RELATION BETWEEN CLASSICAL MODULAR FORMS AND AUTOMORPHIC REPRESENTATIONS

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Contents

1. Hecke operators

1.1. **Classical modular forms.** We recall the definition of holomorphic modular forms with respect to an arithmetic subgroup $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$, introduced many lectures ago.

Definition 1.1. A $f\colon \mathbf{H} \to \mathbf{C}$ is called a *modular form of weight* k with respect to Γ if we have

$$f(\gamma z) = (cz+d)^k f(z) \quad \text{for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \tag{1.1}$$

and also that f is holomorphic at the cusps.

Remark 1.2. For any $g \in \mathrm{GL}_2(\mathbf{R})^+$, we define

$$f|_{k,g}(z) = (\det g)^{k/2}(cz+d)^{-k} f(gz).$$

(Since $\det g \in \mathbf{R}_{>0}$, we can unambiguously take the "positive square root".) The condition (1.1) can then be abbreviated as

$$f|_{\gamma,k} = f \text{ for all } \gamma \in \Gamma.$$

1.2. **Hecke operators.** We will now focus our attention on the case

$$\Gamma = \Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \pmod{N} \right\}.$$

Let $M_k(\Gamma_0(N))$ denote the space of weight k modular forms for $\Gamma_0(N)$, and $S_k(\Gamma_0(N)) \subset M_k(\Gamma_0(N))$ be the subspace of cusp forms. Actually, we will see later that it is more general to consider the spaces $M_k(\Gamma_0(N), \chi) \supset S_k(\Gamma_0(N), \chi)$ where $\chi\colon (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ is a Dirichlet character modulo N; the definition of these is obtained by replacing (1.1) with

$$f(\gamma z) = \chi(a)(cz+d)^k f(z) \text{ for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N).$$

We will define certain *Hecke operators* T_p on $M_k(\Gamma_0(N), \chi)$, which also preserve the subspaces $S_k(\Gamma_0(N), \chi)$.

Definition 1.3. Consider the double coset

$$\Gamma_0(N)\begin{pmatrix}1 & \\ & p\end{pmatrix}\Gamma_0(N) = \bigcup_j \Gamma_0(N)\gamma_j.$$

We define

$$T_p f = \text{“}p^{k/2-1} f|_{k,[\Gamma_0(N)\begin{pmatrix}1 & \\ & p\end{pmatrix}\Gamma_0(N)]}\text{”} = p^{k/2-1}\sum_j \psi(\gamma_j)^{-1} f|_{k,\gamma_j}.$$

Since γ_j is upper-triangular modulo N, we can reasonably define $\psi(\gamma_j)$ to mean "ψ applied to the upper left entry".

Exercise 1.4. Check that we can take the coset representatives

$$\{\gamma_j\} = \left\{\begin{pmatrix}p & \\ & 1\end{pmatrix}\right\} \cup \left\{\begin{pmatrix}1 & b\\ & p\end{pmatrix} : b = 0, \dots, p-1\right\}.$$

Then $\psi(\gamma_j)$ means $\psi(a)$.

Exercise 1.5. Compute the effect of T_p on the Fourier expansion of $f = \sum a_n q^n \in M_k(N,\chi)$. [The correct answer is $T_p f = \sum b_n q^n$ where $b_n = a_{np} + \chi(p)p^{k-1}a_{n/p}$.]

Definition 1.6. Let $f, g \in S_k(\Gamma)$. Then we define the *Petersson inner product*

$$(f,g) := \int_{\mathcal{F}(\Gamma)} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2} \tag{1.2}$$

where $\mathcal{F}(\Gamma)$ is a fundamental domain for Γ acting on $\mathbf{H}$.

Exercise 1.7. Explain why (1.2) is well-defined.

Theorem 1.8. *The T_p for $(p,N) = 1$ generate a commutative subalgebra of $M_k(\Gamma_0(N),\chi)$, and satisfy*

$$(T_p f, g) = \chi(p)(f, T_p g).$$

Theorem 1.8 implies that the Hecke operators T_p for $p \nmid N$ are individually normal, hence diagonalizable. Furthermore, they commute with each other and can therefore be *simultaneously* diagonalized.

Definition 1.9. A *Hecke eigenform* is a modular form $f \in M_k(\Gamma_0(N),\chi)$ which is an eigenvector for all T_p with $p \nmid N$.

Exercise 1.10. Show that if $f = \sum a_n q^n$ is a Hecke eigenform with $a_1 \neq 0$, then its Fourier coefficients a_n for $(n,p) \neq 1$ are determined by a_1.

Example 1.11. Let $N = dM$ and $f \in M_k(\Gamma_0(M),\chi)$. Then we can view $f(z) \in M_k(\Gamma_0(N),\chi)$ and $f(dz) \in M_k(\Gamma_0(N),\chi)$. We claim that these are eigenforms for each T_p with $(p,N) = 1$, with the same eigenvalue as f. For $f(z)$, this is obvious from Exercise (1.4).

