LECTURE 23: GLOBAL AUTOMORPHIC L-FUNCTION AND FUNCTIONAL EQUATION
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1 Introduction: Definition of the L-function

As usual, let k be a global field, $\mathbf{A}$ its adele ring, and $G = \mathrm{GL}_2$. We fix a multiplicative quasi-character χ, i.e. a continuous homomorphism $\chi\colon k^\times\backslash\mathbf{A}^\times \to \mathbf{C}^\times$. We have $\chi = \prod_v \chi_v$ with $\chi_v := \chi|_{k_v}$. For all but finitely many v, χ_v is unramified, i.e. $\chi_v|_{\mathscr{O}_v} \equiv 1$.

We will also fix a non-trivial additive character ψ of the compact abelian group $\mathbf{A}^+/k^+$. Similarly, we have $\psi = \prod_v \psi_v$ with $\psi_v := \psi|_{k_v^+}$. (Here, we regard k_v^+ as embedded into $\mathbf{A}$ by setting all components other than v to 0). For all but finitely many v (including the archimedean places), the largest (fractional) ideal of k_v on which ψ_v vanishes is $\mathscr{O}_v$.[1] The choice of ψ is not such a big deal: given any non-trivial ψ^0, we have an isomorphism $k \xrightarrow{\sim} \widehat{\mathbf{A}^+/k^+}$ by the map $\lambda \mapsto (x \mapsto \psi^0(\lambda x))$.[2] ψ determines a unique Haar measure dx on $\mathbf{A}$ such that the Fourier inversion formula $f(x) = \widehat{\widehat{f}}(-x)$ holds with the Fourier transform $\widehat{f}(\xi) = \int f(x)\psi(-x)\,dx$ defined with respect to ψ. This determines a Haar measure $d^\times x$ on $\mathbf{A}^\times$. These measures are product measures: there are Haar measures $(dx)_v, (d^\times x)_v$ on $k_v^+, k_v^\times$ respectively[3] such that for almost all v, $\int_{\mathscr{O}_v^+}(dx)_v = \int_{\mathscr{O}_v^\times}(d^\times x)_v = 1$, and we have $dx = \prod_v d_v x, d^\times x = \prod_v d_v^\times x$ (in the sense that if we take a compact open subset $U = \prod_v U_v$ of $\mathbf{A}^+$ or $\mathbf{A}^\times$, the two sides of this equation give the same thing evaluated on U: the product on the right is finite since $U_v = \mathscr{O}_v$ or $\mathscr{O}_v^\times$ for all but finitely many v). These measures descend to quotient measures on $\mathbf{A}^+/k^+$ and $k^\times\backslash\mathbf{A}^\times$.

When k is a number field, we also fix a maximal compact subgroup K_∞ inside $\prod_{v|\infty}\mathrm{GL}_2(k_v)$ which is $\mathrm{O}_2(\mathbf{R})$ at the real places and $\mathrm{U}_2(\mathbf{R})$ at the complex places.[4]

Today, we want to define the global L-function $L_\pi(\chi, s)$ attached to an irreducible admissible representation π of $G(\mathbf{A})$.[5] We will also define ϵ-factors $\epsilon_\pi(\chi, s)$. When $\pi \subseteq \mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbf{A}), \omega)$

[1] As far as I can tell, this is not *a priori* obvious (although it is obvious that $\mathscr{O}_v \subseteq \ker\psi_v$ for all but finitely many v). One must construct a particular "standard" character ψ^0 of $\mathbf{A}^+/k^+$ and verify that this condition is true for this character, then use the fact that any other character is of the form $x \mapsto \psi^0(\lambda x)$ for $\lambda \in k$, as discussed above. If someone knows an *a priori* proof, please enlighten me.

[2] Proof: first, note that the Pontryagin dual of the compact abelian group $\mathbf{A}^+/k^+$ is discrete. Self-duality of k_v for all v implies that $\mathbf{A}$ is self-dual as well, so we can think of $\widehat{\mathbf{A}^+/k^+}$ as a subspace of $\mathbf{A}$ (i.e. every character of $\mathbf{A}$ is of the form $x \mapsto \psi^0(ax)$ for some $a \in \mathbf{A}$). Since ψ^0 vanishes on k, so does $\psi^0(\lambda\cdot)$ for any $\lambda \in k$, as multiplication by λ stabilizes k. Thus, it suffices to show that the image of $\widehat{\mathbf{A}^+/k^+}/k^+$ is 0 in the compact group $\mathbf{A}^+/k+$. But it is discrete, hence finite. However, it is a k-vector subspace, since the condition of vanishing on k is preserved by multiplying against k, so it is 0.

[3] $(dx)_v$ is the unique self-dual Haar measure on k_v^+, and $(d^\times x)_v$ is $\frac{(dx)_v}{|x|_v}$ when v is archimedean and $\frac{q_v}{q_v-1}\frac{(dx)_v}{|x|_v}$ when v is non-archimedean with residue field $\mathbf{F}_{q_v}$. These factors are chosen to make $\int_{\mathscr{O}_v^\times}(d^\times x)_v$ for all but finitely many v.

[4] This is so the notion of "$(\mathfrak{g}, K_\infty)$-module", and hence "irreducible admissible $G(\mathbf{A})$-representation" (see the next footnote) makes sense. Note that the subgroup SO_2 is not intrinsic to the $\mathbf{R}$-group GL_2, as it depends on the choice of an inner product; however, any two choices are conjugate.

[5] This is really thought of as a representation of the global Hecke algebra $\mathcal{H}$, so at the archimedean places k_v we just get a $(\mathfrak{g}, K)$-module rather than a representation of $G(k_v)$. We will continue making this abuse of notation everywhere.

is cuspidal automorphic, we will then show that the L-function satisfies the functional equation:

$$L_\pi(\chi, s) = \epsilon_\pi(\chi, s) L_\pi(\omega\chi^{-1}, 1-s) \tag{1}$$

Flath's theorem says that we have a canonical decomposition $\pi \simeq \bigotimes'_v \pi_v$, where the π_v are irreducible admissible representations of the local Hecke algebras $\mathcal{H}_v$. Since we have already defined L-functions for such representations, the natural guess for an Euler product is:

Definition 1.1. Let π be an irreducible admissible representation of $G(\mathbf{A})$ with local components π_v. Let χ be a quasi-character of $k^\times \backslash \mathbf{A}^\times$. Then we may define the L-function of π with respect to χ as a *formal* Euler product:

$$L_\pi(\chi, s) := \prod_v L_{\pi_v}(\chi_v, s) \tag{2}$$

Remark 1.2. Note that this definition does not require π to be cuspidal automorphic. However, as we will see, the good analytic properties of this L-function depend crucially on this condition. Indeed, the converse theorem tells us exactly that if $L_\pi(\chi, s)$ converges to an entire function which is bounded in vertical strips and which satisfies (1), then π is cuspidal automorphic.

