LECTURE 26: CONVERSE THEOREM
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As always, $G = \mathrm{GL}_2$, and we work over a global field k with adele ring $\mathbf{A}$. Today, we will prove the *converse theorem*. We also fix a non-trivial additive character $\psi\colon k^+\backslash\mathbf{A}^+ \to \mathbf{C}$ everywhere. This is a converse to the following theorem, shown in Lecture 23:

Theorem 1. Suppose that $\pi \subseteq \mathcal{A}_{\mathrm{cusp}}(G(\mathbf{Q})\backslash G(\mathbf{A}), \omega)$ is a cuspidal automorphic representation. Then for every character χ of $k^\times\backslash\mathbf{A}^\times$, $L_\pi(\chi, s)$ is entire, bounded in vertical strips, and satisfies the functional equation:

$$L_\pi(\chi, s) = \epsilon_\pi(\chi, s) L_\pi(\omega\chi^{-1}, 1-s) \tag{1}$$

Precisely, the converse theorem states:

Theorem 2 (Converse Theorem)**.** If $\pi = \bigotimes_v' \pi_v$ is an irreducible admissible representation of $G(\mathbf{A})$ such that:

- π has unitary central character $\omega : k^\times\backslash\mathbf{A}^\times \to \mathbf{C}$.
- For each v, π_v is an irreducible admissible representation of $G(k_v)$.
- Each π_v is *pre-unitary*.
- Each π_v is infinite-dimensional (i.e. generic, i.e. admits a Whittaker model).
- π_v is spherical for all but finitely many v.

Then, if $L_\pi(\chi, s)$ is entire, bounded in vertical strips, and satisfies the functional equation (1), π appears in $\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbf{A}), \omega)$.

The first step of the proof will be to produce a map $\pi \to \mathrm{Fun}(G(\mathbf{A}), \mathbf{C})$, and then we will verify that the image lies inside $\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbf{A}), \omega)$. We will only use the L-function to see that the functions in the image are w-invariant for $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in G(k)$.

Now, we build a candidate map. Recall that for $\pi' \subseteq \mathcal{A}_{\mathrm{cusp}}$ and $\varphi \in \pi'$ an automorphic form in π', we have the following identity, coming from Fourier inversion:

$$\varphi(g) = \sum_{\alpha \in k^\times} W_\varphi[\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) g] \tag{2}$$

Here, W_φ is the function on $\mathrm{G}(\mathbf{A})$ corresponding to φ in the Whittaker model of π'.

Note that the terms in the right-hand side of this formula makes sense without assuming that π' appears in the space of automorphic forms, since the Whittaker model is attached intrinsically to the $G(\mathbf{A})$-representation π'. Thus we may use (2) as a candidate formula for our map from π to $\mathcal{A}_{\mathrm{cusp}}(G(\mathbf{Q})\backslash G(\mathbf{A}), \omega)$.

More precisely, since π_v is generic for all v, the local Whittaker models $\mathscr{W}_v$ exist for all v, and therefore we have the global Whittaker model $\pi \simeq \mathscr{W} = \bigotimes_v' \mathscr{W}_v$. For each v such that π_v is spherical and ψ_v has conductor 0, we consider the unique element $W_v^0 \in \mathscr{W}_v$ which is $\mathrm{GL}_2(\mathscr{O}_v)$-invariant and which is uniformly equal to 1 on $\mathrm{GL}_2(\mathscr{O}_v)$. The elements of $\mathscr{W}$ are finite sums of functions $\prod_v W_v$ with $W_v = W_v^0$ for all but finitely many v.

Then, our candidate map $\mathscr{W} \to \mathrm{Fun}(G(\mathbf{A}), \mathbf{C})$ is defined by:

$$W \mapsto \left(\varphi_W \colon g \mapsto \sum_{\alpha \in k^\times} W\big(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) g\big) \right)$$

Once we have shown that this sum converges absolutely to a nice function, cuspidality will follow immediately (indeed, the Fourier expansion corresponding to (2) has no constant term!).

Thus, we need to show:

(1) The series above defining φ_W is absolutely convergent, and the resulting function is smooth with moderate growth.

(2) The function φ_W is left-invariant under $G(k)$.

