LECTURE 4: ADELIC HEIGHTS AND $\mathrm{SL}_2, \mathrm{GL}_2, \mathrm{PGL}_2$ OVER $k_0 = \mathbf{Q}, \mathbf{F}_q(t)$

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We'll follow [2, §1.8-§1.11] for the bulk of today's lecture. Recall the statement of Theorems **C** and F of reduction theory:

Theorem 1 ("Theorem **C**"). For G a connected reductive group over a global field k, $[G] = G(\mathbf{A})^1/G(k)$ is compact if and only if $\mathscr{D}G$ is k-anisotropic ("semi-simple k-rank is 0").

Last time, we proved the forward implication: if $\mathscr{D}G$ contains a nontrivial split k-torus then $[G]$ is non-compact. In particular, this verifies Theorem **C** for any split G over any k, such as $\mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$.

Theorem 2 ("Theorem F"). Assume $\mathscr{D}G$ is k-isotropic. Choose a maximal split k-torus $S \subset G$ and minimal parabolic k-subgroup P with $S \subseteq P \subseteq G$. Then

$$G(\mathbf{A})^1 = K \cdot S(c)P(\mathbf{A})^1 G(k)$$

for some *compact* subset $K \subseteq G(\mathbf{A})^1$ and some $c > 0$, with

$$S(c) := \{s \in S(\mathbf{A}) \cap G(\mathbf{A})^1 \mid |\alpha(s)|_k \leq c \text{ for all } \alpha \in \Delta\}$$

where Δ is the basis for the positive system of roots $\Phi(P, S)$ in the relative root system $\Phi(G, S)$ of non-trivial S-weights on $\mathrm{Lie}(G)$.

In the case $G = \mathrm{SL}_2$ and $k = \mathbf{Q}$, we saw for suitable S and P that the image of $S(c)$ in $[G]/G(\widehat{\mathbf{Z}})$ may be identified with the ray $\{iy \mid y \geq \frac{1}{c^2}\}$ in the upper half-plane.

Remark 3. Last time we saw that once both theorems are proved, we can replace $P(\mathbf{A})^1$ in this expression with a compact subset $K' \subseteq P(\mathbf{A})^1$ by applying Theorem **C** to any Levi factor of P (i.e., a k-subgroup mapping isomorphically onto the connected reductive $P/\mathscr{R}_{u,k}(P)$).

To prove these theorems, we'll start off by showing Theorem F when G is one of SL_2, GL_2, or PGL_2 and k is one of $\mathbf{Q}, \mathbf{F}_q(t)$ (since Theorem **C** for split groups has been settled). We will adapt the classical proof describing the fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ on the upper half-plane into something more group-theoretic (and adelic).

Let V be a non-zero finite-dimensional k-vector space. We'll denote $V \otimes \mathbf{A}$ as $V_{\mathbf{A}}$. Given a point of $V_{\mathbf{A}}$, can we assign this a meaningful "norm"? Consider norms $\|\cdot\|_v$ on $V_v = V \otimes_k k_v$ for each place v; these are required to be compatible with $|\cdot|_v$ on k_v in the sense that $\|ax\|_v = |a|_v \|x\|_v$ and we furthermore ask it to satisfy the ultrametric inequality when v is non-archimedean. We will impose a further compatibility condition: for all but finitely many v, $\|\cdot\|_v$ is the "sup-norm" with respect to a common choice of k-basis.

Remark 4. Why don't we just pick a k-basis for V at the start and ask $\|\cdot\|_v$ to be a sup-norm for this basis for all v? Our definition has the following advantage: if $g \in \mathrm{GL}(V \otimes \mathbf{A}) = \mathrm{GL}(V)(\mathbf{A})$, then $\{\|g(\cdot)\|_v\}$ is another such collection. Indeed, if we choose a k-basis for V to define $k^n \simeq V$ then we get a lattice $L_v \simeq \mathscr{O}_v^{\oplus n} \subseteq V_v$ for each finite v, and this is preserved by g for all but finitely many v; i.e., $g_v \in \mathrm{GL}_n(L_v)$ for all but finitely many v. Thus $\|\cdot\|_v$ is the sup-norm for all but finitely many v, so our notion is preserved by global automorphisms of $V_{\mathbf{A}} := V \otimes \mathbf{A}$.