Remark 1.3. This definition *does not* depend on the choice of ψ, since the definitions of the local L-functions do not depend on a choice of additive character. See §3.

2 Review of global Whittaker models

The first result we need is that when π is cuspidal automorphic, the Euler product defining $L_\pi(\chi, s)$ converges on some right half-plane. It turns out that a much weaker condition suffices:

Definition 2.1. We say that an irreducible admissible representation $\pi \simeq \bigotimes'_v \pi_v$ of $G(\mathbf{A})$ is *pre-unitary* if π_v admits a $\mathcal{H}_v$-invariant hermitian form for each v.

Remark 2.2. The reason we say "*pre*-unitary" instead of "unitary" for such representations is that the latter is reserved for Hilbert space representations.

In what follows, it will be convenient to assume that π_v is infinite-dimensional for all v, i.e. that π_v is not a twist of the determinant character.[6] This condition turns out to be *equivalent* to the existence of a global Whittaker model. This follows from the following theorem, which describes the relationship between local and global Whittaker models:

Proposition 2.3. Let π be an irreducible admissible representation of $G(\mathbf{A})$. Then we have:

- π has a Whittaker model if and only if π_v has a Whittaker model for all v. Equivalently, π has a Whittaker model if and only if for all v, π_v is infinite-dimensional.
- If π has a Whittaker model, it is necessarily unique and spanned by functions of the form $W(g) = \prod_v W_v(g_v)$ with $W_v \in \mathcal{W}(\pi_v)$ such that for all but finitely many non-archimedean v, $W_v = W_v^0$. The function $W_v^0 \in \mathcal{W}(\pi_v)$ is the unique function which is invariant by $\mathrm{GL}_2(\mathscr{O}_v)$ and which satisfies $W_v^0|_{\mathrm{GL}_2(\mathscr{O}_v)} \equiv 1$.

[6] This condition is probably not essential for showing convergence of the L-function, using the GL_1 theory at places where π_v is one-dimensional.

- Moreover, for all $g \in G(\mathbf{A})$, for any $W \in \mathcal{W}(\pi)$, the function $x \mapsto W\big(\left(\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right) \cdot g\big)$ is $O(|x|^{-N})$ for all $N > 0$.[7]

Proof. This is the content of [1, Theorem 3.5.4, p. 326].[8] We will sketch the proof below for convenience.

Assuming that π_v is infinite-dimensional for all v, we have seen that local Whittaker models $\mathcal{W}(\pi_v)$ exist for all v. For all but finitely many non-archimedean v, π_v is spherical and the largest ideal of k_v on which ψ_v is trivial is $\mathscr{O}_v$. For such v, there is a unique $W_v^0 \in \mathcal{W}(\pi_v)$ such that $W_v^0(g) = 1$ for all $g \in \mathrm{GL}_2(\mathscr{O}_v)$: see [2, Theorem 11, p. 1.52]. Thus, it makes sense to consider the restricted product $\mathcal{W}(\pi) := \bigotimes_v' \mathcal{W}(\pi_v)$ with respect to the W_v^0. This is generated by functions $W : g \mapsto \prod_v W_v(g_v)$ for $W_v \in \mathcal{W}(\pi_v)$ for all v and $W_v = W_v^0$ for all but finitely many v. Then it is straightforward to check that this space has the desired properties of a global Whittaker model, so this settles the question of existence in the case that π_v is infinite-dimensional for all v. The rapid descent claim follows from the fact that the Whittaker models at archimedean places satisfy rapid descent (in Lecture 19, this property is shown in the proof of existence of archimedean Whittaker models).

Now, let $\mathcal{W}$ be a Whittaker model for π. First, we construct a vector $\varphi \in \pi$ such that $W_\varphi(1) \neq 0$. For an arbitrary non-zero φ_0, we have $W_{\varphi_0}(g) \neq 0$ for some $g \in G(\mathbf{A})$. Since the action of $g_v \in G(k_v)$ for v non-archimedean on π is given by right-translation in $\mathcal{W}$, we have $W_{g_v^{-1}\cdot\varphi_0}(g) = W_{\varphi_0}(gg_v^{-1})$, so we may assume g is supported at the archimedean places. Then, a similar but slightly tricker argument removes the archimedean parts of g, working first with the representation of K_∞ on π, and then with the representation of $\mathfrak{g}$ on π. If $W_\varphi(1) \neq 0$ for some φ, the functional $\varphi \mapsto W_\varphi(1)$ is non-zero, so we can find some pure tensor on which it does not vanish: thus we may assume $\varphi = \otimes_v'(\varphi_v^0)$. Let $\iota : \pi_v \to \pi$ be given by $\varphi_v \mapsto \varphi_v \otimes (\otimes_{v' \neq v}' \varphi_{v'}^0)$.

Now, we define local Whittaker functions by $W_{\varphi_v}(g_v) = W_{\iota(\varphi_v)}(\iota(g_v))$ where $\iota : G(k_v) \to G(\mathbf{A})$ is given by $g_v \mapsto g_v \times (1)_{v' \neq v}$. These clearly define a $\mathcal{H}_v$-equivariant homomorphism from π_v to a space of Whittaker functions on $G(k_v)$ (i.e. functions that transform appropriately and which have moderate growth - these properties follow immediately from the properties of $\mathcal{W}$). We've seen that $W_{\varphi_v^0}(1) = W_\varphi(1) \neq 0$, so this map is non-zero: since π_v is irreducible, it is an embedding. Thus, we have constructed Whittaker models $\mathcal{W}(\pi_v)$ for all v, so the π_v are infinite-dimensional.

