LECTURE 3: REDUCTION THEORY II

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We will keep our running notation, with k a global field and $\mathbf{A}$ its adele ring.

Recall that last time, in order to control compactness properties, we introduced the modified adelic groups $H(\mathbf{A})^1$. These are defined by

$$H(\mathbf{A})^1 := \{h \in H(\mathbf{A}) \mid |\chi(h)|_k = 1\ \forall \chi\colon H \to \mathbf{G}_m, \chi \text{ is defined over } k\}$$

Note that $H(\mathbf{A})^1 = H(\mathbf{A})$ when H is anisotropic, since this means that H has no k-characters.

Theorem 1. Let $H_1 \hookrightarrow H_2$ be a closed k-subgroup with H_2 a linear algebraic group. Then $[H_1] \to [H_2]$ is a closed embedding, where $[H] = H(k)\backslash H(\mathbf{A}_k)^1$

Remark 2. We saw at the end of the last lecture that this is not true in general if we use $H(\mathbf{A})$ instead of $H(\mathbf{A})^1$: we saw a counterexample with $B \hookrightarrow \mathrm{PGL}_2$ for B a Borel subgroup.

In order to handle reduction theory, even in the case $G = \mathrm{GL}_2$, over arbitrary number fields, we will need to prove statements over $\mathbf{Q}$ or other small number fields to avoid large class number problems. The following example is useful for this:

Example 3. For k'/k a finite separable extension and H any linear algebraic k-group, we have $H \hookrightarrow R_{k'/k}(H_{k'}) =: H'$, where $R_{k'/k}(H_{k'})$ is the *Weil restriction* functor (which is representable by a linear algebraic k-group). This functor is defined by $R_{k'/k}(H_{k'})(A) = H_{k'}(A \otimes_k k')$, so the injection $H \hookrightarrow R_{k'/k}(H_{k'})$ is given by sending a k-morphism $\operatorname{Spec} A \to H$ to its base change by k'.

Then we have $[H'] = H_{k'}(k')\backslash H_{k'}(\mathbf{A}_{k'})^1$, since $R_{k'/k}(X')(\mathbf{A}_k) = X'(\mathbf{A}_{k'} = k' \otimes_k \mathbf{A}_k)$ as *topological spaces*: see [2, Example 2.4]. The fact that this identification sends $R_{k'/k}(X')(\mathbf{A}_k)^1$ to $X'(A_{k'})^1$ comes from the following proposition:

Proposition 4. If G' is a linear algebraic k'-group, then characters $\chi'\colon G' \to \mathbf{G}_m$ defined over k naturally give characters $\chi\colon R_{k'/k}(G') \to \mathbf{G}_m$ by $\chi' \mapsto N_{k'/k} \circ R_{k'/k}(\chi')$, and this map is an isomorphism between the k'-character group of G' and the k-character group of $R_{k'/k}(G')$. See [3, Ch.1, §2].

This proposition can allow certain problems for $[G]$ to be studied in the split case, using $G_{k'}$ for suitable k'/k.

Now, let's prove Theorem 1:

Proof. in order to show that the continuous injective map $[H_1] \hookrightarrow [H_2]$ is a closed embedding, the key fact will be that the image is *closed*. Equivalently, we can show that $H_2(k)H_1(\mathbf{A})^1 \subseteq H_2(\mathbf{A})^1$ is closed. See [1, Lemma 4.2.5] for this reduction step.

Now, we can pick a representation (defined over k) $\rho\colon H_2 \to \mathrm{GL}(V)$ such that $H_1 = \mathrm{Stab}_{H_2}(L)$ for some line $L \subseteq V$. This can always be done: see [4, §14]. In particular, choosing any point $x \in L \setminus \{0\}$, we get a character $\chi\colon H_1 \to \mathbf{G}_m = \mathrm{GL}(L)$ defined functorially by $h \mapsto \frac{h \cdot x}{x}$. On $\mathbf{A}$-points, this gives $\chi_{\mathbf{A}}\colon H_1(\mathbf{A}) \to \mathbf{A}^\times$.

