LECTURE 19: ARCHIMEDEAN WHITTAKER MODELS FOR GL_2

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APRIL 4, 2018
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Let $F = \mathbf{R}$ or $\mathbf{C}$. Let $G = \mathrm{GL}_2(F)$ as a real Lie group and K be the obvious maximal compact; $K = \mathrm{O}(2)$ if $F = \mathbf{R}$ and $K = \mathrm{U}(2)$ if $F = \mathbf{C}$. Let $\mathfrak{g} = (\mathrm{Lie}\, G) \otimes_{\mathbf{R}} \mathbf{C}$. The "representations" of G which we will discuss are actually $(\mathfrak{g}, K)$-modules, i.e. compatible representations of $\mathfrak{g}$ and K.

Recall that we had the principal series: let $\mu_1, \mu_2 \colon \mathrm{F}^\times \to \mathbf{C}$ be quasi-characters. We consider the space $\mathcal{B}(\mu_1, \mu_2)$ of right-K-finite continuous functions $f : G \to \mathbf{C}$ such that

$$f\left(\left(\begin{smallmatrix} a_1 & b \\ 0 & a_2 \end{smallmatrix}\right) g\right) = \mu_1(a_1)\mu_2(a_2) \left|\frac{a_1}{a_2}\right|_F^{1/2} f(g).$$

Here we note that the norm $|\cdot|_F$ is the canonical one for any local field given by $d(ax) = |a|_F dx$, where dx is a Haar measure for $(F, +)$.

We also note that as f is K-finite, theory of representations of compact Lie groups ensures that $f|_K$ is analytic. Since quasi-characters are analytic and $G = BK$ where B is the Borel of invertible upper triangular matrices, we see that all such f are also analytic on G, considering G as a real analytic manifold.

Denote by ρ_{μ_1,μ_2} the $(\mathfrak{g}, K)$-module given by the right translation action on $\mathcal{B}(\mu_1, \mu_2)$; that is to say $U(\mathfrak{g})$ acts by differentiating the usual right translation action. We saw in Lectures 17 and 18 that any irreducible admissible representation $(\mathfrak{g}, K)$-module is a subrepresentation of some ρ_{μ_1,μ_2}.

Let $\psi : F \to \mathbf{C}$ be a non-trivial (unitary) additive character; we will actually pick $\psi(x) = e^{2\pi i x}$ if $F = \mathbf{R}$ and $\psi(x) = e^{4\pi i \,\mathrm{Re}(x)}$ if $F = \mathbf{C}$ to copy formulas from [1]. The space $\mathrm{Ind}_U^G \psi$ of *Whittaker functions* are the space of smooth K-finite functions $W : G \to \mathbf{C}$ such that $W\left(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) g\right) = \psi(b)W(g)$ for any $b \in F$, $g \in G$, and

$$W\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = O(|a|^N) \text{ for some } N \text{ as } a \to +\infty \tag{1}$$

A *Whittaker model* of an irreducible admissible $(\mathfrak{g}, K)$-module π is an embedding $\pi \hookrightarrow \mathrm{Ind}_U^G \psi$ as $(\mathfrak{g}, K)$-modules. The main result today the following statement identical to the non-archimedean case:

Theorem 1. Let π be any (non-zero) irreducible admissible $(\mathfrak{g}, K)$-module. Then π has a Whittaker model if and only if π is infinite-dimensional. In this case, the Whittaker model is unique.

We remark here that the uniqueness theorem of this part is going to have the following application: the Fourier expansion for automorphic forms of GL_2 (analogue of q-expansion for modular forms) gives, after rewritten into adelic language, a global Whittaker model. Now if a global admissible representation of $G(\mathbf{A}_K)$ (not an actual representation of $G(\mathbf{A}_K)$) had multiplicity > 1, i.e. can be embedded into $L^2_{cusp}(G(\mathbf{A}_K))$ in more than one way modulo constants, it would have more than one global Whittaker models and that will contradict the uniqueness of local Whittaker models. Thus, uniqueness of local Whittaker models will help us prove the "multiplicity one" theorem.