Finally, we show that for any pure tensor $\xi = \bigotimes_v' \xi_v \in \pi$, $W_\xi \in \mathcal{W}$ is the function $g \mapsto \prod_v W_{\xi_v}(g_v)$. This argument is given in Lecture 22 (there, π is assumed to be cuspidal automorphic, but this condition is never used in the proof). First, the formula is shown by induction on $|S|$ that for any finite set S of places and $g \in G(\mathbf{A})$ with $g_v = 1$ for $v \notin S$. This induction argument is an application of the local uniqueness results. Then, one passes to the limit by using the fact that if $\bigotimes_v' \xi_v \in \pi$ and $g \in G(\mathbf{A})$, there is a finite set of places S such that ξ_v is $G(\mathscr{O}_v)$-fixed and $g_v \in G(\mathscr{O}_v)$ for all v outside of S.

This settles the question of uniqueness and finishes the proof.

□

[7] This condition is part of the definition of a global Whittaker model given in [2]. However, in [3], the definition only requires that this function be $O(|x|^N)$ for *some* N, i.e. that it is of "moderate growth". This is sufficient for uniqueness, since the local uniqueness theorem at archimedean places only requires moderate growth.

[8] There's a small gap in the proof here, since this book does not prove that Whittaker functions for $\mathrm{GL}_2(\mathbf{C})$ are analytic on the underlying real analytic manifold. This should follow from the same sort of elliptic regularity arguments he gives in the real case.

Corollary 2.4. If $\pi \subseteq \mathcal{A}_{\text{cusp}}(G(k)\backslash G(\mathbf{A}), \omega)$ is cuspidal automorphic, then π_v is infinite-dimensional for all v.

Proof. Via Fourier analysis on the compact abelian group $\mathbf{A}^+/k^+$, we showed in Lecture 22 that π has a Whittaker model. Thus, we may apply the previous proposition.

There is an alternative proof which does not use Whittaker models. This argument should generalize better to more general connected reductive groups, when Whittaker models may not exist.[9] If $\varphi \in \pi$ is any vector, we can think of it as a continuous function on $G(\mathbf{A})$. Thus, it makes sense to consider $\varphi|_{\mathrm{SL}_2(\mathbf{A})}$. Since $\pi_v \simeq (\mathbf{C}, \det\rho)$ for some character ρ, φ satisfies $\varphi(gg_v) = \rho(\det(g_v))\varphi(g)$ for any $g \in G(\mathbf{A}), g_v \in G(k_v)$. In particular, if $g_v \in \mathrm{SL}_2(k_v)$, we have $\varphi(gg_v) = \varphi(g)$. Thus, $\varphi|_{\mathrm{SL}_2(\mathbf{A})}$ is left-invariant by $\mathrm{SL}_2(k)$ and right-invariant by $\mathrm{SL}_2(k_v)$ as well as by some compact open subgroup K. However, by strong approximation for SL_2, we have $\mathrm{SL}_2(\mathbf{A}) = \mathrm{SL}_2(k)\mathrm{SL}_2(k_v)K$, so $\varphi|_{\mathrm{SL}_2(\mathbf{A})}$ is constant. Now, by applying the same argument to the functions $r_g\varphi$ for $g \in G(\mathbf{A})$, we see that φ is actually invariant under left translation by $\mathrm{SL}_2(\mathbf{A})$; since $\mathrm{SL}_2(\mathbf{A})$ is normal in $G(\mathbf{A})$, this is equivalent to φ being invariant under right translation by $\mathrm{SL}_2(\mathbf{A})$. Since this is true for any $\varphi \in \pi$, we see that $\pi|_{\mathrm{SL}_2(\mathbf{A})}$ is the trivial representation. Thus, π_v is trivial on $\mathrm{SL}_2(k_v)$ *for all* v, which implies that π_v is a twist of the determinant for all v, and thus π is one-dimensional. □

3 Review of the local theory

Lectures 14 and 20 develop the theory of local L-functions (in the non-archimedean and archimedean cases respectively), zeta integrals, and ϵ-factors. We will recall the main theorems and definitions here for convenience. The setup is that we have an infinite-dimensional irreducible admissible $G(k_v)$-representation[10] π_v, a quasi-character χ_v of $k_v^\times$, a non-trivial character ψ_v of k_v^+, and a Whittaker model $\pi \xrightarrow{\sim} \mathcal{W}(\pi_v)$ defined with respect to ψ_v.

First, we define zeta integrals:

Definition 3.1. For $W \in \mathcal{W}(\pi_v)$, we define the *zeta integral*:

$$Z_v(W, \chi_v, s) = \int_{(k_v)^\times} W\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right)\chi_v(x)^{-1}|x|_v^{2s-1}\, d^\times x$$

We also have the slight variant:

Definition 3.2. Let $g \in G(k_v)$. Then we have:

$$\zeta_v(W, g; \chi_v, s) = \int_{(k_v)^\times} W\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix} \cdot g\right)\chi_v(x)^{-1}|x|_v^{2s-1}\, d^\times x = Z_v(\rho_g W, \chi_v, s)$$

Here, ρ_g is right-translation by g.

[9]For a general connected reductive group, we can replace the role of SL_2 by $G' = \mathcal{D}\mathscr{G}$, which is semisimple and thus admits no characters. (Note that $\mathcal{D}\mathrm{GL}_2 = \mathrm{SL}_2$). However, this group may fail to be simply connected, preventing the use of a strong approximation argument. Thus, one would instead pass to the group $\widetilde{G'}$, its simply connected central cover. Then one can show that $\widetilde{G}(\mathbf{A}) \to G(\mathbf{A})$ has a commutative cokernel. Thus any representation of $G(\mathbf{A})$ whose pullback to $\widetilde{G}(\mathbf{A})$ is trivial must be a character.

[10]As above, we treat the phrase "irreducible admissible $G(k_v)$-representation" as shorthand for "irreducible admissible $\mathcal{H}_v$-representation". In the non-archimedean case, these are actually representations of $G(k_v)$, but in the archimedean case, they are merely $(\mathfrak{g}, K_\infty)$-modules.

