LECTURE 5: ADELIZATION OF MODULAR FORMS, PART I
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This will be the first part of several lectures discussing how to translate the classical theory of modular forms into the study of appropriate subspaces of the Hilbert space of L^2-functions on certain adelic coset spaces.

1 From half-plane to Lie groups

For any connected reductive group G over a global field k and non-empty finite set S of places of k containing the archimedean places, an S*-arithmetic subgroup* of $G(k)$ (or more accurately: of G) is a subgroup commensureable with $\mathscr{G}(\mathscr{O}_{k,S})$ for a flat affine $\mathscr{O}_{k,S}$-group scheme $\mathscr{G}$ of finite type with generic fiber G (e.g., if $G = \mathrm{SL}_n$ over k then we can take $\mathscr{G}$ to be SL_n over $\mathscr{O}_{k,S}$). Such a $\mathscr{G}$ always exists: the schematic closure of G in the $\mathscr{O}_{k,S}$-group GL_n relative to a closed k-subgroup inclusion of G into the k-group GL_n. (This schematic closure method gives rise to *all* $\mathscr{G}$'s since any flat affine group scheme of finite type over a Dedekind domain is a closed subgroup scheme of some GL_n, by adapting the well-known analogue over fields, using that any finitely generated torsion-free module over a Dedekind domain is projective and hence a direct summand of a finite free module.) The notion of S-arithmeticity is independent of the choice of $\mathscr{G}$ and has reasonable functorial properties in the k-group G; see [7, Ch. 1, 3.1.1(iv), 3.1.3(a)]. In the special case $k = \mathbf{Q}$ and $S = \{\infty\}$ one usually says "arithmetic" rather than "S-arithmetic".

Let Γ be an arithmetic subgroup of $\mathrm{SL}_2(\mathbf{Q})$ (e.g., any congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$). Consider a holomorphic function $f\colon \mathbf{H} \to \mathbf{C}$, where $\mathbf{H}$ is the upper half-plane $\{a + bi \in \mathbf{C} \mid b > 0\}$ (with respect to a fixed choice of $i = \sqrt{-1}$) on which $\mathrm{SL}_2(\mathbf{R})$ acts transitively via linear fractional transformations. Recall that $\mathrm{SL}_2(\mathbf{R})/\mathrm{SO}_2(\mathbf{R}) \simeq \mathbf{H}$ as real-analytic manifolds via $g \mapsto g(i)$.

Definition 1.1. For $g \in \mathrm{GL}_2^+(\mathbf{R})$ (the identity component of $\mathrm{GL}_2(\mathbf{R})$, or equivalently the subgroup of $\mathrm{GL}_2(\mathbf{R})$ with positive determinant), define

$$\left(f|_k g\right)(z) = f(gz)\left(\frac{\sqrt{\det g}}{cz + d}\right)^k.$$

Using this definition, we have:

Definition 1.2. The holomorphic function f is a Γ*-automorphic form of weight* k if $f|_k\gamma = f$ for all $\gamma \in \Gamma$ and f is moreover "holomorphic at the cusps".

Let's review from an "algebraic group" viewpoint what it means that f is holomorphic at the cusps. The action of $\mathrm{SL}_2(\mathbf{Q})$ on $\mathbf{H}$ via linear fractional transformations extends to a transitive action on $\mathbf{P}^1(\mathbf{Q}) = \mathbf{Q} \cup \{\infty\}$ that encodes the transitive conjugation action of $\mathrm{SL}_2(\mathbf{Q})$ on the set of Borel $\mathbf{Q}$-subgroups of SL_2 via the $\mathbf{Q}$-isomorphism $\mathbf{P}^1 \simeq \mathrm{SL}_2/B_\infty$ (inverse to $g \mapsto g(\infty)$) onto the "variety of Borel subgroups" of SL_2, with B_∞ the upper-triangular Borel subgroup (which is the SL_2-stabilizer of ∞). The action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{P}^1(\mathbf{Q}) = \mathbf{P}^1(\mathbf{Z})$ is transitive, so the action of Γ has finitely many orbits.

If we consider the quotient of $\mathbf{H}$ by the action of Γ, it is non-compact but for a suitable "horocycle topology" on $\mathbf{H} \cup \mathbf{P}^1(\mathbf{Q})$ we get a compactification by adding in the finitely many Γ-orbits on $\mathbf{P}^1(\mathbf{Q})$, which we call *cusps* of Γ. The complex structure on $\Gamma\backslash\mathbf{H}$ extends to this compactification in a standard (and even unique) manner, as is explained in many introductory books on classical modular forms. The stabilizer in $\mathrm{SL}_2(\mathbf{R})$ of $s \in \mathbf{P}^1(\mathbf{Q})$ is $B_s(\mathbf{R})$ for the Borel $\mathbf{Q}$-subgroup B_s corresponding to s, so the Γ-stabilizer $\Gamma_s := \Gamma \cap B_s(\mathbf{Q})$ of s is an arithmetic subgroup of B_s (arithmeticity interacts well with passage to algbraic subgroups). Letting U_s denote the unipotent radical of B_s, we have:

Lemma 1.3. If Γ_0 is an arithmetic subgroup of B_s then $\Gamma_0 \cap U_s(\mathbf{Q})$ is infinite cyclic. Moreover, $\Gamma_0 \cap U_s(\mathbf{Q})$ has index at most 2 in Γ_0, with index 2 if and only if Γ_0 contains an element γ whose eigenvalues are equal to -1.

The conjugation action of Γ_0 on $U_s(\mathbf{Q})$ is trivial, and Γ_0 is infinite cyclic except exactly when $-1 \in \Gamma_0$, in which case $\Gamma_0 = \langle -1 \rangle \times (\Gamma_0 \cap U_s(\mathbf{Q}))$.

Proof. The intersection $\Gamma_0 \cap U_s(\mathbf{Q})$ is an arithmetic subgroup of $U_s \simeq \mathbf{G}_a$, so it is commensurable with $\mathbf{G}_a(\mathbf{Z}) = \mathbf{Z}$. A subgroup of $\mathbf{G}_a(\mathbf{Q}) = \mathbf{Q}$ commensurable with $\mathbf{Z}$ is clearly infinite cyclic.

Since $\mathbf{G}_m(\mathbf{Z}) = \mathbf{Z}^\times = \{\pm 1\}$ is the entire torsion subgroup of $\mathbf{G}_m(\mathbf{Q}) = \mathbf{Q}^\times$, the only arithmetic subgroups of $\mathbf{G}_m$ over $\mathbf{Q}$ are the trivial group and $\{\pm 1\}$. In particular, the trivial subgroup of $\mathbf{G}_m(\mathbf{Q})$ is arithmetic, so by [7, 3.1.3(a)] applied to the quotient map $B_s \to B_s/U_s \simeq \mathbf{G}_m$ it follows that any arithmetic subgroup of $B_s(\mathbf{Q})$ contains its intersection with $U_s(\mathbf{Q})$ with finite index. Thus, any element γ of $\Gamma_0 \subset B_s(\mathbf{Q})$ has some positive power belonging to $U_s(\mathbf{Q})$. Since B_s is conjugate to $B_\infty \subset \mathrm{SL}_2$, it follows that the eigenvalues of γ are inverse roots of unity in $\mathbf{Q}^\times$ and hence are either both equal to 1 or both equal to -1, with the former happening if and only if γ is unipotent, which is equivalent to the condition that $\gamma \in U_s(\mathbf{Q})$.

When there exist non-unipotent elements of Γ_0, if γ, γ' are two such elements, then $\gamma'\gamma^{-1}$ is unipotent (as we see by hand upon computing with B_∞, for example), so $\Gamma_0 \cap U_s(\mathbf{Q})$ has index 1 or 2 in Γ_0, with index 2 precisely when Γ_0 contains an element γ whose eigenvalues are -1.