This will suffice for the proof of the converse theorem, since we will have defined a non-zero map of $G(\mathbf{A})$ representations from the irreducible $\mathscr{W}$ to $\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbf{A}), \omega)$. Since we know this latter space decomposes as a direct sum of irreducible representations with multiplicity 1, it will follow that π is uniquely a direct summand of this space.

Proof of (1): We first observe that there is considerable redundancy in the sums defining $\varphi_W(g)$: at least formally, $\varphi_{h\cdot W}(g) = \varphi_W(gh)$ for $h \in K_\infty \times G(\mathbf{A}_f)$. As we shall see later, the Iwasawa decomposition at archimedean places and unitarity of ψ and ω will allow us to simplify even further, reducing to the sums defining $\varphi_W(g)$ for $g = \left(\begin{smallmatrix} x_\infty & 0 \\ 0 & 1 \end{smallmatrix}\right) \times \mathrm{Id}$ for $x_\infty \in \mathbf{A}_\infty^\times$. Thus, we are free to work in this simplifed setting for our initial estimates. In fact, because it will cause no further difficulty and be useful later, we will make our initial estimates when $g = \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for any $x \in \mathbf{A}$. We will also assume $W = \bigotimes W_v$ is a pure tensor.

Thus, we are considering sums of the form

$$\sum_{\alpha \in k^\times} W\big(\left(\begin{smallmatrix} \alpha x & 0 \\ 0 & 1 \end{smallmatrix}\right) \big). \tag{3}$$

We now turn our attention to bounding the summands. We first observe that for $W_v = W_v^0$, we have $W_v^0\big(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)\big) = 0$ if $x \notin \mathscr{O}_v$. In general, W_v has compact support in k_v. Thus, there is a compact open subset $U \subset \mathbf{A}_f$ such that the support of $x \mapsto W\big(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)\big)$ is contained in $(\mathbf{A}_\infty \times U) \cap \mathbf{A}^\times$.

We also need an estimate of $W\big(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)\big)$ that is valid for $|x|$ large (which is possible even if $x_f \in U$). The first step is to give uniform estimates for $W_v^0\Big(\left(\begin{smallmatrix} x_v & 0 \\ 0 & 1 \end{smallmatrix}\right)\Big)$. Since π_v is spherical for the places where we have defined W_v^0, we know that $\pi_v \simeq \pi_{\mu_v,\nu_v}$ is (non-special) unramified principal series (i.e. μ_v, ν_v are unramified characters). Thus, we have the explicit formula:

$$W_v^0\Big(\left(\begin{smallmatrix} x_v & 0 \\ 0 & 1 \end{smallmatrix}\right)\Big) = |x_v|^{1/2} \sum_{\substack{i,j \geqslant 0 \\ i+j=v(x_v)}} \mu_v(\varpi_v^i)\nu_v(\varpi_v^j)$$

Since we assumed that π_v is pre-unitary for all v, there exists $-1 < 0 \leqslant t < 1$ such that $|\mu_v| = |\cdot|^{t/2}$ and $|\nu_v| = |\cdot|^{-t/2}$ (cf. Lecture 16, Theorem 4). Thus, for each term $i, j \geqslant 0$, $i + j = v(x_v)$, we have

$$|\mu_v(\varpi_v)^i \nu_v(\varpi_v)^j|_v \leqslant |x_v|^{-1/2}.$$

Thus,

$$\left|W_v^0\left(\left(\begin{smallmatrix} x_v & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\right| \leqslant (v(x_v)+1)|x_v|_v^{1/2-1/2} = v(x_v) + 1 \leqslant |x_v|^{-1}$$

where here we use that for $v(x_v) \geqslant 0$, $v(x_v) + 1 \leqslant |x_v|^{-1}$.

For any other finite place v, we have:

$$\left|W_v\left(\left(\begin{smallmatrix} x_v & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\right| \leqslant C_v |x_v|^{-1-\sigma_v}$$

for some $C_v > 0, \sigma_v > 0$. There is also a bound available for the archimedean places (due to rapid decay of the Whittaker functions for the archimedean places). Multiplying everything together gives us that:

$$\left|W\left(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\right| \leqslant C_1 |x|^{-1-\sigma} e^{-C_2\|x\|_\infty}$$

for some $C_1, C_2, \sigma > 0$.

