mathfrak{g}, K)$ **representation** is a vector space V with an $\mathfrak{g}$ action $\pi_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(V)$ and an action $\pi_K : K \to \mathrm{GL}(V)$ such that

(1) $V = \oplus_i V_i$ with V_i stable under the K action and each V_i finite dimensional
(2) For $X \in \mathfrak{g}, g \in K$,
$$\pi_{\mathfrak{g}}\left((\mathrm{Ad}\, g)X\right) = \pi_K(g)\pi_{\mathfrak{g}}(X)\pi_K(g^{-1})$$
(3) For any X in the Lie algebra of K,
$$\lim_{t \to 0} \frac{\pi_K(\exp(tX)) - \pi_K(\mathrm{id})}{t} = \pi_{\mathfrak{g}}(X).$$
Note that this limit makes sense by the first property because every $v \in V$ lies in some finite dimensional stable subspace, in which the limit is well defined.

Remark 1.4. In what follows, by abuse of notation, if $\pi_{\mathfrak{g}}$ and π_K form a $(\mathfrak{g}, K)$, representation, we shall often denote both $\pi_{\mathfrak{g}}$ and π_K by the same symbol π.

Definition 1.5. Let V be an $(\mathfrak{g}, K)$ representation and, considering V as a K representation, write $V \simeq \oplus_\sigma V(\sigma)$ for $V(\sigma)$ the K-isotypic components (i.e., σ's parameterize irreducible finite dimensional representations of V. By [KV95, Proposition 1.18] any finite dimensional $V(\sigma)$ representation is a direct sum of irreducible representations, due to compactness of K). We say V is **admissible** if each $V(\sigma)$ is finite dimensional.

Remark 1.6. One can define the Hecke algebra associated to G_F, and there is a bijection between admissible representations of the Hecke algebra and admissible $(\mathfrak{g}, K)$ representations. In what follows we will classify $(\mathfrak{g}, K)$ representations, and hence this will also classify representations of the Hecke algebra associated to G_F. We do not discuss Hecke algebras further in these notes.

Definition 1.7. Define the $(\mathfrak{g}, K)$ representation $\rho(\mu_1, \mu_2)$ on $\mathscr{B}(\mu_1, \mu_2)$ by sending a smooth function $f : G_F \to \mathbf{C}$ and a distribution μ supported on K to the smooth function $\rho(\mu)f$ defined by

$$(\rho(\mu)f)(g) = \int_{G_F} f(gh)d\mu(h),$$

Similarly, define the left action $\lambda(\mu)f$ by

$$(\lambda(\mu)f)(g) = \int_{G_F} f(h^{-1}g)d\mu(h),$$

Alternatively phrased, defining f_g as the function given by $f_g(h) = f(gh)$, we have $(\rho(\mu)f)(g) = \langle \mu, f_g \rangle$, where the pairing means evaluating the distribution μ on the function f_g.

Remark 1.8. For $X \in \mathfrak{g}$, f a smooth function on G_F, and $g \in G$ verify that the $(\mathfrak{g}, K)$ representation is given explicitly by $(\rho(X)f)(g) = \frac{d}{dt} f\left(ge^{tX}\right)|_{t=0}$.

Exercise 1.9. Verify $(\rho(X)f)(g) = \frac{d}{dt} f\left(ge^{tX}\right)|_{t=0}$ by checking the right hand side is G-invariant and agrees with the derivation X at the identity.

Using the above characterization and definitions, we computed the explicit formulas for how the representation $\rho(\mu_1, \mu_2)$ acts. To state these formulas, we introduce the following notation.

Definition 1.10. Take $F = \mathbf{R}$ and for $i = 1, 2$, let $\mu_i : \mathbf{R}^\times \to \mathbf{C}^\times$ be two given quasi-characters. By continuity, the values of the quasi-characters on $\mathbf{R}_{>0}$ are determined by their value on any positive real other than 1, and therefore $\mu_i|_{\mathbf{R}_{>0}}$ must be of the form $\mu_i(t) = t^{s_i}$ for some $s_i \in \mathbf{C}$. Depending on whether $\mu_i(-1) = 1$ or -1, we can then write

$$\mu_i(t) = (\operatorname{sgn} t)^{m_i} |t|^{s_i}$$

for $m_i \in \{0, 1\}, s_i \in \mathbf{C}$.

Define $s := s_1 - s_2, m := |m_1 - m_2|$ so that $\mu_1(t)\mu_2^{-1}(t) = (\operatorname{sgn} t)^m |t|^s$.

Let $n \equiv m \bmod 2$ and define $\phi_n \in \mathscr{B}(\mu_1, \mu_2)$ by

$$\phi_n\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) := \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} e^{in\theta}.$$

Define

$$\begin{aligned}
\varepsilon &:= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\
\kappa_\theta &:= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
U &:= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
J &:= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
V_+ &:= \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \\
V_- &:= \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \\
X_+ &:= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
X_- &:= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
Z &:= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
D &:= X_+X_- + X_-X_+ + \frac{1}{2}Z^2.
\end{aligned}$$

We saw last time the the collection $\{\phi_n\}_{n\equiv m \bmod 2}$ forms a basis for $\mathscr{B}(\mu_1, \mu_2)$. We saw that the elements defined above in Definition 1.10 act via ρ in the following manner.

Lemma 1.11. *Let $F = \mathbf{R}$. With $U, \varepsilon, V_+, V_-, D, J$ as defined in Definition 1.10, their actions under ρ on ϕ_n are given by*

$$\begin{aligned}
\rho(U)\phi_n &= in\phi_n \\
\rho(\varepsilon)\phi_n &= (-1)^{m_1}\phi_{-n} \\
\rho(V_+)\phi_n &= (s+1+n)\phi_{n+2} \\
\rho(V_-)\phi_n &= (s+1-n)\phi_{n-2} \\
\rho(D)\phi_n &= \frac{1}{2}\left(s^2-1\right)\phi_n \\
\rho(J)\phi_n &= (s_1+s_2)\phi_n.
\end{aligned}$$

The main theorem we saw last time was the following.

Theorem 1.12. *For $F = \mathbf{R}$ or $\mathbf{C}$, every irreducible admissible $(\mathfrak{g}, K)$ representation π, there is some pair of quasi-characters μ_1 and μ_2 so that π is realized as a subrepresentation of $\rho(\mu_1, \mu_2)$.*

Remark 1.13. At least we saw a proof of this in the case $F = \mathbf{R}$. The case $F = \mathbf{C}$ was omitted, and I am not sure whether there is a relatively elementary proof along similar lines to the case $F = \mathbf{R}$, or if the only known proof relies of a deep theorem of Harish-Chandra.

By Theorem 1.12, to understand all possible irreducible admissible $(\mathfrak{g}, K)$ representations, it suffices to understand how to decompose $\rho(\mu_1, \mu_2)$ as a sum of irreducible admissible representations, and then understand when two such irreducible admissible subrepresentations are equivalent. We take this for $F = \mathbf{R}$ in § 2 and for $F = \mathbf{C}$ in § 3.

2. Irreducible components of $\rho(\mu_1, \mu_2)$ over $\mathbf{R}$

In this section, we take up the gauntlet of classifying all irreducible admissible $(\mathfrak{g}, K_{\mathbf{R}})$ representations. For the remainder of this section, we work over $F = \mathbf{R}$.

The outline of this section is as follows: In § 2.1 we classify $\mathfrak{g}_{\mathbf{C}}$ stable subspaces of $\mathscr{B}(\mu_1, \mu_2)$. In § 2.2, we prove three results determining when an $\mathfrak{g}$ subrepresentation of a $(\mathfrak{g}, K_{\mathbf{R}})$ representation is actually a $(\mathfrak{g}, K_{\mathbf{R}})$ subrepresentation. Then, in § 2.3 we state the classification statement which is subsequently proven in § 2.4, § 2.5, and § 2.6.

2.1. $\mathfrak{g}_{\mathbf{C}}$ stable subrepresentations.

By Theorem 1.12, in order to understand all irreducible admissible $(\mathfrak{g}, K_{\mathbf{R}})$ representations, it suffices to understand the irreducible sub and quotient representations of $\rho(\mu_1, \mu_2)$.

In this subsection, in particular in Proposition 2.3, we define the sub and quotient representations of $\rho(\mu_1, \mu_2)$, which are $\mathfrak{g}_{\mathbf{C}}$ stable. Later we explain how these connect to $(\mathfrak{g}, K_{\mathbf{R}})$ stable subspaces.

We need the elementary preparatory lemma:

Lemma 2.1. *Any subspace $V \subset \mathscr{B}(\mu_1, \mu_2)$ stable under $\mathfrak{g}_{\mathbf{C}}$ is spanned by $\{\phi_j : \phi_j \in V\}$.*

Proof. To see this, suppose $v = \sum_{j=a}^{b} \alpha_j \phi_j \in V$ with $\alpha_b \neq 0$. It suffices to show $\phi_b \in V$. Suppose there are t nonzero elements in the set $\{\alpha_a, \ldots, \alpha_b\}$.

Exercise 2.2. Verify that the elements $v, \rho(U)v, \ldots, \rho(U)^{t-1}v$ are independent. *Hint:* One can do this explicitly by writing ϕ_i as linear combinations of $\rho(U)^k v$, or in a tricky way (which is really the same way) using that the Vandermonde determinant does not vanish.

Therefore, by the above exercise, the elements $v, \rho(U)v, \ldots, \rho(U)^{t-1}v$ are independent, but also land in the t dimensional space spanned by those ϕ_j for which the corresponding α_j is nonzero. Therefore, ϕ_b lies in the span of $v, \rho(U)v, \ldots, \rho(U)^{t-1}v$, as desired. □

Using Lemma 2.1 and Lemma 1.11, we can verify the following characterization of $\mathfrak{g}_{\mathbf{C}}$ invariant subspace of $\mathscr{B}(\mu_1, \mu_2)$.