The conjugation action of B_s on the commutative $U_s \simeq \mathbf{G}_a$ factors through a character of $B_s/U_s \simeq \mathbf{G}_m$ that is a square in the character lattice (as is well-known for the root system of SL_2 and anyway is seen explicitly for $s = \infty$). That must kill the 2-torsion elements in $\mathbf{G}_m(\mathbf{Q}) = \mathbf{Q}^\times$ and so kills the image of Γ_0 in $(B_s/U_s)(\mathbf{Q})$. Thus, the Γ_0-conjugation on $U_s(\mathbf{Q})$ is trivial. Since $\Gamma_0 \cap U_s(\mathbf{Q})$ is infinite cyclic, the only cases when infinite cyclicity might fail is when there exists a non-unipotent $\gamma \in \Gamma_0$. Assuming such a γ exists, we have just seen that γ commutes with the infinite cyclic $\Gamma_0 \cap U_s(\mathbf{Q})$, so Γ_0 is commutative and generated by γ and $\Gamma_0 \cap U_s(\mathbf{Q})$ with $\gamma^2 \in \Gamma_0 \cap U_s(\mathbf{Q})$. The known structure of finitely generated abelian groups then implies that Γ_0 fails to be infinite cyclic if and only if Γ_0 contains such a γ with order 2. Any such γ must be semisimple and so is geometrically diagonalizable, yet both eigenvalues are -1 and so it is geometrically conjugate to the central element -1. Thus, necessarily $\gamma = -1$, so we are done. $\square$

For each $s \in \mathbf{P}^1(\mathbf{Q})$ there exists $g_s \in \mathrm{SL}_2(\mathbf{Q})$ such that $g_s \cdot \infty = s$. We can even arrange that $g_s \in \mathrm{SL}_2(\mathbf{Z})$ since $\mathbf{P}^1(\mathbf{Q}) = \mathbf{P}^1(\mathbf{Z})$. The holomorphic function $f|_k g_s$ is invariant under the group $g_s^{-1}\Gamma_s g_s \subset B_\infty(\mathbf{Q})$ that is as described in Lemma 1.3; in particular, this group lies inside $\{\pm 1\} \times U_\infty(\mathbf{Q})$. If we take g_s to come from $\mathrm{SL}_2(\mathbf{Z})$ then all choices of g_s are related through $B_\infty(\mathbf{Z})$

whose conjugation action on $U_\infty \simeq \mathbf{G}_a$ is through multiplication against squares in $\mathbf{Z}^\times = \{\pm 1\}$ (and whose conjugation action on -1 is obviously trivial), so under this condition on g_s we can say that $g_s^{-1}\Gamma g_s$ is independent of the choice of g_s. But without such an integrality requirement then varying the choice of g_s has the effect of scaling by $(\mathbf{Q}^\times)^2$ on U_∞.

By Lemma 1.3, $g_s^{-1}\Gamma g_s$ meets $U_\infty(\mathbf{Q})$ with index at most 2 and the standard $\mathbf{Q}$-isomorphism $U_\infty \simeq \mathbf{G}_a$ defined by $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mapsto x$, identifies the intersection of $g_s^{-1}\Gamma_s g_s$ and $U_\infty(\mathbf{Q})$ with the infinite cyclic group generated by a unique $h_s \in \mathbf{Q}_{>0}$.

The function $f|_k g_s$ is invariant under weight-k slashing against $g_s^{-1}\Gamma_s g_s$ since f is Γ-automorphic of weight k, but weight-k slashing against $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in U_\infty(\mathbf{R})$ has nothing to do with k: it is just composition with additive translation by x on $\mathbf{H}$. Hence, this holomorphic function is invariant under $z \mapsto z + h_s$, so it descends to a holomorphic function on the open punctured unit disc Δ^* via the quotient map

$$q_{h_s} = e^{2\pi i z/h_s} : \mathbf{H} \to \mathbf{H}/(z \sim z + h_s) = \Delta^*.$$

As such, we have a Fourier–Laurent expansion $f|_k g_s = \sum_{n\in\mathbf{Z}} a_{n,s}(f) q_{h_s}^n$, and "holomorphicity at s" means that this Fourier expansion has no negative-degree terms; i.e., $a_{n,s}(f) = 0$ for all $n < 0$.

Remark 1.4. Although $f|_k g_s$ is unaffected by replacing g_s with γg_s for $\gamma \in \Gamma$ (and γg_s is a valid choice for $g_{\gamma . s}$ *provided* we don't require g_s to belong to $\mathrm{SL}_2(\mathbf{Z})$, except of course when $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$), the full range of choices of g_s for a given s is given by right multiplication against anything in $B_\infty(\mathbf{Q})$. Making such a change in g_s for a given s changes $f|_k g_s$ by weight-k slashing against an element of $B_\infty(\mathbf{Q})$ and also changes h_s through multiplication against a nonzero rational square. Thus, in general h_s is *not* intrinsic to the orbit Γs but we see that the property "holomorphicity at s" is independent of the choice of g_s and consequently is intrinsic to Γs (since γg_s is a valid choice of $g_{\gamma s}$).

When $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ and we require $g_s \in \mathrm{SL}_2(\mathbf{Z})$ (so γg_s remains a valid choice for $g_{\gamma s}$) then the triviality of $(\mathbf{Z}^\times)^2$ makes h_s intrinsic to Γs and the intervention of $U_\infty(\mathbf{Z})$ descends to rotation of Δ^* via multiplication against $e^{2\pi i m/h_s}$ for $m \in \mathbf{Z}$. Thus, the Fourier expansion of $f|_k g_s$ is intrinsic to the orbit $\Gamma.s$ up to precisely the operation of multiplying the nth coefficient by ζ^n for all $n \in \mathbf{Z}$ with ζ an h_sth root of unity. Continuing to assume $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$, if moreover the period h_s of the cusp is equal to 1 (classicaly s is called a *regular* cusp in this case, and all cusps of $\Gamma_0(N)$ and $\Gamma_1(N)$ are regular except for some specific small N) then there is no ambiguity at all and hence the "q-expansion at s" is really intrinsic to the cusp (i.e., independent of $g_s \in \mathrm{SL}_2(\mathbf{Z})$ and of the representative $s \in \mathbf{P}^1(\mathbf{Q})$ for a given Γ-orbit).

Definition 1.5. If f is a Γ-automorphic form of weight k, we say that it is a *cusp form* if its Fourier expansion at each cusp has vanishing constant term.

One can also express holomorphicity at the cusps as a "moderate growth" condition, which we discuss later. It is this condition which translates more easily to the automorphic setting. In any case, we are most interested in cusp forms; for such forms the "moderate growth" condition will also follow from a representation-theoretic formulation of cuspidality that makes sense in a much wider setting.

Definition 1.6. We define $M_k(\Gamma)$ to be the space of Γ-automorphic forms of weight k, and $S_k(\Gamma)$ to be the space of Γ-cusp forms of weight k.

For a Dirichlet character ψ mod N, $S_k(N,\psi)$ denotes the space of $\Gamma_1(N)$-cusp forms of weight k such that if $g \in \Gamma_0(N)$, then $f|_k g = \psi([g])f$, where $[g]$ is the class of g in $\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbf{Z}/N\mathbf{Z})^\times$.

In order to identify modular forms with certain L^2 functions on adelic coset spaces for certain congruence subgroups $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$, the idea is to first define a map sending Γ-cusp forms to L^2 functions on $\Gamma\backslash\mathrm{SL}_2(\mathbf{R})$, and then use strong approximation for SL_2 to identify $\Gamma\backslash\mathrm{SL}_2(\mathbf{R})$ with $K_\Gamma\backslash\mathrm{GL}_2(\mathbf{A})/\mathrm{GL}_2(\mathbf{Q})$, where $\mathbf{A} = \mathbf{A}_\mathbf{Q}$ is the adele ring of $\mathbf{Q}$ and K_Γ is a specific compact open subgroup of $\mathrm{GL}_2(\widehat{\mathbf{Z}})$. One important part of this identification will be to identify the images of $S_k(\Gamma)$ and $S_k(N,\psi)$ on the adelic side.

2 Function theory on Lie groups

We define a map $M_k(\Gamma) \to C^\infty(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$ by sending f to $\phi_f\colon g \mapsto (f|_k g)(i)$. This is injective because it is easy to verify that

$$f(x+iy) = \phi_f\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2}\end{pmatrix} y^{-k/2}.$$

Proposition 2.1. For $f \in S_k(\Gamma)$ we have $\phi_f \in L^2(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$ when $\Gamma\backslash\mathrm{SL}_2(\mathbf{R})$ is equipped with the $\mathrm{SL}_2(\mathbf{R})$-invariant measure arising from a choice of Haar measure on the unimodular group $\mathrm{SL}_2(\mathbf{R})$. If $f \in M_k(\Gamma)$ and $f \notin S_k(\Gamma)$ then $\phi_f \notin L^p(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$ for all $p \geq 1$ when $k \geq 2$.

Proof. Define $F(z) = |f(z)|(\mathrm{Im}\, z)^{k/2}$. For $g \in \mathrm{SL}_2(\mathbf{R})$, we have:

$$\begin{aligned} |\phi_f(g)| &= \left| f\left(\frac{ai+b}{ci+d}\right)\frac{1}{(ci+d)^k}\right| \\ &= \left| f\left(\frac{ai+b}{ci+d}\right)\right| \left(\frac{1}{|ci+d|^2}\right)^{k/2} \\ &= F(g(i)). \end{aligned}$$

In particular, $|\phi_f|$ is right-invariant by the $\mathrm{SL}_2(\mathbf{R})$-stabilizer $K := \mathrm{SO}_2(\mathbf{R})$ of i which is moreover a (maximal) compact subgroup of $\mathrm{SL}_2(\mathbf{R})$, so the L^2 property can be checked using the induced function on the quotient space

$$\Gamma\backslash\mathrm{SL}_2(\mathbf{R})/K = \Gamma\backslash\mathbf{H}$$

with its induced $\mathrm{SL}_2(\mathbf{R})$-invariant measure arising from the Haar measure, namely a positive constant multiple of $(dx\,dy)/y^2$.

