LECTURE 2: COMPACTNESS AND VOLUME FOR ADELIC COSET SPACES

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OCTOBER 11, 2017
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Let k be a global field with adele ring $\mathbf{A}$, G a connected reductive k-group, Z a maximal central k-torus, and $\mathscr{D}G$ the derived group of G (equal to the commutator $[G(\overline{k}), G(\overline{k})]$ on the level of $\overline{k}$-points, but not necessarily for k-points). Multiplication defines a central k-isogeny:

$$Z \times \mathscr{D}G \to G$$

Since this is an isogeny, this is surjective with finite kernel on $\overline{k}$-points, but over the ring $\mathbf{A}$ this is very far from true. Indeed, we'll see that the image is very "thin" in the topological space $G(\mathbf{A})$, and that the kernel on $\mathbf{A}$-points is compact but usually infinite. For example, $\mu_n(\mathbf{A})$ is certainly infinite!

For example, let $G = \mathrm{GL}_n$. Then we have $Z = \mathbf{G}_m$, i.e. the group of invertible diagonal matrices. Furthermore $G = \mathbf{G}_m \ltimes (\mathscr{D}G)$ and we therefore have $G/\mathscr{D}G \xrightarrow{\sim} \mathbf{G}_m$ via the determinant map. But the map $Z(\mathbf{A}) \to G(\mathbf{A})/(\mathscr{D}G)(\mathbf{A})$ is the map $\mathbf{A}^\times \to \mathbf{A}^\times$ sending t to t^n; so the kernel is $\mu_n(\mathbf{A})$ and the image is the set of n-th powers, which is certainly "thin".

Remark 1. For a *parabolic* k-subgroup $P \subseteq G$, it is however always true that $G(\mathbf{A})/P(\mathbf{A}) \to (G/P)(\mathbf{A})$ is a homeomorphism, and the target is *compact*. Note that here we're using the fact that we have an adelic topology on $X(\mathbf{A})$ for non-affine schemes X. Roughly, this works because the same is true on field-valued points, which is part of the structure theory of reductive groups.

Note that in general, if $H \subseteq G$ is a subgroup, we have an injection of groups $G(R)/H(R) \hookrightarrow (G/H)(R)$ for all k-algebras R, but this is often far from surjective, due to a cohomological obstruction. Sometimes, we can show this obstruction vanishes at least when R is a field via Hilbert's Theorem 90. For example, this occurs if H is a split torus (which easily follows from the classical Hilbert 90, which is for $H = \mathbf{G}_m$).

In automorphic calculations, one often uses the adelic double coset space:

$$G(k)\backslash\big(G(\mathbf{A})/Z(\mathbf{A})\big)$$

Does this have a nice topology or measure? A basic question, discussed below, is this: does $G(k)$ have discrete image in $G(\mathbf{A})/Z(\mathbf{A})$?

Lemma 2. A connected reductive group G is "algebraically unimodular", which means that the left-invariant global top-degree differential forms are also right invariant. Equivalently, the algebraic modulus character $\chi_G : G \to \mathbf{G}_m$ is trivial[1]. This implies (with additional work, such as the analytic inverse function theorem and formulation of the relationship between measures and differential forms over local fields), that for global fields k the topological group $G(\mathbf{A})$ is unimodular in the topological group sense (i.e. the left and right Haar measures agree), and that the same is true for $G(k_v)$ for any place v of k.

[1] See [1, §4.2] for the definition and properties of the invariant differential forms and the modulus character for algebraic groups.

