WHITTAKER MODELS AND MULTIPLICITY ONE

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Contents

1. Cuspidal automorphic representations

Let k be a global field and G a connected reductive group over k. Recall that the cuspidal L^2-spectrum of G decomposes discretely:

Theorem 1.1 ([Howe1], [Howe2]). *There is a decomposition*

$$L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbf{A}_k), \omega) \cong \widehat{\bigoplus_i} V_i.$$

with the V_i being topologically irreducible, closed subrepresentations, with each isomorphism type appearing with finite multiplicity.

One of the main goals of this talk is to prove that for $G = \mathrm{GL}_2$, each isomorphism type appears with *multiplicity one.*

Theorem 1.2 (Multiplicity One for L^2). *Every irreducible component of $L^2_{\mathrm{cusp}}(\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbf{A}_k), \omega)$ has multiplicity one, i.e. is not isomorphic to any other irreducible component.*

We will approach this theorem by passing to the associated admissible automorphic representation. Recall that $\mathcal{A}(G, \omega)$ denotes the space of *automorphic forms* on G, meaning smooth functions $\phi \colon G(\mathbf{A}) \to \mathbf{C}$ such that:

(1) ϕ has central character ω,
(2) ϕ is right K-finite,
(3) ϕ is $Z(U(\mathfrak{g}))$-finite,
(4) ϕ has moderate growth at the cusps.

We let $\mathcal{A}_{\mathrm{cusp}}(G, \omega) := \mathcal{A}(G, \omega) \cap L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbf{A}), \omega)$.

Theorem 1.3 ([Ngo] Theorem 5.4.4). *Let π be an irreducible $G(\mathbf{A})$-invariant subspace of $L^2(G(F)\backslash G(\mathbf{A}), \omega)$. Then $\pi \cap \mathcal{A}(G, \omega)$ is an irreducible admissible $G(\mathbf{A})$-representation.*

Remark 1.4. Recall that to say that $\pi \cap \mathcal{A}(G,\omega)$ is a "$G(\mathbf{A})$-representation" is really an abuse of notation. Really it is a module over the global Hecke algebra, so it has an action of $G(k_v)$ for every non-archimedean v and the structure of a $(\mathfrak{g}, K)$-module at the archimedean places.

It is easy to deduce from this that $\pi \mapsto \pi \cap \mathcal{A}(G,\omega)$ defines a bijection between irreducible summands of the Hilbert space $L^2_{\text{cusp}}(\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbf{A}_k),\omega)$ and irreducible summands of $\mathcal{A}_{\text{cusp}}(G,\omega)$, whose inverse is obtained by taking the closure. Theorem 1.2 then follows from:

Theorem 1.5 (Multiplicity One)**.** *Let (π, V) and (π', V') be two irreducible admissible subrepresentations of $\mathcal{A}_{\text{cusp}}(\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbf{A}_k),\omega)$. If $\pi \cong \pi'$, then $V = V'$.*

We will in fact prove a stronger version of Theorem 4.1, namely that it is enough to have *local* isomorphisms at all but finitely many places. To formulate this, recall Flath's Theorem on decomposing irreducible admissible representations as a restricted tensor product:

Theorem 1.6 (Flath)**.** *Let (π, V) be an irreducible admissible automorphic representation of $G(\mathbf{A}_k)$. Then we have*

$$(\pi, V) \cong \bigotimes_v{}' (\pi_v, V_v)$$

for irreducible admissible representations (π_v, V_v) of $G(k_v)$.

Theorem 1.7 (Strong Multiplicity One)**.** *Let (π, V) and (π', V') be two irreducible admissible subrepresentations of $\mathcal{A}_0(\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbf{A}_k),\omega)$. Assume that $\pi_v \cong \pi'_v$ for all but finitely many non-archimedean v. Then $V = V'$.*

In this talk, we will only prove Theorem 1.7 under the hypothesis that $\pi_v \cong \pi'_v$ at all archimedean places.

Remark 1.8. For almost all v the representation π_v will be unramified, hence described in a straightforward way from its Hecke eigenvalues.

Remark 1.9. These theorems remain true with GL_2 replaced by GL_n. However, they fail in greater generality (e.g. for $G = \mathrm{Sp}_{2g}$).