Since $\mathrm{SL}_2(\mathbf{R})$ acts transitively on $\mathbf{H}$ and ϕ_f is left Γ-invariant, it follows that F is also left Γ-invariant. Hence, F defines a continuous function on the quotient space $\Gamma\backslash\mathbf{H}$ that is compactified with finitely many cusps and has finite volume for its $\mathrm{SL}_2(\mathbf{R})$-invariant measure $(dx\,dy)/y^2$. The L^2-property for ϕ_f on $\Gamma\backslash\mathrm{SL}_2(\mathbf{R})$ is therefore the same as that for F on $\Gamma\backslash\mathbf{H}$ near each of the finitely many cusps relative to the measure $(dx\,dy)/y^2$.

For *any* $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$ the identity

$$(cz+d)^2\,\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = (\mathrm{Im}\, z)\left(\frac{cz+d}{c\overline{z}+d}\right)$$

implies

$$\begin{aligned} F(gz) &= |f(gz)|(\operatorname{Im} gz)^{k/2} \\ &= |(f|_k g)(z)||(cz+d)^k|(\operatorname{Im} gz)^{k/2} \\ &= |(f|_k g)(z)|(\operatorname{Im} z)^{k/2}. \end{aligned}$$

By using the horocycle topology on $\mathbf{H} \cup \mathbf{P}^1(\mathbf{Q})$, to analyze the behavior of F near a cusp on $\Gamma\backslash\mathbf{H}$ is the same as to analyze $F(g_s z)$ with z far up in a vertical strip of bounded width for $g_s \in \mathrm{SL}_2(\mathbf{Q})$ as above (note that typically $g_s \notin \Gamma$!). But this is exactly $|(f|_k g_s)(z)|(\operatorname{Im} z)^{k/2}$, so if f is not "cuspidal at s" then $|(f|_k g_s)(z)|$ approaches a nonzero constant value as $\operatorname{Im} z \to \infty$ so ϕ_f fails to be L^p for any $p \geq 1$ since $y^{k/2}/y^2 \notin L^p((c,\infty))$ for any $c > 0$ when $k \geq 2$. (This calculation that near each cusp on $\Gamma\backslash\mathrm{SL}_2(\mathbf{R})$ we have $\phi_f = O(y^{k/2})$ as $y \to \infty$ will turn into a "moderate growth" condition in the adelic theory.)

Now suppose $f \in S_k(\Gamma)$. The exponential decay of the Fourier expansion of $f|_k g_s$ swamps out the growth of $(\operatorname{Im} z)^{k/2}$ as this imaginary part gets large: by cuspidality the Fourier expansion is bounded in absolute value by a constant multiple of $|e^{2\pi i(x+iy)}| = e^{-2\pi y}$, and $e^{-2\pi y} y^{k/2}$ tends to 0 as $y \to \infty$. Thus, F extends to a continuous function on the compactification of $\Gamma\backslash\mathbf{H}$ by assigning it value 0 at each cusp, so ϕ_f is a bounded function on the space $\Gamma\backslash\mathrm{SL}_2(\mathbf{R})$ that has finite volume. Hence, ϕ_f belongs to $L^\infty \subset L^2$. □

We conclude that the injective map $M_k(\Gamma) \to C^\infty(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$ defined by $f \mapsto \phi_f$ has image that meets $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$ in exactly $S_k(\Gamma)$. Let's see how the properties of $f \in M_k(\Gamma)$ translate into properties of ϕ_f.

- Let $s \in \mathbf{P}^1(\mathbf{Q})$ represent a cusp of Γ, and let U_s be the unipotent radical of the corresponding Borel $\mathbf{Q}$-subgroup of SL_2, so by Lemma 1.3 we know that $\Gamma \cap U_s(\mathbf{R})$ is a subgroup of the s-stabilizer Γ_s with index at most 2 and is an infinite cyclic subgroup of $U_s(\mathbf{R}) \simeq \mathbf{R}$. We claim that f is "cuspidal at s" (i.e., its Fourier expansion at s has vanishing constant term, a property intrinsic to the cusp rather than depending on the representative s) if and only if $\int_{(\Gamma\cap U_s(\mathbf{R}))\backslash U_s(\mathbf{R})} \phi_f(ug)\,du = 0$ for all $g \in \mathrm{SL}_2(\mathbf{R})$ (using any Haar measure du on $U_s(\mathbf{R})$, the choice of which clearly doesn't matter).

  Once this is shown, it follows that an element $f \in M_k(\Gamma)$ is cuspidal if and only if for the unipotent radical U of *every* Borel $\mathbf{Q}$-subgroup of SL_2 we have

  $$\int_{U_\Gamma\backslash U(\mathbf{R})} \phi_f(ug)\,du = 0$$

  with $U_\Gamma := \Gamma \cap U(\mathbf{Q})$ an arithmetic subgroup (so infinite cyclic in $U(\mathbf{R})$). This vanishing condition for a given U only depends on the Γ-conjugacy class of U (since ϕ_f is left Γ-invariant and we vary through all $g \in \mathrm{SL}_2(\mathbf{R})$), and its validity for all U will be called *cuspidality* for ϕ_f (to be appropriately adapted to the L^2 adelic theory).

  To verify the asserted reformulation of cuspidality at s, we will calculate with the Fourier expansion at s. Note that $U_s = g_s \cdot U_\infty \cdot g_s^{-1}$ for a choice of $g_s \in \mathrm{SL}_2(\mathbf{Q})$ satisfying $g_s(\infty) = s$. Write $g_s^{-1} g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ so for any function $F : \mathbf{H} \to \mathbf{C}$ we have $(F|_k(g_s^{-1}g))(w) = (cw+d)^{-k} F((g_s^{-1}g) \cdot w)$. Let $z = (g_s^{-1}g) \cdot i$ and $\alpha = (ci+d)^{-k}$, so for any function

$F : \mathbf{H} \to \mathbf{C}$ we have $(F|_k(g_s^{-1}g))(i) = \alpha F(z)$. Finally, define $f_s = f|_k g_s$. Now, we may calculate the integral via the Fourier expansion at s as:

$$\begin{aligned}
\int_{(\Gamma\cap U_s(\mathbf{R}))\backslash U_s(\mathbf{R})} \phi_f(ug)\,du &= \int_{((g_s^{-1}\cdot\Gamma_s g_s)\cap U_\infty(\mathbf{R}))\backslash U_\infty(\mathbf{R})} \phi_f(g_s u g_s^{-1} g)\,du \\
&= \int_{((g_s^{-1}\cdot\Gamma_s g_s)\cap U_\infty(\mathbf{R}))\backslash U_\infty(\mathbf{R})} \big((f_s)|_k u)|_k(g_s^{-1}g)\big)(i)\,du \\
&= \alpha \int_{((g_s^{-1}\cdot\Gamma_s g_s)\cap U_\infty(\mathbf{R}))\backslash U_\infty(\mathbf{R})} ((f_s)|_k u)(z)\,du \\
&= \alpha \int_{h_s\mathbf{Z}\backslash\mathbf{R}} f_s(z+x)\,dx \\
&= \alpha \int_0^{h_s} f_s(z+x)\,dx \\
&= \alpha \int_0^{h_s} \sum_{n\geq 0} a_{n,s}(f)(e^{2\pi i(z+x)/h_s})^n\,dx \\
&= \alpha \cdot h_s \cdot a_{0,s}(f)
\end{aligned}$$

Note that $\alpha h_s \in \mathbf{C}^\times$, so this equals 0 if and only if $a_{0,s}(f) = 0$, which amounts to the element $f \in M_k(\Gamma)$ being cuspidal at s.

- Let $K = \mathrm{SO}_2(\mathbf{R})$ be the $\mathrm{SL}_2(\mathbf{R})$-stabilizer of $i \in \mathbf{H}$; this is a maximal compact subgroup of $\mathrm{SL}_2(\mathbf{R})$. Defining
$$r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$
it is easy to check from the definition of ϕ_f in terms of f that $\phi_f(g\cdot r(\theta)) = \phi_f(g)e^{-ik\theta}$; this is an important refinement of the fact seen in the proof of Proposition 2.1 that $|\phi_f|$ is right K-invariant. For $f \in S_k(\Gamma)$, this says that ϕ_f is an eigenfunction for K under the right regular representation R of $\mathrm{SL}_2(\mathbf{R})$ on $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$ defined by $(R(g)\cdot\phi)(h) = \phi(hg)$ (which makes sense on the L^2-space since $\mathrm{SL}_2(\mathbf{R})$ is unimodular), with eigencharacter $r(\theta) \mapsto e^{-ik\theta}$ encoding the *weight* k of f.