Proof. This follows from the fact that $G = Z \cdot \mathscr{D}G$ with Z central and $\mathscr{D}G$ killed by any character, so $\chi_G|_Z$ and $\chi_G|_{\mathscr{D}G}$ both vanish and thus χ_G vanishes. $\square$

Remark 3. There is a *canonical* measure on $G(\mathbf{A}_k)$, called the *Tamagawa measure*. How does this work? Via an invariant top-degree differential form G, we get a measure on $G(k_v)$ for each local field v; via the implicit function theorem for k_v-analytic manifolds this measure is induced from the measure on k_v determined by the normalized valuation. But in order to build a measure on $G(\mathbf{A}_k)$, we need to pick normalizing factors to make the infinite product converge. We can try to do this via an integral structure of G over $\mathscr{O}_{k,S}$, normalizing to make the measure of $G(\mathscr{O}_{k_v})$ equal to 1, but this is not sufficiently canonical (for example, G does not have a preferred smooth model over some $\mathscr{O}_{k,S}$) to expect it to have good properties. The actual construction is more complicated. In any case, for our purposes, it suffices to work with any given Haar measure.[2]

Up to scaling, there is a unique right-invariant measure satisfying reasonable regularity properties on any locally compact Hausdorff topological group, a construction of which for $G(\mathbf{A})$ was described above. See [5]. This induces a measure on $G(k)\backslash G(\mathbf{A})/Z(\mathbf{A})$ in accordance with (a version of) Fubini's Theorem[3]. What can we say about this? Does it have finite volume? Is it compact?

Lemma 4. The image of the natural map $G(k) \to G(\mathbf{A})/Z(\mathbf{A})$ is discrete.

Proof. Consider the diagram

$$\begin{tikzcd} G(k)/Z(k) \arrow[r] \arrow[d, hook] & G(\mathbf{A})/Z(\mathbf{A}) \arrow[d, "\alpha"] \\ (G/Z)(k) \arrow[r] & (G/Z)(\mathbf{A}) \end{tikzcd}$$

It will be enough to show that α is a closed embedding, since $(G/Z)(k)$ is discrete in $(G/Z)(\mathbf{A})$ (this just follows from discreteness of k in $\mathbf{A}$ via the definition of the topology on the adelic points of G/Z).

By some general considerations about topological group actions, as developed in Bourbaki, it suffices to show that the image of α is closed. To do this, we can first spread out the situation to $\mathscr{Z} \hookrightarrow \mathscr{G}$ with $\mathscr{Z}, \mathscr{G}$ each $\mathscr{O}_{k,S}$-group schemes with connected fibers.

Now, we have, as in Weil's approach to defining the adelic topology on adelic points of schemes:

$$(G/Z)(\mathbf{A}) = \bigcup_{S' \supseteq S} \left(\prod_{v \in S'} (G/Z)(k_v) \times \prod_{v \in S'} (\mathscr{G}/\mathscr{Z})(\mathscr{O}_v) \right)$$

We can apply a theorem of Lang about the existence of k_v rational points, plus the fact that $\mathscr{O}_v$ is henselian to see that

$$\prod_{v \in S'} (\mathscr{G}/\mathscr{Z})(\mathscr{O}_v) = \prod_{v \notin S'} \mathscr{G}(\mathscr{O}_v)/\mathscr{Z}(\mathscr{O}_v)$$

[2]For discussion and proofs of some of these statements, look at [8, Lecture 3], which in turn draws from [6].

[3]See [4]. However, Lang's version of "coset Fubini" is for continuous compactly supported functions; typically, one is interested in more general L^1 functions, so you need to take a limit to extend the result to this case.

If $\xi = (\xi_v)_v \in (G/Z)(\mathbf{A}_k) \subseteq \prod_v (G/Z)(k_v)$ has $\xi_v \in \mathrm{im}(G(k_v))$ for all v, then ξ is in the image of $G(\mathbf{A}_k)$. Thus, it suffices to show that $G(k_v) \to (G/Z)(k_v)$ has closed image for each v. Now, since Z is smooth, $G \to G/Z$ is smooth, so by the Zariski-local structure theorem for smooth maps and the analytic inverse function theorem, $G(k_v) \to (G/Z)(k_v)$ is a k_v-analytic submersion. In particular, the image is an open subgroup and therefore closed. □

Let $S \subseteq Z$ be a maximal k-split sub-torus, which is equivalently described by the maximal split central k-torus in G. Then we can define $\overline{G} := G/S$, and its maximal central torus is $\supseteq Z/S =: \overline{Z}$, which is anisotropic. Since this might not be trivial, G is not semisimple yet, but at least it has no characters: as we saw before, characters of a connected reductive group are the same thing as characters of its central torus, and anisotropic tori have no characters. In other words, $X_k(\overline{G}) = \{1\}$.