We call $\xi \in V_{\mathbf{A}}$ *primitive* if $\xi \in \mathrm{GL}(V)(\mathbf{A}) \cdot (V - \{0\}) \subseteq V_{\mathbf{A}}$. For example, in the case $V = k$, a primitive adele is just an idele by another name. These adelic vectors have a very concrete description that also explains the terminology:

Lemma 5. Fix a k-basis $\mathbf{e} = \{e_i\}$ of V. An element $\xi = (\xi_v) \in V_{\mathbf{A}} \subset \prod_v V_v$ is primitive if and only if every ξ_v is nonzero and for all but finitely many v the nonzero ξ_v has $\mathbf{e}$-coordinates belonging to $\mathscr{O}_v$ with at least one coordinate belonging to $\mathscr{O}_v^{\times}$. (In other words, for all but finitely many v, ξ_v is a primitive vector in the $\mathscr{O}_v$-lattice $\oplus_i \mathscr{O}_v e_i \subset \oplus_i k_v e_i = V_v$.

Proof. It is clear that $\mathrm{GL}(V_{\mathbf{A}})$ preserves the class of vectors satisfying the proposed characterization of primitivity, and every element of $V - \{0\}$ obviously satisfies those conditions, so by definition of primitivity it follows that the proposed characterization is a necessary condition for primitivity. To check sufficiency, consider $\xi = (\xi_v)$ satisfying these conditions, so there is a finite set S of places of k containing all archimedean places such that for all $v \notin S$ the vector ξ_v is primitive in the $\mathscr{O}_v$-lattice $\oplus_i \mathscr{O}_v e_i$. Since $\mathrm{GL}_n(\mathscr{O}_v)$ acts transitively on the set of primitive vectors in $\mathscr{O}_v^n$ for all $v \notin S$ (even all non-archimedean v) and $\mathrm{GL}(V_v)$ acts transitively on $V_v - \{0\}$ for all $v \in S$ (even all v), we see that $\mathrm{GL}(V_{\mathbf{A}})$ carries ξ to e_1 (and so $\xi \in \mathrm{GL}(V_{\mathbf{A}})(V - \{0\})$ as desired). □

For primitive ξ, $\|\xi_v\|_v \neq 0$ for all v, and $\|\xi_v\|_v = 1$ for almost all v, so $\prod_v \|\xi_v\|_v$ is a product of finitely many non-zero terms and hence trivially converges to a positive number. For such ξ, we define the *adelic height*[1] $\|\xi\|$ to be $\prod_v \|\xi_v\|_v$. By Remark 4, for any $g \in \mathrm{GL}(V_{\mathbf{A}})$, $\|g(\cdot)\|$ also gives a height on primitive vectors, and the ratio of $\|g(\cdot)\|$ to $\|\cdot\|$ is clearly bounded above and below by positive constants.

Proposition 6 (Properties of Heights). (i) $\|t\xi\| = |t|_k \cdot \|\xi\|$ for $t \in \mathbf{A}^{\times}$ and ξ primitive.

(ii) For $\|\cdot\|, \|\cdot\|'$ two heights, there is some $c, C > 0$ such that

$$c \leq \|\xi\|'/\|\xi\| \leq C$$

for all primitive ξ.

(iii) If $\{\xi_n\}$ are primitive and $\xi_n \to 0$ in $V_{\mathbf{A}}$, then $\|\xi_n\| \to 0$.

(iv) (Approximate converse to (iii)) If $\{\xi_n\}$ are primitive with $\|\xi_n\| \to 0$, then there exist $\lambda_n \in k^{\times}$ such that $\lambda_n \xi_n \to 0$.

[1]This has nothing to do with other notions of height in arithmetic geometry!

Remark 7. Note that (iv) is as good as we can hope for, since if $\|\xi_n\| \to 0$, then by (i) $\|\lambda_n \xi_n\| = |\lambda_n|_k \|\xi_n\| = \|\xi_n\| \to 0$ as well by the product formula. Certainly multiplying by such elements could destroy the property that $\xi_n \to 0$ in $V_{\mathbf{A}}$.

The first two properties in Proposition 6 are easy: (ii) follows from the easy analysis fact that all norms on a finite-dimensional topological vector space over a locally compact field are metrically equivalent (i.e. the analogous inequality as in the statement of (ii) holds), and (i) is immediate from the construction.