Combined this estimate with the vanishing outside of $\mathbf{A}_\infty \times U$, we conclude that for

$$F = C_1 \cdot \chi_{\mathbf{A}_\infty \times U} \cdot e^{-C_2\|x\|_\infty},$$

a function on $\mathbf{A}$, we have

$$\left|W\left(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\right| \leqslant |x|^{-1-\sigma} F(x)$$

for all $x \in \mathbf{A}^\times$.

Note that F is a positive Bruhat-Schwarz function on $\mathbf{A}$. Using this, we can estimate (3) in terms of F:

$$\begin{aligned}
\left|\varphi_W\left(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\right| &\leqslant \sum_{\alpha \in k^\times} \left|W\left(\left(\begin{smallmatrix} \alpha x & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\right| \\
&\leqslant \sum_{\alpha \in k^\times} |\alpha x|^{-1-\sigma} F(\alpha x) \\
&\leqslant |x|^{-1-\sigma} \sum_{\alpha \in k^\times} F(\alpha x)
\end{aligned}$$

(where in the last step we have used the product formula $|\alpha| = 1$ for $\alpha \in k^\times$.) Because F is Bruhat-Schwarz, $\sum_{\alpha \in k^\times} F(\alpha x)$ is rapidly decreasing as $|x| \to \infty$, and we conclude not only that the sum converges absolutely but that $\varphi_W(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right))$ is rapidly decreasing as $|x| \to \infty$.

We can also use this estimate to control the behavior as $|x| \to 0$: using Poisson summation, we find

$$\sum_{\alpha \in k^\times} F(\alpha x) \leqslant \sum_{\alpha \in k} F(\alpha x) = |x|^{-1} \sum_{\alpha \in k} \widehat{F}(x^{-1}\alpha)$$

Since $\widehat{F}$ is also Bruhat-Schwarz, as $|x| \to 0$ the sum is dominated by the term when $\alpha = 0$ (the difference is rapidly decreasing), and we conclude

$$\left|\varphi_W(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right))\right| = O(|x|^{-2-\sigma}).$$

Taking everything together, we've now established:

Lemma 3. For W a pure tensor and $x \in \mathbf{A}^\times$, the series defining $\varphi_W(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right))$ is absolutely convergent, and satisfies the growth conditions:

(a) For *any* N we have, as $|x| \to \infty$,

$$|\varphi_W\big(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)\big)| = O(|x|^{-N})$$

(b) There exists a $q > 0$ such that as $|x| \to 0$,

$$|\varphi_W\big(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)\big)| = O(|x|^{-q})$$

Using these estimates, we can now show

Theorem 4. For any g and $W \in \mathscr{W}$, the sum defining $\varphi_W(g)$ is absolutely convergent. Moreover, φ_W is a C^∞ function on $G(\mathbf{A})$, and for any $g \in G(\mathbf{A})$,

(a) For any $N > 0$ we have, as $|x| \to \infty$,

$$|\varphi_W\big(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right) g\big)| = O(|x|^{-N})$$

(b) There exists a $q > 0$ such that as $|x| \to 0$,

$$|\varphi_W\big(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right) g\big)| = O(|x|^{-q})$$

Proof. For any $h \in K_\infty \times G(\mathbf{A}_f)$, the series defining $\varphi_W(g)$ is the same as that defining $\varphi_{h \cdot W}(gh^{-1})$, so, to prove $\varphi_W(g)$ is well-defined and smooth at g, by changing W it suffices to assume $g = g_\infty \times \mathrm{Id}$ and that $g_\infty = u_\infty m_\infty$ for u upper nilpotent and m in the diagonal torus (here we have also used the Iwasawa decomposition at each archimedean place). Similarly, it suffices to prove the desired estimates for such a g and any W.

We start by proving absolute convergence. Since any W is a finite sum of pure tensors, it suffices to suppose W is a pure tensor. We write $u_\infty \times 1 = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ and $m_\infty \times 1 = \left(\begin{smallmatrix} xz & 0 \\ 0 & z \end{smallmatrix}\right)$ so that

$$g = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} xz & 0 \\ 0 & z \end{smallmatrix}\right).$$

Now we observe that (using the ψ-invariance of W under left multiplication by upper unipotents and the central character of W):

$$W\big(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} xz & 0 \\ 0 & z \end{smallmatrix}\right)\big) = \psi(u\alpha)\omega(z) W\big(\left(\begin{smallmatrix} \alpha x & 0 \\ 0 & 1 \end{smallmatrix}\right)\big)$$

Thus, because ψ and ω are unitary,

$$|W\big(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) g\big)| = |W\big(\left(\begin{smallmatrix} \alpha x & 0 \\ 0 & 1 \end{smallmatrix}\right)\big)|.$$

Thus, we obtain absolute convergence from Lemma 3.