By applying Hilbert's Theorem 90 plus some additional care, we can see that:

$$G(k)\backslash G(\mathbf{A})/Z(\mathbf{A}) \simeq \overline{G}(k)\backslash\overline{G}(\mathbf{A})/\overline{Z}(\mathbf{A})$$

and this isomorphism respects the topology and the measures. In order to see why this should be true, note that $Z(\mathbf{A}) \supseteq S(\mathbf{A}) \supseteq S(k)$.

Here is a useful feature of the anisotropic situation with G replaced by $\overline{G}$:

Proposition 5. $\overline{Z}(k)\backslash\overline{Z}(\mathbf{A})$ is *compact*.

This follows from the fact that $\overline{Z}$ is k-anisotropic. To show this for any k-anisotropic torus, one shows that the compactness is unaffected by passing to a k-isogenous torus, and then use arguments with Galois lattices reduce to the case where the anisotropic torus is the kernel of the norm map $N_{k'/k} : R_{k'/k}(\mathbf{G}_m) \to \mathbf{G}_m$ for a finite separable extension k'/k.

Since $\overline{Z}(k)\backslash\overline{Z}(\mathbf{A})$ is compact, the volume-finiteness of $G(k)\backslash G(\mathbf{A})/Z(\mathbf{A})$ is equivalent to the volume-finiteness of $\overline{G}(k)\backslash\overline{G}(\mathbf{A})$.

Example 6. Let $G = \mathrm{SL}_n$ and $k = \mathbf{Q}$. Since SL_n is semisimple, $G = \overline{G}$ here. Does $\mathrm{SL}_n(\mathbf{Q})\backslash\mathrm{SL}_n(\mathbf{A})$ have finite volume? Since $\mathrm{SL}_n(\widehat{\mathbf{Z}})$ is compact and open in the finite adelic part $\mathrm{SL}_n(\mathbf{A}_f)$, the volume-finiteness of this space amounts to showing the volume-finiteness of $\mathrm{SL}_n(\mathbf{Q})\backslash\mathrm{SL}_n(\mathbf{A})/\mathrm{SL}_n(\widehat{\mathbf{Z}})$, which is isomorphic to $\mathrm{SL}_n(\mathbf{Z})\backslash\mathrm{SL}_n(\mathbf{R})$. We discussed this last bijection last time via strong approximation for SL_n; it is easily seen to be a homeomorphism, and respects the measures of interest up to a scaling factor.

In [7, Theorem 10.4] one finds the classical volume computation of the "$\mathbf{Z}$-structure measure" of $\mathrm{SL}_n(\mathbf{Z})\backslash\mathrm{SL}_n(\mathbf{R})$ as $V = \zeta(2)\zeta(3)\cdots\zeta(n)$.

The volume-finiteness for the double-coset spaces will be useful in many places later on: for example, on a space with finite measure, any L^2 function is L^1. In order to avoid the problems arising from the presence of non-trivial characters, we'll introduce the following variant of $G(\mathbf{A})$:

Definition 7.

$$G(\mathbf{A})^1 = \Big\{ g \in G(\mathbf{A}) \mid \big|\chi(g)\big| = 1 \;\forall \chi \in X_k(G) \Big\}$$

Note that $G(k) \subseteq G(\mathbf{A})^1$ by the product formula. In addition, the formation of $G(\mathbf{A})^1$ is *functorial* in G: if $G_1 \to G_2$ is a homomorphism, then any character of G_2 pulls back to a character of G_1.

Example 8.