Next note that

$$f|_{k,\mathrm{diag}(d,1)\gamma_j} = f_{k,\mathrm{diag}(d,1)\gamma_j \mathrm{diag}(d^{-1},1)\mathrm{diag}(d,1)}$$

Since we can write

$$f(dz) = d^{-k/2} f|_{k,\mathrm{diag}(d,1)},$$

the fact that conjugation by $\begin{pmatrix}d & \\ & 1\end{pmatrix}$ permutes the coset representatives in Exercise 1.4 implies the claim for $f(dz)$ as well.

1.3. **Automorphic forms.** Recall that $\mathcal{A}_0(G,\omega)$ is equipped with an action of the Hecke algebra

$$\mathcal{H}(G(\mathbf{A})) = \bigotimes{}' (\mathcal{H}(G(\mathbf{Q}_p)), e_K)$$

where $e_K = \mathbf{I}_{G(\mathbf{Z}_p)}$ is the indicator function of a hyperspecial maximal compact subgroup K. The action is via convolution, meaning $\varphi \in \mathcal{H}(G(\mathbf{Q}_p))$ takes

$$\phi \mapsto \phi * \check{\varphi}(y) = \int_{G(\mathbf{Q}_p)} \phi(xg)\varphi(g^{-1})\, dg.$$

Definition 1.12. Define $\widetilde{T}_p \in \mathcal{H}(G(\mathbf{Q}_p))$ to be the indicator of the double coset

$$K_p \begin{pmatrix} 1 & \\ & p \end{pmatrix} K_p.$$

We will use the same notation for the induced operator (with appropriate twist to account for the central character). We will only ever apply $\widetilde{T}_p$ on a right K_p-invariant function ϕ, on which it acts as

$$\widetilde{T}_p\phi(g) = \int_{K_p} \phi(gk_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix})\omega^{-1}(k_p)\, dk_p.$$

Remark 1.13. Note that this definition is purely local. In other words, in terms of the tensor product decomposition

$$\pi \cong \bigotimes{}' \pi_p$$

of an automorphic representation, the action of $\mathcal{H}(G(\mathbf{Q}_p))$ is only on the component π_p.

There is a correspondence between modular forms and (certain) automorphic forms, which we denoted $f \leftrightarrow \phi_f$. Recall that (in our normalization) this was defined by

$$\phi_f(g) = f|_{k,g}(i)$$

and

$$f(x+iy) = y^{k/2}\phi_f \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \tag{1.3}$$

How do the two kinds of Hecke operators match up under this correspondence?

Lemma 1.14. *We have*

$$p^{k/2-1}\widetilde{T}_p(\phi_f) = \phi_{T_p(f)}.$$

Exercise 1.15. Prove Lemma 1.14.

1.4. **Spherical representations.** Let F be a non-archimedean local field. Recall that we say that a representation (π, V) of $\mathrm{GL}_2(F)$ is *spherical* if $V^{\mathrm{GL}_2(\mathcal{O}_F)} \neq 0$. Recall the classification of spherical representations from [Zav].

Theorem 1.16. *Let F be a non-archimedean local field. If (V,π) is an irreducible admissible representation of $\mathrm{GL}_2(F)$, which is spherical, then either:*

(1) $V \cong \pi(\chi_1,\chi_2)$ is a (non-special, so $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$) principal series representations with χ_1,χ_2 being unramified characters, or
(2) (V,π) is an unramified character.

Furthermore, we saw that if V is spherical then V^K is 1-dimensional, so the Hecke algebra $\mathcal{H}_K(G(\mathbf{Q}_p)) := \mathcal{H}(K\backslash G(\mathbf{Q}_p)/K) = e_K\mathcal{H}(G(\mathbf{Q}_p))e_K$ acts through a character. Regarding $\widetilde{T}_p \in \mathcal{H}_K(G(\mathbf{Q}_p))$, we have that the spherical representation $\pi(\chi_1, \chi_2)$ where $\chi_i = |\cdot|^{s_i}$ corresponds to Hecke eigenvalue $p^{s_1} + p^{s_2}$ for $\widetilde{T}_p$. Since the central character pins down $s_1 + s_2$, we conclude:

Corollary 1.17. *There is a unique (up to isomorphism) spherical representation of* $\mathrm{GL}_2(F)$ *with given central character and eigenvalue for* $\widetilde{T}_p$*.*

2. From classical modular forms to automorphic representations

2.1. **The automorphic representation associated to a modular form.** Let $f \in S_k(\Gamma_0(N), \chi)$. We let $\pi_f \subset \mathcal{A}_{\mathrm{cusp}}(G, \omega)$ be the automorphic representation generated by f.