Proposition 2.3. *Let μ_1, μ_2 be two quasi-characters, and define s and m as in Definition 1.10. If $s - m$ is even, then $\mathscr{B}(\mu_1, \mu_2)$ is irreducible as a $\mathfrak{g}_{\mathbf{C}}$ representation. If $s - m$ is odd and $s > 0$, then the nonzero proper irreducible $\mathfrak{g}_{\mathbf{C}}$ stable subspaces are*

$$\mathscr{B}_1(\mu_1, \mu_2) := \bigoplus_{\substack{n \geq s+1 \\ n \equiv s+1 \bmod 2}} \mathbf{C} \cdot \phi_n$$

$$\mathscr{B}_2(\mu_1, \mu_2) := \bigoplus_{\substack{n \leq -s-1 \\ n \equiv s+1 \bmod 2}} \mathbf{C} \cdot \phi_n$$

$$\mathscr{B}_s(\mu_1, \mu_2) := \mathscr{B}_1(\mu_1, \mu_2) + \mathscr{B}_2(\mu_1, \mu_2)$$

If $s - m$ is odd and $s = 0$, then the nonzero proper irreducible $\mathfrak{g}_{\mathbf{C}}$ stable subspaces are

$$\mathscr{B}_1(\mu_1, \mu_2) := \oplus_{\substack{n \le -s-1 \\ n \equiv s+1 \bmod 2}} \mathbf{C} \cdot \phi_n$$
$$\mathscr{B}_2(\mu_1, \mu_2) := \oplus_{\substack{n \ge s+1 \\ n \equiv s+1 \bmod 2}} \mathbf{C} \cdot \phi_n$$

If $s - m$ is odd and $s < 0$, then the nonzero proper irreducible $\mathfrak{g}_{\mathbf{C}}$ stable subspaces are

$$\mathscr{B}_1(\mu_1, \mu_2) := \oplus_{\substack{n \ge s+1 \\ n \equiv s+1 \bmod 2}} \mathbf{C} \cdot \phi_n$$
$$\mathscr{B}_2(\mu_1, \mu_2) := \oplus_{\substack{n \le -s-1 \\ n \equiv s+1 \bmod 2}} \mathbf{C} \cdot \phi_n$$
$$\mathscr{B}_f(\mu_1, \mu_2) := \mathscr{B}_1(\mu_1, \mu_2) \cap \mathscr{B}_2(\mu_1, \mu_2).$$

Proof. By Lemma 2.1, any $\mathfrak{g}_{\mathbf{C}}$ invariant subspace of $\mathscr{B}(\mu_1, \mu_2)$ is spanned by those ϕ_i is contains.

Suppose first $s - m$ is even. Then by Lemma 1.11, we know $\rho(V_\pm)\phi_n = (s + 1 \pm n)\phi_{n\pm2}$. Because $s - m$ is even and $n - m$ is even as n varies over all $\phi_n \in \mathscr{B}(\mu_1, \mu_2)$ (as the set of ϕ_n for which $n \equiv m \bmod 2$ form a basis for $\mathscr{B}(\mu_1, \mu_2)$) it follows that $s + 1 \pm n$ is odd, hence never 0. This implies $\rho(\mu_1, \mu_2)$ is irreducible.

To conclude, we only need deal with the case $s - m$ is odd. As in the even case, if we have some $\mathfrak{g}_{\mathbf{C}}$ invariant subspace $V \subset \mathscr{B}(\mu_1, \mu_2)$, V is determined by those $\phi_n \in V$, and by applying $\rho(V_\pm)$ to ϕ_n, we have $\phi_{n\pm2} \in V$ so long as $s + 1 \pm n \neq 0$. Therefore, using $\rho(V_+)\phi_n = (s + 1 + n)\phi_{n+2}$, if $\phi_n \in V$ then $\phi_{n+2} \in V$, unless possibly $s + 1 = -n$. Similarly, using $\rho(V_-)\phi_n = (s + 1 - n)\phi_{n-2}$, if $\phi_n \in V$, then $\phi_{n-2} \in V$, unless possibly $s + 1 = n$.

There are three cases depending on the sign of s. We only address $s > 0$ since the cases $s = 0$ and $s < 0$ are analogous. If there is some $\phi_n \in V$ with $s + 1 \ge -n$, then by Lemma 2.1 and the preceding paragraph, V contains $\mathscr{B}_2$. Similarly, if there is some $\phi_n \in V$ with $n \ge s + 1$, then $\mathscr{B}_1 \subset V$. Finally, if there is some $\phi_n \in V$ with $-s - 1 < n < s + 1$, then necessarily $V = \mathscr{B}(\mu_1, \mu_2)$. Therefore, we obtain $\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_s$ as the only possible nonzero proper invariant subspaces. ☐

2.2. **Relating $\mathfrak{g}$ representations to $(\mathfrak{g}, K_{\mathbf{R}})$ representations.** Having understood the possible $\mathfrak{g}_{\mathbf{C}}$ invariant subspaces, we next turn to describing the possible $(\mathfrak{g}, K_{\mathbf{R}})$ subrepresentations. For this, we need to understand the possible forms of the restriction of an irreducible $(\mathfrak{g}, K_{\mathbf{R}})$ representation to an $\mathfrak{g}$ representation. The two results we will need from this section in future sections are Corollary 2.10 and Lemma 2.11. We suggest that the reader read these to statements and proceed to the next section.

Definition 2.4. Recall ε as defined in Definition 1.10. For (π, V) an $\mathfrak{g}$ representation, let $((\mathrm{Ad}\,\varepsilon)\,\pi, V)$ denote the representation sending $X \in V$ to $\pi\left((\mathrm{Ad}\,\varepsilon)\,X\right)$. In the case that π is the restriction of a $(\mathfrak{g}, K_{\mathbf{R}})$ representation, we have $\pi\left((\mathrm{Ad}\,\varepsilon)\,X\right) = \pi(\varepsilon)\pi(X)\pi(\varepsilon^{-1})$.

The following lemma determines the restriction of an irreducible $(\mathfrak{g}, K_{\mathbf{R}})$ representation to $\mathfrak{g}$.

Proposition 2.5. *Suppose (π, V) is an irreducible $(\mathfrak{g}, K_{\mathbf{R}})$ representation. Then, either*

(1) $(\pi|_{\mathfrak{g}}, V)$ is irreducible and is equivalent to $((\mathrm{Ad}\,\varepsilon)\,\pi, V)$

(2) *There is an isomorphism $V \simeq V_1 \oplus V_2$ with V_i stable under $\mathfrak{g}$ so that the $\mathfrak{g}$ representations $\pi_i := \pi|_{V_i}$ on V_i are irreducible. Further, (π_1, V_1) is not equivalent to (π_2, V_2) but $((\operatorname{Ad}\varepsilon)\,\pi_1, V_1)$ is equivalent to (π_2, V_2).*

Proof. First, suppose $(\pi|_{\mathfrak{g}}, V)$ is irreducible. Then, it is equivalent to $((\operatorname{Ad}\varepsilon)\,\pi, V)$ with equivalence given by the operator $X \mapsto \pi(\varepsilon)X$. Indeed, using that $\varepsilon^2 = \mathrm{id}$, for $X \in \mathfrak{g}, v \in V$,

$$\begin{aligned}\pi(X)\pi(\varepsilon)v &= \pi(\varepsilon)\pi(\varepsilon)\pi(X)\pi(\varepsilon)v \\ &= \pi(\varepsilon)\pi\left((\operatorname{Ad}\varepsilon)\,\pi(X)\right)v.\end{aligned}$$

So, suppose instead that $(\pi|_{\mathfrak{g}}, V)$ is not irreducible. Let $V_1 \subset V$ be a proper $\mathfrak{g}$-irreducible subspace and define $V_2 := \pi(\varepsilon)V_1$. To start, we will show $V_1 \oplus V_2 = V$ (implicit here is that $V_1 \cap V_2 = 0$).

By the definition of $(\mathfrak{g}, K_{\mathbf{R}})$ representation, the $\mathfrak{g}$ representation determines the representation of the Lie algebra of K, and then because we can exponentiate Lie algebra elements over $\mathbf{R}$, the Lie algebra representation determines the representation of $K^0_{\mathbf{R}}$ (the connected component of the identity in $K_{\mathbf{R}}$). Any $(\mathfrak{g}, K_{\mathbf{R}})$ representation is determined by its restriction to $\mathfrak{g}$ together with the action of the element $\varepsilon \in K_{\mathbf{R}}$. It follows that $V_1 + V_2$ is $(\mathfrak{g}, K_{\mathbf{R}})$ stable. Similarly, $V_1 \cap V_2$ is $(\mathfrak{g}, K_{\mathbf{R}})$ stable. However, since V_1 is not all of V, $V_1 \cap V_2$ must be a proper subspace of V. Therefore, $V_1 \cap V_2 = 0$, being a proper subrepresentation of an irreducible $\mathfrak{g}$ representation. Hence, $V_1 + V_2 = V_1 \oplus V_2$. Further, $V = V_1 \oplus V_2$ because the latter is nonzero and $(\mathfrak{g}, K_{\mathbf{R}})$ stable, hence equal to all of V, as V was $(\mathfrak{g}, K_{\mathbf{R}})$ irreducible.

To conclude, we only need show (π_1, V_1) is not equivalent to (π_2, V_2) but $((\operatorname{Ad}\varepsilon)\,\pi_1, V_1)$ is equivalent to (π_2, V_2).