For $x \in L \setminus \{0\}$, consider the morphism $H_2 \to \underline{V}$ given functorially on points over any k-algebra by $h \mapsto h \cdot x$, where $\underline{V}$ is the affine k-variety associated to V, i.e. we have $V(A) = V \otimes_k A$ for any k-algebra A. Taking $A = \mathbf{A}$, we get an orbit map $H_2(\mathbf{A}) \to \underline{V}(\mathbf{A})$. Consider $Z := (\mathbf{A}^\times)^1 \cdot H_2(k) \cdot x \subseteq V(\mathbf{A})$, and consider $Z' = H_2(\mathbf{A}) \times_{V(\mathbf{A})} Z$ inside $H_2(\mathbf{A})$. We can check that $Z' = H_2(k)\chi_{\mathbf{A}}^{-1}(\mathbf{A}^\times)^1$ (if $h \in H_2(\mathbf{A})$ is such that $h \cdot x \in (\mathbf{A}^\times)^1 \cdot x$, then in particular h stabilizes L, so $h \in H_1$). This contains $H_2(k) \cdot H_1(\mathbf{A})^1$ as a *closed* subgroup, since $h \in H_1(\mathbf{A})^1$ implies that $|\chi_{\mathbf{A}}(h)| \leq 1$. Thus, it suffices to show that Z is closed in $V(\mathbf{A})$.

Let's see that $Z \subseteq V(\mathbf{A})$ is closed. Recall from algebraic number theory that the norm-one part of the idèle class group $k^\times \backslash (\mathbf{A}^\times)^1$ is *compact*. So, for a suitable compact subset K of $(\mathbf{A}^\times)^1$, we have $(\mathbf{A}^\times)^1 = K \cdot k^\times$. Thus, we have $(\mathbf{A}^\times)^1 \cdot H_2(k) \cdot x = K \cdot \big(H_2(k)k^\times \cdot x\big)$. But since $H_2(k) \cdot k^\times \cdot x \subseteq V(k)$ and $V(k)$ is *discrete* in $V(\mathbf{A})$, the fact that $K \cdot (H_2(k) \cdot k^\times \cdot x)$ is closed in $V(\mathbf{A})$ follows from the fact that a *compact* subspace times a discrete subspace is closed. □

Now, we want to address the following via *reduction theory*:

Questions. For G a connected reductive group over k:

1. When is $[G]$ compact?
2. When it is non-compact, is there some approximate fundamental domain which can be used to show that $\mathrm{vol}([G]) < \infty$.

Reduction theory will give us the following theorems:

Theorem 5 ("Theorem C": Mostow-Tamagawa, Harder)**.** For G a connected reductive group over a global field k, $[G]$ is compact iff $\mathscr{D}G$ is k-anisotropic (i.e. it has no non-trivial split tori, or equivalently it has no proper parabolic k-subgroups).

Remark 6. For $k = \mathbf{Q}$ and G semisimple and simply connected, the strong approximation theorem implies that $[G]$ is compact iff $G(\mathbf{R})/G(\mathbf{Z})$ is.[1] The latter statement is how people would have stated this in the 1950's, with no adelic points in sight.

Let's see an example where the simply-connected assumption may be relaxed:

Example 7. Let (L, q) be a quadratic lattice over $\mathbf{Z}$ such that $L_{\mathbf{Q}}$ is non-degenerate and $\mathbf{Q}$-anisotropic, meaning that it has no non-trivial rational zeros.[2] Then $\mathrm{SO}(q)(\mathbf{R})/\mathrm{SO}(q)(\mathbf{Z}) = \mathrm{SO}(L_{\mathbf{R}})/\mathrm{SO}(L)$ is *compact*.

Indeed, the "universal cover" $\mathrm{Spin}(q) \to \mathrm{SO}(q)$ is a central isogeny of $\mathbf{Q}$-groups, so $\mathrm{Spin}(q)$ is $\mathbf{Q}$-anisotropic[3] and simply connected. Thus by Theorem C and the remark following it, $\mathrm{Spin}(q)(\mathbf{R})/\mathrm{Spin}(q)(\mathbf{Z})$ is compact, but it has "finite index" image in X.