Remark 2.1. Let us first clarify how ω and χ correspond to each other. We regard the central character ω of $Z(K)\backslash Z(\mathbf{A})$ as a Hecke character on $K^\times\backslash\mathbf{A}_K^\times$. Then it can can be regarded as a Dirichlet character on $(\mathbf{Z}/N\mathbf{Z})^\times$. We claim that (with the conventions made here), that character is χ^{-1}.

Why? If ϕ_f transform according to the central character ω then

$$\phi_f(g\iota_p \mathrm{diag}(z, z)) = \omega(z)\phi_f(g)$$

where ι_p is the inclusion of $\mathrm{GL}_2(\mathbf{Q}_p)$ into $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$. By Strong approximation and smoothness of ϕ_f, the action of $\iota_p\mathrm{diag}(z, z)$ coincides with that of $\iota_p\mathrm{diag}(z', z')$ for $z' \in \mathbf{Q}$ which is sufficiently congruent to z modulo p. (See Example 2.6 for a more formulation of this claim.) When we use an element of $Z(\mathbf{Q})$ to cancel out $\iota_p\mathrm{diag}(z', z')$, the archimedean coordinate is multiplied by $\iota_\infty\mathrm{diag}(z', z')^{-1}$.

Theorem 2.2. *Let* $f \in S_k(\Gamma_0(N), \chi)$ *be an eigenform with respect to almost all Hecke operators. Then* π_f *is irreducible.*

Proof Sketch. The basic idea is as follows. Suppose π_f decomposes into a sum of irreducible representations. For any constituent, the local components are almost all unramified, and then their spherical vectors are obtained from f, so their Hecke eigenvalues are all the same. By the classification of spherical representations, this implies that they are isomorphic. Also, since f is of weight k by inspection of the eigenvalue for the Laplacian, and f is killed by the lowering operator, the same holds for a vector in the archimedean component ∞ in all of the irreducible constituents, so they are also isomorphic at ∞. Being isomorphic at ∞ and almost all non-archimedean places, the Strong Multiplicity One theorem ([Feng]) implies that the constituents are globally isomorphic, hence by Multiplicity One are the same space. □

We now give a more careful development of the proof. Let $\widehat{\pi_f}$ denote the closure of π_f in $L^2_{\mathrm{cusp}}(\mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}_\mathbf{Q}), \omega)$.

Lemma 2.3. *We have a Hilbert direct decomposition*

$$\widehat{\pi_f} = \widehat{\bigoplus_{i\in I}} \widehat{\pi}_i$$

where π_i *is an admissible automorphic representation.*

Proof. We use that $\widehat{\pi_f}$ is a closed subrepresentation of $L^2_{\mathrm{cusp}}(\mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A_Q}),\omega)$, which decomposes discretely by [Howe1], [Howe2]. In fact we claim that any sub or quotient representation of a Hilbert direct sum of irreducible representations is again of this form. By duality, it suffices to prove the statement for quotients.

Let $V = \widehat{\bigoplus} V_i \twoheadrightarrow W$ be such a quotient. By Zorn's lemma, the collection of irreducible mutually orthogonal subrepresentations of W has a maximal element; we have to see that there is no non-zero vector orthogonal to their Hilbert direct sum $W' \subset W$. Otherwise, the image of some V_i is non-zero in W/W', hence gives an irreducible subrepresentation of $(W')^{\perp}$, which is a contradiction. □

As indicated in the proof sketch, it will suffice to show that the π_i are all isomorphic. By Flath's theorem, we have

$$\pi_i \cong {\bigotimes}' \pi_{i,p}$$

and by Strong Multiplicity One ([Feng], Theorem 1.7) it will suffice to show that for any i, j we have $\pi_{i,p} \cong \pi_{j,p}$ for almost all finite p, and at ∞.

We have

$$\phi_f = \sum_{i \in I} \phi_i, \quad \phi_i \in \pi_i.$$

We must have $\phi_i \neq 0$, since the projection map is $G(\mathbf{A})$-equivariant and the translates of ϕ_f generate π_f. Then ϕ_i is a Hecke eigenvector for T_p with Hecke eigenvalue $a_p(f)$.

We can write ϕ_i as a finite sum of pure tensors, almost all of whose local components are the spherical vector in $\pi_{i,p}$. Hence for almost all p, the pure tensor has the same component in at $\pi_{i,p}$. Since the Hecke algebra acts locally, this shows that the spherical vector has eigenvalue $a_p(f)$ for almost all p.