By (ii), in order to prove (iii) it is harmless to fix a k-basis of V and to let $\|\cdot\|_v$ be the sup norm with respect to this basis for all v. Then, it is easy to verify that $\xi_n \to 0$ implies that the sequence $\|\xi_n\|$ is at least bounded. To prove (iii), we give the following simple argument suggested by Zev Rosengarten during the lecture that is much simpler than Springer's suggested argument. Upon fixing a k-basis of V we may assume that $\|\cdot\|_v$ is the sup-norm with respect to this basis for all v. Then the condition that $\xi_n \to 0$ in $V_{\mathbf{A}}$ implies that for large enough n, $\|\xi_n\|_v \leq 1$ for all v and that there is some finite set of places $v_1, \ldots, v_n$ such that $\|\xi_n\|_{v_i} \to 0$ for each i. Then we can see directly that this implies that the product $\|\xi_n\| = \prod_v \|\xi_n\|_v$ goes to 0, as desired.

To prove (iv), the key step is to reduce to the case $\dim_k V = 1$ (so in effect, the case $V = k$), for which Springer gives no argument beyond asserting it is true. I realized during the lecture that the argument I had in mind for proving (iv) was wrong, and got myself completely confused (and began to have doubts if (iv) is true), but Sheela Devadas pointed out Lemma 5 and how this does the job as follows.

Fix a k-basis $\mathbf{e} = \{e_i\}$ of V. Choose $\xi \in V_{\mathbf{A}} = \oplus_i \mathbf{A} e_i$ such that $\xi_v \neq 0$ for all v. Writing $\xi = \sum_i \xi_i e_i$ with $\xi_i \in \mathbf{A}$, for each place v we let $c_v \in k_v^\times$ be an element satisfying $|c_v|_v = \sup_i |\xi_i|_v \neq 0$. For non-archimedean v, clearly c_v is well-defined up to $\mathscr{O}_v^\times$-multiple, so the adele $[\xi] := (c_v) \in \mathbf{A}$ is well-defined (for a fixed choice of $\mathbf{e}$!) up to multiplication against $\prod_{v|\infty} k_v^\times \times \prod_{v\nmid\infty} \mathscr{O}_v^\times$.

The crucial observation (whose proof led to noting Lemma 5) is that an element $\xi \in V_{\mathbf{A}}$ is primitive if and only if $\xi_v \neq 0$ for all v and the associated adele $[\xi] = (c_v) \in \mathbf{A}$ is an idele (i.e., $[\xi]$ is a primitive adele). In particular, by design we see that when we use the height defined by sup-norm relative to $\mathbf{e}$ at every place of v then any primitive ξ satisfies $\|\xi\| = |[\xi]|_k$ (the right side being the idelic norm of $[\xi]$, which is independent of the choice of c_v's with $\mathbf{e}$ fixed).

It is clear from the definition of the adelic topology on $V_{\mathbf{A}}$ described in terms of $\mathbf{e}$ and from the definition of $[\xi]$ that if $\{\xi_n\}$ is a sequence of primitive vectors and $[\xi_n] \to 0$ in $\mathbf{A}$ then $\xi_n \to 0$ in $V_{\mathbf{A}}$. For any $\lambda \in k^\times$ and primitive ξ we have $[\lambda\xi] = \lambda[\xi]$ in $\mathbf{A}^\times$, so to prove property (iv) of adelic heights it suffices (in view of the settled properties (i) and (ii)!) to treat the case $V = k$.

The proof of property (iv) of adelic heights is now reduced to a statement about the idele group equipped with its usual idelic norm:

Lemma 8. Let $\{\xi_n\}$ be a sequence in $\mathbf{A}^\times$ such that $|\xi_n|_k \to 0$. There exist elements $\lambda_n \in k^\times$ such that $\lambda_n \xi_n \to 0$ in $\mathbf{A}$.

Proof. Let S be a non-empty finite set of places containing the archimedean places and big enough so that the ring $\mathscr{O}_{k,S}$ of S-integers has trivial class group. This says $\mathbf{A}^\times = k^\times(\prod_{v\in S} k_v^\times)U_S$ for $U_S := \prod_{v\notin S} \mathscr{O}_v^\times$. Multiplying each ξ_n by an arbitrary element of U_S is harmless (since $\mathbf{A}$ has a base of neighborhoods of 0 stable under multiplication by U_S and all elements of U_S have idelic norm equal to 1). Thus, we may arrange that $\xi_n \in k_S^\times$ for all n, where $k_S := \prod_{v \in S} k_v$. Multiplying each ξ_n by an element of $\mathscr{O}_{k,S}^\times$ introduces factors at places outside S, but a further multiplication by elements of U_S gets rid of that. Hence, it suffices to find elements $\lambda_n \in \mathscr{O}_{k,S}^\times$ such that $\lambda_n \xi_n \to 0$ in k_S (rather than in $\mathbf{A}$). Our task has now been "localized" into the more concrete topological k-algebra k_S (for which $k_S^\times$ has the subspace topology, in contrast with $\mathbf{A}^\times \subset \mathbf{A}$!).