To show $((\operatorname{Ad}\varepsilon)\,\pi_1, V_1)$ is equivalent to (π_2, V_2), note that $\pi(\varepsilon)$ is an intertwining operator. Indeed, for $X \in \mathfrak{g}, v \in V_1$,

$$\pi_2(X)\pi(\varepsilon)v_1 = \pi(\varepsilon)\pi\left((\operatorname{Ad}\varepsilon)\,\pi_1\right)(X)v_1,$$

analogously to the case that $(\pi|_{\mathfrak{g}}, V)$ was irreducible above.

To finish the proof, we verify (π_1, V_1) is not equivalent to (π_2, V_2). Suppose they are equivalent, so there is some linear $A : V_1 \to V_2$ with $A\pi_1(X) = \pi_2(X)A$ for all $X \in \mathfrak{g}$.

We next claim that the map

$$\begin{aligned}\phi\colon V &\to V \\ v_1 + v_2 &\mapsto A^{-1}v_2 + Av_1\end{aligned}$$

commutes with the action of $(\mathfrak{g}, K_{\mathbf{R}})$. If this is the case, then we will reach a contradiction, since this commutation, by a version of Schur's lemma, implies the map ϕ must be a scalar, but it is also not a scalar, as it permutes the subspaces V_1 and V_2.

We check ϕ commutes with the action of $(\mathfrak{g}, K_{\mathbf{R}})$. To see this, it suffices to check it commutes with the action of $\mathfrak{g}$ and ε, because commuting with $\mathfrak{g}$ implies the action commutes with $K^0_{\mathbf{R}}$ via exponentiation, and commuting with ε then implies it commutes with all of $K_{\mathbf{R}}$. To see it commutes with the action of $\mathfrak{g}$, note that for $X \in \mathfrak{g}$, and $v_1 \in V_1$ and $v_2 \in V_2$

$$\pi_1(X)A^{-1}v_2 + \pi_2(X)Av_1 = A^{-1}\pi_2(X)v_2 + A\pi_1(X)v_1,$$

so the actions commute.

Finally, it remains to check that ϕ commutes with the action of ε. For this, we must first verify, after possibly rescaling A that $(A^{-1}\pi(\varepsilon))^2 = 1$. The key to showing this is the following lemma.

Lemma 2.6. *With notation as above, $\left(A^{-1}\pi(\varepsilon)\right)^2$ commutes with $\mathfrak{g}$.*

Proof. Observe $A^{-1}\pi_2(X) = A\pi_1(X)$, and recall that we showed above $\pi(\varepsilon)\pi_1(X) = \pi_2((\operatorname{Ad}\varepsilon)X)\pi(\varepsilon)$. Combining these, we find, for $X \in \mathfrak{g}, v \in V_1$,

$$\begin{aligned} A^{-1}\pi(\varepsilon)\pi_1(X)v &= A^{-1}\pi_2\left((\operatorname{Ad}\varepsilon)X\right)\pi(\varepsilon)v \\ &= \pi_1\left((\operatorname{Ad}\varepsilon)X\right)A^{-1}\pi(\varepsilon)v. \end{aligned}$$

Therefore,

$$\begin{aligned} \left(A^{-1}\pi(\varepsilon)\right)^2\pi_1(X)v &= \pi_1\left((\operatorname{Ad}\varepsilon)^2 X\right)(A^{-1}\pi(\varepsilon))^2 v \\ &= \pi_1(X)\left(A^{-1}\pi(\varepsilon)\right)^2 v \end{aligned}$$

and so $\left(A^{-1}\pi(\varepsilon)\right)^2$ commutes with $\mathfrak{g}$. □

By a version of Schur's lemma, Lemma 2.6 means $\left(A^{-1}\pi(\varepsilon)\right)^2$ is a scalar. By absorbing a square root of that scalar into A, we can assume the scalar is 1, so $\left(A^{-1}\pi(\varepsilon)\right)^2 = 1$.

Hence, after rescaling, we find $A^{-1}\pi(\varepsilon) = \pi(\varepsilon)A$ and $\pi(\varepsilon)A^{-1} = A\pi(\varepsilon)$, implying

$$\pi(\varepsilon)Av_1 + \pi(\varepsilon)A^{-1}v_2 = A^{-1}\pi(\varepsilon)v_1 + A\pi(\varepsilon)v_2,$$

and so ϕ commutes with the action of ε. □

We define the twist of a $(\mathfrak{g}, K_{\mathbf{R}})$ representation by the sign character. One can more generally define the twist by an arbitrary character, as in [Sno06, 6.1.6], but since we will only need to twist by sign, we do not concern ourselves with that definition here.

Definition 2.7. For π a $(\mathfrak{g}, K_{\mathbf{R}})$ representation, define $\operatorname{sgn}\otimes\pi$ to be the $(\mathfrak{g}, K_{\mathbf{R}})$ representation so that $\operatorname{sgn}\otimes\pi(X) = \pi(X)$ for $X \in \mathfrak{g}$ and $\operatorname{sgn}\otimes\pi(M) = \operatorname{sgn}(\det M)\cdot\pi(M)$ for $M \in K_{\mathbf{R}}$.

Exercise 2.8. For π a $(\mathfrak{g}, K_{\mathbf{R}})$ representation, verify $\operatorname{sgn}\otimes\pi$ is indeed a $(\mathfrak{g}, K_{\mathbf{R}})$ representation.

To state the next corollary, we also need to define the notion of an admissible $\mathfrak{g}$ representation.

Definition 2.9. Let π be an $\mathfrak{g}$ representation. Let κ_n denote the representation of $\operatorname{Lie}(K_{\mathbf{R}})$ obtained by differentiating the $\mathrm{SO}_2(\mathbf{R})$ 1-dimensional representation

$$\begin{aligned} K_n \colon \mathrm{SO}_2(\mathbf{R}) &\to \mathbf{C}^\times \\ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} &\mapsto e^{in\theta}. \end{aligned}$$

We say π is **admissible** if the restriction of $\pi|_{\operatorname{Lie}(K_{\mathbf{R}})} \simeq \oplus_{n\in\mathbf{Z}} V_n$ where each V_n is a direct sum of finitely many copies of κ_n.

Corollary 2.10. *Suppose (π_1, V_1) is an irreducible admissible $\mathfrak{g}$ representation which is not equivalent to $(\mathrm{Ad}\,\varepsilon)\,\pi_1$ and let $(\pi_2, V_2) := ((\mathrm{Ad}\,\varepsilon)\,\pi_1, V_1)$. Let $V = V_1 \oplus V_2$. Then,*

(1) There is an irreducible admissible $(\mathfrak{g}, K_{\mathbf{R}})$ representation π on V restricting to $\pi_1 \oplus \pi_2$.
(2) If ν is an irreducible admissible $(\mathfrak{g}, K_{\mathbf{R}})$ representation so that $\pi_1 \subset \nu|_{\mathfrak{g}}$ as $\mathfrak{g}$ representations then ν is equivalent to π as $(\mathfrak{g}, K_{\mathbf{R}})$ representations. In particular, $\mathrm{sgn} \otimes \pi$ is equivalent to π.

Proof. (1) To construct such a representation, we take $\pi(X) = \pi_1(X) \oplus \pi_2(X)$ and admissibility yields a representation of $\mathrm{SO}(2, \mathbf{R})$ on V, so it suffices to define $\pi(\varepsilon)$. Indeed, define

$$\pi(\varepsilon)\,(v_1, v_2) := (v_2, v_1).$$

Then, this uniquely defines a $(\mathfrak{g}, K_{\mathbf{R}})$ representation extending the $\mathfrak{g}$ representation. Since π_i are both irreducible $\mathfrak{g}$ representations, π is an irreducible $(\mathfrak{g}, K_{\mathbf{R}})$ representation. Further, π is admissible because its restriction to $\mathfrak{g}$ is admissible.

(2) Since the restriction of π to $\mathfrak{g}$ is not irreducible, by Proposition 2.5, the representation π must be given by $\pi_1 \oplus \pi_2$ with π_2 equivalent to $(\mathrm{Ad}\,\varepsilon)\,\pi_1$. Hence, it is uniquely determined to be the representation constructed in the proof of the first part. In particular, $\mathrm{sgn} \otimes \pi$ restricts to π_1 on $\mathfrak{g}$, and is therefore equivalent to π.

□

Lemma 2.11. *Suppose (π_0, V) is an irreducible admissible $\mathfrak{g}$ representation which is equivalent to $((\mathrm{Ad}\,\varepsilon)\,\pi_0, V)$. Then*

(1) There is an irreducible admissible $(\mathfrak{g}, K_{\mathbf{R}})$ representation π restricting to π_0
(2) The $(\mathfrak{g}, K_{\mathbf{R}})$ representations π and $\mathrm{sgn} \otimes \pi$ are not equivalent
(3) Any $(\mathfrak{g}, K_{\mathbf{R}})$ representation restricting to π_0 is either equivalent to π or $\mathrm{sgn} \otimes \pi$.

Proof. (1) First, take $\pi(X) := \pi_0(X)$ for $X \in \mathfrak{g}$, and the definition of admissibility uniquely determines an $\mathrm{SO}(2, \mathbf{R})$ representation. To define an $(\mathfrak{g}, K_{\mathbf{R}})$ representation, it suffices to define the action of ε. For this, recall our assumption that π_0 is equivalent to $(\mathrm{Ad}\,\varepsilon)\,\pi_0$, and so there is some $A : V \to V$ with $A\pi_0(X) = \pi_0\left((\mathrm{Ad}\,\varepsilon)\,X\right) A$ for $X \in \mathfrak{g}$. This implies $A^2\pi_0(X) = \pi_0\left((\mathrm{Ad}\,\varepsilon)^2\,X\right) A^2 = \pi_0(X)A^2$, and so A^2 commutes with $\pi_0(X)$. By Schur's lemma, A^2 is a scalar. By absorbing the square root of that scalar into A, we may assume $A^2 = 1$. Then, we may define $\pi(\varepsilon) := A$, and this defines an $(\mathfrak{g}, K_{\mathbf{R}})$ representation.