The property of f that we still need to capture is that it is a *holomorphic* function on $\mathbf{H}$. There is a unique left-invariant tensor field of type (0,2) on $\mathrm{SL}_2(\mathbf{R})$ inducing a given quadratic form $q : \mathfrak{sl}_2(\mathbf{R}) \to \mathbf{R}$. To further impose right-invariance of the tensor field under K can be arranged via averaging over K, in which case it descends to such a tensor field on $\mathrm{SL}_2(\mathbf{R})/K$ whose effect on the tangent space at the identity is invariant under the adjoint action of K. By consideration of K-weights in the complexification, we see that the space of K-invariants in $\mathrm{Sym}^2((\mathfrak{sl}_2(\mathbf{R})/\mathfrak{k})^*)$ is 1-dimensional. But we can make such a K-invariant quadratic form that is *positive-definite* by choosing an initial positive-definite quadratic form q_0 on $\mathfrak{sl}_2(\mathbf{R})/\mathfrak{k}$ and averaging it under the adjoint representation of K (such averaging preserves positive-definiteness, whereas it may not preserve non-degeneracy if we pick a non-degenerate q_0 with mixed signature). Thus, up to $\mathbf{R}^\times$-multiple there is a *unique* nonzero left-invariant tensor field of type (0,2) on $\mathrm{SL}_2(\mathbf{R})/K$ and up to sign it is a *Riemannian* metric tensor.

To any oriented pseudo-Riemannian manifold $(M, \mathrm{d}s^2)$ with pure dimension n there is a canonically associated Laplace–Beltrami operator on the space $\Omega^p(M)$ of smooth global p-forms on M given by

$$\Delta_p := \delta_{p+1} \circ d_p + d_{p-1} \circ \delta_p$$

where $\delta_p : \Omega^p(M) \to \Omega^{p-1}(M)$ is the operator $(-1)^{n(p+1)+1} *_{n-p+1} d_{n-p} *_p$ with $*_j : \Omega^j(M) \to \Omega^{n-j}(M)$ arising from the usual Hodge star on each fiber $\wedge^j(\mathrm{T}^*_m(M))$ defined by the induced non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle_{j,m}$ on $\wedge^j(\mathrm{T}^*_m(M))$ arising from $\mathrm{d}s^2(m)$ on $\mathrm{T}_m(M)$ and the orientation of $\wedge^n(\mathrm{T}_m(M))$ (i.e., $\omega \wedge (*_m \eta) = \langle \omega, \eta\rangle_{j,m}\mu(m)$ for $\omega, \eta \in \wedge^j(\mathrm{T}^*_m(M))$ and $\mu(m)$ the unique vector in the positive half-line of $\wedge^n(\mathrm{T}^*_m(M))$ satisfying $\langle \mu(m), \mu(m)\rangle_{n,m} = \pm 1$. Since $*_{n-j} *_j = (-1)^{j(n-j)}$, so $(-1)^{n(p+1)+1} *_{n-p+1} *_{p-1} = (-1)^p$, we can equivalently write $\delta_p = (-1)^p *_{p-1}^{-1} d_{n-p} *_p$ For $p = 0$ this is $\delta_1 d_0 = - *_0^{-1} d *_1 d$ on smooth functions.

Changing the orientation of M has no effect on Δ_p (as it suffices to check this on each connected component, where there is just a sign change possible on the orientation; such a sign change negates the Hodge-star in *every* degree, and so visibly has no effect on Δ_p from the definition involving two Hodge-stars), so via globalization from orientable open subsets it can be defined without needing $(M, \mathrm{d}s^2)$ to even admit an orientation, let alone for an orientation to be chosen.

Example 2.2. Let's compute $\Delta_0(f)$ relative to an *oriented* local coordinate system $\{y_1, \dots, y_n\}$. Defining $g_{ij} = \langle \partial_{y_i}, \partial_{y_j}\rangle$ as usual, so $\mathrm{d}s^2 = \sum_{ij} g_{ij} dx_i \otimes dx_j$, dy_i is dual to $\sum_k g^{ik}\partial_{y_k}$ for the matrix (g^{ij}) inverse to (g_{ij}). Hence, we compute (via symmetry of the metric tensor) that the induced bilinear form on the cotangent bundle satisfies

$$\langle dy_i, dy_j\rangle = \sum_{k,\ell} g^{ik} g_{k\ell} g^{j\ell} = g^{ji} = g^{ij},$$

so orientedness of the coordinate system implies that the oriented unit top-degree differential form is $\mu := |\det(g_{ij})|^{1/2} dy_1 \wedge \cdots \wedge dy_n$. Also, by definition clearly $\star(\mu) = (-1)^r$ where $(n-r, r)$ is the signature of the metric tensor (equivalently, this is the sign $(-1)^r$ of $\det(g_{ij})$). The determination of μ yields that

$$dy_i \wedge \star_1(dy_j) = g^{ij} |\det(g_{k\ell})|^{1/2} dy_1 \wedge \cdots \wedge dy_n$$

for all i, so

$$\star(dy_j) = \sum_i (-1)^{i-1} g^{ij} |\det(g_{k\ell})|^{1/2} dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_n.$$

Hence,

$$\star(df) = \sum_{ij} g^{ij} |\det(g_{k\ell})|^{1/2} \partial_{y_j}(f) dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_n,$$

so

$$d \star (df) = \sum_{ij} \partial_{y_i}(g^{ij} |\det(g_{k\ell})|^{1/2} \partial_{y_j}(f)) dy_1 \wedge \cdots \wedge dy_n = \sum_{ij} \frac{\partial_{y_i}(g^{ij} |\det(g_{k\ell})|^{1/2} \partial_{y_j}(f)}{|\det(g_{k\ell})|^{1/2}} \mu.$$

We noted above that $\star(\mu) = (-1)^r$, so

$$\begin{aligned}\Delta_0(f) = -\star d \star (df) &= (1)^{r+1} \sum_{ij} \frac{\partial_{y_i}(g^{ij} |\det(g_{k\ell})|^{1/2} \partial_{y_j}(f)}{|\det(g_{k\ell})|^{1/2}} \\ &= (-1)^{r+1} \sum_{ij} (g^{ij} \partial^2_{y_i y_j}(f) + \partial_{y_i}(g^{ij}) \partial_{y_j}(f) + (1/2\Delta) \partial_{y_i}(\Delta) g^{ij} \partial_{y_j}(f)\end{aligned}$$

for $\Delta := |\det(g_{k\ell})|$. In other words:

$$\Delta_0 = (-1)^{r+1} \sum_{ij} (g^{ij} \partial^2_{y_i y_j} - (\partial_{y_i}(g^{ij}) + (1/2\Delta) \partial_{y_i}(\Delta) g^{ij}) \partial_{y_j}).$$

This is a differential operator on $C^\infty(M)$ of order exactly 2 everywhere, with no "constant term", and by inspection it is elliptic if and only if the symmetric inverse matrix (g^{ij}) is definite, which is to say that the metric tensor is Riemannian up to an overall sign. (In the general Riemannian case the operator Δ_p on $\Omega^p(M)$ is elliptic for all p [10, 6.34–6.36].) The messy coefficient of the degree-1 part of Δ_0 can be expressed in terms of the Christoffel symbols of the metric tensor, but we don't need that and so don't address it here. (In the special case of a flat metric we can pick local oriented coordinates for which the matrix (g_{ij}) is constant and even $g_{ij} = \varepsilon_i \delta_{ij}$ for signs $\varepsilon_i = \pm 1$, so then $\Delta_0 = (-1)^{r+1} \sum_i \varepsilon_i \partial^2_{y_i}$ with r equal to the number of negative ε_i's. This recovers the familiar Laplacian on R^n up to an overall sign.)

Example 2.3. For any Lie group G with finitely many connected components and maximal compact subgroup K (so K meets every connected component of G and $K^0 = K \cap G^0$, with K^0 maximal compact in G^0), we can find a K-invariant positive-definite quadratic form q on $\mathfrak{g}/\mathfrak{k}$ (unlike for $G = \mathrm{SL}_2(\mathbf{R})$, there is generally more than a ray's worth of such q's). From this we get an associated left-invariant Riemannian metric on G/K and hence a left-invariant degree-2 elliptic Laplace–Beltrami operator on $\Omega^p(G/K)$ for all $p \geq 0$.