The zeta integrals satisfy:

Proposition 3.3. For any $W \in \mathcal{W}(\pi)$, the zeta integrals converge for $\mathrm{Re}(s) \gg 0$ and admit analytic continuations to meromorphic functions with at most 2 poles.

These transform according to the *gamma factors*:

Proposition 3.4. There is some meromorphic function $\gamma_v(\chi_v, s)$, which is in $\mathbf{C}(q_v^{-s})$ in the non-archimedean case, such that for any $W \in \mathcal{W}(\pi)$:

$$Z_v(\widetilde{W}, \omega_v\chi_v^{-1}, 1-s) = \gamma_v(\chi, s)Z_v(W, \chi, s)$$

Here, $\widetilde{W}(g) := W(gw)$ with $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ the Weyl group element, so $\widetilde{W} = \rho_w W$. We could reword this as:

$$\zeta_v(W, wg; \omega_v\chi_v^{-1}, 1-s) = \gamma_v(\chi, s)\zeta_v(W, g; \chi, s)$$

Furthermore, these satisfy the functional equation:

$$\gamma_v(\chi, s)\gamma_v(\omega_v\chi_v^{-1}, 1-s) = \omega_\pi(-1)$$

Now, we can define the L-functions. First, we need to define $L_v(\rho, s)$ for a character ρ of $k_v^\times$. This is:

Definition 3.5. Let ρ be a character of $k_v^\times$. First, let k_v be non-archimedean. We define the L-function:

$$L_v(\rho, s) = \begin{cases} 1 & \rho \text{ ramified} \\ (1-\rho(\varpi_v)q_v^{-s})^{-1} & \rho \text{ unramified} \end{cases}$$

Now, if $k_v = \mathbf{R}$ and $\rho(x) = |x|_v^r(\mathrm{sgn}(x))^m$, $\psi_v(x) = e^{2\pi iux}$, we define:

$$L_{\mathbf{R}}(\rho, s) = \pi^{-\frac{s+r+m}{2}}\Gamma\left(\frac{s+r+m}{2}\right)$$

If $k_v = \mathbf{C}$, $\rho(x) = |x|^r x^m \overline{x}^n$, and $\psi_v(x) = e^{4\pi i\,\mathrm{Re}(wx)}$, we define:

$$L_{\mathbf{C}}(\rho, s) = 2(2\pi)^{-(s+r+m+n)}\Gamma(s+r+m+n)$$

This lets us define the L-function for the irreducible admissible representation π_v:

Definition 3.6. When k_v is non-archimedean, we define $L_{\pi_v}(\chi_v, s)$ as:

$$L_{\pi_v}(\chi_v, s) = \begin{cases} 1 & \pi_v \text{ cuspidal} \\ L(\chi_v^{-1}\mu_v, 2s-\frac{1}{2}) \cdot L(\chi_v^{-1}\nu_v, 2s-\frac{1}{2}) & \pi_v = \pi_{\mu_v,\nu_v}, \mu_v/\nu_v \neq |\cdot|^{\pm 1} \\ L(\chi_v^{-1}\mu_v, 2s-\frac{1}{2}) & \pi_v = \pi_{\mu_v,\nu_v}, \mu_v/\nu_v = |\cdot| \\ L(\chi_v^{-1}\nu_v, 2s-\frac{1}{2}) & \pi_v = \pi_{\mu_v,\nu_v}, \mu_v/\nu_v = |\cdot|^{-1} \end{cases}$$

When $k_v = \mathbf{R}$ or $\mathbf{C}$ and $\pi_v = \pi_{\mu_v,\nu_v}$ is a principal series, we define:

$$L_{\pi_v}(\chi_v, s) = L(\chi_v^{-1}\mu_v, 2s-\frac{1}{2}) \cdot L(\chi_v^{-1}\nu_v, 2s-\frac{1}{2})$$

When $k_v = \mathbf{R}$ and $\pi_v = \pi(\rho)$ for ρ a quasi-character of $\mathbf{C}$, we define:

$$L_{\pi_v}(\chi_v, s) = L(\rho, s)$$

Remark 3.7. As we can see directly, the definition of the local L-functions does not depend on ψ_v. However, the zeta integrals do depend on ψ_v, via the dependence of the measure and the Whittaker model.

These are related to the zeta integrals by:

Proposition 3.8.

$$\{Z_v(W,\chi_v,s) \mid W \in \mathcal{W}(\pi_v)\} = L_{\pi_v}(\chi_v,s)\cdot \mathbf{C}[q^{-2s},q^{2s}]$$

The L-functions transform according to the ϵ-factors:

Definition 3.9. Define:

$$\epsilon_v(\chi_v,s) = \gamma_v(\chi_v,s)\cdot \frac{L_{\pi_v}(\chi_v,s)}{L_{\pi_v}(\chi_v^{-1}\omega_v,1-s)}$$

We can eliminate the γ factor from this definition to get:

$$\epsilon_{\pi_v}(\chi_v,s) := \frac{Z_v(\widetilde{W},\omega_{\pi_v}\chi_v^{-1},1-s)}{Z_v(W,\chi_v,s)}\cdot\frac{L_{\pi_v}(\chi_v,s)}{L_{\pi_v}(\chi_v^{-1}\omega_{\pi_v},1-s)}$$

whenever $Z_v(W,\chi_v,s)$ is not the 0 function.

The functional equation for the γ factors immediately gives us:

Proposition 3.10. The local ϵ-factors satisfy the functional equation

$$\epsilon_v(\chi_v,s)\cdot\epsilon_v(\chi_v^{-1}\omega_v,1-s) = \omega_v(-1)$$

4 Convergence

Now, we want to show that the Euler product defining $L_\pi(\chi,s)$ converges for sufficiently nice π. It turns out that the following condition is sufficient:

Theorem 4.1. Let π be an irreducible admissible representation of $G(\mathbf{A})$ which is moreover pre-unitary, such that π_v is infinite-dimensional for all v. Then the Euler product defining $L_\pi(\chi,s)$ converges for $\mathrm{Re}(s) \gg 0$.

Note that this implies convergence of the L-function when π is cuspidal automorphic, since $\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbf{A}),\omega)$ is contained in the unitary Hilbert $G(\mathbf{A})$-representation $L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbf{A}),\omega)$.