(2) The representation $\mathrm{sgn} \otimes \pi$ is given by the same representation on $\mathfrak{g}$ as described in the proof of the first part, but with $\mathrm{sgn} \otimes \pi(\varepsilon) = -A$ while $\pi(\varepsilon) = A$. If π and $\mathrm{sgn} \otimes \pi$ were equivalent, their restriction to $\mathfrak{g}$ would be equivalent. But the only possible such equivalences, by Schur's lemma, are scalars. Therefore, the intertwining operator would necessarily be a nonzero scalar, but this would not intertwine $\pi(\varepsilon)$ and $(\mathrm{sgn} \otimes \pi)\,(\varepsilon)$ because for any nonzero scalar c, $c \neq -c$.

(3) We saw in the proof of the first part that any representation π restricting to π_0 is determined uniquely by $\pi(\varepsilon)$ and further that we may assume $\pi(\varepsilon)^2 = 1$. By Schur's lemma, any such isomorphism must be given by $c \cdot A$ for a scalar c. Further, $c^2 = 1$, so $c = \pm 1$ meaning $\pi(\varepsilon) = \pm A$ in which case the corresponding representation is either π or $\mathrm{sgn} \otimes \pi$.

□

2.3. **The classification statement over the reals.** In order to state the classification of admissible irreducible $G_{\mathbf{R}}$ representations, we first define the following sub and quotient representations of $\mathscr{B}(\mu_1, \mu_2)$.

Definition 2.12. Let $\mu_1, \mu_2 : \mathbf{R}^\times \to \mathbf{C}^\times$ be two quasi-characters.

(1) If $s - m$ is even, let $\pi(\mu_1, \mu_2)$ denote $\rho(\mu_1, \mu_2)$.
(2) If $s - m$ is odd and $s > 0$, let $\sigma(\mu_1, \mu_2) := \rho(\mu_1, \mu_2)|_{\mathscr{B}_s(\mu_1,\mu_2)}$.
(3) If $s - m$ is odd and $s > 0$, let $\pi(\mu_1, \mu_2) := \rho(\mu_1, \mu_2)|_{\mathscr{B}(\mu_1,\mu_2)/\mathscr{B}_s(\mu_1,\mu_2)}$.
(4) If $s - m$ is odd and $s < 0$, let $\pi(\mu_1, \mu_2) := \rho(\mu_1, \mu_2)|_{\mathscr{B}_f(\mu_1,\mu_2)}$.
(5) If $s - m$ is odd and $s < 0$, let $\sigma(\mu_1, \mu_2) := \rho(\mu_1, \mu_2)|_{\mathscr{B}(\mu_1,\mu_2)/\mathscr{B}_f(\mu_1,\mu_2)}$.

We call the representations $\pi(\mu_1, \mu_2)$ are *principal series representations*. We call the representations $\sigma(\mu_1, \mu_2)$ are *special representations*.

Theorem 2.13. *Let μ_1 and μ_2 be two quasi-characters of $\mathbf{R}^\times$.*

(1) If $\mu_1\mu_2^{-1}$ is not of the form $t \mapsto t^p \operatorname{sgn} t$ for $p \neq 0$, then $\pi(\mu_1, \mu_2)$ is irreducible as a $(\mathfrak{g}, K_{\mathbf{R}})$ representation.
(2) If $\mu_1\mu_2^{-1}$ is of the form $t \mapsto t^p \operatorname{sgn} t$ for $p > 0$, then $\sigma(\mu_1, \mu_2)$ is the only irreducible $(\mathfrak{g}, K_{\mathbf{R}})$ subrepresentation, and it is infinite dimensional with finite codimension.
(3) If $\mu_1\mu_2^{-1}$ is of the form $t \mapsto t^p \operatorname{sgn} t$ for $p < 0$, then $\pi(\mu_1, \mu_2)$ is the only irreducible $(\mathfrak{g}, K_{\mathbf{R}})$ subrepresentation, and it is finite dimensional with infinite codimension.
(4) $\pi(\mu_1, \mu_2)$ is equivalent as a $(\mathfrak{g}, K_{\mathbf{R}})$ representation to $\pi(\mu_1', \mu_2')$ if $\{\mu_1, \mu_2\} = \{\mu_1', \mu_2'\}$. Furthermore, $\sigma(\mu_1, \mu_2)$ is equivalent to $\sigma(\mu_1', \mu_2')$ if $\{\mu_1, \mu_2\}$ is equal to either $\{\mu_1', \mu_2'\}$ or $\{\operatorname{sgn} \mu_1', \operatorname{sgn} \mu_2'\}$.
(5) The equivalences listed in (4) are the only equivalences among the above representations listed in (1), (2), and (3).

The proof of the theorem is given below in § 2.4, § 2.5, and § 2.6.

2.4. **Classifying subrepresentations.** Before proving parts (1), (2), and (3), we record the following lemma, crucially testing how much we have retained from high school level algebra:

Lemma 2.14. *Define m and s as in Definition 1.10 so that $\mu_1\mu_2^{-1}$ is of the form $(\operatorname{sgn} t)^m |t|^s$. Then, $s - m$ is odd if and only if $\mu_1\mu_2^{-1}(t) = t^s \operatorname{sgn} t$.*

Proof. Indeed, $\mu_1\mu_2^{-1}(t) = t^s \operatorname{sgn} t$. always holds for positive t, and is possibly off by a uniform sign for negative t, i.e., if it is off by a sign, then it is off by that same sign for all negative t. So to verify this claim, we only need to check it in the case $t = -1$. One can then easily verify this by examining the four cases depending on whether m is even or odd and s is even or odd. For example, if m is even and s is odd, then $(\operatorname{sgn} -1)^m |-1|^s = 1 \cdot 1 = -1 \cdot -1 = (-1)^s (\operatorname{sgn} -1)$. The other three cases are similar. □

Proof of (1), (2), and (3) of Theorem 2.13. Define m and s as in Definition 1.10. To start, we deal with the case $s - m$ is odd. Using Lemma 2.14, we see this is the case if and only if $\mu_1\mu_2^{-1}(t) = t^s \operatorname{sgn} t$. Then, by Proposition 2.3, we see that in this case, $\pi(\mu_1, \mu_2)$ is even an irreducible $\mathfrak{g}_{\mathbf{C}}$ representation, and hence an irreducible $(\mathfrak{g}, K_{\mathbf{R}})$ representation.

To conclude, we deal with the case $s - m$ is even. We now split into two cases, depending on whether $p = s$ is 0, positive or negative. First, suppose $p \geq 0$. With notation as in Proposition 2.3, observe that $\mathscr{B}_1(\mu_1, \mu_2) \simeq (\operatorname{Ad} \varepsilon)\mathscr{B}_2(\mu_1, \mu_2)$ essentially because conjugating by ε interchanges the actions of V_+ and V_-, using the formulas in Lemma 1.11. Furthermore, $\mathscr{B}_1(\mu_1, \mu_2)$ is not equivalent to the irreducible $\mathfrak{g}$ representation $\mathscr{B}_2(\mu_1, \mu_2)$ because in one of the representations, every nonzero vector is killed by a finite power of $\rho(V_+)$ but not by any power of $\rho(V_-)$, while in the other every nonzero vector is killed by a finite power of $\rho(V_-)$ but not by any power of $\rho(V_+)$.

Therefore, by Corollary 2.10, $\mathscr{B}_s(\mu_1, \mu_2) = \mathscr{B}_1(\mu_1, \mu_2) + \mathscr{B}_2(\mu_1, \mu_2)$ determines an irreducible representation of $(\mathfrak{g}, K_{\mathbf{R}})$. By Proposition 2.3, this is the only nonzero irreducible representation, other than possibly $\rho(\mu_1, \mu_2)$ itself. In the case $p = 0$, these are the same representation, but when $p > 0$, these are distinct. By definition, $\mathscr{B}_s(\mu_1, \mu_2)$ has infinite dimension and finite codimension in $\mathscr{B}(\mu_1, \mu_2)$.

The situation for $p = s < 0$ is similar. In this case, the representation on $\mathscr{B}_f(\mu_1, \mu_2) = \mathscr{B}_1(\mu_1, \mu_2) \cap \mathscr{B}_2(\mu_1, \mu_2)$ is equivalent to $(\operatorname{Ad} \varepsilon)\mathscr{B}_f(\mu_1, \mu_2)$ with intertwining operator $\pi(\varepsilon)$ and hence by Lemma 2.11 is the restriction of an irreducible $(\mathfrak{g}, K_{\mathbf{R}})$ representation with the same underlying vector space. However, $\mathscr{B}_i(\mu_1, \mu_2)$ is not equivalent to $(\operatorname{Ad} \varepsilon)\mathscr{B}_i(\mu_1, \mu_2)$ for $i \in \{1, 2\}$ and is in fact equivalent to $(\operatorname{Ad} \varepsilon)\mathscr{B}_{3-i}(\mu_1, \mu_2)$. Therefore, using Proposition 2.3, Corollary 2.10, and Lemma 2.11, we find that $\mathscr{B}_f(\mu_1, \mu_2)$ is the only nonzero proper subrepresentation of $\rho(\mu_1, \mu_2)$. □

2.5. **Demonstrating equivalences.** We have enumerated all irreducible admissible subrepresentations. Namely, as we vary quasi-characters μ_1 and μ_2, these representations are the $\pi(\mu_1, \mu_2)$ and $\sigma(\mu_1, \mu_2)$ (when defined). It remains to identify when two such representations are equivalent.

We start by showing that the claimed equivalences of Theorem 2.13(4) hold.