We now show smoothness. Since W is fixed by a compact open in $G(\mathbf{A}_f)$, so is φ_W, and thus it remains only to show that as we vary g_∞ in a small neighborhood we obtain a smooth function. Note that W lives in a finite dimensional subspace of $\mathscr{W}$ on which K_∞ acts by analytic characters, i.e.

$$(k_\infty \times \mathrm{Id}) \cdot W = \chi_1(k) a_1 W_1 + \ldots + \chi_n(k) a_n W_n$$

for characters χ_i of K_∞, $a_i \in \mathbf{C}$, and a basis W_i for the finite dimensional space. Thus, we find

$$\begin{aligned}\varphi_W(u_\infty m_\infty k_\infty) &= \varphi_{k_\infty \times \mathrm{Id} \cdot W}(u_\infty m_\infty) \\ &= \sum_{i=1}^{n} \chi_i(k) a_i \cdot \varphi_{W_i}(u_\infty m_\infty)\end{aligned}$$

On the other hand, using the coordinates x, z, and u as above, we find

$$\varphi_{W_i}(u_\infty m_\infty) = \sum_{\alpha \in k^\times} \psi(u\alpha)\omega(z) W_i\left(\left(\begin{smallmatrix} \alpha x & 0 \\ 0 & 1 \end{smallmatrix}\right)\right).$$

Putting these together, we find that the absolute convergence is uniform, and thus it converges to a smooth (in fact analytic) function because the terms are also smooth (in fact analytic).

Finally, the estimates are immediate by following the proof of Lemma 3 using the same termwise removal of u_∞ and z_∞ as above (the characters then disappear when we take the absolute value of a term). □

Proof of (2): Now, we need to show that our function φ_W on $G(\mathbf{A})$ is left $G(k)$-invariant. First, we will show that φ_W is left-invariant by $b \in B(k)$, the upper-triangular Borel subgroup. For $b = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, we use the fact that W is a Whittaker function and that ψ is trivial on k. For $b = \left(\begin{smallmatrix} * & 0 \\ 0 & 1 \end{smallmatrix}\right)$, left-invariance is built into the definition since we are adding $W\left(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) g\right)$ over all $\alpha \in k^\times$. Finally, when $b = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right)$, we get invariance because ω is trivial on $k^\times$.

Now, by Bruhat decomposition, we have boiled down the question of verifying left $G(k)$-invariance to showing the following identity:

$$\varphi_W(g) = \varphi_W(wg)$$

for $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Instead of trying to prove this directly for a single value of g, it will be helpful to consider the values on a vertical line through x. Precisely, we define two functions on $\mathbf{A}^\times$:

$$F_1(x) := \varphi_W\left(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right) g\right), \qquad F_2(x) := \varphi_W\left(w \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right) g\right) = \omega(x)\varphi_W\left(\left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) wg\right).$$

Clearly it suffices to show that $F_1(x) = F_2(x)$ (take the value at 1).

Now, we can take "Mellin transforms" so we can relate this discussion to the L-functions. Recall from Lecture 23 that we studied the ζ-integral:

$$\zeta(W, g; \chi, s) = \int_{k^\times \backslash \mathbf{A}^\times} W\left(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right) g\right) \chi(x)^{-1} |x|^{2s-1}\, d^\times x$$

Thus, we have:

$$\zeta(W, g; \chi, s) = \int_{\mathbf{A}^\times} W\big(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)\big)\chi(x)^{-1}|x|^{2s-1}\, d^\times x = \int_{k^\times\backslash\mathbf{A}^\times} F_1(x)\chi(x)^{-1}|x|^{2s-1}\, d^\times x$$

where the right hand side converges for $\mathrm{Re}\, s >> 0$.