These notes are concerned with the proof of Theorem 1.7, and will proceed as follows.

(1) We will begin by introducing the notion of the "Fourier expansion" of an automorphic form. We will find that the "coefficients" of the Fourier expansion define a certain "Whittaker functional", which is a concept that you already met a long time ago.
(2) We will state a local uniqueness theorem for Whittaker functionals, and deduce a global uniqueness theorem.
(3) We will conclude Theorem 4.1 from the uniqueness of global Whittaker functionals. We will indicate partial progress towards Theorem 4.1.
(4) We will return to discuss the proof of the uniqueness of local Whittaker functionals in the non-archimedean case (the archimedean case having been handled already in [Tsai1]).

2. The Fourier expansion of a cusp form

Let π be a cuspidal automorphic representation for GL_2. Fix $g \in \mathrm{GL}_2(\mathbf{A})$ and $\varphi \in \pi$, and consider the function

$$\varphi_g(x) := \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) : k\backslash \mathbf{A}_k \to \mathbf{C}.$$

By Fourier analysis on locally compact abelian groups, noting that $k\backslash\mathbf{A}_k$ is actually compact, we have

$$\varphi_g(x) = \sum_{\psi \in \widehat{k\backslash\mathbf{A}_k}} \varphi_{N,\psi}(g)\psi(x), \tag{2.1}$$

where

$$\widehat{\varphi}_{N,\psi}(g) := \int_{k\backslash\mathbf{A}_k} \varphi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right)\psi(-u)du.$$

We can simplify this as follows. By fixing a basepoint ψ_0, we obtain an isomorphism $\widehat{k\backslash\mathbf{A}_k} \cong k$ by identifying $\lambda \in k$ with the character $\psi_\lambda \colon x \mapsto \psi_0(\lambda x)$. (A good analogy is that $k\backslash\mathbf{A}_k \cong \mathbf{Z}\backslash\mathbf{R}$.) In these terms, we have

$$\varphi_{N,\lambda\psi_0}(g) = \int_{k\backslash\mathbf{A}_k} \varphi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right)\psi_0(-\lambda u)du.$$

The cuspidality of φ implies that the "constant" Fourier coefficient $\varphi_{N,1}$, corresponding to the trivial character, vanishes. Therefore, we may restrict our attention to $\lambda \neq 0$. Then we may we implement a change of variables $\lambda u \mapsto u$. Since $\lambda \in k$, the product formula implies that this actually preserves the Haar measure. It takes the argument of φ to

$$\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} \mapsto \begin{pmatrix}1 & u\lambda^{-1}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\lambda^{-1} & \\ & 1\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda & \\ & 1\end{pmatrix}.$$

Hence by left invariance of φ, we have

$$\varphi_{N,\lambda\psi_0}(g) = \int_{k\backslash\mathbf{A}_k} \varphi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right)\psi_0(-\lambda u)du = \int_{k\backslash\mathbf{A}_k} \varphi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda & \\ & 1\end{pmatrix} g\right)\psi_0(-u)du.$$

Setting $x = 0$ in (2.1), we have proved:

Theorem 2.1. *For a cuspidal automorphic form φ, we have the Fourier expansion*

$$\varphi(g) = \sum_{\lambda \in k^\times} \int_{k\backslash\mathbf{A}_k} \varphi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda & \\ & 1\end{pmatrix} g\right)\psi_0(-u)du.$$

Remark 2.2. What happens in general? We used a special feature of GL_2 here, namely that the unipotent radical N of a Borel is *abelian*. We used this in applying Fourier analysis, which only detects functions on abelian groups. In general, you might expect that the Fourier coefficients of φ only carry information about the abelianization of N.