Proof. Flath's theorem tells us that π_v is *spherical* for all but finitely many non-archimedean v,[11] i.e. $\pi_v^{\mathrm{GL}_2\mathscr{O}_v} \neq \{0\}$. (Indeed, these spaces are then necessarily one-dimensional and are then used to *define* the restricted tensor product $\bigotimes'\pi_v \simeq \pi$). Thus, we may choose a finite set of places S_0 such that for all $v \notin S_0$, we have:

[11] Note that $\mathrm{GL}_2(\mathscr{O}_v)$ is not actually intrinsic to the k-group GL_2. However, given an arbitrary choice of $\mathscr{O}_{k,S}$-model of GL_2, it is isomorphic to the standard integral model $\mathscr{GL}_2$ away from finitely many places. Then any vector in the admissible π is fixed by some open compact subgroup K_f of $\mathrm{GL}_2(\mathbf{A}_f)$, and with respect to this arbitrary choice of model, K_f is equal to $\mathrm{GL}_2(\mathscr{O}_v)$ at all but finitely many places. Thus, our claim does not depend on a choice of integral model, but the specific set of places S_0 is not intrinsically determined. Similar comments apply whenever we discuss $\mathrm{GL}_2(\mathscr{O}_v)$.

- v is non-archimedean.
- π_v is spherical.
- χ_v is unramified.
- The largest ideal of k_v on which ψ_v vanishes is $\mathscr{O}_v$.
- $\int \mathscr{O}_v dx = \int_{\mathscr{O}_v^\times} d^\times x = 1$.

We know by the classification of spherical representations given in Lecture 15 that for all such v, π_v is either a twist of the determinant character or $\pi_v \simeq \pi_{\mu_v,\nu_v}$ with μ_v, ν_v unramified quasi-characters of $k_v^\times$ with $\mu_v/\nu_v \neq |\cdot|^{\pm 1}$. By Proposition 2.3, we do not need to worry about the former possibility. Then, we have:

$$L_{\pi_v}(\chi_v, s) = L(\mu_v \chi_v^{-1}, s') L(\nu_v \chi_v^{-1}, s')$$

Here, $s' := 2s - \frac{1}{2}$. Now, since μ_v, ν_v, χ_v are all unramified, we have:

$$L_{\pi_v}(\chi_v, s) = (1 - \mu_v \chi_v^{-1}(\varpi_v) q_v^{-s'})^{-1} (1 - \nu_v \chi_v^{-1}(\varpi_v) q_v^{-s'})^{-1}$$

Here, ϖ_v is the uniformizer of $\mathscr{O}_v$ and q_v is the cardinality of the residue field at v. (We assume of course that the valuation $|\cdot|$ on k_v is normalized so $|\varpi_v| = q^{-1}$).

Since π_v is pre-unitary, it follows from Lecture 16 that either μ_1, μ_2 are both unitary or $\mu_2 = \overline{\mu_1}^{-1}$ and $\mu := \mu_1 \mu_2^{-1} = |\mu_1|^2 = |\mu_2|^{-2} = |\cdot|^{\sigma_v}$ for some $0 < \sigma_v < 1$. Thus, in either case, we have:

$$|\mu_1(x)| = |x|^{\sigma_v/2}, \qquad |\mu_2(x)| = |x|^{-\sigma_v/2}$$

for some $0 \leqslant \sigma_v < 1$ (the case that μ_1, μ_2 are both unitary is the case $\sigma_v = 0$). Applying this to $x = \varpi_v$, we have:

$$|\mu_1(\varpi_v)| = q_v^{-\sigma_v/2}, \qquad |\mu_2(\varpi_v)| = q_v^{\sigma_v/2}$$

Thus, we may reduce the convergence of the Euler product in (2) to the convergence of:

$$\prod_{v \notin S_0} \frac{1}{(1 - \chi_v^{-1}(\varpi_v)\mu_v(\varpi_v) q_v^{-s'})(1 - \chi_v^{-1}(\varpi_v)\nu_v(\varpi_v) q_v^{-s'})}$$

This in turn reduces to convergence of the sums:

$$\sum_{v \notin S_0} \chi_v^{-1}(\varpi_v)\mu_v(\varpi_v) q_v^{-s'}, \qquad \sum_{v \notin S_0} \chi_v^{-1}(\varpi_v)\nu_v(\varpi_v) q_v^{-s'}$$

Since χ is a quasi-character of $k^\times \backslash \mathbf{A}^\times$, we must have $|\chi(x)| = |x|^t$ for some $t \in \mathbf{R}$ and all $x \in \mathbf{A}^\times$. Thus, we have $|\chi(\varpi_v)| = |\chi_v(\varpi_v)| = q_v^{-t}$. Therefore, taking absolute values in the above sums reduces our convergence question to that of:

$$\sum_{v \notin S_0} q_v^{-(\mathrm{Re}(s') + t \pm \sigma_v)} \leqslant \sum_{v \notin S_0} q_v^{-(\mathrm{Re}(s') + t - 1)}$$

Finally, this converges for $\mathrm{Re}(s) \gg 0$: it is essentially the logarithm of the ζ-function of k. $\square$

5 ϵ-factors

Now, we must define the global ϵ-factor $\epsilon_\pi(\chi, s)$. Since we use Whittaker models for the definition, we will assume that π_v is infinite-dimensional for all v in this section: by Corollary 2.4, this is the case when π is cuspidal automorphic.

Recall that we have:

$$\epsilon_{\pi_v}(\chi_v, s) = \frac{Z_v(\widetilde{W}, \omega_{\pi_v}\chi_v^{-1}, 1-s)}{Z_v(W, \chi_v, s)} \cdot \frac{L_{\pi_v}(\chi_v, s)}{L_{\pi_v}(\chi_v^{-1}\omega_{\pi_v}, 1-s)}$$

for all $W \in \mathcal{W}(\pi_v)$ such that $Z_v(W, \chi_v, s)$ is not identically 0.