If we have a Whittaker model $\pi \hookrightarrow \mathrm{Ind}_U^G \psi$ as $v \mapsto W_v$, then $v \mapsto W_v\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\big)$ gives a *Whittaker functional*, i.e. a function ϕ on π that satisfies $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right).\phi(v) = \psi(b)\phi(v)$. When π is finite-dimensional this is not possible as the nilpotent element of $\mathfrak{gl}_2$ given by $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) = \left[\left(\begin{smallmatrix} 1/2 & 0 \\ 0 & -1/2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)\right]$ has to act by a nilpotent operator in the finite-dimensional $\mathfrak{gl}_2$-representation determined by π. Thus, $W_v\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\big) \equiv 0$, so $\pi = 0$. This settles the "only if" part of Theorem 1.

To prove the existence of Whittaker model when π is infinite-dimensional, we will construct an embedding $\pi \hookrightarrow \mathrm{Ind}_U^G \psi$ by factoring it as $\pi \hookrightarrow \rho_{\mu_1,\mu_2} \hookrightarrow \mathrm{Ind}_U^G \psi$. We have seen the tautological map

$$\mathrm{Hom}_G(\rho_{\mu_1,\mu_2}, \mathrm{Ind}_U^G \psi) \to \mathrm{Hom}_U(\mathrm{Res}_U^G(\rho_{\mu_1,\mu_2}), \psi) \tag{2}$$

As we have $(\mathfrak{g}, K)$-modules but not actual representations of G, to go from the right of (2) to the left will require an integration. For heuristic let us pretend (2) is an isomorphism for a moment. Then $\mathrm{Res}_U^G(\rho_{\mu_1,\mu_2})$ appears as $\mathrm{Res}_U^G \mathrm{Ind}_B^G(\cdot)$ and we will find $\mathrm{Hom}_U(\mathrm{Res}_U^G(\rho_{\mu_1,\mu_2}), \psi)$ by looking at the "open cell" of $B\backslash G/U$. Identifying $B\backslash G$ with $\mathbf{P}^1$, the open Bruhat cell may be identified with $\mathbf{A}^1$ via $x \mapsto \left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$, and U acts by $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \cdot x = x + b$. We want to construct a $\mathbf{C}$-valued function on $B\backslash G$ which is equivariant with U. This gives the following element in $\mathrm{Hom}_U(\mathrm{Res}_U^G(\rho_{\mu_1,\mu_2}), \psi)$:

$$v \mapsto W_v\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\big) = \int_F f\Big(\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\Big)\psi(x)dx.$$

If we begin by an embedding $\rho_{\mu_1,\mu_2} \hookrightarrow \mathrm{Ind}_U^G \psi$, we may then attempt construct an element in $\mathrm{Hom}_G(\rho_{\mu_1,\mu_2}, \mathrm{Ind}_U^G \psi)$ given by $f \mapsto W_f$ as follows: for any $f \in \rho_{\mu_1,\mu_2}$ we have

$$W_f(g) := \int_F f\Big(\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) g\Big)\psi(x)dx. \tag{3}$$

We have

$$f\Big(\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) g\Big) = f\Big(\left(\begin{smallmatrix} x^{-1} & 1 \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & x^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) g\Big) \tag{4}$$

$$= \mu_1^{-1}(x)\mu_2(x)|x|_F^{-1} f\Big(\left(\begin{smallmatrix} 0 & -1 \\ 1 & x^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) g\Big) \tag{5}$$

$$= \mathrm{O}\big(|\mu_1^{-1}(x)\mu_2(x)| \cdot |x|_F^{-1}\big) \text{ as } x \to \infty. \tag{6}$$

Let $s_1, s_2 \in \mathbf{R}$ be such that $|\mu_i(x)| = |x|^{s_i}$. If $s_1 - s_2 > 0$, then the integral (3) converges absolutely thanks to (4). According to the classification (is there an intrinsic reason for this other than Whittaker model?), our irreducible π either appears as $\pi = \rho_{\mu_1,\mu_2} = \rho_{\mu_2,\mu_1}$ for a unique set of $\{\mu_1, \mu_2\}$, or appears as a subquotient in some ρ_{μ_1,μ_2} and ρ_{μ_1,μ_2} for a unique set of $\{\mu_1, \mu_2\}$.