However, for GL_n this can be salvaged using the *mirabolic subgroup*

$$P_n = \begin{pmatrix}\boxed{\mathrm{GL}_{n-1}} & *\\ 0 & 1\end{pmatrix} \subset \mathrm{GL}_n,$$

which can be viewed as the stabilizer of a vector in the standard representation. We inductively prove that

$$f(g) = \sum_{\gamma \in N_{n-1}(\mathbf{Q})\backslash \mathrm{GL}_{n-1}(\mathbf{Q})} f_{N,\psi_0}\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right).$$

Note that $P_n = GL_{n-1} \times U_n$. By Fourier analysis on U_n, we have

$$f(g) = \sum_{\gamma \in P_{n-1}(\mathbf{Q})\backslash \mathrm{GL}_{n-1}(\mathbf{Q})} f_{U_n,\psi_0}(\gamma g). \tag{2.2}$$

Now, viewing $\gamma \mapsto f_{U_n,\psi_0}(\gamma g)$ as a function on $\mathrm{GL}_{n-1}(\mathbf{A})$, we can apply the induction hypothesis to conclude that

$$f_{U_n,\psi_0}(\gamma g) = \sum_{\eta \in N_{n-2}(\mathbf{Q})\backslash \mathrm{GL}_{n-2}(\mathbf{Q})} (f_{U_n,\psi_0})_{N_{n-1},\psi_0}(\eta\gamma g). \tag{2.3}$$

Combining (2.2) and (2.3), we find that

$$\begin{aligned} f(g) &= \sum_{\gamma \in P_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} \sum_{\eta \in N_{n-2}(k)\backslash \mathrm{GL}_{n-2}(k)} \int_{[U_n]\times[N_{n-1}]} f(u\eta\gamma g)\psi_0(-u)\,du \\ &= \sum_{\gamma \in N_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} \int_{N_n} f(u\gamma g)\psi_0(-u)\,du \end{aligned}$$

Let's summarize what has happened in this section. We have associated to $\varphi \in \pi$ and $g \in G(\mathbf{A})$ a "Fourier coefficient"

$$\varphi_{N,\lambda\psi_0}(g) := \int_{k\backslash \mathbf{A}_k} \varphi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda & \\ & 1 \end{pmatrix} g\right)\psi_0(-u)du.$$

Focus on the case $\lambda = 1$. We can view $\varphi \mapsto \varphi_{N,\psi_0}$ as a map $V \to \mathrm{Fun}(G(\mathbf{A}), \mathbf{C})$ which has the property that replacing g by $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g$ transforms it as

$$\begin{aligned} \varphi_{N,\psi_0}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) &= \int_{k\backslash \mathbf{A}_k} \varphi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\psi_0(-u)du \\ &= \int_{k\backslash \mathbf{A}_k} \varphi\left(\begin{pmatrix} 1 & x+u \\ 0 & 1 \end{pmatrix} g\right)\psi_0(-x-u)\psi_0(x)du \\ &= \psi_0(x)\varphi_{N,\psi_0}(g) \end{aligned}$$

In other words, we obtain a map

$$V \to \mathrm{Fun}_{N(\mathbf{A})}(G(\mathbf{A}), (\mathbf{C}, \psi_0)).$$

Definition 2.3. Let ψ be a nondegenerate character of $N(\mathbf{A})$. A *(global) Whittaker model* for V is a subrepresentation of $\mathrm{Fun}_{N(\mathbf{A})}(G(\mathbf{A}), (\mathbf{C}, \psi))$ which is isomorphic to V, consisting of functions f with moderate growth, i.e. such that

$$f\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) = O(|x|^{-T}) \text{ for all } T.$$

Note that this definition forces the Whittaker model to also have central character ω.

To check that $\varphi \mapsto \varphi_{N,\psi}$ defines a Whittaker model, it only remains to explain why $\varphi_{N,\psi}$ has moderate growth. But since $\varphi_{N,\psi}$ is obtained from φ by integrating against a character over a compact set, this follows immediately from the fact that φ has moderate (even rapid, since we assumed cuspidality) growth.

Corollary 2.4 (Global existence of Whittaker models)**.** *A cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbf{A})$ *has a Whittaker model.*

Remark 2.5. We emphasize again that this fact is specific to GL_n.

Remark 2.6. Note that by Frobenius reciprocity, a Whittaker model induces an $N(\mathbf{A})$-equivariant map $V \to (\mathbf{C}, \psi)$.