For any Lie group G, each $X \in \mathfrak{g}$ gives rise to a visibly left-invariant $\mathbf{R}$-linear derivation of $C^\infty(G)$ via the recipe

$$(Xf)(g) = \partial_t|_{t=0}(f(g \exp_G(tX))).$$

The unique smooth vector field giving rise to this derivation must be left-invariant by uniqueness, so it is the left-invariant vector field extending X at the identity since $\exp_G(tX)$ is a parametric curve through the identity with velocity X at the identity. The interaction of such a differential operator X with the right regular representation on smooth functions is given by

$$\begin{aligned}(X \circ R(g_0))(f) = X(R(g_0)(f)) : g &\mapsto \partial_t|_{t=0}((R(g_0)(f))(g \exp_G(tX))) \\ &= \partial_t|_{t=0}(f(g \exp_G(tX) g_0)) \\ &= \partial_t|_{t=0}(f(g g_0 \exp_G(t \mathrm{Ad}_G(g_0^{-1})(X)))),\end{aligned}$$

which is to say

$$X \circ R(g_0) = R(g_0) \circ \mathrm{Ad}_G(g_0^{-1})(X). \tag{1}$$

The identity $X(Yf) - Y(Xf) = [X, Y](f)$ is proved in [2, III, §3.7, Def. 7, Prop. 27(ii),(iii)]; this uses crucially that the definition of $(Xf)(g)$ involves *right* multiplication by $\exp_G(tX)$, rather than left multiplication; see [2, III, §3.7, Def. 7(b)]. Thus, the operations $f \mapsto Xf$ uniquely extend to an action of the associative universal enveloping algebra $U(\mathfrak{g})$ on $C^\infty(G)$. The Poincaré-Birkhoff-Witt theorem giving the structure of $U(\mathfrak{g})$ thereby identifies the space $\mathscr{D}(G)$ of left-invariant differential operators on $C^\infty(G)$ with $U(\mathfrak{g})$. We want to relate this to the space $\mathscr{D}(G/K)$ of left G-invariant differential operators on $C^\infty(G/K)$. This is a special case of the following more general considerations.

Let G be any Lie group, and H a closed subgroup. There is a natural inclusion $C^\infty(G/H) \hookrightarrow C^\infty(G)^H$, and since $\pi : G \to G/H$ is an H-bundle it is easy to see that this inclusion is an equality. Letting $\mathscr{D}(G)$ and $\mathscr{D}(G/H)$ respectively denote the spaces of left invariant differential operators on G and on G/H (left-invariant for the G-action in the latter case), we want to describe $\mathscr{D}(G/H)$ in terms of the space $\mathscr{D}(G)^H$ of elements of $\mathscr{D}(G)$ invariant for the natural action of H through *right* multiplication on G. Since elements of $\mathscr{D}(G)$ are left-invariant by G, the identification of $\mathscr{D}(G)$ with $U(\mathfrak{g})$ carries $\mathscr{D}(G)^H$ over to $U(\mathfrak{g})^H$ where H acts through its Ad_G on $\mathfrak{g}$.

For any $D \in \mathscr{D}(G)^H$ and $f \in C^\infty(G/H)$, the smooth function $D(f \circ \pi)$ on G is invariant for the right multiplication by H since $D \in \mathscr{D}(G)^H$; i.e., $D(f \circ \pi) \in C^\infty(G)^H = C^\infty(G/H)$. Hence, $D(f \circ \pi) = (\overline{D}f) \circ \pi$ for a unique $\overline{D}f \in C^\infty(G/H)$.

Lemma 2.4. For $D \in \mathscr{D}(G)^H$, the $\mathbf{R}$-linear map $C^\infty(G/H) \to C^\infty(G/H)$ defined by $f \mapsto \overline{D}f$ is a differential operator on G/H. If D has order $\le d$ then $\overline{D}$ has order $\le d$.

Proof. This problem is local over G/H, since the formation of Df is clearly of local nature over G/H (i.e., its restriction to an open $U \subset G/H$ only depends on $f|_U$). Since the left G-action is just coming along for the ride and not really doing anything, using that $G \to G/H$ is an H-bundle thereby reduces our task to the analogue where $G \to G/H$ is replaced by $H \times V \to V$ for an open subset $V \subset \mathbf{R}^n$ and where we work with $f \in C_c^\infty(V)$. Letting $e : V \to H \times V$ be $v \mapsto (1, v)$, we see that $\overline{D}f = D(f \circ \pi) \circ e$ for $f \in C_c^\infty(V)$. Now H has done its job (to provide the product structure), and to exhibit this as a differential operator can replace H with a coordinate neighborhood of 1. In this case everything is obvious (including that $\overline{D}$ has order $\le d$ if D does). □

Proposition 2.5 (Helgason)**.** If $\mathfrak{h}$ admits an H-stable complement in $\mathfrak{g}$ under Ad_G then the natural map of associative $\mathbf{R}$-algebras $U(\mathfrak{g})^H = \mathscr{D}(G)^H \to \mathscr{D}(G/H)$ defined by $D \to \overline{D}$ is surjective. Moreover, an element of $\mathscr{D}(G/H)$ with order d can be lifted to an element of $\mathscr{D}(G)^H$ with order d.

Note that the hypothesis on H is satisfied when H is compact, since the finite-dimensional continuous linear representations of a compact group are semisimple.

Proof. The proof we give is a streamlined version of the proof of [6, Lemma 16]. Choose $D' \in \mathscr{D}(G/H)$ (i.e., a differential operator on G/H invariant under the left G-action). We seek $D \in \mathscr{D}(G)^H$ such that $\overline{D} = D'$. Let $m = \dim G$ and $n = \dim(G/H)$, and pick a basis $\{v_1, \ldots, v_n, \ldots, v_m\}$ of $\mathfrak{g}$ whose first n elements span an $\mathrm{Ad}_G|_H$-stable complement to $\mathfrak{h}$. For $U \subset \mathbf{R}^n$ a sufficiently small open neighborhood of 0, there is a diffeomorphism of U onto an open neighborhood N of $\xi_0 := \pi(1) \in G/H$ given by

$$(t_1, \ldots, t_n) \mapsto \pi(\exp_G(t_1X_1 + \cdots + t_nX_n)).$$

Since D' is a differential operator over G/H, for any $f' \in C^\infty(G/H)$ the restriction $(D'f)|_U \in C^\infty(U)$ is given by a standard differential operator in the coordinates $\{t_1, \ldots, t_n\}$. That is, there exists a polynomial $P \in C^\infty(U)[\partial_{t_1}, \dots, \partial_{t_n}]$ independent of f such that

$$(D'f)|_U = P(f|_U)$$

(really $f|_U$ should be transported to a smooth function on N via the diffeomorphism $U \simeq N$ mentioned above). Evaluating both sides at e_0 turns the coefficients functions of P into real numbers, so we get $P_0 \in \mathbf{R}[y_1, \dots, y_n]$ such that

$$(D'f)(\xi_0) = (P_0(\partial_{t_1}, \dots, \partial_{t_n})((f \circ \pi)(\exp_G(t_1 v_1 + \cdots + t_n v_n))))|_{t=0}.$$

A general point $\xi \in G/H$ has the form $g(\xi_0)$ for some $g \in G$, so the assumed (!) left-invariance of D' under the G-action on G/H implies $(D'f)(\xi) = (D'(f \circ \lambda_g))(\xi_0)$ with $\lambda_g : G/H \to G/H$ the left-multiplication action of g. Hence,

$$(D'f)(\xi) = (P_0(\partial_{t_1}, \dots, \partial_{t_n})((f \circ \pi)(g \exp_G(t_1 v_1 + \cdots + t_n v_n))))|_{t=0}.$$

A coordinate system on G near g is given by

$$(t_1, \dots, t_m) \mapsto g \exp_G(t_1 v_1 + \cdots + t_n v_n + \cdots + t_m v_m)$$

for $(t_1, \dots, t_m)$ belonging to a fixed open $\Omega \subset U \times \mathbf{R}^{m-n}$ around 0 *independent* of g. Thus, even though $\exp_G : \mathfrak{g} \to G$ is rarely a homomorphism, we are motivated to consider the construction D assigning to each $F \in C^\infty(G)$ the function $DF : G \to \mathbf{R}$ defined by

$$\begin{aligned} (DF)(g) &= (P_0(\partial_{t_1}, \dots, \partial_{t_n})(F(g \exp_G(t_1 v_1 + \cdots + t_n v_n + \dots t_m v_m))))|_{t=0} \\ &= (P_0(\partial_{t_1}, \dots, \partial_{t_n})(F(g \exp_G(t_1 v_1 + \cdots + t_n v_n))))|_{t=0} \end{aligned}$$

(the second equality since $P_0(\partial_{t_1}, \dots, \partial_{t_n})$ commutes with specializing any of $t_{n+1}, \dots, t_m$ to 0). It is obvious by consideration of the above local coordinate system near a chosen but arbitrary $g_0 \in G$ that DF is smooth and that $F \mapsto DF$ is a differential operator on $C^\infty(G)$. Note that if D' has order d then we can arrange that P_0 has total degree d, so D clearly has order at most d. We shall prove that D is left G-invariant and right H-invariant, and that $\overline{D} = D'$ (so we would be done: in particular, D must have order exactly d, since otherwise it would have order at most $d-1$ and hence so would $\overline{D} = D'$, contrary to how d was defined).