In the latter case, $s_1 - s_2$ must be a non-zero integer and π appears as a subrepresentation iff either $\dim \pi = \infty$ and $s_1 - s_2 \in \mathbf{Z}_{>0}$ or $\dim \pi < \infty$ and $s_1 - s_2 \in \mathbf{Z}_{<0}$. In other words, by assuming π is infinite-dimensional we may assume that either $s_1 - s_2 \geq 0$ or that $\pi = \rho_{\mu_1,\mu_2}$ and $s_1 - s_2$ is not a strictly negative integer. In the latter case, by writing $\mu_2(x) = |x|^s \cdot \mu_2'(x)$ for $s \in \mathbf{C}$ where μ_2' is some quasi-character with $|\mu_2'| = |\mu_1|$, it is possible to define (3) with analytic continuation in s past the $\mathrm{Re}(s) = 0$ line except at $s = -1, -2, \ldots$ where one encounters poles. For any infinite-dimensional π we never encounter $s \in \mathbf{Z}_{<0}$ as discussed above, and this gives a construction of the Whittaker model.

One may check that when we define W_f this way, it satisfies the condition that $W_f\left(\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)g\right) = \psi(b)W_f(g)$ and the condition of being right K-finite. However, it is not clear how to settle the issue of moderate growth, i.e. the condition that $W_f\left(\left(\begin{smallmatrix}a & 0\\ 0 & 1\end{smallmatrix}\right)\right) = \mathrm{O}(|a|^N)$.

The analytic continuation argument is however a bit nasty to write down, and we will instead follow a trick of Jacquet-Langlands (which Godement reproduced) which will also help simplify other arguments.

To begin with, how do we write vectors in $\mathcal{B}_{\mu_1,\mu_2}$? Let $K_1 = \mathrm{SO}(2)$ when $F = \mathbf{R}$ and $K_1 = \mathrm{SU}(2)$ when $F = \mathbf{C}$. We still have $G = BK_1$, and thus an $f \in \mathcal{B}_{\mu_1,\mu_2}$ is determined by $f|_{K_1}$. The set K_1 can be conveniently described as

$$K_1 = \{\left(\begin{smallmatrix}a & -\bar{b}\\ b & \bar{a}\end{smallmatrix}\right) \mid a, b \in F,\ |a|^2 + |b|^2 = 1\}$$

By studying the representation theory of K or K_1, one sees that the space of K-finite continuous functions on K_1 are given by polynomials in a, b, $\overline{a}$ and $\overline{b}$. We will however like to do Fourier transform later and will prefer the whole F^2. Consider the space $\mathcal{S}(F^2)^{\mathrm{adm}}$ of Schwartz functions on F^2 that are K-finite, where K acts on F^2 through the standard representation of $G = \mathrm{GL}(F^2)$. Note that the map $g \mapsto g.\begin{pmatrix}1\\0\end{pmatrix}$ restricts to a diffeomorphism between K_1 and the unit sphere in F^2. We have

$$B \cap K_1 = \{\left(\begin{smallmatrix}c & 0\\ 0 & c^{-1}\end{smallmatrix}\right) \mid c \in F,\ |c| = 1\}.$$

Given a function $\phi \in \mathcal{S}(F^2)^{\mathrm{adm}}$ with the property that $\phi(ca, c^{-1}b) = \mu_1(c)\mu_2^{-1}(c)\phi(a,b)$ for any $c \in F$, $|a| = 1$ and $a, b \in F$, we may restrict it to the unit sphere, pull it back to K_1 and extend it to a function $f'_\phi \in \mathcal{B}_{\mu_1,\mu_2}$.