3. Uniqueness of Whittaker models

3.1. **Non-archimedean case.** We'll recall some results stated earlier, and defer their proofs to later. In this section K is a non-archimedean local field. Fix a non-trivial additive character ψ of K, and view it as a character of $N(K)$ in the obvious way. We will choose a model for the induced representation $\mathrm{Ind}_{N(K)}^{G(K)}\psi$:

$$\mathrm{Ind}_{N(K)}^{G(K)}\psi = \{f \in C^\infty(G(K), \mathbf{C}) \mid f(ng) = \psi(n)g\}.$$

Definition 3.1. Let (π, V) be an irreducible admissible representation of $G(K)$. A *(local) Whittaker model* for π is subspace of $\mathrm{Ind}_{N(K)}^{G(K)}\psi$ which is isomorphic to π.

Remark 3.2. A Whittaker model is the same as a non-zero map in

$$\mathrm{Hom}_{G(K)}(V, \mathrm{Ind}_{N(K)}^{G(K)}\psi) = \mathrm{Hom}_{N(K)}(V, \psi)$$

by Frobenius reciprocity. An element of the latter space is called a *Whittaker functional.*

Theorem 3.3 (Local uniqueness of Whittaker models)**.** *Let (π, V) be an irreducible admissible representation of $G(K)$. Then (π, V) has at most one (local) Whittaker model, i.e. any two Whittaker functionals on V are proportional.*

Remark 3.4. This result holds in great generality (i.e. whenever it makes sense).

Definition 3.5. Let (π, V) be an irreducible admissible representation of $G(K)$. We say that (π, V) is *generic* if it admits a local Whittaker model.

3.2. **Archimedean case.** In the archimedean case, it is necessary to pose some growth conditions. Let (π, V) be an irreducible admissible representation of $\mathcal{H}_G(K)$.

Definition 3.6. A *Whittaker* model is a space of functions

$$\{W_\xi \colon G(K) \to \mathbf{C} \colon \xi \in V\}$$

such that

(1) $W_\xi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \psi(x)W_\xi(g)$,
(2) Each W_ξ is C^∞ and $W_{\pi(X)\xi} = W_\xi * \check{X}$.
(3) For each $\xi \in V$, there exists T such that

$$W_\xi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = O(|a|^T) \text{ as } |a| \to \infty.$$

Theorem 3.7 (Local uniqueness of Whittaker models)**.** *Let (π, V) be an irreducible admissible representation of $G(K)$. Then (π, V) has at most one (local) Whittaker model, i.e. any two Whittaker functionals on V are proportional.*

Proof. The proof was covered in [Tsai1]. □

3.3. **Global uniqueness.** Using local uniqueness for Whittaker models, we can deduce global uniqueness.

Theorem 3.8 (Global uniqueness for Whittaker models)**.** *Let (π, V) be an irreducible cuspidal automorphic representation of GL_2. If (π, V) has a Whittaker model, then it is unique.*

Proof. Let W be a Whittaker model for (π, V), so W is generated by W_ξ for $\xi \in V$. With respect to a decomposition $V \cong \bigotimes' V_v$, we may assume that there exists a pure tensor $\xi^0 = (\otimes \xi_v^0)$ such that $W_{\xi^0}(e) \neq 0$ since $W_{\xi^0}(g) = W_{g\cdot\xi^0}(e)$. (This requires a bit of care at the Archimedean places, since there we do not get an action of $G(k_\infty)$.)

For $\xi_S = (\xi_v)_{v\in S} \in \otimes_{v\in S} V_v$ define $\iota_S(\xi_S) = (\xi_v) \otimes (\xi^0_{w\neq v}) \in V$. Similarly, for $g_S = (g_v)_{v\in S} \in \prod_{v\in S} G(k_v)$ define $\iota_S(g_S) \in G(\mathbf{A}_k)$.

Define a representation W_v of $G(k_v)$ which is generated by $W_{\xi_v}(g_v) := W_{\iota(\xi_v)}(\iota_v(g_v))$ as ξ_v ranges over V_v. This is easily checked to be a Whittaker model for V_v. By our normalization, we have $W_{\xi_v^0}(1) = 1$. Then the restricted tensor product $\otimes' W_v$ is defined (with respect to the $W_{\xi_v^0}$), and we claim that

$$\prod W_{\xi_v^0}(g_v) = W_{\xi^0}(g). \tag{3.1}$$

We first prove the claim for all $g \in \mathrm{GL}_2(\mathbf{A}_{k,S})$ by induction on $|S|$. Picking $w \in S$ and fixing ξ_v^0, g_v for $v \notin S$, both sides define Whittaker models for V_w (as ξ_w varies and the result is viewed as a function of g_w), hence the functions in question are proportional. The constant of proportionality can be computed by taking $g_w = e$, where we find that it is $\prod_{v\neq w} W_{\xi_v^0}(g_v)$ by the induction hypothesis. This completes the case where $g \in G(\mathbf{A}_{k,S})$ for some finite set S.