If $F = f \circ \pi$ for $f \in C^\infty(G/H)$ then

$$(DF)(g) = (P_0(\partial_{t_1}, \dots, \partial_{t_n})((f \circ \pi)(g \exp_G(t_1 v_1 + \cdots + t_n v_n))))|_{t=0} = (D'f)(g(\xi_0)),$$

so if it is shown that $D \in \mathcal{D}(G)^H$ then $(\overline{D}f)(gH) = (DF)(g) = (D'f)(g(\xi_0))$, which is to say $\overline{D} = D'$ as desired. Hence, our task is reduced to checking that the differential operator D on G is left G-invariant and right H-invariant. Left G-invariance of D says $(DF) \circ \lambda_{g_0} = D(F \circ \lambda_{g_0})$ for all $g_0 \in G$ and $F \in C^\infty(G)$ (where λ_{g_0} is left multiplication by g_0 on G), and this is identity is obvious from how DF is defined. Right H-invariance of D says $(DF) \circ \rho_h = D(F \circ \rho_h)$ for all $h \in H$ and $F \in C^\infty(G)$ (where ρ_h is right multiplication by h on G), and proving this will require more effort (in effect, establishing an H-invariance property of P_0).

By definition of D we have

$$\begin{aligned} (DF)(gh) &= (P_0(\partial_{t_1}, \dots, \partial_{t_n})(F(g \exp_G(\mathrm{Ad}_G(h)(t_1 v_1 + \cdots + t_n v_n))h)))|_{t=0} \\ &= (P_0(\partial_{t_1}, \dots, \partial_{t_n})((F \circ \rho_h)(g \exp_G(\mathrm{Ad}_G(h)(t_1 v_1 + \cdots + t_n v_n)))))|_{t=0}, \end{aligned}$$

so for right H-invariance it suffices (upon renaming $F \circ \rho_h$ as F) to show that

$$(P_0(\partial_{t_1}, \dots, \partial_{t_n})(F(g \exp_G(\mathrm{Ad}_G(h)(t_1 v_1 + \cdots + t_n v_n)))))|_{t=0}$$

is independent of $h \in H$. Since $\{v_1, \dots, v_n\}$ spans an $\mathrm{Ad}_G|_H$-stable subspace $V \subset \mathfrak{g}$, if $M_h \in \mathrm{GL}_n(\mathbf{R})$ is the matrix for $\mathrm{Ad}_G(h)$ relative to the ordered basis $\{v_1, \dots, v_n\}$ of V, this final expression is the same as

$$(P_0(M_h^{-1}(\partial_{t_1}), \dots, M_h^{-1}(\partial_{t_n}))(F(g \exp_G(t_1 v_1 + \cdots + t_n v_n))))|_{t=0}.$$

Thus, the right H-invariance is reduced to showing that $P_0 : \mathbf{R}^n \to \mathbf{R}$ is invariant under composition with the matrices $M_h^{-1} = M_{h^{-1}}$ for all $h \in H$. Since $(D'f)(g(\xi_0))$ for $f \in C^\infty(G/H)$ is unaffected by multiplication on g by anything in H on the right and likewise such $f \circ \pi$ is unaffected by right H-multiplication on G, the formula for $(D'f)(g(\xi_0))$ in terms of P_0 implies the desired H-invariance for P_0 by plugging gh in place of h in the formula for $(D'f)(g(\xi_0))$ similarly to the preceding calculation of $(DF)(gh)$. □

We conclude that the Laplace-Beltrami operator $\Delta_K \in \mathscr{D}(\mathrm{SL}_2(\mathbf{R})/K)$ arises from an element of $U(\mathfrak{sl}_2(\mathbf{R}))^K$ with degree 2, and this latter element must have no "constant part" since Δ_K kills constant functions by design. Hence, Δ_K arises from a sum $D_1 + D_2$ for $D_j \in U(\mathfrak{sl}_2(\mathbf{R}))^K$ with degree j (possibly $D_1 = 0$, but definitely $D_2 \neq 0$). By Poincaré-Birkhoff-Witt, the natural map

$$\mathrm{Sym}^j(\mathfrak{g}) \to U_{\le j}(\mathfrak{g})/U_{<j}(\mathfrak{g})$$

is an isomorphism for all $j > 0$ and any finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0. Thus,

$$U_{\le j}(\mathfrak{sl}_2(\mathbf{R}))^K / U_{<j}(\mathfrak{g})^K \simeq \mathrm{Sym}^j(\mathfrak{sl}_2(\mathbf{R}))^K,$$

and for $j = 2$ we have already noted that the right side is 2-dimensional (with an evident line arising from $U_{\le 2}(\mathfrak{k})/U_{<2}(\mathfrak{k})$) whereas for $j = 1$ the right side is 1-dimensional (arising from the line $\mathfrak{k} \subset U_{\le 1}(\mathfrak{k}) = \mathbf{R} \oplus \mathfrak{k}$).

Via the decomposition $\mathrm{SL}_2(\mathbf{R}) = NAK$, it is clear that the image of $U(\mathfrak{k}) \subset U(\mathfrak{sl}_2(\mathbf{R}))^K$ in $\mathfrak{D}(\mathrm{SL}_2(\mathbf{R})/K)$ vanishes since a K-invariant smooth function on $\mathrm{SL}_2(\mathbf{R}) = NA \times K$ is pulled back from a smooth function on NA. It follows that the image of $U_{\le 2}(\mathfrak{sl}_2(\mathbf{R}))^K$ in $\mathscr{D}(\mathrm{SL}_2(\mathbf{R})/K)$ is *at most* 1-dimensional, coinciding with the image of any single line not arising from $U(\mathfrak{k})$. We now construct a canonical such line more broadly.

For any finite-dimensional semisimple Lie algebra $\mathfrak{g}$ over a field F of characteristic 0 the *canonical* symmetric bilinear Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to F$ defined by $\kappa(X, Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ is non-degenerate and equivariant for the adjoint representation of $\mathfrak{g}$ on itself. Thus, this gives rise to a canonical nonzero central element C homogenous of degree 2 in the center $\mathfrak{z}$ of $U(\mathfrak{g})$ called the *Casimir element*; this is defined in [2, Ch. I, §3.7].

For a connected semisimple F-group G with Lie algebra $\mathfrak{g}$, the Casimir element C cannot lie in $\mathrm{Lie}(T)$ for a maximal F-torus T since centrality of C would then force it to lie in the Lie algebra of every maximal torus over $\overline{F}$ yet the intersection of such Lie algebras coincides with $\mathrm{Lie}(Z_G) = 0$ (as $\mathrm{char}(F) = 0$). Hence, the image in $\mathscr{D}(\mathrm{SL}_2(\mathbf{R})/K)$ of $U_{\le 2}(\mathfrak{sl}_2(\mathbf{R}))^K$ that is at most 1-dimensional is spanned by the image of C. We know that Δ_K is a nonzero element in this image, so Δ_K is the effect of some $\mathbf{R}^\times$-multiple of C!

Scaling the Riemannian metric by $a > 0$ scales the Hodge-star by $1/a$ and so (by its definition) scales Δ_K by $1/a^2$. Thus, we can scale the metric so that Δ_K is the effect of $\pm 2C$ for a unique sign. In Appendix A we will see that the sign comes out to be negative; i.e., $\Delta_K = -2C$ (for a suitable choice of the metric that we will work out in Appendix A too).

3 Characterization of cusp forms

If $g \in \mathrm{SL}_2(\mathbf{R})$, we can use the NAK decomposition to write

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} 0 & \\ 0 & y^{-1/2} \end{pmatrix} \cdot r(\theta).$$

Let's use these coordinates to write down the element of $\mathscr{D}(\mathrm{SL}_2(\mathbf{R}))$ corresponding to $-2C$ for the Casimir element C of $U(\mathfrak{sl}_2(\mathbf{R}))$. In Appendix A (the discussion preceding Lemma A.1) we work it out. Denoting that differential operator as Δ (since its composition with $C^\infty(\mathrm{SL}_2(\mathbf{R})/K) \to C^\infty(\mathrm{SL}_2(\mathbf{R}))$ recovers the Laplace-Beltrami operator Δ_K, as discussed in Appendix A), we get:

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - y \frac{\partial^2}{\partial x \, \partial \theta}$$

in $\mathscr{D}(\mathrm{SL}_2(\mathbf{R}))$.