It will actually be more convenient to consider the subspace $\mathcal{S}_1(F^2) \subset \mathcal{S}(F^2)^{\mathrm{adm}}$ of functions $\mathcal{S}_1(F^2) = \{e^{-|a|^2-|b|^2}P(a,b,\overline{a},\overline{b})\}$, where P is some polynomial. For $\phi \in \mathcal{S}_1(F^2)$, the above idea motivates us to consider instead the function $f_\phi \in \mathcal{B}_{\mu_1,\mu_2}$ defined by

$$f_\phi(g) := \mu_2(\det(g)) \cdot |\det g|_F^{-1/2} \cdot \int_F \phi\left(g^{-1} \cdot \begin{pmatrix}t\\0\end{pmatrix}\right)\mu_1(t)\mu_2(t)^{-1}\,dt$$

as long as $s = s_1 - s_2 > -1$ so that this converges near $t = 0$.

Taking $\phi \in \mathcal{S}_1(F^2)$ with $\phi = e^{-|a|^2-|b|^2}P(a,b,\overline{a},\overline{b})$ and P is a non-zero monomial, we may check that $f_\phi|_{K_1}$ is a multiple of P, and the multiple is non-zero if and only if P satisfies $P(ca, c^{-1}b, c^{-1}\overline{a}, c\overline{b}) = \mu_1(c)\mu_2(c)^{-1}P(a,b,\overline{a},\overline{b})$ for $c \in F$, $|c| = 1$. This is (and should be) exactly the earlier condition for a function on the unit sphere to extend to a function in $\mathcal{B}_{\mu_1,\mu_2}$. Since every K-finite function on K_1 is a polynomial in $a, b, \overline{a}, \overline{b}$, all functions in $\mathcal{B}_{\mu_1,\mu_2}$ arises as some f_ϕ for some $\phi \in \mathcal{S}_1(F^2)$.

Now, we are led to write:

$$
\begin{aligned}
W_\phi(g) &:= W_{f_\phi}(g) && (7)\\
&= \mu_2(\det(g)) \cdot |\det g|_F^{-1/2} \cdot \int_F \int_F \phi\left(g^{-1} \cdot \begin{pmatrix} -tx \\ -t \end{pmatrix}\right) \mu_1(t)\mu_2(t)^{-1}\psi(x)\, dx\, dt && (8)\\
&= \mu_2(\det(g)) \cdot |\det g|_F^{-1/2} \cdot \int_F \mu_1(t)\mu_2(t)^{-1}|t|_F^{-1} \int_F \phi\left(g^{-1} \cdot \begin{pmatrix} -y \\ -t \end{pmatrix}\right) \psi(y/t)\, dy\, dt && (9)
\end{aligned}
$$

As a Fourier transform of a Schwartz function in any of its (linear) variable is a Schwartz function, the function

$$(t_1, t_2) \mapsto \int_F \phi\left(g^{-1} \cdot \begin{pmatrix} -y \\ -t_2 \end{pmatrix}\right) \psi(yt_1)\, dy$$

has exponential decay when either t_1 or $t_2 \to \infty$. In other words, the inner integral in (7)

$$\int_F \phi\left(g^{-1} \cdot \begin{pmatrix} -y \\ -t \end{pmatrix}\right) \psi\left(\frac{y}{t}\right) dy$$

has exponential decay both when $t \to 0$ and $t \to \infty$. Consequently the outer integral in (7) converges absolutely for any μ_1, μ_2. For any $f \in \mathcal{B}_{\mu_1,\mu_2}$ we can thus find some ϕ with $f = f_\phi$ and define $W_f = W_\phi$. We remind that our definitions of f_ϕ and W_ϕ all have μ_1, μ_2 in them and everything depends on μ_1, μ_2. We have to check that this is well-defined, i.e. we don't have any ϕ with $f_\phi = 0$ but $W_\phi \neq 0$.[1] We have:

[1]As discussed in the seminar, this should more-or-less follow from that $f_\phi|_{K_1}$ is a multiple of ϕ restricted to the unit sphere (the unit sphere is then identified as K_1). There was a mistake of Cheng-Chiang, that $f_\phi|_{K_1}$ is only a multiple of the restriction of $\phi = e^{-|a|^2-|b|^2}P(a, b, \overline{a}, \overline{b})$ when P is a monomial. In general, $f_\phi|_{K_1}$ will be a sum of different multiples of monomials. If one had carefully imposed the condition that P has to have to correct "degree," i.e. satisfies $P(ca, c^{-1}b, c^{-1}\overline{a}, c\overline{b}) = \mu_1(c)\mu_2(c)^{-1}P(a, b, \overline{a}, \overline{b})$ for $c \in F$, $|c| = 1$, then the constants in the multiples will be non-zero, and one has $f_\phi = 0 \Rightarrow \phi = 0$. By doing so, the heavy computation on the first half of the next page can be skipped.

$$f_\phi\Big(\left(\begin{smallmatrix}1&0\\x&1\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)g\Big) = \mu_2(\det(g))\cdot|\det(g)|_F^{-1/2}\cdot\int_F\phi\left(g^{-1}\cdot\begin{pmatrix}tx\\-t\end{pmatrix}\right)\mu_1(t)\mu_2(t^{-1})dt$$

$$= \mu_2(\det(g))\cdot|\det(g)|_F^{-1/2}\cdot\int_F\left(\int_F\left(\int_F\phi\left(g^{-1}\cdot\begin{pmatrix}y\\-t\end{pmatrix}\right)\bar{\psi}(ya)\,dy\right)\psi(txa)\,da\right)\mu_1(t)\mu_2(t^{-1})\,dt$$

$$= \mu_2(\det(g))\cdot|\det(g)|_F^{-1/2}\cdot\int_F\left(\int_F\left(\int_F\phi\left(g^{-1}\cdot\begin{pmatrix}y\\-t\end{pmatrix}\right)\bar{\psi}(ya/t)\,dy\right)\psi(xa)\,da\right)|t|_F^{-1}\mu_1(t)\mu_2(t^{-1})\,dt$$

$$= \mu_2(\det(g))\cdot|\det(g)|_F^{-1/2}\cdot\int_F\left(\int_F\left(\int_F\phi\left(g^{-1}\cdot\begin{pmatrix}y\\-t\end{pmatrix}\right)\bar{\psi}(ya/t)\;\,dy\right)|t|_F^{-1}\mu_1(t)\mu_2(t^{-1})dt\right)\psi(xa)\,da$$

$$= \mu_2(\det(g))\cdot|\det(g)|_F^{-1/2}\cdot$$

$$\int_F\left(\int_F\left(\int_F\phi\left(g^{-1}\cdot\begin{pmatrix}-y/a\\-t\end{pmatrix}\right)\psi(y/t)\,dy\right)|t|_F^{-1}\mu_1(t)\mu_2(t^{-1})\,dt\right)|a|_F^{-1}\psi(xa)\,da$$

$$= \int_F W_\phi\big(\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)g\big)\mu_2^{-1}(a)|a|_F^{-1/2}\psi(xa)\,da$$

Here in the third inequality we again use that the Fourier transform $\int_F\phi\left(g^{-1}\cdot\begin{pmatrix}y\\-t\end{pmatrix}\right)\overline{\psi}(ya/t)\,dy$ has exponential decay when either a or $t\to\infty$. What we have done is that we have re-written f_ϕ as a Fourier transform of W_ϕ, i.e. we just carried out a Fourier inversion formula to (3). If $s=s_1-s_2>0$, then this will be a direct consequence of (3) by the usual Fourier inversion formula. Now the formula works for $s>-1$ (which was needed to define f_ϕ). This shows that if $f_\phi\equiv 0$ (say as a function of x in the above formula), then $W_\phi\equiv 0$ since Fourier transform is an isomorphism.