It remains to identify which of the principal series representations $\pi(\mu_1, \mu_2)$ is equivalent to which other $\pi(\mu_1', \mu_2')$ and similarly for the special representations $\sigma(\mu_1, \mu_2)$.

Proof of Theorem 2.13(4). First, we can see $\sigma(\mu_1, \mu_2)$ is equivalent to $\sigma(\operatorname{sgn} \mu_1, \operatorname{sgn} \mu_2)$ by the second part of Corollary 2.10 using that the $\mathfrak{g}$ representation ν_1 on $\mathscr{B}_1(\mu_1, \mu_2)$ is not equivalent to the corresponding representation ν_2 on $\mathscr{B}_2(\mu_1, \mu_2)$ but ν_1 is equivalent to $(\operatorname{Ad} \varepsilon)\, \nu_2$.

To conclude, it suffices to show $\pi(\mu_1, \mu_2)$ is equivalent to $\pi(\mu_2, \mu_1)$ and similarly $\sigma(\mu_1, \mu_2)$ is equivalent to $\sigma(\mu_2, \mu_1)$. To show both of these equivalences, it suffices to construct an operator $T : \mathscr{B}(\mu_1, \mu_2) \to \mathscr{B}(\mu_2, \mu_1)$, nonzero on the given subrepresentation (either $\pi(\mu_1, \mu_2)$ or $\sigma(\mu_1, \mu_2)$) commuting with the action of $(\mathfrak{g}, K_{\mathbf{R}})$. Indeed, in this case, the operator T will then restrict to an intertwining operator on the given subrepresentations.

Let ϕ_n denote the basis of $\mathscr{B}(\mu_1, \mu_2)$ from Definition 1.10 and ϕ_n' the basis of $\mathscr{B}(\mu_2, \mu_1)$. Because ϕ_n spans the n-eigenspace of $\rho(U)$ and ϕ_n' spans the n-eigenspace of $\rho'(U)$, T must take $\phi_n \mapsto a_n \phi_n'$ for some $a_n \in \mathbf{C}$.

Because T commutes with the operator $\rho(V_+)$, we see $(s + 1 + n)\, a_{n+2} = (-s + 1 + n)\, a_n$. Because T commutes with the operator $\rho(V_-)$, we see $(s + 1 - n)\, a_{n-2} = (-s + 1 - n)\, a_n$. Because T commutes with the operator $\rho(\varepsilon)$, we see $(-1)^{m_2}\, a_n = (-1)^{m_1}\, a_{-n}$.

We leave the remainder of the proof as a straightforward exercise in high school algebra:

Exercise 2.15. Verify that these three conditions together determine uniquely the collection of values a_n. *Hint:* It may help to separate into cases based on whether $s-m$ is even or odd, and then based on the sign of s. Essentially, you will find that for the smallest nonzero irreducible subrepresentation of ρ, the a_n will be supported on the basis for that subrepresentation. The third condition that $(-1)^{m_2} a_n = (-1)^{m_1} a_{-n}$ is mostly redundant except in the case that $s-m$ is odd and $s>0$.

□

Remark 2.16. It is possible to give explicit formulas for the a_n in terms of the Γ function. For example, when $s-m$ is even, we can take

$$a_n = \frac{\Gamma\left(\frac{1}{2}(-s+1+n)\right)}{\Gamma\left(\frac{1}{2}(s+1+n)\right)}.$$

2.6. **Showing we have found all possible equivalences.** Finally, we prove the Theorem 2.13(5), completing the proof of the theorem.

Lemma 2.17. *For quasi-characters $\mu_1, \mu_2, \mu_1', \mu_2'$, $\pi(\mu_1,\mu_2)$ is not equivalent to $\sigma(\mu_1',\mu_2')$.*

Proof. First, note that $\sigma(\mu_1',\mu_2')$ is always infinite dimensional, so it is only possibly equivalent to an infinite dimensional $\pi(\mu_1,\mu_2)$. However, infinite dimensional $\pi(\mu_1,\mu_2)$ always have in for all integers n in the spectrum of the corresponding operator $\rho(U)$, while $\rho(U)$ acting on $\sigma(\mu_1',\mu_2')$ is always missing some in, for some $n \in \mathbf{Z}$, from its spectrum. □

Proof of Theorem 2.13(5). We have already shown in Lemma 2.17, there are no equivalences between $\pi(\mu_1,\mu_2)$ and $\sigma(\mu_1',\mu_2')$.

Let ν and ν' denote two equivalent irreducible admissible representations with quasi-characters μ_1,μ_2 for ν and μ_1',μ_2' for ν' (so that either $\nu=\pi(\mu_1,\mu_2)$ or $\nu=\sigma(\mu_1,\mu_2)$ and similarly for ν'). Let T denote the corresponding equivalence so that $T\nu(X)=\nu'(X)T$ for all $X\in K_{\mathbf{R}}$ and $X\in\mathfrak{g}$. Let $\mu_i(t)=(\operatorname{sgn} t)^{m_i}|t|^{s_i}, \mu_i'(t)=(\operatorname{sgn} t)^{m_i'}|t|^{s_i'}, s=s_1-s_2, m=|m_1-m_2|, s'=s_1'-s_2', m'=|m_1'-m_2'|$. Let ϕ_n denote a basis for ν and ϕ_n' a basis for ν'.

As a first step, we show $m=m'$. As shown in the proof of Theorem 2.13(4), T must send the ϕ_n to $a_n\phi_n'$ because ϕ_n and ϕ_n' span the 1-dimensional eigenspaces with eigenvalue n. By Lemma 1.11, and the fact that $T\nu(\varepsilon)=\nu(\varepsilon)T$, we see

$$a_n(-1)^{m_1'} = a_{-n}(-1)^{m_1}.$$

Using the equivalence between $\pi(\mu_1,\mu_2)$ and $\pi(\mu_2,\mu_1)$ as well as the equivalence between $\sigma(\mu_1,\mu_2)$ and $\sigma(\mu_2,\mu_1)$ we similarly find

$$a_n(-1)^{m_2'} = a_{-n}(-1)^{m_2}.$$

Combining these two identities yields $(-1)^{m_1-m_2}=(-1)^{m_1'-m_2'}$ so $m=m'$.

Next, using that $T\nu(D)=\nu'(D)T$, we find $s^2-1=(s')^2-1$ so $s=\pm s$. Also using $T\nu(J)=\nu'(J)T$, we find $s_1+s_2=s_1'+s_2'$. These conditions imply $\{s_1,s_2\}=\{s_1',s_2'\}$ and either $\{m_1,m_2\}=\{m_1',m_2'\}$ or $\{m_1,m_2\}=\{1-m_1',1-m_2'\}$. Phrased another way, we have either $\{\mu_1,\mu_2\}=\{\mu_1',\mu_2'\}$ or $\{\mu_1,\mu_2\}=\{\operatorname{sgn}\mu_1',\operatorname{sgn}\mu_2'\}$.

The only remaining equivalence to rule out is to show that $\pi(\mu_1,\mu_2)$ is not equivalent to $\pi(\operatorname{sgn}\mu_1,\operatorname{sgn}\mu_2)=\operatorname{sgn}\otimes\pi(\mu_1,\mu_2)$. However, because $\pi(\mu_1,\mu_2)$ is equivalent to

$(\mathrm{Ad}\,\varepsilon)\,\pi(\mu_1,\mu_2)$, it follows $\pi(\mu_1,\mu_2)$ is not equivalent to $\mathrm{sgn}\otimes\pi(\mu_1,\mu_2)$ by Lemma 2.11, as desired. □

3. Irreducible components of $\rho(\mu_1,\mu_2)$ over $\mathbf{C}$

We next describe the case that $F = \mathbf{C}$. For the remainder of this section we work over $F = \mathbf{C}$. Retain the notation discussed at the beginning of § 1. That is, we let $G_{\mathbf{C}}$ denote $\mathrm{GL}_2(\mathbf{C})$, $K_{\mathbf{C}} := U(2), \mathfrak{g} := Lie(G_{\mathbf{C}})$ viewed as a real Lie algebra, $\mathfrak{g}_{\mathbf{C}} := \mathfrak{g}\otimes_{\mathbf{R}}\mathbf{C}$, and $\mathscr{U}$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$. Also, let ρ_n denote the unique irreducible $n+1$-dimensional representation of $\mathrm{SU}(2)$. Explicitly, this is given as follows. We can identify the $n+1$ dimensional complex vector space with degree n homogeneous polynomials in two variables. Let these variables be denoted by the vector z, so $z = (z_1, z_2)$. Then, for $p(z)\in\rho_n$, $M\in\mathrm{SU}(2)$ acts by

$$(M\cdot p)(z) := p(Mz).$$

The classification of irreducible admissible representations is, in many respects similar to the case that $F = \mathbf{R}$. However, some parts become slightly more complicated and confusing for several reasons. The main reason is that we no longer have an explicit basis of $\mathscr{B}(\mu_1,\mu_2)$ given by the functions ϕ_n (as although they are $K_{\mathbf{R}}$ finite the ϕ_n are not generally $K_{\mathbf{C}}$ finite). Another reason for added confusion is that we view $\mathfrak{g}$ as a real Lie algebra, so when we complexify, we will have two different "actions of i", one from the complexification and one from the action of the multiplication by i matrix (which we call $\mathscr{J}$) on $\mathfrak{g}_{\mathbf{C}}$. Admittedly this second issue does not require significant new ideas to deal with, unlike the first issue which requires real ingenuity.

In what follows, we first state some notation in § 3.1. Then we state the main classification theorem over $F=\mathbf{C}$, Theorem 3.5 in § 3.3. Following this, we sketch a proof in § 3.3. Finally provide a proof of [JL70, Lemma 6.1] in § 3.4, which is fairly elementary, but central to the proof of Theorem 3.5.