For a choice of $v_0 \in S$ and $c_0 \in k_{v_0}^\times$ with $|c_0|_{v_0} < 1$, via the evident inclusion $k_{v_0}^\times \hookrightarrow k_S^\times$ we can write $\xi_n = c_0^{a_n} \xi_n'$ for integers a_n and $\xi_n' \in k_S^\times$ such that $|c_0|_{v_0} \le \|\xi_n'\|_S \le 1$. In particular, $c_0^{a_n} \to 0$ in k_S, so it suffices that every ξ_n' admits an $\mathscr{O}_{k,S}^\times$-multiple contained within a common compact subset of k_S. In fact, this can be done using a common compact subset of $k_S^\times$. provided that the norm $k_S^\times / \mathscr{O}_{k,S}^\times \to \mathbf{R}_{>0}$ is a proper map. But it is classical that this norm map is topologically identified with the quotient map modulo $(k_S^\times)^1 / \mathscr{O}_{k,S}^\times$ (followed by the harmless inclusion of a discrete subgroup into $\mathbf{R}_{>0}$ in the function field case), and the the S-unit theorem gives the compactness of $(k_S^\times)^1 / \mathscr{O}_{k,S}^\times$. □

Here is an overview of the ideas in the reduction of Theorems C and F for general groups G over general k to the special cases $\mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$ over finite Galois extensions of k. At a crucial step, adelic heights will be used.

Consider some connected reductive G over k. If G has positive semisimple k-rank (so the relative root system is non-empty), let S be a maximal k-split torus. For $\alpha \in \Phi(G, S)$, we have $S_\alpha = (\ker \alpha)^0_{\mathrm{red}} \subseteq S$ is a codimension-1 torus killed by $\alpha \colon S \twoheadrightarrow \mathbf{G}_m$. Then $Z_G(S_\alpha)$ is a connected reductive group with semisimple k_0-rank 1. When G is split, so is $Z_G(S_\alpha)$. Via extensive use of the serious structure theory of reductive groups over fields, one can reduce Theorem F for G to the corresponding theorems for all of the $Z_G(S_\alpha)$'s (which are split when G is split); this is [2, 2.3-2.4]. I will later try to find time to write up an exposition of how this is done; it is not at all obvious. In this way, Theorem F over any k is reduced to the case of semisimple k-rank equal to 1; the special feature of such cases is that all proper parabolic k-subgroups are minimal, and so all such k-subgroups constitute a single $G(k)$-conjugacy class. This reduction step for split k-groups remains within the split setting, so the case of Theorem F for split G over any k is reduced to split G of semisimple k-rank equal to 1. We have also noted above that Theorem C for split G is already settled.

The following fundamental fact is a special role for the groups $\mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$ in the general structure theory (and is an immediate consequence of the general structure theory of split connected reductive groups over fields):

Lemma 9. Over any field, the split connected reductive G of semisimple rank 1 are precisely the groups $H \times T$ for a split torus T and $H = \mathrm{SL}_2, \mathrm{GL}_2, \mathrm{PGL}_2$.

Thus, for split G of semisimple k-rank equal to 1 we have $[G] = [H] \times [T]$ for one of those 3 possibilities for H, and it is classical that $[T]$ is compact (the adelic synthesis of finiteness of generalized class groups and the S-unit theorem). Theorem F for general split G over a general k (nothing remains to be done for Theorem C for such G) thereby reduces to the cases of $\mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$ over k. Let's grant these special cases over *general* k, so Theorem F is settled in the general split case over general k; we also already settled Theorem C in the general split case over general k.