Lastly, we observe

$$W_\phi\big(\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)\big) = \mu_2(a)\cdot|a|_F^{1/2}\cdot\int_F\mu_1(t)\mu_2(t)^{-1}|t|_F^{-1}\int_F\phi\begin{pmatrix}-a^{-1}y\\-t\end{pmatrix}\psi(y/t)\,dydt$$

$$= \mu_2(a)\cdot|a|_F^{-1/2}\cdot\int_F\mu_1(t)\mu_2(t)^{-1}|t|_F^{-1}\int_F\phi\begin{pmatrix}-y\\-t\end{pmatrix}\psi(ay/t)dydt$$

again has exponential decay as $a\to\infty$. This verifies (1), and proves the existence of Whittaker model for infinite-dimensional irreducible admissible $(\mathfrak{g},K)$-modules.

It remains to prove that the Whittaker model is unique when it exists. We have $G = ZUAK$ where Z is the center, U is the group of strictly upper triangular matrices, and

$$A = \left\{ \left(\begin{smallmatrix} a^{1/2} & 0 \\ 0 & a^{-1/2} \end{smallmatrix} \right) \mid a \in \mathbf{R}_+ \right\}$$

Thus a Whittaker model $v \mapsto W_v$ is determined by $w_v(a) := W_v\left(\left(\begin{smallmatrix} a^{1/2} & 0 \\ 0 & a^{-1/2} \end{smallmatrix} \right) \right)$ for $a \in \mathbf{R}_+$. One uses the classification of irreducible admissible $(\mathfrak{g}, K)$-modules to write down some explicit basis $\{$various $v\}$, and write down the corresponding differential equation for $w_v(a)$. The point is that the Casimir operator will give a second-order differential equation for $w_v(a)$. For example, when $F = \mathbf{R}$ and $v = v_n$ is the vector in π on which K_1 acts by weight n, the differential equation is of the form

$$w_v''(a) = \left(\frac{s^2 - 1}{4a^2} - \frac{2n\pi}{a} + 4\pi^2 \right) w_v(a).$$

The solutions are the classical Whittaker functions (actually studied by Whittaker). The point is then that as there is no first-order term, if one solutions decays exponentially then any other has to grow exponentially, and thus not giving a Whittaker model. To be precise, the matrix describing the ODE

$$\begin{pmatrix} w_v''(a) \\ w_v'(a) \end{pmatrix} = \begin{pmatrix} 0 & * \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_v'(a) \\ w_v(a) \end{pmatrix}$$

has trace zero. If w_1, w_2 are two linearly independent solutions of the ODE, then

$$\frac{d}{da} \left(\begin{smallmatrix} w_1'(a) & w_2'(a) \\ w_1(a) & w_2(a) \end{smallmatrix} \right) = \left(\begin{smallmatrix} 0 & * \\ 1 & 0 \end{smallmatrix} \right) \left(\begin{smallmatrix} w_1'(a) & w_2'(a) \\ w_1(a) & w_2(a) \end{smallmatrix} \right)$$

and thus the Wronskian $\det \left(\begin{smallmatrix} w_1'(a) & w_2'(a) \\ w_1(a) & w_2(a) \end{smallmatrix} \right) =: \det W$ is constant in a, since $\frac{d}{da} \det W = \operatorname{tr} \left(\begin{smallmatrix} 0 & * \\ 1 & 0 \end{smallmatrix} \right) \det W$. This proves the uniqueness, via the uniqueness theorem for solutions to linear ODE. When $F = \mathbf{C}$ the ODE is more complicated (see [2, §6].) and the trace function will not be identically zero, but it will be a function with a zero at ∞ which will imply that $\det \left(\begin{smallmatrix} w_1'(a) & w_2'(a) \\ w_1(a) & w_2(a) \end{smallmatrix} \right)$ has at most polynomial growth, and thus still the result we need.

References

[1] Godement, R. *Notes on Jacquet-Langlands' Theory*, The Institute for Advanced Study, 1970, available at http://math.stanford.edu/ conrad/conversesem/refs/godement-ias.pdf.

[2] Jacquet, H.; Langlands, R.P. Automorphic forms on $\mathrm{GL}(2)$. Lecture NOtes in Mathematics, Vol. 114. *Springer-Verlag, Berlin-New York*, 1970.