Next we move on to the archimedean place. The point is that the holomorphicity of f is equivalent to ϕ_f being killed by the lowering operator L, while the fact that f has weight k corresponds to

$$\phi_f(gk_\theta) = e^{ik\theta}\phi_f(g), \quad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2(\mathbf{R}),$$

or in other words that ϕ_f is the k-weight space of $U(\mathfrak{g})$. Therefore each $\pi_{i,\infty}$ has a non-zero vector, namely ϕ_i, which is killed by the lowering operator and in the k-weight space. But by the classification of irreducible admissible $(\mathfrak{g}, K)$-modules, this property characterizes the discrete series of weight k, i.e. it pins down $\pi_{i,\infty}$ up to isomorphism. This completes the proof. □

We can put an equivalence relation on Hecke eigenforms by saying that $f \sim f'$ if f and f' share almost all Hecke eigenvalues. Then the proof of Theorem 2.2 shows that:

Corollary 2.4. *There is a bijection between equivalence classes of $f \in M_k(\Gamma_0(N), \omega)$ and irreducible summands $\pi_f \subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_2, \omega)$.*

2.2. **Newforms.** In Example 1.11 we saw that there are maps

$$M_k(\Gamma_0(M), \chi) \rightrightarrows M_k(\Gamma_0(Md), \chi) \tag{2.1}$$

sending $f(z) \mapsto f(z)$ and $f(z) \mapsto f(dz)$.

Definition 2.5. The span $M_k^{\mathrm{old}}(\Gamma_0(N))$ of the images of (2.1) for all $1 < d \mid M$ in $M_k(\Gamma_0(N))$ is called the *space of old forms.* The *newforms* $M_k^{\mathrm{new}}(\Gamma_0(N))$ are the orthogonal complement of $M_k^{\mathrm{old}}(\Gamma_0(N), \chi)$ under the Petersson inner product.

Example 2.6. We already know from Example 1.11 that the process of producing oldforms doesn't alter Hecke eigenvalues, so they must lie in the same automorphic representation. Let's see this explicitly.

Recall from (1.3) that the modular form attached to an automorphic form is

$$f(z) = y^{-k/2}\Phi\left(\begin{pmatrix} y^{1/2} & y^{-1/2}x \\ & y^{-1/2}\end{pmatrix}\right)$$

Let's see how the action of $G(\mathbf{A}_f)$ turns newforms into oldforms. The action is to right multiply by

$$\iota_p\begin{pmatrix} p^{-1} & \\ & 1\end{pmatrix}$$

where ι_p is the inclusion of $G(\mathbf{Q}_p)$ into $G(\mathbf{A_Q})$. By Strong Approximation for SL_2 (plus the fact that $\mathbf{Q}$ has class number one), we can write any adele $g \in \mathrm{GL}_2(\mathbf{A})$ as

$$g = \gamma g_\infty k, \quad \gamma \in \mathrm{GL}_2(\mathbf{Q}), g_\infty \in \mathrm{GL}_2(\mathbf{R}), k \in K_0(N)$$

where $K_0(N)$ is the compact open subgroup corresponding to $\Gamma_0(N)$. Then

$$\gamma g_\infty k\left(\iota_p\begin{pmatrix} p^{-1} & \\ & 1\end{pmatrix}\right) = \gamma'\left(\iota^p\begin{pmatrix} p & \\ & 1\end{pmatrix}\right)g_\infty k',$$

where ι^p is the diagonal inclusion of $\mathbf{G}(\mathbf{Q})$ into $G(\mathbf{A}^p_{\mathbf{Q}})$. So we see that if Φ corresponds to $f(z)$, then $\iota_p\begin{pmatrix} p^{-1} & \\ & 1\end{pmatrix}\Phi$ essentially corresponds to the oldform $f(pz)$.

Theorem 2.7 ("Classical multiplicity one")**.** *Suppose $f, f' \in M_k^{\mathrm{new}}(\Gamma_0(N))$ are Hecke eigenforms which are newforms, and that $a_p(f) = a_p(f')$ for almost all p. Then f and f' are proportional.*

Remark 2.8. As Example 1.11 shows, this is definitely false without the assumption of both f and f' being newforms.

2.3. **The conductor of a local representation.**

Theorem 2.9 (Casselman, Novodvorskii)**.** *Let F be a non-archimedean local field with uniformizer ϖ and (V,π) be an admissible infinite-dimensional representation of $\mathrm{GL}_2(F)$, with central character ω. There exists a largest ideal $\mathrm{cond}(\pi)$ of $\mathcal{O}_F$ such that the space*

$$\left\{v \colon \pi(g)v = \omega(a)v \text{ for all } g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(\mathrm{cond}(\pi))\right\} \tag{2.2}$$

is non-zero, and moreover it is 1-dimensional.

Remark 2.10. The result was generalized to GL_n by Jacquet-Piatetski-Shapiro-Shalika.