3.1. Further notations for $F = \mathbf{C}$.

We introduce some further notations for $F = \mathbf{C}$, analogous to the case $F = \mathbf{R}$.

Definition 3.1. Let μ_1 and $\mu_2 : \mathbf{C}^\times \to \mathbf{C}^\times$ denote two continuous quasi-characters. Note that any quasi-character can be written in the form

$$\mu_i = (z\overline{z})\,\frac{z^{a_i}\overline{z}^{b_i}}{(z\overline{z})^{\frac{1}{2}(a_i+b_i)}}.$$

with $s_i\in\mathbf{C}, a_i\in\mathbf{Z}_{\geq 0}, b_i\in\mathbf{Z}_{\geq 0}$ and either $a_i = 0$ or $b_i = 0$. Let $\mu = \mu_1\mu_2^{-1}$ and write μ in the form $\mu = (z\overline{z})\,\frac{z^a\overline{z}^b}{(z\overline{z})^{\frac{1}{2}(a+b)}}$ with $s\in\mathbf{C}, a\in\mathbf{Z}_{\geq 0}, b\in\mathbf{Z}_{\geq 0}$, and either $a=0$ or $b=0$.

Exercise 3.2. Verify that every quasi-character of $\mathbf{C}^\times$ can indeed be written in the above form. *Hint:* A quasi-character of $\mathbf{C}^\times$ is determined by its values on $\mathbf{R}_{\geq 0}$ and the unit circle. The value on $\mathbf{R}_{\geq 0}$ is determined by its value on a single number other than 1, as in the real case. On the unit circle, the quasi-character must be given by raising to some integer power.

Definition 3.3. Let $\mathscr{B}(\mu_1, \mu_2)$ denote the space of complex valued functions $f : G_{\mathbf{C}} \to \mathbf{C}$ which are $K_{\mathbf{C}}$ finite and satisfy the transformation property

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} f(g).$$

Then, $\mathscr{U}$ acts on $\mathscr{B}(\mu_1, \mu_2)$ by right action ρ, defined as in Definition 1.7, see also Remark 1.8 for an explicit description of how elements of the Lie algebra act.

We let $\mathscr{B}(\mu_1, \mu_2; \rho_n) \subset \mathscr{B}(\mu_1, \mu_2)$ denote the subspace such that $\mathscr{B}(\mu_1, \mu_2)|_{\mathrm{SU}(2)}$ lies in the ρ_n isotypic component when viewed as a $\mathrm{SU}(2)$ representation.

3.2. **The statement in the case $F = \mathbf{C}$.** We are ready to state the main classification theorem for irreducible admissible representations of $G_{\mathbf{C}}$. This is quite similar in spirit to the analogous classification of Theorem 2.13 of $G_{\mathbf{R}}$.

Remark 3.4. The following theorem statement is essentially taken from [JL70, Theorem 6.2], but we replace parts $(iv) - (vi)$ there with the slightly more precise statement (iv) below, and then number the following statement (vii) to match the corresponding one in [JL70, Theorem 6.2(vii)].

Theorem 3.5. *Let μ_1, μ_2, μ be as in Definition 3.1.*

(i) Suppose μ is not of the form $z \mapsto z^p \overline{z}^q$ or $z \mapsto z^{-p}\overline{z}^{-q}$ for $p \geq 1, q \geq 1$. Then $\rho(\mu_1, \mu_2)$ is irreducible. We let $\pi(\mu_1, \mu_2)$ denote such a representation.

(ii) Suppose μ is of the form $z \mapsto z^p \overline{z}^q$ with $p \geq 1, q \geq 1$. Then,

$$\mathscr{B}_s := \bigoplus_{\substack{n \geq p+q \\ n \equiv p+q \bmod 2}} \mathscr{B}(\mu_1, \mu_2; \rho_n)$$

is the unique proper stable subspace of $\mathscr{B}(\mu_1, \mu_2)$. We let $\sigma(\mu_1, \mu_2)$ denote any representation equivalent to the restriction of $\rho(\mu_1, \mu_2)$ to $\mathscr{B}_s(\mu_1, \mu_2)$ and let $\pi(\mu_1, \mu_2)$ denote any representation equivalent to the representation

$$\mathscr{B}_f(\mu_1, \mu_2) := \mathscr{B}(\mu_1, \mu_2)/\mathscr{B}_s(\mu_1, \mu_2)$$

induced by $\rho(\mu_1, \mu_2)$.

(iii) Suppose μ is of the form $z \mapsto z^{-p}\overline{z}^{-q}$ with $p \geq 1, q \geq 1$. Then,

$$\mathscr{B}_f := \bigoplus_{\substack{|p-q| \leq n \leq p+q \\ n \equiv p+q \bmod 2}} \mathscr{B}(\mu_1, \mu_2; \rho_n)$$

is the unique proper stable subspace of $\mathscr{B}(\mu_1, \mu_2)$. We let $\pi(\mu_1, \mu_2)$ denote any representation equivalent to the restriction of $\rho(\mu_1, \mu_2)$ to $\mathscr{B}_f(\mu_1, \mu_2)$ and let $\sigma(\mu_1, \mu_2)$ denote any representation equivalent to the representation

$$\mathscr{B}_s(\mu_1, \mu_2) := \mathscr{B}(\mu_1, \mu_2)/\mathscr{B}_f(\mu_1, \mu_2)$$

induced by $\rho(\mu_1, \mu_2)$.

(iv) The equivalences between the representations π and σ for varying quasi-characters are precisely the following (i.e., there are no others)

(a) The representation $\pi(\mu_1, \mu_2)$ is equivalent to $\pi(\mu_1', \mu_2')$ if $\{\mu_1, \mu_2\} = \{\mu_1', \mu_2'\}$.

(b) The representation $\sigma(\mu_1, \mu_2)$ is equivalent to $\sigma(\mu_1', \mu_2')$ if $\{\mu_1, \mu_2\} = \{\mu_1', \mu_2'\}$.

(c) For some pair of quasi-characters ν_1, ν_2 with $\nu_1\nu_2^{-1}(z) = z^p\overline{z}^q$ with $p \geq 1, q \geq 1$, we have $\sigma(\nu_1, \nu_2)$ is equivalent to $\pi(\nu_1', \nu_2')$ if $\nu_1\nu_2 = \nu_1'\nu_2'$ and $\nu_1'(\nu_2')^{-1}$ is either given by $z \mapsto z^p\overline{z}^{-q}$ or $z \mapsto z^{-p}\overline{z}^q$.

(d) *For some pair of quasi characters ν_1, ν_2 with $\nu_1\nu_2^{-1}(z) = z^{-p}\overline{z}^{-q}$ with $p \geq 1, q \geq 1$, we have $\sigma(\nu_1, \nu_2)$ is equivalent to $\pi(\nu_1', \nu_2')$ with $\nu_1\nu_2 = \nu_1'\nu_2'$ and $\nu_1'(\nu_2')^{-1}$ either of the form $z \mapsto z^p\overline{z}^{-q}$ or $z \mapsto z^{-p}\overline{z}^q$.*

(vii) *Every irreducible admissible representation is equivalent to some $\pi(\mu_1, \mu_2)$.*

Remark 3.6. In the statement of Theorem 3.5, similarly to the real case in Theorem 2.13, when $\rho(\mu_1, \mu_2)$ is not irreducible, we use $\pi(\mu_1, \mu_2)$ to denote the irreducible subquotients of finite dimension and $\sigma(\mu_1, \mu_2)$ to denote the irreducible subquotients of finite codimension.

3.3. **Sketch of the proof of Theorem 3.5.** We now describe the idea of proof of Theorem 3.5. For a complete proof, we refer the reader to the well-exposited [JL70, Theorem 6.2]. Following this description, we provide a proof of [JL70, Lemma 6.1], which was omitted from the original [JL70].

3.3.1. *Proving (i), (ii), and (iii).* We sketch a proof of (i), (ii), and (iii). For details we refer the reader to [JL70, p. 112-114].

One first shows:

Fact 3.7 ([JL70, Lemma 6.2.1])**.** If $\mathscr{B}(\mu_1, \mu_2)$ has a finite dimensional nonzero proper subrepresentation, then that subrepresentation is $\mathscr{B}_f(\mu_1, \mu_2)$ and $\mu_1\mu_2^{-1}$ is given by $z \mapsto z^{-p}\overline{z}^{-q}$ for $p \geq 1, q \geq 1$.

The key ingredient here to show this is Proposition 3.22, stated and proven below (corresponding to [JL70, Lemma 6.1(ii)]).

One next shows via explicit computations on the real Lie algebra of $\mathrm{GL}_2(\mathbf{C})$:

Fact 3.8 ([JL70, Lemma 6.2.2])**.** If $V \subset \mathscr{B}(\mu_1, \mu_2)$ is a nonzero proper invariant subspace then either V contains a finite dimensional invariant subspace or else

$$V = \sum_{n \geq n_0} \mathscr{B}(\mu_1, \mu_2; \rho_n),$$

where n_0 is the smallest value of k so that the subspace $\mathscr{B}(\mu_1, \mu_2, \rho_k) \subset \mathscr{B}(\mu_1, \mu_2)$ is nonzero.

Using the above two facts (corresponding to [JL70, Lemmas 6.2.1 and 6.2.2]) one can employ the duality between $\mathscr{B}(\mu_1, \mu_2)$ and $\mathscr{B}(\mu_1^{-1}, \mu_2^{-1})$ where the pairing is explicitly given by sending $(f_1, f_2) \mapsto \int_{K_{\mathbf{C}}} f_1(\kappa)f_2(\kappa)d\kappa$. Indeed, suppose V is some proper invariant subspace of $\mathscr{B}(\mu_1, \mu_2)$. By the first fact above, both $\mathscr{B}(\mu_1, \mu_2)$ and $\mathscr{B}(\mu_1^{-1}, \mu_2^{-1})$ cannot contain a finite dimensional invariant subspace, and hence by the second fact, one of the subspaces has finite codimension and the other has finite dimension. This shows that the only possible invariant subspaces are $\mathscr{B}_f$ and $\mathscr{B}_s$ in the various cases.