[1] Strictly speaking, $G(\mathbf{Z})$ is of course meaningless since G is a $\mathbf{Q}$-group. But thinking of G as a subgroup of some GL_n, we can take a Zariski closure and thus get a fixed $\mathbf{Z}$-model to keep in mind.

[2] However, we will allow L to be indefinite over $\mathbf{R}$, i.e. it could have real zeros.

[3] $\mathrm{SO}(q)$ is k-anisotropic in the sense of not containing any non-trivial k-split torus iff q is k-anisotropic in the nineteenth century sense of having no zeros over k: see [4, Homework 6, Problem 5]

Theorem 8 ("Theorem F": Gauß-Minkowski)**.** If $\mathscr{D}G$ is k-isotropic and $S \subseteq G$ is a maximal k-split torus and $P \supseteq S$ is a minimal parabolic k-subgroup, then there exists a compact subset $K \subseteq G(\mathbf{A})^1$ and some $c > 0$ such that $G(\mathbf{A})^1 = K \cdot S(c)P(\mathbf{A})^1 \cdot G(k)$, where we define:

$$S(c) = \{s \in S(\mathbf{A}) \cap G(\mathbf{A})^1 \mid |\alpha(s)|_k \leq c\, \forall \alpha \in \Delta = \text{ basis of } \Phi(P, S) \subseteq \Phi(G, S)\}$$

In English, this is the set of points of $s \in S(\mathbf{A})$ such that $|\chi(s)| \leq 1$ for all k-characters χ of G, and $|\alpha(s)| \leq c$ for α running over a basis of the roots of P.

Remark 9. Theorem F implies that $\mathrm{vol}\big([G]\big) < \infty$ always; this is due to Borel–Harish-Chandra and Harder. Brian will write something up about this.

Example 10. Let $G = \mathrm{SL}_2, k = \mathbf{Q}$. Then via strong approximation, we have:

$$G(\mathbf{Q})\backslash G(\mathbf{A})/G(\widehat{\mathbf{Z}}) \cdot K_\infty = \mathrm{SL}_2(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{R})/K_\infty$$

Here, K_∞ is a maximal compact subgroup of $G(\mathbf{R})$, e.g. we could take $K_\infty = \mathrm{SO}_2(\mathbf{R})$, and we take the obvious $\mathbf{Z}$-structure on G.

Now, we can use the classical "NAK-decomposition" of $\mathrm{SL}_2(\mathbf{R})$: this says that $\mathrm{SL}_2(\mathbf{R})$ is diffeomorphic to the product $N \times A \times K_\infty$ with $N = \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\}$ is the unipotent radical of the upper Borel subgroup, A is the split maximal torus $\{\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)\}$, and K_∞ is the maximal compact subgroup $\mathrm{SO}_2(\mathbf{R})$ as we noted already. Then we can obtain a diffeomorphism $\mathrm{SL}_2(\mathbf{R})/K_\infty \simeq \mathfrak{h}_i$, the upper half-plane, via the map $g \mapsto g(i) = x + it^2$ (i.e. $g \in \mathrm{SL}(2)$ acts on $\mathfrak{h}_i$ by fractional linear transformations).

Now, let's see how to interpret Theorem F in this classical context. Since G is $\mathbf{Q}$-split and $G = \mathscr{D}G$, in particular $\mathscr{D}G$ is $\mathbf{Q}$-isotropic. Then we can pick $S = A$ the diagonal maximal torus, $P = B^- \supseteq S$, the lower triangular Borel subgroup. Then $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \mapsto t^{-2}$ is a basis for $\Phi(P, S)$, so $S(c)$ is the set of $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ with $t^{-2} \leq c$, i.e. the set with $t^2 \geq \frac{1}{c}$. In the upper half-plane this corresponds to $\{yi \mid y \geq \frac{1}{c}\}$.