The proof, which we defer to §2.9, will be by case analysis. For the principal series and special representations, one has an explicit model of the representation. For supercuspidal representations, one can use the Kirillov model for this purpose, but that analysis relies on more theory. A corollary of the *proof* of Theorem 2.9 is a more precise estimate on the number of fixed vectors as the level increases:

Corollary 2.11. *We have*

$$\dim\{v \colon \pi(g)v = \omega(g)v \text{ for all } g \in \Gamma_0(\mathrm{cond}(\pi)\varpi^i\} = i+1.$$

Definition 2.12. The $\mathrm{cond}(\pi)$ as in Theorem 2.9 is called the *conductor* of π.

Example 2.13. If π is spherical, then $\mathrm{cond}(\pi) = \mathcal{O}_F$.

2.4. **The conductor of an automorphic representation.** Now we turn our attention to the global case.

Definition 2.14. Let π be a (global) automorphic representation of GL_2. Then we have by Flath's theorem

$$\pi \cong \bigotimes_p{}' \pi_p.$$

Define the *conductor* of π to be

$$\mathrm{cond}(\pi) := \prod \mathrm{cond}(\pi_p).$$

By Example 2.13, this is well-defined.

Theorem 2.9 leads to a distinguished vector in the representation π_f. Indeed, for the infinite place we take the lowest weight vector, and for each finite place we take a vector in the 1-dimensional space (2.2) for the conductor $\mathrm{cond}(\pi_p)$. As π_p is unramified for almost all p, (2.2) becomes simply the 1-dimensional space of spherical functions, so we get a well-defined element of $\bigotimes' \pi_v$.

Proposition 2.15. *The preceding construction gives a bijection between newforms and constituents of* π_f.

Proof. We first need to check that the distinguished vector ϕ induces a newform f_ϕ. Let $N = \mathrm{cond}(\pi)$. It is clear from the definition that π_f cannot contain any oldforms of level N, as an oldform would give a vector invariant by a bigger group. Now, f_ϕ is evidently a Hecke eigenform of level $\Gamma_0(N)$, so by the orthogonality of different cuspidal automorphic representations it must be a newform. This shows at least that we have a well defined association from cuspidal automorphic representations to newforms.

Next we need to rule out the possibility that there are two different newforms f, f' in π. By the one-dimensionality in Theorem 2.9, if this were not the case then without loss of generality we may assume that $\mathrm{cond}(f) \mid \mathrm{cond}(f')$. By Corollary 2.11, the oldforms $f(p^i z)$ account for all functions in (2.2) transforming by ω under $\Gamma_0(\mathrm{cond}(f'))$.

□

Proof of Theorem 2.7. Theorem 2.15 plus Strong Multiplicity One for cuspidal automorphic representations (and the fact that modular forms of weight k have the same infinity type, as was explained in the proof of Theorem 2.2) immediately imply the classical Strong Multiplicity One Theorem for modular forms, Theorem 2.7. □

2.5. **L-functions.** We now compare the classical L-function attached to a modular form with the L-function of its associated automorphic representation.

Definition 2.16. Let $f \in M_k(\Gamma_0(N), \chi)$ be a modular form. We define its *L-function* to be

$$L(f, s) := \int_0^\infty f(iy) y^s \, \frac{dy}{y}.$$

Explicitly, if

$$f = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

then

$$L(f, s) = (2\pi)^{-s} \Gamma(s) \sum a_n n^{-s}.$$

If f is a newform, then we have $T_p(f) = a_p f$ for *every* p, hence

$$L(s,f) = (2\pi)^{-s}\Gamma(s)\prod_p \left(1 - a_p p^{-s} + p^{k-1-2s}\right)^{-1}$$

Corollary 2.17. *If $f \in S_k(\Gamma_0(1))$ is a cuspidal eigenform of level one, then we have $L(f,s) = L(\pi_f, s - \frac{k-1}{2})$.*

Proof. We have $T_{p^n} f = a_{p^n} f$. Since f is an eigenform, and T_n, T_m commute if $(m,n) = 1$ we have (at least formally)

$$\sum a_n n^{-s} = \prod_p \left(\sum a_{p^k} p^{-ks}\right).$$

From the relation

$$T_{p^n} T_p = T_{p^{n+1}} + p^{k-1} T_{p^{n-1}}$$

we find that (for $\operatorname{Rep} s \gg 0$)

$$\left(\sum a_{p^n} p^{-ns}\right) = (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

Therefore, we have

$$L(f,s) = (2\pi)^{-s}\Gamma(s)\prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

On the other hand,

$$L(\pi_f, s) = (2\pi)^{-s-\frac{k-1}{2}}\Gamma(s + \frac{k-1}{2}) \prod_{p<\infty} (1 - p^{s_1} p^{-s})^{-1}(1 - p^{s_2} p^{-s})^{-1}.$$

Since $a_p = p^{\frac{k-1}{2}}(p^{s_1} + p^{s_2})$, this can be rewritten as

$$L(\pi_f, s) = (2\pi)^{-s-\frac{k-1}{2}}\Gamma(s + \frac{k-1}{2}) \prod_{p<\infty} (1 - a_p p^{-s-\frac{k-1}{2}} + p^{-2s})^{-1}.$$

□

Remark 2.18. It is true more generally that if $f \in M_k(\Gamma_0(N))$ is a newform, then $L(f,s) = L(\pi_f, s)$. See [Gelb] §6.