Now a general $(g_v) \in G(\mathbf{A}_k)$ has the property that $g_v\xi_v^0 = \xi_v^0$ for all but finitely many v, so we may replace g_v by e to reduce to the case already proven; the claim (3.1) follows.

We conclude that the vector W_{ξ^0} (which is obviously non-zero since it evaluates to 1 on e) is common to W and $\bigotimes' W_v$ within $\mathrm{Ind}_{N(\mathbf{A})}^{G(\mathbf{A})} \psi$. Hence these two (irreducible!) representations must coincide for any Whittaker model W. But the W_v were unique by local uniqueness, proving the result. □

4. Multiplicity One

We first prove a weaker version where we demand an isomorphism at *all* local places.

Theorem 4.1 (Weak Multiplicity One)**.** *Let (π, V) and (π', V') be two irreducible admissible subrepresentations of $\mathcal{A}_0(\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbf{A}_k), \omega)$. Assume that $\pi_v \cong \pi'_v$ for all v. Then $V = V'$.*

Proof. As discussed in §2, the Fourier coefficient corresponding to ψ furnishes non-vanishing Whittaker models on V and V'. By Theorem 3.8 these Whittaker models coincide; call the common Whittaker model W.

The assumption gives an abstract isomorphism $\theta \colon V \to V'$. Let $\varphi \in V$ and $\varphi' = \theta(\varphi)$. We have the diagram (by uniqueness of Whittaker models)

$$\begin{array}{ccc} V & \xrightarrow[\sim]{\theta} & V' \\ & {\scriptstyle \varphi\mapsto\varphi_{N,\psi}}\searrow\sim \quad \sim\swarrow{\scriptstyle \varphi'\mapsto\varphi'_{N,\psi}} & \\ & W & \end{array}$$

By irreducibility of W the images of φ and φ' in W must be proportional, so the Fourier expansions are proportional. Hence we find that every $\varphi \in V$ lies in V' and vice versa, so they are the same space. □

To proceed to the proof of the *strong* Multiplicity One theorem, we now assume that $\pi_v \cong \pi'_v$ for all v outside a finite set S of places, excluding *all* the archimedean places.

Remark 4.2. The assumption on Archimedean places is unnecessary. After we develop the theory of L-functions attached to automorphic representations, we'll be able to give a proof of the full strong Multiplicity One Theorem.

It will suffice to produce a single non-zero function common to V and V', which we will do by writing a Fourier expansion. Since $\pi_v \cong \pi'_v$ outside S, they have the same Whittaker model W_v and we pick a function $f_v \in W_v$ which is required to be the spherical function normalized to value 1 on K_v if W_v is spherical.

For the $v \in S$, we have $W_v, W'_v \supset C_c^\infty(k_v^\times)$ by the theory of the Kirillov model. Therefore, we can choose f_v, f'_v to agree on $\begin{pmatrix} k_v^\times & \\ & 1 \end{pmatrix}$. Set $f = \bigotimes f_v$, $f' = \bigotimes f'_v$. Then we have the Fourier series

$$\varphi(g) = \sum_{a \in k^\times} f\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

$$\varphi'(g) = \sum_{a \in k^\times} f'\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

By construction, φ and φ' agree on $G(k)G(k_\infty)B(\mathbf{A})K$, so it will suffice to see that

$$G(k)G(k_\infty)B(\mathbf{A})K = G(\mathbf{A}). \tag{4.1}$$

We'll deduce this from Strong Approximation, which implies that $\mathrm{SL}_n(k)\,\mathrm{SL}_n(k_\infty)$ is dense in $\mathrm{SL}_n(\mathbf{A}_k)$, hence in turn that $\mathrm{SL}_n(k)\,\mathrm{SL}_n(k_\infty)U = \mathrm{SL}_n(\mathbf{A}_k)$ for any open compact subgroup $U \subset \mathrm{SL}_n(\mathbf{A}_k)$. Consider the short exact sequence

$$1 \to \mathrm{SL}_n(\mathbf{A}) \to \mathrm{GL}_n(\mathbf{A}) \to \mathbf{A}^\times \to 1.$$