Now, assume that $v \notin S_0$. By [2, Theorem 11, p. 1.52], there is a uniquely determined spherical function $W_v^0 \in \mathcal{W}(\pi_v)$ with $W_v^0|_{G(\mathscr{O}_v)} \equiv 1$. We have the following result:

Proposition 5.1. For any $v \notin S_0$, we have:

$$Z_v(W_v^0, \chi_v, s) = L_{\pi_v}(\chi_v, s)$$

Proof. We can compute the zeta integral for W_v^0 as:

$$\begin{aligned}
Z_v(W_v^0, \chi_v, s) &= \int_{F^\times} W_v^0\big(\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)\big)\chi(x)^{-1}|x|^{2s-1}\, d^\times x \\
&= \sum_{n=-\infty}^{\infty} \int_{\mathscr{O}_v^\times} W_v^0\Big(\left(\begin{smallmatrix} \varpi^n u & \\ & 1 \end{smallmatrix}\right)\Big)\chi(\varpi^n u)^{-1}|\varpi^n u|^{2s-1}\, d^\times u \\
&= \sum_{n=-\infty}^{\infty} \int_{\mathscr{O}_v^\times} W_v^0\Big(\left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right)\Big)\chi(\varpi)^{-n}q^{-n(2s-1)}\, d^\times u \\
&= \sum_{n=-\infty}^{\infty} W_v^0\Big(\left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right)\Big)\chi(\varpi)^{-n}q^{-n(2s-1)}
\end{aligned}$$

In [2, §3.16, (268)], Godement computes that (here, $d = 0$ since $\mathscr{O}_v$ is the largest ideal of k_v contained in $\ker \psi_v$)

$$W_v^0\Big(\left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right)\Big) = \begin{cases} q^{-n/2}\sum_{i+j=n}\mu_v(\varpi^i)\nu_v(\varpi^j) & n \geqslant 0 \\ 0 & n < 0 \end{cases}$$

Thus, we have:

$$\begin{aligned}
Z(W_v^0, \chi_v, s) &= \sum_{n=-\infty}^{\infty} W_v^0\Big(\left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right)\Big)\chi_v(\varpi)^{-n} q^{-n(2s-1)} \\
&= \sum_{n=0}^{\infty} \chi_v(\varpi)^{-n} q^{-n(2s-\frac{1}{2})} \sum_{i+j=n} \mu_v(\varpi^i)\nu_v(\varpi^j) \\
&= \left(\sum_{i=0}^{\infty} \mu_v(\varpi^i)\chi_v(\varpi^{-i}) q_v^{-is'}\right)\left(\sum_{j=0}^{\infty} \nu_v(\varpi^j)\chi_v(\varpi^{-j}) q_v^{-js'}\right) \\
&= \frac{1}{1-\mu_v\chi_v^{-1}(\varpi)q^{-s'}} \frac{1}{1-\nu_v\chi_v^{-1}(\varpi)q^{-s'}} \\
&= L(\mu_v\chi_v^{-1}, s')L(\nu_v, \chi_v^{-1}, s') \\
&= L_{\pi_v}(\chi_v, s)
\end{aligned}$$

□

Here, $s' = 2s - \frac{1}{2}$. Now, since W_v^0 is right-invariant by $G(\mathscr{O}_v)$ and $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in G(\mathscr{O}_v)$, we have $\widetilde{W_v^0} = W_v^0$. Thus, we can compute the ϵ-factor for $v \notin S_0$ as:

$$\begin{aligned}
\epsilon_{\pi_v}(\chi_v, s) &= \frac{Z(\widetilde{W_v^0}, \omega_{\pi_v}\chi_v^{-1}, 1-s)}{Z(W_v^0, \chi_v, s)} \cdot \frac{L_{\pi_v}(\chi_v, s)}{L_{\pi_v}(\chi_v^{-1}\omega_{\pi_v}, 1-s)} \\
&= \frac{Z(W_v^0, \omega_{\pi_v}\chi_v^{-1}, 1-s)}{Z(W_v^0, \chi_v, s)} \cdot \frac{L_{\pi_v}(\chi_v, s)}{L_{\pi_v}(\chi_v^{-1}\omega_{\pi_v}, 1-s)} \\
&= \frac{L_{\pi_v}(\chi_v^{-1}\omega_{\pi_v}, 1-s)}{L_{\pi_v}(\chi_v, s)} \cdot \frac{L_{\pi_v}(\chi_v, s)}{L_{\pi_v}(\chi_v^{-1}\omega_{\pi_v}, 1-s)} \\
&= 1
\end{aligned}$$

Thus, the following definition makes sense:

Definition 5.2. Let π be an irreducible admissible representation of $G(\mathbf{A})$ with π_v infinite-dimensional for all v. Then the ϵ-factor of π is defined by the Euler product:

$$\epsilon_\pi(\chi, s) := \prod_v \epsilon_{\pi_v}(\chi_v, s)$$

We saw above that $\epsilon_{\pi_v}(\chi_v, s)$ is the constant function 1 for all but finitely many v, so convergence of this product is automatic.

Remark 5.3. Unlike the L-function, the definition of the ϵ factor here appears to depend on ψ, since the local ϵ factors legitimately do depend on ψ. However, if π is cuspidal automorphic, the functional equation $L_\pi(\chi, s) = \epsilon_\pi(\chi, s)L_\pi(\omega\chi^{-1}, 1-s)$ shows that the global $\epsilon_\pi(\chi, s)$ is actually independent of ψ.

We can multiply together the local functional equations for the ϵ-factors to get:

Theorem 5.4. If π is an irreducible admissible representation of $G(\mathbf{A})$ such that π_v is infinite-dimensional for all v, we have:

$$\epsilon_\pi(\chi, s)\epsilon_\pi(\chi^{-1}\omega_\pi, 1-s) = \omega_\pi(-1)$$

Proof. Indeed, for each place v, Proposition 3.10 says that we have a functional equation of the above form for the local ϵ-factors, and we have $\prod_v \omega_{\pi_v}(-1) = \omega_\pi(-1)$. □

Remark 5.5. When π is cuspidal automorphic, the central character ω_π is $k^\times$-invariant, so this says:

$$\epsilon_\pi(\chi, s)\epsilon_\pi(\chi^{-1}\omega_\pi, 1-s) = 1$$

This also follows from the functional equation.