In the general case of semisimple k-rank 0 (there is nothing to do for Theorem F in such cases), we want to show that $[G]$ is compact (so Theorem C would be settled in general). Let $T \subset G$ be a maximal k-torus containing a split maximal k-torus S. (We have S central in G since we're in the case of semisimple k-rank equal to 0.) Pick a finite Galois extension k'/k splitting T. The natural map

$$[G] \hookrightarrow [\mathrm{R}_{k'/k}(G_{k'})] = [G_{k'}]$$

is a *closed embedding* as we saw Lecture 2! Since $G_{k'}$ is split (as $T_{k'}$ is a split maximal k'-torus) and Theorem F is settled in the split case in general (conditional on the cases of $\mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$ still to be done!), we obtain a description $G_{k'}(\mathbf{A}_{k'})^1 = K \cdot T(c')P(\mathbf{A}_{k'})^1 G(k')$ for a compact set K. One can then use the general relationship of $\Phi(G, S)$ and $\Phi(G_{k'}, T_{k'})$ applied in this setting with empty $\Phi(G, S)$ to deduce enough information about Tits' $*$-action of $\mathrm{Gal}(k'/k)$ on $\Phi(G_{k'} T_{k'})$ to see that the way $[G]$ lies inside $[G_{k'}]$ is controlled by the compact part of $T(c')$, from which the desired compactness of $[G]$ follows; this is [2, 3.5] (which rests on the subtle [2, 2.6]); I will try to write up an exposition of the details on this later. In this way Theorem C is settled in general (once again, conditional on Theorem F for $\mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$ over general k).

For the case of semisimple rank 1 (and Theorem F), we choose a faithful representation $G \hookrightarrow \mathrm{GL}(V)$ such that P is the stabilizer of a line. The condition of having semisimple rank 1 ensures that there is only one $G(k)$-conjugacy class of proper parabolic k-subgroups of G (in other words, they are all minimal). One has to use lots of arguments with adelic heights on this high-dimensional V (including property (iv)) to harness the explicit description of $[G_{k'}]$ for a finite Galois extension k'/k splitting G (using Theorems C and F for the split $G_{k'}$ over k') to get the desired description of $[G]$ as in Theorem F. This is [2, 3.6]; I will try to write up an exposition of the details on this later.

So far we have relied on a lot of serious structure theory of reductive groups (which we have admittedly swept under the rug here, ultimately just pointing to the places in [2] where it is used) to bring the general task over general k down to the special cases of $\mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$ over general k. We saw in the last lecture how to reduce the Theorems C and F in general over k to the same results for the Weil restriction down to any k_0 over which k is finite separable. Such a k_0 can always be found that is either $\mathbf{Q}$ or some $\mathbf{F}_q(t)$. However, if we start with GL_2 over some general k, its Weil restriction to k_0 is never split when $k \neq k_0$ (though its semisimple k_0-rank is equal to 1)!

Now let's finally do something real: use adelic heights and their properties in the case that $\dim V = 2$ and $k_0 = \mathbf{Q}, \mathbf{F}_q(t)$ to study $\mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$ for these special fields. (We have seen that adelic heights play a crucial role for more general V in the treatment of more general G too.) Fix a basis of the 2-dimensional V. Take the height to rest on the sup-norm with respect to this basis for all non-archimedean v and (for $k_0 = \mathbf{Q}$) use the usual Euclidean length for $v = \infty$.

Define a (maximal) compact subgroup of $\mathrm{GL}(k_v)$ given by

$$K_v = \{g \in \mathrm{GL}(V_v) \mid \|g(\cdot)\|_v = \|\cdot\|_v\} = \begin{cases} \mathrm{GL}_2(\mathscr{O}_v), v \neq \infty; \\ \mathrm{O}_2(\mathbf{R}), v = \infty, \end{cases}$$

Taking the direct product, we get a compact subgroup $K = \prod_v K_v \subseteq \mathrm{GL}(V_{\mathbf{A}})$.

For $B = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ the upper-triangular Borel subgroup of GL_2, we have $\mathrm{GL}_2/B \simeq \mathbf{P}^1$ by sending

g to $g(\infty)$. A mild argument shows that there is a topological isomorphism

$$\mathrm{GL}_2(\mathbf{A})/B(\mathbf{A}) \simeq (\mathrm{GL}_2/B)(\mathbf{A}) = \mathbf{P}^1(\mathbf{A}) = \prod_v \mathbf{P}^1(k_v)$$

This uses that for $v \neq \infty$, by the valuative criterion of properness (or just clearing denominators of homogeneous coordinates) we have $\mathbf{P}^1(k_v) = \mathbf{P}^1(\mathscr{O}_v)$. Hence, as an exercise we get $\mathrm{GL}_2(\mathbf{A}) = K \cdot B(\mathbf{A})$.