Remark 3.1. On any connected Lie group G with semisimple Lie algebra, the Killing form defines a non-degenerate quadratic form that is (negative-)definite if and only if the group is compact. By non-degeneracy it always defines a left-invariant pseudo-Riemannian metric tensor, and hence s Laplace-Beltrami operator on each $\Omega^p(G)$. But these operators are *never* elliptic when the metric is not Riemannian (i.e., when G is non-compact). The operator Δ above on $C^\infty(\mathrm{SL}_2(\mathbf{R}))$ is non-elliptic by inspection of its principal symbol.

Proposition 3.2. If $f \in S_k(\Gamma)$, then $\Delta \phi_f = -(k/2)((k/2) - 1)\phi_f$

Proof. By definition,

$$\phi_f(g) = \phi_f\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{smallmatrix}\right)\right)e^{-ik\theta} y^{k/2} f(x+iy) e^{-ik\theta}.$$

This gives us that

$$\begin{aligned} \Delta \phi_f &= (-y^{k/2+2}e^{-ik\theta})\frac{\partial^2 f}{\partial x^2} + (-y^{k/2+2}e^{-ik\theta})\frac{\partial^2 f}{\partial y^2} + (-y^{k/2+1}e^{-ik\theta} \cdot k)\frac{\partial f}{\partial y} \\ &\quad - \frac{k}{2}\left(\frac{k}{2} - 1\right) y^{k/2} f(x+iy) e^{-ik\theta} + (-y^{k/2+1}e^{-ik\theta} \cdot (-ik))\frac{\partial f}{\partial x} \\ &= -\frac{k}{2}\left(\frac{k}{2} - 1\right)\phi_f, \end{aligned}$$

where the last equality follows from the Cauchy-Riemann equations characterizing holomorphicity of f (recall that $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$). □

Proposition 3.3. The image of $S_k(\Gamma)$ in $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$ is exactly characterized as the set of all ϕ such that:

- $R(r(\theta))(\phi) = e^{-ik\theta}\phi$; i.e., ϕ is an eigenfunction for K under the right regular representation, with "weight" k under the standard parameterization $r : \mathbf{R}/2\pi\mathbf{Z} \simeq K$, or in other words ϕ is an L^2-section of a hermitian line bundle on $\Gamma\backslash\mathrm{SL}_2(\mathbf{R})/K$.

- the pullback of ϕ to a Γ-invariant locally L^1 function on $\mathrm{SL}_2(\mathbf{R})$ is an eigenfunction for Δ in the distributional sense, with eigenvalue $-(k/2)(k/2-1)$ (so ϕ is smooth by elliptic regularity: the function $\widetilde{\phi} = e^{ik\theta}\phi$ satisfies $R(\theta_0)(\widetilde{\phi}) = \widetilde{\phi}$ for all θ_0, which is to say that the L^2-function $\widetilde{\phi}$ is right K-invariant, so by "coset Fubini" $\widetilde{\phi}$ is the pullback of an L^2-function on $\mathrm{SL}_2(\mathbf{R})/K = \mathbf{H}$ that is likewise killed by Δ_K that is the *elliptic* Laplace-Beltrami operator on $\mathbf{H}$),

- it satisfies the "cuspidality condition"

$$\int_{(\Gamma\cap U(\mathbf{R}))\backslash U(\mathbf{R})} \phi(ug)\,du = 0$$

  for all $g \in \mathrm{SL}_2(\mathbf{R})$ and all unipotent radicals U of Borel $\mathbf{Q}$-subgroups of SL_2.

Remark 3.4. In these applications of ellipticity one even gets real-analyticity and not just smoothness for ϕ: see [1, Appendix, Ch. 4, Part II] for an elegant proof of inhertiance of real-analyticity in the elliptic regularity theorem.

For ϕ satisfying the conditions of Proposition 3.3, so ϕ is smooth, the formula for f in terms of ϕ_f motivates making the definition

$$f(x+iy) := \phi\left(\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix}\right) y^{-k/2}$$

for $x \in \mathbf{R}$ and $y > 0$. This is smooth function on $\mathbf{H}$. It's straightforward to check that $f|_k\gamma = f$ for $\gamma \in \Gamma$ precisely because of the left Γ-invariance of ϕ, and that if f is holomorphic then its zeroth Fourier coefficient vanishes at each cusp of Γ precisely because of the cuspidality condition.

Thus, $f \in S_k(\Gamma)$ as soon as we know f is *holomorphic* with $f|_k g_s$ bounded near s for each $s \in \mathbf{P}^1(\mathbf{Q})$ (with $g_s \in \mathrm{SL}_2(\mathbf{Q})$ carrying ∞ to s, the choice of which we have seen doesn't matter). This boundedness property for all s can be deduced from a "moderate growth" condition on ϕ that is a consequence of the cuspidality condition (see Remark 3.7), but the proof of holomorphicity (in effect, that f satisfies the Cauchy-Riemann equations $\partial f/\partial\overline{z} = 0$) lies rather deeper: it requires some input from representation theory. Moreover, this deduction of holomorphicity for f built from such ϕ seems to be omitted from many standard references discussing the representation-theoretic aspects of classical modular forms (such references generally only prove the much easier converse direction), so we give a proof in Appendix A using the representation theory of $\mathfrak{sl}_2(\mathbf{R})$.

Proposition 3.3 motivates the following definition:

Definition 3.5. A Γ-*cusp form* on $G = \mathrm{SL}_2(\mathbf{R})$ is a smooth function $\phi\colon G \to \mathbf{C}$ such that:

(1) $\phi(\gamma \cdot g) = \phi(g)$ for all $\gamma \in \Gamma$

(2) ϕ is "K-finite" for a maximal compact subgroup K of G; i.e., the span of $\{\kappa \cdot \phi\}_{\kappa \in K}$ is finite-dimensional (this property is independent of the choice of K since all such K's are related to each other through G-conjugation);

(3) ϕ is an eigenfunction of Δ, [1]

(4) ϕ is cuspidal (i.e., $\int_{(\Gamma \cap U(\mathbf{R}))\backslash U(\mathbf{R})} \phi(ug)du = 0$ for all $g \in G$ and unipotent radicals U of Borel $\mathbf{Q}$-subgroups of SL_2).

Proposition 3.3 says that if such a ϕ is an eigenfunction for K with eigencharacter $r(\theta) \mapsto e^{-ik\theta}$ and the eigenvalue of Δ on ϕ is $-(k/2)(k/2-1)$ then we get exactly the functions ϕ_f for $f \in S_k(\Gamma)$. However, there are lots of cusp forms for Γ that are eigenfunctions for K with an eigencharacter that has nothing to do with its Laplacian eigenvalue, so these don't arise from $S_k(\Gamma)$. Such additional ϕ arise from non-holomorphic Maass forms. An example of a non-cuspidal Maass form (although Maass cusp forms also exist) is the Eisenstein series:

$$E\left(z, \frac{1}{2} + \frac{it}{2}\right) = \sum_{\substack{(c,d)=1 \\ c,d \in \mathbf{Z}}} \frac{(\mathrm{Im}\, z)^{1/2+it/2}}{|cz+d|^{1+it}}$$

Note that $\Delta E = \frac{1+t^2}{4} E$. Although we are focused on cusp forms, we take this opportunity to define more general automorphic forms:

Definition 3.6. A Γ-*automorphic form* on $G = \mathrm{SL}_2(\mathbf{R})$ is a smooth function $\phi : G \to \mathbf{C}$ satisfying (1), (2), and (3) of Definition 3.5, plus

(4$'$) ϕ is of "moderate growth" at the cusps of Γ: there is an $A > 0$ such that for each cusp s of Γ and $g_s \in G$ such that $g_s(\infty) = s$,

$$\left|\phi\left(g_s \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}\right)\right| \ll y^A.$$

(4$''$) As a function on $\Gamma\backslash G$, ϕ is square-integrable.

Remark 3.7. First, Proposition 2.1 shows that ϕ_f is even bounded, hence of moderate growth. In general, the cuspidality condition for all cusps implies the moderate growth condition at all cusps. Morally, having a negative Fourier coefficient at a cusp s prevents us from being square-integrable near that cusp, but see [8, Cor. 3.4.3] for a rigorous argument (which has the benefit of being more generalizable).