- If $X_k(G) = 1$, then $G(\mathbf{A})^1 = G(\mathbf{A})$, so in particular $\overline{G}(\mathbf{A})^1 = \overline{G}(\mathbf{A})$.
- On the other extreme, if $G = \mathbf{G}_m$, note that $G(k)\backslash G(\mathbf{A})$ is certainly non-compact due to the fact that the norm map has image $\mathbf{R}_+$. However, it's a classical theorem of algebraic number theory that $G(k)\backslash G(\mathbf{A})^1 = k^\times\backslash(\mathbf{A}^\times)^1$, i.e. the group of norm-one idéles, is compact.

Proposition 9. If $X_k(G) \neq 1$, then $G(k)\backslash G(\mathbf{A})$ has infinite volume.[4]

Proof. The existence of characters implies that we have a non-trivial maximal central split k-torus $S \subseteq G$. Let $\overline{G} = G/S$, so

$$G(k)\backslash G(\mathbf{A}) \twoheadrightarrow \overline{G}(k)\backslash\overline{G}(\mathbf{A})$$

is a fibration with fibers isomorphic to $S(k)\backslash S(\mathbf{A})$[5], and $S(k)\backslash S(\mathbf{A}) = \left(k^\times\backslash\mathbf{A}^\times\right)^n$ for some n, which has infinite volume. This fibration is compatible with the algebraic invariant differentials, since $\det T_e(G) = \det T_e(\overline{G}) \otimes \det T_e(S)$. Thus, we can run a Fubini argument to show that the total space has infinite volume. In other words, if $1 \in L^1\big(G(k)\backslash G(\mathbf{A})\big)$ then we can deduce that $1 \in L^1\big(S(k)\backslash S(\mathbf{A})\big)$, which is a contradiction. □

Now, we want to focus on $[G] := G(k)\backslash G(\mathbf{A})^1$ and characterize when this has finite volume or is compact. Note that formation of $[G]$ is functorial in G since $G(\mathbf{A})^1$ is. This helps with our original problem, since $[\overline{G}] = \overline{G}(k)\backslash\overline{G}(\mathbf{A})^1$. So if we show that $[\overline{G}]$ has finite volume or is compact, the same is true for $G(k)\backslash G(\mathbf{A})/Z(\mathbf{A})$.

Theorem 10. For $H_1 \hookrightarrow H_2$ a closed immersion of linear algebraic k-groups, then $[H_1] \to [H_2]$ is a closed embedding.

Returning to our "puzzle" from the end of the last lecture, let's show that we can't drop the $(\cdot)^1$ aspect:

Proposition 11. Let B be a Borel k-subgroup of PGL_2. The natural continuous injective map $B(k)\backslash B(\mathbf{A}) \to \mathrm{PGL}_2(k)\backslash\mathrm{PGL}_2(\mathbf{A})$ is not a topological embedding.

It is a purely formal consequence of general facts concerning orbits of locally compact Hausdorff groups on locally compact Hausdorff spaces (see [2, Theorem 4.2.1]) that failure to be a topological embedding forces the image to be non-closed, but in Remark 12 we show more directly that the image of this map is also not closed.

Proof. Using inversion on adelic points, it is the same to show that the natural continuous map

$$B(\mathbf{A})/B(k) \to \mathrm{PGL}_2(\mathbf{A})/\mathrm{PGL}_2(k)$$

is not a topological embedding. Define $B(\mathbf{A}) \times^{B(k)} \mathrm{PGL}_2(k)$ to be the topological quotient of $B(\mathbf{A}) \times \mathrm{PGL}_2(k)$ (where $\mathrm{PGL}_2(k)$ is given the discrete topology) by the equivalence relation

[4]This implies that it's non-compact, but it's better to give a direct proof of non-compactness.