Strong approximation implies that $G(k)G(k_\infty)B(\mathbf{A})K \supset \mathrm{SL}_n(A)$. On the other hand, $B(\mathbf{A})$ surjects onto $\mathbf{A}^\times$, so we deduce (4.1). □

Exercise 4.3. Formulate and prove an analogous statement over global function fields. If you are familiar with Weil's interpretation of adeles in terms of Bun_G, give a direct *geometric* proof that

$$B(k)\backslash B(\mathbf{A})/U \cap B(\mathbf{A}) \xrightarrow{\sim} B(k)\backslash G(\mathbf{A})/U$$

which implies (4.1).

□

5. Proof of uniqueness of local Whittaker models

5.1. **Toy case: finite fields.** Let's momentarily consider the toy case of finite fields. We want to argue that every representation in $\mathrm{Ind}_{N(\mathbf{F}_q)}^{G(\mathbf{F}_q)} \psi$ shows up with multiplicity one. By an elementary exercise (which appeared on a recent qualifying exam!), this is equivalent to showing that

$$\mathrm{Hom}_{G(\mathbf{F}_q)}(\mathrm{Ind}_{N(\mathbf{F}_q)}^{G(\mathbf{F}_q)} \psi, \mathrm{Ind}_{N(\mathbf{F}_q)}^{G(\mathbf{F}_q)} \psi)$$

is commutative. By Mackey Theory, this coincides with the Hecke algebra

$$\mathbf{C}[N, \psi\backslash G/N, \psi] = \{\Delta \colon G \to \mathbf{C} \colon \Delta(ngn') = \psi(n)\Delta(g)\psi(n')\}.$$

So we need to prove that $\mathbf{C}[N, \psi\backslash G/N, \psi]$ is commutative.

Let's write down double coset representatives for $N\backslash G/N$. We know from Bruhat decomposition that

$$B\backslash G/B = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cup \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

Therefore

$$N\backslash G/N = \bigcup_{a,b\in \mathbf{F}_q^\times} \begin{pmatrix} a & \\ & b \end{pmatrix} \cup \bigcup_{a,b\in \mathbf{F}_q^\times} \begin{pmatrix} & b \\ a & \end{pmatrix}.$$

We seek functions $\Delta\colon G \to \mathbf{C}$ such that

$$\Delta(ngn') = \psi(n)\Delta(g)\psi(n').$$

We can define Δ separately on each double coset by specifying its value on a representative, but the condition of being well-defined is that Δ annihilate the stabilizer.

It is easy to check by hand that

$$\mathrm{Stab}_{N\times N}\left(\begin{pmatrix} a & \\ & b \end{pmatrix}\right) = \left\{\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} : ax = by\right\}.$$

If $a \neq b$, then $x \neq y$ so that ψ is forced to annihilate such a double coset.

Exercise 5.1. Show more abstractly that the stabilizer in $N \times N$ of NgN is $N \cap g^{-1}Ng$. Use this to give a proof of the above fact without doing any computations with 2×2-matrices.

Now, to show commutativity we will use Gelfand's trick: we will write down an anti-involution that preserves the space in question. Indeed, let $w_0 = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$ and define the anti-involution

$$\iota(g) = w_0 g^t w_0^{-1}.$$

By inspection, ι preserves N and the double cosets that have not been ruled out.

Exercise 5.2. Why doesn't this proof work over local fields?

5.2. **Proof for non-archimedean local fields.** We'll take as our starting point:

Proposition 5.3. *Let V be an admissible irreducible representation. If V admits a Whittaker model then so does its contragredient $\widetilde{V}$.*

Proof. This follows from a theorem of Gelfand and Kazdhan; see [Ngo] Proposition 4.6.2. □

Recall the Hecke algebra $\mathcal{H}_G(K) = C_c^\infty(G(K))$. We have actions ℓ_g and r_g on $\mathcal{H}_G(K)$, given by $\ell_g f(x) = f(g^{-1}x)$ and $r_g f(x) = f(xg)$.