To conclude the proof of $(i), (ii)$ and (iii) it suffices to show $\mathscr{B}_f(\mu_1, \mu_2)$ and $\mathscr{B}_s(\mu_1, \mu_2)$ are the only possible invariant sub and quotient spaces. By the duality mentioned above, it suffices to verify this in the case of $\mathscr{B}_f(\mu_1, \mu_2)$ with $\mu_1\mu_2^{-1}(z) = z^{-p}\overline{q}^{-q}$. One can then explicitly compute that $\mathscr{B}_f(\mu_1, \mu_2)$ is invariant using that we understand the restriction to $\mathrm{SU}(2)$ via Proposition 3.22 below.

3.3.2. *Proving that the equivalences in (iv) are the only possible ones.* We explain how to show that the equivalences stated in (iv) are the only possible ones. Indeed, the proof of this part is quite analogous to the proof of Theorem 2.13(4). In the real case, one uses the identities given in Lemma 1.11. For the complex case, one uses analogous identities for central elements of the complexification of the Lie algebra for $\mathrm{GL}_2(\mathbf{C})$, as stated and proven below in Proposition 3.13 and Proposition 3.17 (corresponding to [JL70, Lemma 6.1(i)]). For the details, we refer the reader to [JL70, p. 114-115].

3.3.3. *Explicit intertwiners in (iv).* To complete the sketch of (iv), we explain why the claimed equivalences between π's and σ's actually exist. For this, it suffices to produce explicit intertwiners. Indeed, these are given by the operator $M(s)$ defined by

$$(M(s)f)(g) := \int_{\mathbf{C}} f\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx.$$

Note that the dependence on s is given through f where $f \in \mathscr{B}(\mu_1, \mu_2)$, with s as in Definition 3.1. In the real case, these are described in [Bum97, Equation (6.7) p. 227, ff.], but the analogous statements carry over to the complex case.

Remark 3.9. In [God74, p. 2-13, lines 8-9], Godement remarks that someone should explicitly construct the isomorphisms between $\sigma(\mu_1, \mu_2)$ and $\pi(\nu_1, \nu_2)$. The isomorphism is given explicitly by the above intertwining operator.

This completes the proof sketch of $(i) - (iv)$.

3.3.4. *Proving (vii).* This follows from the previous parts once we know every irreducible admissible representation is indeed a subrepresentation of some $\mathscr{B}(\mu_1, \mu_2)$, which was shown last time, see Theorem 1.12.

3.4. **Proving** [JL70, Lemma 6.1]**.** In the remainder of this section, we provide a proof of [JL70, Lemma 6.1]. This was stated with only a brief indication of the proof in [JL70], and the proof is fairly straightforward. Nevertheless, we include the proof for completeness.

Definition 3.10. Let $\mathscr{J}$ be the multiplication by i operator on $\mathfrak{g}$, defined by

$$\mathscr{J}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ia & ib \\ ic & id \end{pmatrix}.$$

Remark 3.11. Letting $\mathfrak{g}$ denote the Lie algebra of $\mathrm{GL}_2(\mathbf{C})$ considered over the real numbers, and $\mathfrak{g}_{\mathbf{C}} := \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} \simeq \mathfrak{g} \oplus i\mathfrak{g}$ its complexification. we obtain an isomorphism

$$\begin{aligned} L\colon \mathfrak{g} \oplus i\mathfrak{g} &\to \mathfrak{g} \oplus \mathfrak{g} \\ X + iY &\mapsto (X + JY, X - JY). \end{aligned}$$

with J as defined above, see [Kna86, Proposition 2.5].

Then, under this identification, letting $\mathscr{U}_1$ denote the Lie algebra of $\mathfrak{g}$ considered over the reals, the universal enveloping algebra $\mathscr{U}$ of $\mathfrak{g}_{\mathbf{C}}$ is identified with $\mathscr{U}_1 \otimes \mathscr{U}_1$ via extending the map $\mathfrak{g}_{\mathbf{C}} \to \mathfrak{g} \oplus \mathfrak{g}$ to the corresponding universal enveloping algebras.

Definition 3.12. Let D and J be as in Definition 1.10 and define the elements of $\mathscr{U}$, thought of as elements of $\mathscr{U}_1 \otimes \mathscr{U}_1$, for $\mathscr{U}_1$ the universal enveloping algebra of $\mathrm{GL}_2(\mathbf{C})$ under the identification of Remark 3.11

$$\begin{aligned} J_1 &:= J \otimes 1 \\ J_2 &:= 1 \otimes J \\ D_1 &:= D \otimes 1 \\ D_2 &:= 1 \otimes D. \end{aligned}$$

We next embark on a hefty computation to determine the action by ρ of the above elements in Proposition 3.13 and Proposition 3.17

Proposition 3.13. *The action $\rho(J_1)$ and $\rho(J_2)$ on $\mathscr{B}(\mu_1, \mu_2)$ is given by multiplication by the scalars*

$$\begin{aligned} \rho(J_1) &= s_1 + s_2 + \frac{1}{2}(a_1 - b_1 + a_2 - b_2) \\ \rho(J_2) &= s_1 + s_2 + \frac{1}{2}(b_1 - a_1 + b_2 - a_2) \end{aligned}$$

Proof. We will compute $\rho(J_1)$, as the computation for $\rho(J_2)$ is completely analogous, and has large overlap with the computation for $\rho(J_1)$. Under the identification of Remark 3.11 we have that J_1, thought of as an element in $\mathfrak{g} \oplus \mathfrak{g}$, corresponds to the element $\frac{1}{2}J + \frac{i}{2}\mathscr{J} J$, where $\mathscr{J}$ is the multiplication by i operator defined in Definition 3.10. In order to continue the proof, we state and prove two computational sublemmas.

Lemma 3.14. *With notations as above, we claim that $(Jf)(g) = 2(s_1 + s_2)f(g)$.*

Proof. To see this, we compute

$$\begin{aligned} (Jf)(g) &= \frac{\partial}{\partial t} f(g e^{tJ})|_{t=0} \\ &= \frac{\partial}{\partial t} f\left(g \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}\right)|_{t=0} \\ &= \frac{\partial}{\partial t} f\left(\begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} g\right)|_{t=0} \\ &= \frac{\partial}{\partial t} \mu_1(e^t)\mu_2(e^t) f(g). \end{aligned}$$

So, to conclude $(Jf)(g) = (s_1 + s_2)f(g)$, it suffices to verify

$$\frac{\partial}{\partial t}\mu_1(e^t)\mu_2(e^t) = (s_1 + s_2)\mu_1(e^t)\mu_2(e^t),$$

as we can then reverse the above computation. Indeed,

$$\frac{\partial}{\partial t}\mu_1(e^t)\mu_2(e^t) = (s_1 + s_2)\mu_1(e^t)\mu_2(e^t)$$

is easily obtained by plugging e^t into the definition of μ_i of Definition 3.1 and differentiating. □

Lemma 3.15. *With notations as above, we claim that $((\mathscr{J} J)f)(g) = a_1 - b_1 + a_2 - b_2$.*

Proof. Indeed, this follows from an analogous computation to the preceding lemma, using that

$$\mathscr{J} J = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

and so

$$e^{t\mathscr{J} J} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{it} \end{pmatrix}.$$

□

Putting the above two lemmas together, we have that J_1 acts by multiplication by

$$\begin{aligned} \rho(J_1) &= \frac{1}{2}\rho(J) + \frac{i}{2}\rho(\mathscr{J} J) \\ &= \frac{1}{2}2(s_1 + s_2) - \frac{i}{2} \cdot i\,(a_1 - b_1 + a_2 - b_2) \\ &= s_1 + s_2 + \frac{1}{2}\,(a_1 - b_2 + a_2 - b_2)\,. \end{aligned}$$

This concludes our calculation of $\rho(J_1)$.

Exercise 3.16. Verify $\rho(J_2) = s_1 + s_2 + \frac{1}{2}\,(b_1 - a_1 + b_2 - a_2)$ similarly, reusing many of the above computations.

□

We next compute the action of D_1 and D_2. These are similar but slightly trickier because D_i live in the second homogeneous piece of the universal enveloping algebra.

Proposition 3.17. *The action $\rho(D_1)$ and $\rho(D_2)$ on $\mathscr{B}(\mu_1, \mu_2)$ is given by multiplication by the scalars*

$$\begin{aligned} \rho(D_1) &= \frac{1}{2}\left(s + \frac{1}{2}\,(a - b)\right)^2 - \frac{1}{2} \\ \rho(D_2) &= \frac{1}{2}\left(s + \frac{1}{2}\,(b - a)\right)^2 - \frac{1}{2} \end{aligned}$$

Proof. We will compute $\rho(D_1)$, as the computation for $\rho(D_2)$ is completely analogous, and has large overlap with the computation for $\rho(D_1)$. Recall that in the universal enveloping algebra of $\mathfrak{g}$ we have

$$D = X_+X_- + X_-X_+ + \frac{1}{2}Z^2 = 2X_-X_+ + Z + \frac{1}{2}Z^2,$$

using the relation

$$[X_+, X_-] = Z$$

in the universal enveloping algebra.