This shows that we can think of $\zeta(W, g; \chi, s)$ as the Mellin transform of F_1, and similarly that we can think of $\zeta(W, wg; \omega\chi^{-1}, 1-s)$ as the Mellin transform of F_2 when $\mathrm{Re}\, s << 0$.

Now, where defined, these ζ integrals are related to the L-functions by:

$$\zeta(W, g; \chi, s) = L_\pi(\chi, s) \prod_v \frac{\zeta_v(W_v, g_v; \chi_v, s)}{L_{\pi_v}(\chi_v, s)}$$

Here, the factors in the product on the right are 1 whenever $W_v = W_v^0$, so it is a finite product. Similarly:

$$\zeta(W, wg; \omega\chi^{-1}, 1-s) = L_\pi(\omega\chi^{-1}, 1-s) \prod_v \frac{\zeta_v(W_v, wg_v; \omega_v\chi_v^{-1}, s)}{L_{\pi_v}(\omega_v\chi_v^{-1}, s)}$$

Now, because we have assumed L_π is entire, and by local considerations the terms of the finite products are entire, we find that the Mellin transforms of F_1 and F_2 are both entire. Moreover, the functional equations we have assumed for L_π combined with the local functional equations show that these entire functions agree, i.e. the analytic continuation of the Mellin transform of F_1 equals the analytic continuation of the Mellin transform of F_2.

Note, however, that this does not immediately imply $F_1 = F_2$. One must show the entire Mellin transform is bounded in vertical strips (using the assumed fact for L_π and an analysis of the local factors), and then use an argument with Mellin inversion for $F \star F_1$ and $F \star F_2$ for F compactly supported to show that $F_1 = F_2$. The boundedness in vertical strips will be necessary to compare integrals in the two inversion formulae, one of which will be taking place along a vertical line with very postive real value, and the other along a vertical line with very negative real value; we refer the reader to [1] for a more thorough discussion. This finishes the proof of (2), and thus of the converse theorem itself!

We give an example application:

Theorem 5. The elliptic curve $E : y^2 = x^3 - x$ over $\mathbf{Q}$ is modular.

To prove this, we will use the following:

Exercise 6. The L-function of $E/\mathbf{Q}$ is the same as the L-function for the Großencharacter of the associated CM field $\mathbf{Q}(i)$ sending $\mathfrak{p}$ to the unique generator of $\mathfrak{p}$ which is congruent to 1 mod $(1+i)^3$.

So, the L-function for E is equal to the L-function for an automorphic form on $k^\times\backslash\mathbf{A}^\times$ for $k = \mathbf{Q}(i)$, which has good analytic behavior by Tate's thesis. The same holds for all the twists, given by composition of an idele class character of $\mathbf{Q}$ with the norm map. In order to apply the Converse theorem, we now need to show that this L-function is equal to $L_\pi(\chi, s)$ for some irreducible admissible $G(\mathbf{A})$-representation π (and similarly for the twists). We will do this by building π_v separately for each v and taking $\pi = \bigotimes_v' \pi_v$.

In other words, we need to show that for each place v of $\mathbf{Q}$, we can find a π_v with the same L-factor and ϵ-factor as the Großencharacter. We can check that at ∞, we can take the discrete series of weight 2. At $p \nmid 2 \cdot (\infty)$, we can take an unramified principal series, and at $p = 2$, we have a supercuspidal representation. All of these are more or less explicit in terms of the given data (there is a bit of work to be done at two since I don't think we have talked about the relations between characters of quadratic extensions and supercuspidals in this seminar).

Remark 7. This example illustrates a general principle about applications of the converse theorem to functoriality (another example to which this principle applies is the Jacquet-Langlands transfer from automorphic representations of quaternion algebras to automorphic representations GL_2.) To apply the converse theorem to prove a global functoriality, one needs two ingredients: the first is a good understanding of the relevant local functorialities (in the above example, constructing the local representations of $\mathrm{GL}_2/\mathbf{Q}$ whose L and ϵ-factors match with the L and ϵ-factors of the local restrictions of the idele character for $\mathbf{Q}(i)$), and the second is the ability to prove good analytic properties of the global L-function in the initial setting (in the above example, the application of Tate's thesis).

References

[1] Godement, R. *Notes on Jacquet-Langlands' Theory*, The Institute for Advanced Study, 1970, available at http://math.stanford.edu/ conrad/conversesem/refs/godement-ias.pdf.