6 Functional Equation

Now, we will assume that $\pi \subseteq \mathcal{A}_{\text{cusp}}(G(k)\backslash G(\mathbf{A}), \omega)$, i.e. that π is a cuspidal automorphic irreducible admissible representation of $G(\mathbf{A})$. In this case, the L-function $L_\pi(\chi, s)$ satisfies all of the good analytic behavior we might ask for:

Theorem 6.1. Let $\pi \subseteq \mathcal{A}_{\text{cusp}}(G(k)\backslash G(\mathbf{A}), \omega)$ be an irreducible admissible cuspidal automorphic representation of $G(\mathbf{A})$. Then for every quasi-character χ of $k^\times\backslash\mathbf{A}^\times$, the function $L_\pi(\chi, s)$ is entire, bounded in every vertical strip, and satisfies the functional equation:

$$L_\pi(\chi, s) = \epsilon_\pi(\chi, s)L_\pi(\omega\chi^{-1}, 1-s)$$

Proof. Let $\varphi \in \pi$. Then for all $g \in G(\mathbf{A})$, we have the Fourier expansion:

$$\varphi(g) = \sum_{\alpha \in k^\times} W_\varphi\big(\left(\begin{smallmatrix} \alpha & \\ & 1 \end{smallmatrix}\right) \cdot g\big)$$

Here, $\varphi \mapsto W_\varphi$ is the (unique!) Whittaker model $\pi \xrightarrow{\sim} \mathcal{W}(\pi)$. We can identify W_φ concretely as the "Fourier coefficient" of φ with respect to ψ: $W_\varphi(g) = \int_{\mathbf{A}^+/k^+} \varphi\big(\left(\begin{smallmatrix} 1 & u \\ & 1 \end{smallmatrix}\right) \cdot g\big)\psi(-u)\, du$.

Now, we have the zeta integral:

$$\zeta(\varphi, g; \chi, s) := \int_{k^\times\backslash\mathbf{A}^\times} \varphi\big(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot g\big)\chi(x)^{-1}|x|^{2s-1}d^\times x \tag{3}$$

Since $\varphi \in \mathcal{A}_{\text{cusp}}(G(k)\backslash G(\mathbf{A}), \omega)$, the function $x \mapsto \varphi\big(\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right) \cdot g\big)$ is rapidly decreasing as $|x| \to \infty$, so the $|x| \geqslant 1$ part of the integral converges absolutely for any s. On the other hand, we have the identity:

$$\begin{aligned} \varphi\Big(\left(\begin{smallmatrix} x^{-1} & \\ & 1 \end{smallmatrix}\right) \cdot g\Big) &= \omega_\pi(x)^{-1}\varphi\big(\left(\begin{smallmatrix} 1 & \\ & x \end{smallmatrix}\right) \cdot g\big) \\ &= \omega_\pi(x)^{-1}\varphi\big(w^{-1}\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right) \cdot wg\big) \\ &= \omega(x)^{-1}\varphi\big(\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right) \cdot wg\big) \end{aligned}$$

The first inequality follows from the fact that the central character of π is ω_π, and the third follows from the fact that φ is left-invariant by $G(k)$.

Thus, $\varphi\Big(\big(\begin{smallmatrix} x^{-1} & \\ & 1 \end{smallmatrix}\big) \cdot g\Big)$ is rapidly decreasing as $|x| \to \infty$ as well. This means that $\varphi\big(\big(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\big) \cdot g\big)$ is rapidly decreasing as $|x| \to 0$, so the $|x| \leqslant 1$ part of the integral (3) converges absolutely for all s as well. Now, we apply the Fourier expansion inside the integral (3):

$$\begin{aligned}\zeta(\varphi, g; \chi, s) &\sim \sum_{\alpha \in k^\times} \int_{k^\times \backslash \mathbf{A}^\times} W_\varphi\big(\big(\begin{smallmatrix} \alpha x & \\ & 1 \end{smallmatrix}\big) \cdot g\big)\chi(\alpha x)^{-1}|\alpha x|^{2s-1} d^\times x \\ &\sim \int_{\mathbf{A}^\times} W_\varphi\big(\big(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\big) \cdot g\big)\chi(x)^{-1}|x|^{2s-1} d^\times x =: \widetilde{\zeta}\end{aligned}$$

This identity holds as long as the right-hand side $\widetilde{\zeta}$ is convergent. By one of the defining properties of the global Whittaker model, the function $x \mapsto W\big(\big(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\big) \cdot g\big)$ is rapidly decreasing as $|x| \to \infty$, so the $|x| > 1$ part of the integral converges for all s. For $|x| \leqslant 1$, we claim that that W_φ at least *bounded* on $G(\mathbf{A})$, so this integral converges for $\mathrm{Re}(s) \gg 0$.

To see this claim, first note that φ is bounded on $G(\mathbf{A})$: in fact, it is even rapidly decreasing on $G(\mathbf{A})$, by definition of cusp forms. Now, the Fourier coefficient

$$W_\varphi(g) = \int_{\mathbf{A}^+/k^+} \varphi\big(\big(\begin{smallmatrix} 1 & u \\ & 1 \end{smallmatrix}\big) \cdot g\big)\psi(-u)\, du$$

is the integral of a bounded function (ψ, being a character of a compact group, is unitary) over the *compact* group $\mathbf{A}^+/k^+$, so it is bounded as well.

Now, let S be a finite set of places containing S_0. We have:[12]

$$\widetilde{\zeta} = \lim_{S \supseteq S_0} \prod_{v \in S} \int_{k_v^\times} W_v\big(\big(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\big) \cdot g_v\big)\chi_v(x)^{-1}|x|_v^{2s-1} d^\times x \times \prod_{v \notin S} \int_{\mathscr{O}_v^\times} W_v\big(\big(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\big) \cdot g_v\big)\chi_v(x)^{-1}|x|_v^{2s-1} d^\times x$$

For large enough S, for any $v \notin S$, we have $W_v\big(\big(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\big) \cdot g_v\big) = \chi_v(x) = |x|_v = 1$ for all $x \in \mathscr{O}_v^\times$: indeed, it suffices for v to be a non-archimedean place such that χ_v is unramified, $g_v \in G(\mathscr{O}_v)$, and $W_v = W_v^0$.