For $c > 0$, we let:

$$B(c) = \{ \left(\begin{smallmatrix} t_1 & * \\ 0 & t_2 \end{smallmatrix}\right) \in B(\mathbf{A}) \mid |t_1/t_2|_k \leq c \}$$

Proposition 10. For any $\epsilon_0 > 0$ and $c = (2/\sqrt{3}) + \epsilon_0$ we have:

$$\mathrm{GL}_2(\mathbf{A}) = K \cdot B(c) \cdot \mathrm{GL}_2(k)$$

This implies Theorem F for GL_2 (over $k = k_0$) by restricting to the "norm-1" parts of $\mathrm{GL}_2(\mathbf{A})$ and $B(c)$, i.e. by restricting to $g \in \mathrm{GL}_2(\mathbf{A})$ such that $|\det g|_k = 1$ and $b \in B(c)$ such that $|t_1/t_2|_k = 1$. The case of SL_2 is treated by a variant of the same method, and the case of PGL_2 can be deduced from that of GL_2 (details left to the reader). Springer states the result with $c = 2/\sqrt{3}$ (no doubt inspired by the classical case of SL_2 over $\mathbf{Q}$), but we'll see in the proof below that non-discreteness of adelic heights for number fields makes it unclear how to really achieve that value for c; our later needs don't require such a specific value (any $c > 2/\sqrt{3}$ is sufficient too).

Proof. This proof is inspired by classical arguments with $\mathrm{SL}_2(\mathbf{Z})$ acting on the upper half-plane, such as those appearing in [1, Chapter VII]. The strategy will be to take a point in $\mathrm{GL}_2(\mathbf{A})$, vary it across the entire orbit of $G(k)$, show that the height is bounded away from 0, and pick an element which is close to the infimum. We shall use the adelic height as above defined with respect to the standard basis of k^2.

Choose $g \in \mathrm{GL}_2(\mathbf{A})$. We seek $\gamma \in \mathrm{GL}_2(k)$ with $g \cdot \gamma \in K \cdot B(c)$. For varying $\xi \in k^2 - \{0\}$ (which is acted on transitively by $\mathrm{GL}_2(k)$), we claim that the height $\|g(\xi)\|$ is bounded away from 0. If not, pick some sequence $\{\xi_n\}$ with $\|g(\xi_n)\| \to 0$. By Proposition 6 (iv), we get $\lambda_n \in k^\times$ such that $\lambda_n g(\xi_n) = g(\lambda_n \xi_n) \to 0$. Since g is fixed, this means that $\lambda_n \xi_n \in k^2 - 0$ goes to 0 in $\mathbf{A}^2$, which is a contradiction since k^2 is discrete in $\mathbf{A}^2$ and $\lambda_n \xi_n \neq 0$ for all n.

Fix a small $\epsilon > 0$. By approximating an infimum, we may pick ξ_0 such that $\|g(\xi_0)\| \leq (1+\epsilon)\inf_\xi \|g(\xi)\|$. (The non-discreteness of adelic heights for number fields makes it unclear how to avoid ϵ in the number field case.) By right multiplication on g by an element γ_0 of $\mathrm{GL}_2(k)$, we can change the basis to reduce to the case $\xi_0 = e_1$. Thus, we are in the situation where:

$$\text{for all } \xi \in k^2 - \{0\}, \|g(e_1)\| \leq (1+\epsilon)\|g\xi\| \tag{1}$$

This property is invariant under *left* multiplication on g by $K = \prod_v K_v$ since for $v \neq \infty$, $\mathrm{GL}_2(\mathscr{O}_v)$ preserves the sup norm $\|\cdot\|_v$ and $\mathrm{O}_2(\mathbf{R})$ preserves the Euclidean norm. We want to show that the inequality (1) implies that $g \in K \cdot B(c)$. Since $\mathrm{GL}_2(\mathbf{A}) = K \cdot B(\mathbf{A})$, we know that $g = mb$ for $m \in K$, $b = \left(\begin{smallmatrix} t_1 & t_1 u \\ 0 & t_2 \end{smallmatrix}\right) \in B(\mathbf{A})$ with $t_1, t_2 \in \mathbf{A}^\times, u \in \mathbf{A}$. Without loss of generality, we can replace g with $m^{-1}g = b$. Now, since left multiplication by K preserves the property (1), we know that:

$$|t_1|_k = \|ge_1\| \leq (1+\epsilon)\|g(\lambda e_1 + \mu e_2)\| \quad \text{for all } (\lambda, \mu) \in k^2 - 0$$

Note that $g(\lambda e_1 + \mu e_2) = (\lambda + \mu u)t_1 e_1 + \mu t_2 e_2$. Fix $\mu = 1$.