How does the volume work out? We can see that $NA = \{\left(\begin{smallmatrix} t & z \\ 0 & t^{-1} \end{smallmatrix}\right)\}$ has left Haar measure $\frac{dt}{t^2}\, dz$. But $z = \frac{x}{t}$ in the coordinates (x, t) for N and A, so $\frac{dt}{t^2}\, dz = \frac{dt}{t^2}\, d\big(\frac{x}{t}\big) = \frac{dx\, dy}{y^2}$, using $y = t^2$. Then, the volume-finiteness reduces to the integral (since we know the integration in the x-direction is over a finite interval, e.g. because $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ sends x to $x + 1$):

$$\int_{\frac{1}{\sqrt{c}}}^{\infty} \frac{dt}{t^2} < \infty$$

Remark 11. The proof of Theorems C and F for $\mathrm{GL}_2, \mathrm{SL}_2$, and PGL_2 with $k = k_0 \in \{\mathbf{Q}, \mathbf{F}(t)\}$ is done by hand, via "adelizing Gauß". To avoid complications with class numbers, the general case over any global field k is *reduced* to the case $k = k_0$. Then *that* case is further reduced to $\mathrm{GL}_2, \mathrm{SL}_2$, and PGL_2 cases, up to giving a direct proof that compactness of $[G]$ implies $\mathscr{D}G$ is k-anisotropic.[4]

Remark 12. Granting Theorems C and F, we can see that the formula

$$G(\mathbf{A})^1 = K \cdot S(c) \cdot P(\mathbf{A})^1 \cdot G(k)$$

[4] Direct proofs are possible, but this is more uniform.

can be massaged further. We can apply Theorem C to the Levi factor P/U, with $U = \mathscr{R}_{k,u}(P)^5$, $L = Z_G(S) \subseteq P = L \ltimes U$. Then since the maximal k-split torus S is central in L, $\mathscr{D}L$ is k-anisotropic, so we may apply Theorem **C** so see that $[L]$ is compact. We have:

$$[L] = L(k)\backslash L(\mathbf{A})^1 = P(k)\backslash P(\mathbf{A})^1/U(\mathbf{A})$$

Because U is k-*split* unipotent, $U(\mathbf{A})/U(k)$ is also compact: the k-split property says exactly that U has a composition series with $\mathbf{G}_a$-quotients, and one can show (for example, in [1]) that compactness of $\mathbf{G}_a(\mathbf{A})/\mathbf{G}_a(k)$, which is a classical algebraic number theory fact, then implies compactness of $U(\mathbf{A})/U(k)$. Also, $P(\mathbf{A})/P(k) \to L(\mathbf{A})/L(k)$ is a "$U(\mathbf{A})/U(k)$-fibration". Thus, $P(\mathbf{A})^1/P(k)$ is compact, so we have a compact subset $K' \subseteq P(\mathbf{A})$ with $K' \cdot P(k) = P(\mathbf{A})$.

Therefore, we can write:

$$G(\mathbf{A})^1 = K \cdot S(c) \cdot K' \cdot G(k)$$

Here, we absorbed the $P(k)$ into $G(k)$.

Let's show one direction of Theorem C:

Lemma 13. If $\mathscr{D}G$ is isotropic, then $[G]$ is *not* compact.

Proof. Pick a faithful representation $G \longrightarrow \mathrm{GL}(V)$ and a non-trivial split torus $S \subset \mathscr{D}G$. Such an S is *not* central, this representation has a non-trivial S-weight $\chi\colon S \to \mathbf{G}_m$ on V. We pick some $\xi \in V_\chi \setminus \{0\}$. Since $\chi\colon S \to \mathbf{G}_m$ splits back up to isogeny, it is surjective on N-th powers for some N. Now, we can pick a sequence $s_n \in S(\mathbf{A})$ such that $\chi(s_n) \to 0$ in $\mathbf{A}$, by picking some sequence in $\mathbf{A}^\times$ which limits to 0 in $\mathbf{A}$, then lifting after taking some powers. Then, we have $s_n \cdot \xi \to 0 \in V \otimes \mathbf{A}$ as $n \to \infty$. Since $S \subseteq \mathscr{D}G$ and $\mathscr{D}G = (\mathscr{D}G)^1$ (it has no characters), we have $S(\mathbf{A}) \subseteq (\mathscr{D}G)(\mathbf{A}) \subseteq G(\mathbf{A})^1$, so $s_n \cdot \xi \in \big(G(\mathbf{A})^1 \cdot V\big) \setminus \{0\}$, and this goes to 0 as $n \to \infty$.