Therefore, we will instead compute $\rho((2X_-X_+ + Z + \frac{1}{2}Z^2) \otimes 1)$. However, using that D_1 is central in the universal enveloping algebra, left action agrees with right action, and so

$$\begin{aligned}\rho((2X_-X_+ + Z + \frac{1}{2}Z^2)\otimes 1) &= \lambda\left((2X_-X_+ + Z + \frac{1}{2}Z^2)\otimes 1\right)\\ &= \lambda\left(2X_-X_+\otimes 1\right) + \lambda(Z\otimes 1) + \lambda\left(\frac{1}{2}Z^2\otimes 1\right).\end{aligned}$$

where λ denotes left action as defined in Definition 1.7 (instead of right action from ρ). Note that under the identification of Remark 3.11 the element $2X_-X_+\otimes 1$ corresponds to $X_-X_+ - i\mathscr{J}X_-X_+$.

Lemma 3.18. *We claim*

$$\lambda(X_-X_+ - i\mathscr{J}X_-X_+) = 0.$$

Proof. To verify this, it suffices to check $\lambda(X_+) = 0$. Indeed, we see

$$\begin{aligned}(\lambda(X_+)f)(g) &= \frac{\partial}{\partial t}f(e^{-tX_+}g)|_{t=0}\\ &= \frac{\partial}{\partial t}f\left(\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}g\right)|_{t=0}\\ &= \frac{\partial}{\partial t}f(g)\\ &= 0.\end{aligned}$$

□

So, it suffices to compute $\lambda(Z\otimes 1) + \lambda\left(\frac{1}{2}Z^2\otimes 1\right)$. We see that under the identification of Remark 3.11 $Z\otimes 1$ corresponds to $\frac{1}{2}Z - \frac{i}{2}\mathscr{J}Z$. Therefore $\frac{1}{2}Z^2$ corresponds to $\left(\frac{1}{2}Z - \frac{i}{2}\mathscr{J}Z\right)^2$.

Lemma 3.19. *We have*

$$\begin{aligned}\lambda(Z) &= -2s-2\\ \lambda(\mathscr{J}Z) &= i(b-a).\end{aligned}$$

Proof. To compute $\lambda(Z)$, note

$$\begin{aligned}\lambda(Z) &= \frac{\partial}{\partial t}f(e^{-tZ}g)|_{t=0}\\ &= \frac{\partial}{\partial t}\mu_1(e^{-t})\mu_2(e^{t})\left|e^{-2t}\right|^{1/2}f(g)|_{t=0}\\ &= \frac{\partial}{\partial t}e^{(-2s-2)t}f(g)|_{t=0}\\ &= (-2s-2)e^{(-2s-2)t}f(g)|_{t=0}\end{aligned}$$

Exercise 3.20. Verify the analogous computation for $\lambda(\mathscr{J}Z)$.

□

Plugging in the result of Lemma 3.19, we find

$$\begin{aligned}\lambda(Z\otimes 1) &= \lambda\left(\frac{1}{2}Z-\frac{i}{2}\mathscr{J}Z\right)\\ &= -(s+1)+\frac{1}{2}\left(b-a\right).\end{aligned}$$

It follows

$$\lambda(\frac{1}{2}Z^2\otimes 1)=\frac{1}{2}\left(-(s+1)+\frac{1}{2}\left(b-a\right)\right)^2.$$

Therefore,

$$\begin{aligned}\rho(D_1) &= \lambda(D_1)\\ &= \lambda(2X_-X_+\otimes 1)+\lambda(Z\otimes 1)+\lambda(\frac{1}{2}Z^2)\\ &= 0-(s+1)+\frac{1}{2}\left(b-a\right)+\frac{1}{2}\left(-(s+1)+\frac{1}{2}\left(b-a\right)\right)^2\\ &= \left(s+\frac{1}{2}(a-b)\right)^2-\frac{1}{2},\end{aligned}$$

as claimed. To simplify the algebra calculation above, one can use the identity $(x-1)+\frac{1}{2}\left(x-1\right)^2=\frac{1}{2}x^2-\frac{1}{2}$, taking $x=-s+\frac{1}{2}(b-a)$. This completes the computation of $\rho(D_1)$

Exercise 3.21. Verify $\rho(D_2)=\frac{1}{2}\left(s+\frac{1}{2}\left(b-a\right)\right)^2-\frac{1}{2}$ similarly, reusing many of the above computations.

□

Having computed the explicit actions of central elements in the universal enveloping algebra, we next verify admissibility of the representations $\rho(\mu_1,\mu_2)$.

Proposition 3.22. *The representation $\rho(\mu_1,\mu_2)$ is admissible. Furthermore, $\rho(\mu_1,\mu_2)|_{\mathrm{SU}(2)}$ contains ρ_n as a subrepresentation if and only if $n\geq a+b$ and $n\equiv a+b \mod 2$ (with a,b as in Definition 3.1) and in this case ρ_n occurs with multiplicity* 1.

Proof. We wish to show

$$\dim Hom_{\mathrm{SU}(2)}(\rho_n,\rho(\mu_1,\mu_2)|_{\mathrm{SU}(2)})=1 \tag{3.1}$$

if and only if $n\geq a+b$.

In particular, this will also prove $\rho(\mu_1,\mu_2)$ is admissible because $K_{\mathbf{C}}=U(2)$ is a semidirect product of SU(2) with $U(1)\simeq S^1$, and since S^1 representations are completely decomposable into 1-dimensional representations, if the restriction $\rho(\mu_1,\mu_2)|_{\mathrm{SU}(2)}$ has finite dimensional isotypic components the same will be true of $\rho(\mu_1,\mu_2)|_{K_{\mathbf{C}}}$.

We proceed to prove Equation 3.1 Recall that by definition, $\rho(\mu_1,\mu_2)$ is a representation induced from the Borel $B_{\mathbf{C}}$ up to $G_{\mathbf{C}}$. That is,

$$\rho(\mu_1,\mu_2)=\mathrm{Ind}_B^G((\mu_1\boxtimes\mu_2)\otimes\Delta_B^{-1/2})$$

where Δ_B is the modulus character given by

$$\Delta_B\left(\begin{pmatrix}\alpha & * \\ 0 & \beta\end{pmatrix}\right) := \left|\frac{\beta}{\alpha}\right|.$$

Define the subgroup $U(1) \subset \mathrm{SU}(2)$ as those matrices of the form

$$U(1) := \left\{\begin{pmatrix}\alpha & 0 \\ 0 & \alpha^{-1}\end{pmatrix} : |\alpha| = 1\right\}.$$

Restricting to $\mathrm{SU}(2)$, and noting that $\mathrm{SU}(2) \cap B = U(1)$ we see that

$$\rho(\mu_1, \mu_2)|_{\mathrm{SU}(2)} = \mathrm{Ind}_{U(1)}^{\mathrm{SU}(2)}\left((\mu_1 \boxtimes \mu_2) \otimes \Delta_B^{-1/2}\right)|_{U(1)}.$$

Therefore, by Frobenius reciprocity, we have

$$\begin{aligned} Hom_{\mathrm{SU}(2)}(\rho_n, \rho(\mu_1, \mu_2)|_{\mathrm{SU}(2)}) &= Hom_{\mathrm{SU}(2)}\left(\rho_n, \mathrm{Ind}_{U(1)}^{\mathrm{SU}(2)}\left((\mu_1 \boxtimes \mu_2) \otimes \Delta_B^{-1/2}\right)|_{U(1)}\right) \\ &= Hom_{U(1)}\left(\rho_n|_{U(1)}, \left((\mu_1 \boxtimes \mu_2) \otimes \Delta_B^{-1/2}\right)|_{U(1)}\right). \end{aligned}$$

We can now compute the dimension of this last vector space. Although $(\mu_1 \boxtimes \mu_2) \otimes \Delta_B|_{U(1)}^{-1/2}$ may at first seem quite scareful, it is actually the down to earth quasi-character given by

$$\begin{aligned} \left((\mu_1 \boxtimes \mu_2) \otimes \Delta_B^{-1/2}\right)|_{U(1)}\left(\begin{pmatrix}\alpha & 0 \\ 0 & \alpha^{-1}\end{pmatrix}\right) &= \mu_1(\alpha)\mu_2(\alpha^{-1})\left|\frac{\alpha}{\alpha^{-1}}\right|^{1/2} \\ &= \mu_1(\alpha)\mu_2(\alpha^{-1}) \\ &= \alpha^{a-b} \end{aligned}$$

since $|\alpha| = 1$. Therefore, the dimension of $Hom_{\mathrm{SU}(2)}(\rho_n, \rho(\mu_1, \mu_2)|_{\mathrm{SU}(2)})$ is at most 1, and equal to 1 if and only if $a - b$ appears as an eigenvalue in the action of $U(1)$ on ρ_n.

To conclude, we compute precisely when $a - b$ appears as an eigenvalue. using our explicit description of ρ_n. Namely, we can write any element v in the underlying vector space of ρ_n as a polynomial

$$v = \sum_{i=0}^{n} c_i x^i y^{n-i}.$$

The action of $U(1)$ is given by

$$\begin{aligned} \begin{pmatrix}\alpha & 0 \\ 0 & \alpha^{-1}\end{pmatrix}\sum_{i=0}^{n} c_i x^i y^{n-i} &= \sum c_i(\alpha x)^i\left(\alpha^{-1}y\right)^{n-i} \\ &= \sum c_i \alpha^{2i-n} x^i y^{n-i}. \end{aligned}$$

Hence, by inspection, the weights are $-n, -n+2, -n+4, \ldots, n$ (i.e., the range between $-n$ and n, increasing 2 at a time) with eigenvectors given by the $n+1$ functions $x^i y^{n-i}$ for $0 \le i \le n$. This shows that $a - b$ is weight if and only if $-n \le a - b \le n$. However, either $a = 0$ or $b = 0$, and both are nonnegative. Hence, $-n \le a - b \le n$ if and only if $a + b \le n$, as desired. □

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