However, if f is not a newform but comes from a newform f_0, then we will obviously have $L(\pi_f, s) = L(\pi_{f_0}, s) = L(f_0, s)$, which is *not* equal to $L(f,s)$.

2.6. **Proof of Theorem 2.9.** We proceed in cases, using the classification of irreducible admissible representations over a local field. The proof follows [Cass].

2.6.1. *Principal series.* By the Iwasawa decomposition, any $f \in \pi(\mu_1, \mu_2)$ is determined by its restriction to $\mathrm{GL}_2(\mathcal{O}_F)$. The central character is $\mu_1\mu_2$. Since a function in $\pi(\mu_1, \mu_2)$ is determined by its restriction to $\mathrm{GL}_2(\mathcal{O}_F)$, the space

$$\left\{ v \colon \pi(g)v = \omega(a)v \text{ for all } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\varpi^i) \right\} \tag{2.3}$$

is the same as the space of functions $f \colon \mathrm{GL}_2(\mathcal{O}_F) \to \mathbf{C}$ such that

$$f\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} g \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) = \mu_1(a)\mu_2(d) f(g) \mu_1\mu_2(a'),$$

$$\text{for all } g \in \mathrm{GL}_2(\mathcal{O}_F), \begin{pmatrix} a & b \\ & d \end{pmatrix} g \in B(\mathbf{Z}_p), \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0(\varpi^i)).$$

To do this, we want to compute a convenient set of coset representatives for

$$B(\mathcal{O}_F)\backslash \mathrm{GL}_2(\mathcal{O}_F)/\Gamma_0(\varpi^i).$$

Note that $\Gamma_0(\varpi^i) \supset \Gamma(\varpi^i)$, the kernel of reduction mod ϖ^i, so that

$$B(\mathcal{O}_F)\backslash \mathrm{GL}_2(\mathcal{O}_F)/\Gamma_0(\varpi^i) = B(\mathcal{O}_F/\varpi^i)\backslash \mathrm{GL}_2(\mathcal{O}_F/\varpi^i)/B(\mathcal{O}_F/\varpi^i). \tag{2.4}$$

Viewing $\mathrm{GL}_2(\mathcal{O}_F/\varpi^i)/B(\mathcal{O}_F/\varpi^i) = \mathbf{P}^1(\mathcal{O}_F/\varpi^i)$, the orbits of the left $B(\mathcal{O}_F/\varpi^i)$-action are represented by

$$(1,1),(1,\varpi),(1,\varpi^2),\dots(1,\varpi^i).$$

Tracing through what this means, it tell us that coset representatives for (2.4) are

$$\{\begin{pmatrix} 1 & 0 \\ \varpi^j & 1 \end{pmatrix} : j = 0,\dots,i\}.$$

Now we compute the stabilizer: we need to solve for a,b,d,a',b',d' such that

$$\begin{pmatrix} a & b \\ & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^j & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varpi^j & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ & d' \end{pmatrix}$$

This leads to the system of equations (mod ϖ^i):

$$\begin{aligned} a' &= a + \varpi^j b \\ b' &= b \\ \varpi^j a' &= \varpi^j d \\ d &= d' + \varpi^j b' \end{aligned}$$

We can use this to eliminate a, b', and d', so the stabilizer is parametrized by a', b, d subject to the one equation

$$\varpi^j a' = \varpi^j d.$$

The issue is then to solve for the number of j such that

$$\mu_1(a' + \varpi^j b)\mu_2(d) = \mu_1(a')\mu_2(a').$$

for all permissible a', b', d. Therefore we must have $\mathrm{cond}(\mu_1) \mid \varpi^j$. Since we are given that $d' \equiv a \pmod{\varpi^i}$, we next need that $\mathrm{cond}(\mu_2) \mid \varpi^{i-j}$. There is a unique solution for $\mathrm{cond}(\mu_1)\,\mathrm{cond}(\mu_2) = \varpi^i$, $j = \mathrm{cond}(\mu_1)$. Furthermore, we see that the number of solutions then increases by 1 when we increment i.