Since $\mathscr{O}_v^\times$ has volume 1 with respect to the Haar measure $d^\times x$ for all $v \notin S_0$, this gives:

$$\widetilde{\zeta} = \lim_{S \supseteq S_0} \prod_{v \in S} \zeta_v(W_v, g_v; \chi_v, s) = \prod_v \zeta_v(W_v, g_v; \chi_v, s)$$

Here, the local zeta integral $\zeta_v(W_v, g_v; \chi_v, s)$ is defined as:

$$\zeta_v(W_v, g_v; \chi_v, s) := \int_{k_v^\times} W_v\big(\big(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\big) \cdot g_v\big)\chi_v(x)^{-1}|x|_v^{2s-1} d^\times x = Z_v(\rho_{g_v} W_v; \chi_v, s)$$

with ρ_{g_v} the right translation operator on $\mathcal{W}(\pi_v)$. Now, fix some g_v. We claim that $\zeta_v(W_v, g_v; \chi_v, s)$ is equal to $L_{\pi_v}(\chi_v, s)$ for all but finitely many v. Indeed, for all but finitely many v, the function $\rho_{g_v} W_v \in \mathcal{W}(\pi_v)$ is equal to W_v^0. Then, if additionally $v \notin S_0$, we have:

$$\zeta_v(W_v, g_v; \chi_v, s) = Z_v(\rho_{g_v} W_v; \chi_v, s) = Z_v(W_v^0; \chi_v, s) = L_{\pi_v}(\chi_v, s)$$

[12]Strictly speaking, it does not make sense to write W_v, since φ may not be a pure tensor in π. However, W is a finite linear combination of functions of the form $g \mapsto \prod_v W_v(g)$, so the integral for $\zeta(\varphi, g; \chi, s)$ breaks into a finite sum of integrals of this form. All formulas involving W_v below should be interpreted as such a sum.

by Proposition 5.1.

Thus, by Theorem 4.1, for any particular g, the infinite product defining $\widetilde{\zeta}$ converges for $\mathrm{Re}(s) \gg 0$. Thus, we have proven:

Proposition 6.2. For any cusp form φ, $g \in G(\mathbf{A})$, and quasi-character χ, the zeta integral $\zeta(\varphi, g; \chi, s)$ admits an Euler product:

$$\zeta(\varphi, g; \chi, s) = \prod_v \zeta_v(W_v, g_v; \chi_v, s)$$

for $\mathrm{Re}(s) \gg 0$.

By the local theory, as developed in Lectures 14 and 20, we have:

$$\{Z(W_v; \chi_v, s) \mid W_v \in \mathcal{W}(\pi_v)\} = L_{\pi_v}(\chi_v, s)\mathbf{C}[q^{-2s}, q^{2s}]$$

Thus, for each $v \in S_0$ we may choose W'_v such that $\zeta_v(W_v, e; \chi_v, s) = Z(W_v, \chi_v, s) = L_{\pi_v}(\chi_v, s)$. By Proposition 5.1, for $v \notin S_0$, $\zeta_v(W_v^0, e; \chi_v, s) = L_{\pi_v}(\chi_v, s)$. Thus, the function

$$W' := \prod_{v \in S_0} W'_v \times \prod_{v \notin S_0} W_v^0$$

is in $\mathcal{W}(\pi)$. There is a unique $\xi \in \pi$ mapping to W' under the isomorphism $\pi \xrightarrow{\sim} \mathcal{W}(\pi)$. Thus, we may apply Proposition 6.2 and Theorem 4.1 to see that for $\mathrm{Re}(s) \gg 0$:

$$L_\pi(\chi, s) = \prod_v L_{\pi_v}(\chi_v, s) = \prod_v \zeta_v(W'_v, g_v; \chi_v, s) = \zeta(\xi, e; \chi, s)$$

Thus, the integral $\zeta(\xi, e; \chi, s)$ defines an analytic continuation of $L_\pi(\chi, s)$. Since the integral defining $\zeta(\xi, e; \chi, s)$ converges absolutely for all $s \in \mathbf{C}$, it defines an entire function. The dependence on s in the integrand is only via the factor $|x|^{2s-1}$, which is an entire function in s. Since its absolute value only depends on $\mathrm{Re}(s)$, we see that $L_\pi(\chi, s)$ is also bounded in vertical strips.

Now, we are ready to prove the functional equation. Via Proposition 6.2 and Theorem 4.1, we have, for $\mathrm{Re}(s) \gg 0$ and any $g \in G(\mathbf{A})$:

$$\frac{\zeta(\varphi, g; \chi, s)}{L_\pi(\chi, s)} = \prod_v \frac{\zeta_v(W_v, g_v; \chi_v, s)}{L_{\pi_v}(\chi_v, s)}$$

Here, the product on the right-hand side is *finite*. Thus, both sides define meromorphic functions of s, so they are equal everywhere.

Thus, we can multiply together finitely many local functional equations to get:

$$\frac{\zeta(\varphi, wg; \omega\chi^{-1}, 1-s)}{L_\pi(\omega\chi^{-1}, 1-s)} = \epsilon_\pi(\chi, s)\frac{\zeta(\varphi, g; \chi, s)}{L_\pi(\chi, s)} \tag{4}$$

Now, by left $G(k)$-invariance of φ, we have $\varphi(wg) = \varphi(g)$, so:

$$
\begin{aligned}
\zeta(\varphi, wg; \omega\chi^{-1}, 1-s) &= \int_{k^\times\backslash\mathbf{A}^\times} \varphi\big(\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right) \cdot wg\big)\omega^{-1}\chi(x)|x|^{1-2s}\, d^\times x \\
&= \int_{k^\times\backslash\mathbf{A}^\times} \varphi\big(\left(\begin{smallmatrix} 1 & \\ & x \end{smallmatrix}\right) \cdot g\big)\omega^{-1}\chi(x)|x|^{1-2s}\, d^\times x \\
&= \int_{k^\times\backslash\mathbf{A}^\times} \varphi\Big(\left(\begin{smallmatrix} x^{-1} & \\ & 1 \end{smallmatrix}\right) \cdot g\Big)\chi(x)|x|^{1-2s} d^\times x \\
&= \zeta(\varphi, g; \chi, s)
\end{aligned}
$$

Thus, we may cancel the ζ terms in (4) to complete the proof.

□

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