Theorem 5.4 (Invariance of Bessel distributions, [Bump] Theorem 4.4.2)**.** *Let*

$$\Delta\colon \mathcal{H}(\mathrm{GL}_2(k_v)) \to \mathbf{C}$$

be a distribution such that

$$\Delta(\ell_n r_{n'} f) = \psi(n)\Delta(f)\psi(n').$$

Then Δ is preserved by ι.

Theorem 5.5. *Let V be an irreducible admissible representation of $\mathrm{GL}_2(k_v)$ and $\widetilde{V}$ its contragredient representation. Then*

$$\dim \mathrm{Hom}_N(V, (\mathbf{C}, \psi)) \cdot \dim \mathrm{Hom}_N(\widetilde{V}, (\mathbf{C}, \psi)) \leq 1.$$

Proof. Let $\lambda^\vee \colon V \to (\mathbf{C}, \psi)$ be a non-zero Whittaker functional and $\lambda \colon \widetilde{V} \to (\mathbf{C}, \psi)$ a non-zero Whittaker functional. We will show that $\lambda^\vee$ determines λ, and vice versa.

If λ and $\lambda^\vee$ were smooth then we would have a Bessel distribution

$$g \mapsto \langle \lambda, \widetilde{\pi}(g)\lambda^\vee \rangle$$

to which we could apply Theorem 5.4. However, there is no reason that this should be the case, so we will have to work a bit harder. What we do get from λ and $\lambda^\vee$ are G-equivariant maps

$$\begin{aligned} \phi_{\lambda^\vee} \colon \mathcal{H}(G) &\to \widetilde{V} \\ f &\mapsto \int_G f(h)(h^{-1} \cdot \ell^\vee)\, dh \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \phi_{\lambda} \colon \mathcal{H}(G) &\to V \\ f &\mapsto \int_G f(h)(h^{-1} \cdot \ell)\, dh. \end{aligned} \tag{5.2}$$

From the explicit formulas we easily check that

$$\begin{aligned} \phi_{\lambda^\vee}(r_g \cdot f) &= g \cdot \phi_{\lambda^\vee}(f) \\ \phi_{\lambda^\vee}(\ell_n \cdot f) &= \psi(n)\phi_{\lambda^\vee}(f) \\ \phi_{\lambda}(r_g \cdot f) &= g \cdot \phi_{\lambda}(f) \\ \phi_{\lambda}(\ell_n \cdot f) &= \psi(n)\phi_{\lambda}(f) \end{aligned}$$

The pairing $\widetilde{V} \otimes V \to \mathbf{C}$ then pulls back to a pairing

$$\mathcal{H}(G) \otimes \mathcal{H}(G) \to \widetilde{V} \otimes V \to \mathbf{C}.$$

The map $G \times G \to G$ given by $(g_1, g_2) \mapsto g_1^{-1} g_2$ induces a map

$$\mathcal{H}(G) \otimes \mathcal{H}(G) \to \mathcal{H}(G)$$

by integrating over the fibers, which are orbits for the diagonal G action by (r_g, r_g). So the G-equivariance implies that this descends to a map

$$\begin{array}{ccccc} \mathcal{H}(G) \otimes \mathcal{H}(G) & \longrightarrow & \widetilde{V} \otimes V & \longrightarrow & \mathbf{C} \\ \downarrow & & & \overset{\theta}{\nearrow} & \\ \mathcal{H}(G) & & & & \end{array}$$

such that $\theta(ngn') = \psi(n)\theta(g)\psi(n')$. Hence Theorem 5.4 implies that

$$\langle f_1, f_2 \rangle = \theta(f_1 * f_2) = \theta(\iota(f_2) * \iota(f_1)) = \langle \iota(f_2), \iota(f_1) \rangle. \tag{5.3}$$

Since the pairing between V and $\widetilde{V}$ is perfect, (5.3) shows that if f_2 is in the kernel of (5.2) (i.e. (5.3) vanishes for all f_1) then $\iota(f_2)$ is in the kernel of (5.1). In other words, (5.1) determines the kernel of (5.2), hence the map (5.2) up to scalar (by irreducibility), and vice versa. Since $\lambda^\vee$ determined (5.1) and λ determined (5.2), this shows that λ pins down $\lambda^\vee$ and vice versa. $\square$

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