2.6.2. *Special representations.* Next we consider the special representations $\sigma(\mu|\cdot|^{\pm 1/2}, \mu|\cdot|^{\mp 1/2})$. The same analysis as before shows that the induced representation has conductor equal to $\mathrm{cond}(\mu)^2$. However, we have to worry whether or our-dimensional space contributes to $\sigma(\mu|\cdot|^{\pm 1/2}, \mu|\cdot|^{\mp 1/2})$ or to the 1-dimensional subquotient. It is easy to check that this can only happen if $\mathrm{cond}(\mu) = \mathcal{O}_F^\times$, and more work shows that it does in fact happen in this case. Therefore, we find that

$$\mathrm{cond}(\pi) = \begin{cases} \mathrm{cond}(\mu)^2 & \mathrm{cond}(\mu) = \mathcal{O}_F^\times \\ (\varpi) & \text{otherwise} \end{cases}$$

2.6.3. *Supercuspidal representation.* We will use the theory of the Kirillov model. Fix an additive character $\psi\colon F \to \mathbf{C}^\times$ with conductor $\mathcal{O}_F$. Recall that a supercuspidal representation has a unique realization on $C_c^\infty(F^\times)$ such that

$$\left(\rho\begin{pmatrix} a & b \\ & d\end{pmatrix} f\right)(\alpha) = \omega(d)\psi\left(\frac{b}{d}\alpha\right) f\left(\frac{a}{d}\alpha\right) \text{ for all } \begin{pmatrix} a & b \\ & d\end{pmatrix} \in B(F). \tag{2.5}$$

Since $\mathrm{GL}_2(F)$ is generated by $B(F)$ and $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$, the representation is completely determined once we specify $\rho(w)$.

It is more convenient to describe $\rho(w)$ in terms of the Fourier transform, and it is convenient to normalize things in the following way. For $f \in C_c^\infty(F^\times)$ and a character $\nu\colon \mathcal{O}_F^\times \to \mathbf{C}^\times$, we define

$$f_n(\nu) = \int_{\mathcal{O}_F^\times} f(u\varpi^n)\nu(u)\,du.$$

(Here the Haar measure on $\mathcal{O}_F^\times$ is normalized to have volume 1.) Define the formal power series

$$f(\nu, t) = \sum t^n f_n(\nu).$$

Since f is compactly supported, only finitely many terms can be non-zero. To get the Fourier transform with respect to a character $\widetilde{\nu}$ of $F^\times$, we set $t = \widetilde{\nu}(\varpi)$ and $\nu = \widetilde{\nu}|_{\mathcal{O}_F^\times}$.

The action of $\rho(w)$ is closely related to a Fourier transform. Its effect on the Fourier expansion is governed by the following Proposition.

Proposition 2.19 ([JL] p.48, 90-91)**.** *For every ν there is a formal power series $C(\nu, t)$ such that*

$$(\rho(w)f)(\nu, t) = C(\nu, t) f(\nu^{-1}\omega_0^{-1}, t^{-1}z_0^{-1})$$

for every $f \in C_c^\infty(F^\times)$, where $\omega_0 = \omega|_{\mathcal{O}_F^\times}$ and $z_0 = \omega(\varpi)$. Moreover, $C(\nu, t)$ is a monomial $C_0(\nu)t^{n_\nu}$ for some $n_\nu \leq -2$.

Now, we want to find the v such that

$$\rho\begin{pmatrix} a & b \\ c & d\end{pmatrix} v = \omega(a)v \text{ for all } \begin{pmatrix} a & b \\ & d\end{pmatrix} \in \Gamma_0(\varpi^i). \tag{2.6}$$

Let $H = \left(\begin{smallmatrix} 0 & 1 \\ -\varpi^i & 0\end{smallmatrix}\right)$. Note that

$$H^{-1}\begin{pmatrix} a & b \\ & d\end{pmatrix} H = \begin{pmatrix} a & \varpi^{-m}c \\ \varpi^m b & d\end{pmatrix}.$$

Therefore, $B(\mathcal{O}_F)$ and $HB(\mathcal{O}_F)H^{-1}$ generate $\Gamma_0(\varpi^i)$. So (using that conjugation by H swaps a and d), (2.6) is equivalent to the following system of equations:

$$\rho\begin{pmatrix} a & b \\ c & d\end{pmatrix} v = \omega(a)v \text{ for all } \begin{pmatrix} a & b \\ & d\end{pmatrix} \in B(\mathcal{O}_F). \tag{2.7}$$

$$\rho\begin{pmatrix} a & b \\ c & d\end{pmatrix} Hv = \omega(d)Hv \text{ for all } \begin{pmatrix} a & b \\ & d\end{pmatrix} \in B(\mathcal{O}_F). \tag{2.8}$$

Now let's find what constraints (2.7) imposes on f. According to the transformation property (2.5) of the Kirillov model, (2.7) is equivalent to

$$\omega(d)\psi\left(\frac{b}{d}\alpha\right) f\left(\frac{a}{d}\alpha\right) = \omega(a)f(\alpha) \text{ for all } \begin{pmatrix} a & b \\ & d\end{pmatrix} \in B(\mathcal{O}_F).$$

or simplifying,

$$f\left(\frac{a}{d}\alpha\right) = \omega\left(\frac{a}{d}\right)\psi\left(-\frac{b}{d}\alpha\right) f(\alpha) \text{ for all } \begin{pmatrix} a & b \\ & d \end{pmatrix} \in B(\mathcal{O}_F).$$

Since ψ has conductor $\mathcal{O}_F$ and b/d varies freely within $\mathcal{O}_F$, this is equivalent to:

(1) f vanishes outside $\mathcal{O}_F$, and
(2) $f(u\alpha) = \omega(u) f(\alpha)$ for all $u \in \mathcal{O}_F^\times$.