Suppose $[G] = G(\mathbf{A})^1/G(k)$ *were* compact. We'll show that this implies 0 is isolated in $G(\mathbf{A})^1 \cdot V \subseteq V \otimes \mathbf{A}$, giving a contradiction. Compactness of $[G]$ gives us a compact subset $K \subseteq G(\mathbf{A})^1$ such that $G(\mathbf{A})^1 = K \cdot G(k)$, so $G(\mathbf{A})^1 \cdot V = K \cdot V$, since $G(k)$ acts on V. We know that $V \subseteq V \otimes \mathbf{A}$ is discrete. Then we can pick an open subset $U \subseteq V \otimes \mathbf{A}$ around 0 such that $U \cap V = \{0\}$. Now, by compactness of K we can pick some open $U' \subseteq U$ around 0 so $K^{-1} \cdot U' \subseteq U$. Then $(K \cdot V) \cap U' \subseteq K \cdot \big(V \cap K^{-1} \cdot U'\big) \subseteq K \cdot (V \cap U) = K \cdot 0 = 0$, so 0 is discrete in $K \cdot V = G(\mathbf{A})^1 \cdot V$. □

Example 14. This shows that $[G]$ is not compact for $G = \mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$, etc., or more generally for any k-split G which is not a torus.

Now, let's see how to reduce the remaining direction of Theorem C and Theorem F to the case $k = k_0 \in \{\mathbf{Q}, \mathbf{F}_q(t)\}$:

In general, we can write k as a finite separable extension of k_0. Let $G_0 = R_{k/k_0}(G)$. Now, we've seen that we have a canonical topological identification $G(\mathbf{A}_k) = G_0(\mathbf{A}_{k_0})$ which respects the subspaces $G(\mathbf{A}_k)^1$ and $G_0(\mathbf{A}_{k_0})^1$, so $[G] = [G_0]$. If $\mathscr{D}G$ is k-anisotropic, then the same must be true for G_0, so we see that Theorem C for G and Theorem C for G_0 are equivalent (since we already proved the converse direction in general).

[5] Since G is reductive, $(\mathscr{R}_{k,u}(P))_{\overline{k}} = \mathscr{R}_{\overline{k},u}(P_{\overline{k}})$, so L is reductive

For Theorem F, pick some split torus S and minimal parabolic subgroup $P \supseteq S$. Taking Weil restrictions, we get $G_0 \supseteq P_0 \supseteq R_{k/k_0}(S) \supseteq S_0$ with $P_0 = R_{k/k_0}(P)$, which is still minimal parabolic (Weil restriction induces an inclusion-preserving bijection between parabolic subgroups of G and G_0), and $S_0 \subseteq R_{k/k_0}(S)$ a maximal split torus of the same rank as S (i.e. using the fact that $S = \prod \mathbf{G}_m$, we can take products of the canonical inclusion $\mathbf{G}_m \hookrightarrow R_{k/k_0}(\mathbf{G}_m)$ which we considered earlier).

For any $c > 0$, after some thought relating $\Phi(G, S)$ and $\Phi(G_0, S_0)$, we have $S_0(c) \subseteq S(c')$ for some $c' > 0$, where we consider both of these as subsets of $G_0(\mathbf{A}_{k_0}) = G(\mathbf{A}_k)$. Thus, Theorem F also reduces to working over k_0. In the end, the *split* case over k_0 reduces to the $\mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$ cases, and this implies the general split case.

References

[1] Brian Conrad, *Finiteness theorems for algebraic groups over function fields*, Compositio Mathematica **148** (2012), no. 2, pp. 555 - 639.

[2] ———, *Weil and Grothendieck approaches to adelic points*, Enseign. Math. (2) **58** (2012), no. 1-2, 61–97.

[3] Joseph Oesterlé, *Nombres de Tamagawa et groupes unipotents en caractéristique* p, Invent. Math. **78** (1984), no. 1, 13–88.

[4] `http://math.stanford.edu/~conrad/252Page/`. Notes from a graduate course on linear algebraic groups, taught by Brian Conrad at Stanford University.