[5]This is plausible but not obvious, since this coset space is not a group! See [2, Lemma A.2.1]

$(xb, y) \sim (x, b^{-1}y)$ for $b \in B(k)$ and $(x, y) \in B(\mathbf{A}) \times \mathrm{PGL}_2(k)$. It is easy to check via the discreteness of the topology on $\mathrm{PGL}_2(k)$ that the commutative diagram of continuous maps

$$\begin{tikzcd} B(\mathbf{A}) \times^{B(k)} \mathrm{PGL}_2(k) \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & \mathrm{PGL}_2(\mathbf{A}) \arrow[d] \\ B(\mathbf{A})/B(k) \arrow[r] & \mathrm{PGL}_2(\mathbf{A})/\mathrm{PGL}_2(k) \end{tikzcd}$$

(with the top map induced by multiplication) is cartesian; i.e., the upper left term is identified with the topological fiber product along the bottom and right maps. Hence, if the bottom map *were* a topological embedding then the top map would be too.

It therefore suffices (for an argument by contradiction) to show that the natural map

$$B(\mathbf{A}) \times^{B(k)} \mathrm{PGL}_2(k) \to \mathrm{PGL}_2(\mathbf{A})$$

is not a topological embedding. Suppose this were a topological embedding. Then the same would hold for the quotient throughout by the left $B(\mathbf{A})$-action, which is exactly the map

$$B(k)\backslash\mathrm{PGL}_2(k) \to B(\mathbf{A})\backslash\mathrm{PGL}_2(\mathbf{A})$$

where the left side is discrete. But the target is topologically $\mathbf{P}^1(\mathbf{A})$ by [3, Theorem 4.5] (going beyond the affine setting!), so we're reduced to checking that the natural inclusion

$$\mathbf{P}^1(k) \to \mathbf{P}^1(\mathbf{A})$$

with countably infinite discrete source and compact Hausdorff target is not a topological embedding. In general a compact Hausdorff space *can* have infinite non-closed subsets whose subspace topology is the discrete topology, such as $\{1, 1/2, 1/3, 1/4, \dots\}$ inside $[0, 1]$, so to get a contradiction we need more information about our specific situation. Fortunately, [3, Example 4.3] provides what we need: it gives a proof that $\mathbf{P}^n(k)$ is dense in $\mathbf{P}^n(\mathbf{A})$ (using S-integral weak approximation for adele rings), and that argument shows more specifically that any open around any point of $\mathbf{P}^n(\mathbf{A})$ contains infinitely many points of $\mathbf{P}^n(k)$. Thus, the subspace topology on $\mathbf{P}^n(k)$ is not the discrete topology. $\square$

Remark 12. The non-closedness of the image of the map in the preceding proposition can be seen directly as follows. We want to show that $\mathrm{PGL}_2(k) \cdot B(\mathbf{A}) \to \mathrm{PGL}_2(\mathbf{A})$ has non-closed image. Since $\mathbf{P}^1(k)$ is not closed (it is even dense) in $\mathbf{P}^1(\mathbf{A})$, we can pick a sequence $\{\xi_n\}$ in $\mathbf{P}^1(k)$ converging to $\xi \in \mathbf{P}^1(\mathbf{A}) - \mathbf{P}^1(k)$. Pick $g \in \mathrm{PGL}_2(\mathbf{A})$ over ξ and $\gamma_n \in \mathrm{PGL}_2(k)$ over ξ_n. Note that $g \notin \mathrm{PGL}_2(k) \cdot B(\mathbf{A})$ since $\xi \notin \mathbf{P}^1(k)$. Since $\mathbf{P}^1(\mathbf{A}) = \mathrm{PGL}_2(\mathbf{A})/B(\mathbf{A})$ with the quotient topology, we can find points $b_n \in B(\mathbf{A})$ such that $g_n b_n \to g$ as $n \to \infty$. This exhibits explicitly that $\mathrm{PGL}_2(k) \cdot B(\mathbf{A})$ is not closed in $\mathrm{PGL}_2(\mathbf{A})$.

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[8] `http://math.stanford.edu/~conrad/BSDseminar/`. Webpage from Stanford's 2015-2016 Number Theory Learning Seminar, with notes contributed from various participants.