We can divide by $|t_2|_k$, and get, defining $\alpha = t_1/t_2$:

$$x := |\alpha|_k \leq (1+\epsilon)\|(\lambda+u)\alpha \cdot e_1 + e_2\|$$

for all $\lambda \in k$ and some $u \in \mathbf{A}$. We want to show that this implies $|x| \leq (2/\sqrt{3}) + \epsilon'$ with $\epsilon' \to 0$ as $\epsilon \to 0$. We'll handle the two cases $k = \mathbf{Q}, k = \mathbf{F}_q(t)$ separately, starting with the case $k = \mathbf{Q}$.

It is harmless to scale α by $\mathbf{Q}^\times$, so we may assume (by strong approximation) that $\alpha \in \mathbf{R}_{>0} \times \widehat{\mathbf{Z}}^\times$. This means that $|\alpha|_\mathbf{Q} = |\alpha_\infty|$. We can choose $\lambda \in \mathbf{Q}$ such that $|\lambda + u|_v \leq 1$ for all $v \neq \infty$, and $|\lambda + u|_v \leq \frac{1}{2}$ for $v = \infty$. This uses the fact that the map $[0,1) \times \widehat{\mathbf{Z}} \to \mathbf{A}_\mathbf{Q}/\mathbf{Q}$ induced from the injection $[0,1) \times \widehat{\mathbf{Z}} \hookrightarrow \mathbf{A}_\mathbf{Q}$ is bijective since the "polar parts" of elements of $\mathbf{Q}_p$ are in $\mathbf{Q}$, and only finitely many are non-zero.

Then, since $\|\cdot\|$ is the product of the sup-norms at each finite place with the Euclidean norm at ∞, we see that since $|\alpha_v|_v = 1$, $\|(\lambda+u)\alpha e_1 + e_2\|_v \leq 1$ for each finite v, so we get from (1):

$$x = |t_1/t_2|_k = |\alpha|_\infty \leq (1+\epsilon)\|(\lambda+u)\alpha e_1 + e_2\|_\infty \leq (1+\epsilon)\sqrt{\frac{x^2}{4}+1}$$

and this implies that $x \leq \frac{2}{\sqrt{3}} + o(\epsilon)$. (The same deduction with $\epsilon = 0$ is exactly the algebraic calculation giving rise to the exact bounds for a fundamental domain in the classical setting with SL_2 over $\mathbf{Q}$.)

When $k = \mathbf{F}_q(t)$, we can do this the same way with $\mathbf{F}_q[t]$ in the role of $\mathbf{Z}$. We can scale α by $\mathbf{F}_q(t)$ so that $|\alpha|_v \leq 1$ for $v \neq \infty$ and pick $\lambda \in \mathbf{F}_q(t)$ such that $|\lambda + \mu|_v \leq 1$ for all $v \neq \infty$ and $|\lambda + \mu|_v \leq \frac{1}{q}$ at ∞. Then we get that the sup norm $\|(\lambda+u)\alpha e_1 + e_2\|$ is at most $\|\lambda + u\| \leq 1$ at each place $v \neq \infty$ and it is $\max 1, \frac{x}{q}$ at $v = \infty$, so we get $x \leq \max(1, x/q)$, so we must have $x \leq 1$ (since $x \not\leq x/q$ for $x > 0$). $\square$

There remains finally the task of how to settle $\mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$ over general k! It is remarkable that these core cases (with general k, not just k_0 as above) control the fate of the most general case. At a later time these notes will be expanded to discuss this important point (the exposition in [2] appears to be obscure about it). For example, Lemma 10.1 in Jacquet-Langlands' book on automorphic forms seems likely to give what we need via a direct argument over any global field; I'll come back to this.

References

[1] J.-P. Serre, *A Course in Arithmetic*, Springer-Verlag, New York-Heidelberg, 1973.

[2] T. A. Springer, *Reduction theory over global fields*, Proc. Indian Acad. Sci. Math. Sci. **104** (1994), no. 1, 207–216.