The second condition implies that the Fourier coefficients of F are supported on a particular family of characters. Considering

$$\int_{\mathcal{O}_F^\times} f(\varpi^n u)\nu(u)\, du = f(\varpi^n) \int_{\mathcal{O}_F^\times} \omega(u)\nu(u)\, du$$

we see that (2.7) is equivalent to:

$$f_n(\nu) \text{ is supported on } \nu = \omega^{-1}|_{\mathcal{O}_F^\times} \text{ and } n \geq 0. \tag{2.9}$$

We apply a similar analysis to (2.8): it tells us that

$$\omega(d)\psi\left(\frac{b}{d}\alpha\right)(Hf)\left(\frac{a}{d}\alpha\right) = \omega(d)(Hf)(\alpha), \text{ for all } \begin{pmatrix} a & b \\ & d \end{pmatrix} \in B(\mathcal{O}_F).$$

which is equivalent to

$$(Hf)_n(\nu, t) \text{ is supported on } n \geq 0 \text{ and } \nu \text{ trivial.} \tag{2.10}$$

Let's translate this into a statement about f. We can write

$$H = \begin{pmatrix} 0 & 1 \\ -\varpi^i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \varpi^i & 0 \\ 0 & 1 \end{pmatrix}$$

so we can write $\rho(H)f = \rho(w)([\varpi^i]f)$. Then

$$\begin{aligned} (Hf)(1,t) &= (w([\varpi^i]f))(1,t) \\ &= \underbrace{C_0(1)t^{n_1}}_{C(1,t)}([\varpi^i]f)(\omega^{-1}, z_0^{-1}t^{-1}) \\ &= C_0(1)t^{n_1} f(\omega^{-1}, t^{-1}z_0^{-1})t^{-i}z_0^{-i}. \end{aligned}$$

So what we've found is that (2.10) is equivalent to:

$$f(\nu, t) \text{ is supported on } \nu = \omega \text{ and } n_1 - n - i \geq 0. \tag{2.11}$$

Combining (2.9) and (2.11), we find that the dimension of the space is $\#\{i \colon 0 \leq n \leq i + n_1\}$.

2.7. **Local components.** Let $f \in S_k^{\text{new}}(\Gamma_0(N))$ be an eigenform (note that ask for trivial nebentypus). It is natural to ask what we can say about the automorphic representation π_f. For example, we have a classification of local representations; can we say what the local components of π_f are? All all the places are easy: for $p = \infty$, the infinity type is the discrete series of weight k. For $p \nmid N$, π_f is spherical, and can therefore be pinpointed from its Hecke eigenvalues.

It is difficult to say what happens at the primes $p \mid N$ in general, but we now know enough to answer this in a special case. In the course of our analysis in the proof of Theorem 2.9, we saw the following outcomes concerning the conductors:

Representation	Conductor
$\pi(\mu_1, \mu_2)$	$c(\mu_1)c(\mu_2)$
$\pi(\mu_1, \mu_2)$	$c(\mu)^2$ or (ϖ)
supercuspidal	≥ 2

Suppose $N = \prod p_i$ is a product of *distinct* primes. Then π_f has conductor (ϖ) at p_i, so it cannot be supercuspidal, since we saw in the proof of Theorem 2.9 that the conductor of a supercuspidal representation is contained in ϖ^2. If π_{f,p_i} were principal series, then its conductor would be an even power of ϖ since the triviality of the central character implies $\mathrm{cond}(\mu_1) = \mathrm{cond}(\mu_2)$. Therefore π_{f,p_i} must be a special representation. In summary, we have the following:

Theorem 2.20. *Suppose $N = \prod p_i$ is a product of* distinct *primes. If $f \in S_k(\Gamma_0(N))$ is a newform and $T(p)f = a_p f$ for all p, then π_f has the following local components:*

(1) π_∞ is the discrete series representation of weight k.
(2) If $(p, N) = 1$ then π_p is the principal series representation $\pi(\mu_1, \mu_2)$ where $|\mu_i| = |t|^{s_i}$ is determined by the conditions

$$\mu_1\mu_2 = 1$$
$$a_p = p^{\frac{k-1}{2}}(p^{s_1} + p^{s_2})$$

(3) If $p = p_i$, then π_p is the special representation with trivial central character χ_p.

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