Second, as we've phrased it, it's not clear how to generalize to other reductive groups G the statement itself of the moderate growth condition. We note (without proving, although it follows from the Iwasawa decomposition) that it is equivalent to the following suggestive reformulation. The group $\mathrm{SL}_2(\mathbf{R})$, as a closed subspace of $\mathrm{Mat}_2(\mathbf{R})$, inherits the sup norm $\|\cdot\|$; i.e. $\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \max(|a|, |b|, |c|, |d|)$. Then ϕ as above is of moderate growth at each cusp if and only if there exists an $A' > 0$ such that $\phi(g) \ll \|g\|^{A'}$. See [4] for more details and the generalization to reductive G.

[1] To generalize beyond $\mathrm{SL}_2(\mathbf{R})$, one needs to replace the eigenfunction condition for Δ with one for the entire center of the universal enveloping algebra acting naturally through differential operators on smooth functions; in the SL_2-case, this center is generated over $\mathbf{R}$ by Δ.

4 More spaces of forms and functions

What about $S_k(N,\psi)$? Fix a Dirichlet character $\psi\colon (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$. We can consider ψ as a unitary character on the idèles in the usual way, via the identification $\mathbf{A}^\times/\mathbf{Q}^\times\mathbf{R}^\times = \widehat{\mathbf{Z}}^\times$.

Let $G = \mathrm{GL}_{2/\mathbf{Q}}$, and for finite places v define $K_v = \mathrm{GL}_2(\mathscr{O}_v)$, a maximal compact subgroup of $\mathrm{GL}_2(\mathbf{Q}_v)$. Also, let Z denote the scheme-theoretic center of G (i.e. the diagonal scalar matrices). For all $v \mid N$, we set

$$K_v^N = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_v \mid \ c \equiv 0 \bmod N\}$$

By strong approximation for SL_2 we can see that:

$$\mathrm{SL}_2(\mathbf{A}) = \mathrm{SL}_2(\mathbf{Q}) \cdot \mathrm{SL}_2(\mathbf{R}) \cdot \prod_{v\mid N} \mathrm{SL}_2(\mathscr{O}_v)^N \cdot \prod_{v\nmid N\cdot\infty} \mathrm{SL}_2(\mathscr{O}_v),$$

where $\mathrm{SL}_2(\mathscr{O}_v)^N$ consists of elements of $\mathrm{SL}_2(\mathscr{O}_v)$ which are upper triangular modulo N. Then observing that the left side is the kernel of $G(\mathbf{A})$ under the determinant map, the right side is the kernel of the determinant map on

$$G(\mathbf{Q}) \cdot \mathrm{GL}_2^+(\mathbf{R}) \cdot \prod_{v\mid N} K_v^N \cdot \prod_{v\nmid N\cdot\infty} K_v$$

and both $G(\mathbf{A})$ and the product above yield the same image (namely, all ideles) under the determinant map. Hence,

$$G(\mathbf{A}) = G(\mathbf{Q}) \cdot \mathrm{GL}_2^+(\mathbf{R}) \cdot \prod_{v\mid N} K_v^N \cdot \prod_{v\nmid N\cdot\infty} K_v.$$

Hence, any $g \in G(\mathbf{A})$ may be written as $g = \gamma \cdot g_\infty \cdot \kappa_0$ with $\gamma \in G(\mathbf{Q}), g_\infty \in \mathrm{GL}_2^+(\mathbf{R}), \kappa_0 \in \prod_{v\mid N} K_v^N \cdot \prod_{v\nmid N\cdot\infty} K_v =: K_0^N$.

We define a map $S_k(N,\psi) \to L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$ by setting $\phi_f(g) = \big(f|_k g_\infty\big)(i)\psi(\kappa_0)$. The notation $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$ (used here and throughout) is misleading. The elements thereof are not square-integrable on $G(\mathbf{Q})\backslash G(\mathbf{A})$; rather, they are required by definition to admit a unitary (idèlic) central character (i.e. defined on $Z(\mathbf{A})$ and trivial on $Z(\mathbf{Q})$) and to have their resulting absolute value function on $(Z(\mathbf{A}) \cdot G(\mathbf{Q}))\backslash G(\mathbf{A})$ be square-integrable (for the measure arising from the Haar measure on the unimodular $G(\mathbf{A})$).

Similarly to the previous case, we have:

Proposition 4.1. For $f \in S_k(N,\psi)$, we have the following properties:

- ϕ_f is well-defined [2] and an eigenfunction for the right regular action of $Z(\mathbf{A})$ with unitary central character ψ; i.e., $\phi_f(gz) = \psi(z)\phi_f(g)$ for any $z \in Z(\mathbf{A})$ and $g \in G(\mathbf{A})$ (so $|\phi_f|$ descends to a continuous function on $Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})$),

- ϕ_f belongs to $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$ (which makes sense by the previous condition) [3],

[2] This follows from $G(\mathbf{Q}) \cap \Big(\mathrm{GL}_2^+(\mathbf{R}) \cdot \prod_{v\mid N} K_v^N \cdot \prod_{v\nmid N\cdot\infty} K_v\Big) = \Gamma_0(N)$ and the action of $\Gamma_0(N)$ on $S_k(N,\psi)$.

[3] This follows from "Theorem F"' in the lectures on reduction theory applied to GL_2 and Proposition 2.1. In particular, volume-finiteness of $Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})$ follows from volume-finiteness of $[\mathrm{GL}_2]$.

- $\phi_f : G(\mathbf{A}) \to \mathbf{C}$ is smooth (i.e., it is locally constant in the finite-adelic part for fixed archimedean component and it is smooth in the archimedean component for fixed finite-adelic part),
- ϕ_f is a "weight k" eigenfunction for the right regular action of K (i.e., $\phi_f(g \cdot r(\theta)) = e^{-ik\theta}\phi_f(g)$);
- ϕ_f is cuspidal in the sense that $\int_{U(\mathbf{Q})\backslash U(\mathbf{A})} \phi_f(ug)du = 0$ for all $g \in G(\mathbf{A})$ and all unipotent radicals U of Borel $\mathbf{Q}$-subgroups of GL_2 (this integral makes sense since $U(\mathbf{Q})\backslash U(\mathbf{A}) = \mathbf{Q}\backslash\mathbf{A}$ is compact, and since we allow variation across all g it follows from the left $G(\mathbf{Q})$-invariance of ϕ_f that it is enough to check this vanishing property for *one* U because the collection of such U is a single $G(\mathbf{Q})$-conjugacy class);
- $\Delta \cdot \phi_f|_{\mathrm{GL}_2^+(\mathbf{R})} = -\frac{k}{2}\left(\frac{k}{2} - 1\right)\phi_f|_{\mathrm{GL}_2^+(\mathbf{R})}$, where Δ is the order-2 left-invariant differential operator arising from the Casimir element of $U(\mathfrak{sl}_2(\mathbf{R})) \subset U(\mathfrak{gl}_2(\mathbf{R}))$;
- $\phi_f(g \cdot \kappa_0) = \phi_f(g) \cdot \psi(\kappa_0)$ for any $\kappa_0 \in K_0^N$.

Furthermore, any element $\phi \in L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$ which satisfies the above properties is of the form ϕ_f for a uniquely-determined $f \in S_k(N, \psi)$.

Again the key point is to verify holomorphicity; one applies the argument from Appendix A *mutatis mutandi*.

Remark 4.2. To give a definition of cuspidality that is more intrinsic to $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$, it is better to speak in terms of "almost every g" rather than consider each g in isolation (since it doesn't make sense to integrate an L^p-function along a measure-0 subset); there are subtleties in doing so, discussed in the Appendix of [9].

Definition 4.3. A *cuspidal automorphic form* on $G = \mathrm{GL}_2$ is a smooth[4] function $\phi\colon G(\mathbf{A}) \to \mathbf{C}$ such that:

(1a) $\phi(\gamma \cdot g) = \phi(g)$ for all $\gamma \in G(\mathbf{Q})$ and all $g \in G$;

(1b) There exists a unitary character ψ on $Z(\mathbf{Q})\backslash Z(\mathbf{A}) = \mathbf{Q}^\times\backslash\mathbf{A}^\times$ such that $\phi(gz) = \phi(g) \cdot \psi(z)$ for all $z \in Z(\mathbf{A})$ and $g \in G(\mathbf{A})$;

(2a) ϕ is invariant under a compact open subgroup of $G(\mathbf{A}_f)$;

(2b) ϕ is K_∞-finite;

(3) ϕ is $\mathfrak{z}$-finite, where $\mathfrak{z}$ is the center of the universal enveloping algebra of $\mathrm{Lie}(G_\infty)$ (it doesn't matter in this definition if one takes $\mathfrak{z}$ over $\mathbf{R}$ or over $\mathbf{C}$, but in practice one always takes it over $\mathbf{C}$ for convenience beyond the setting of split reductive groups such as SL_2);

(4) ϕ is cuspidal.

[4]This means that it is smooth at the archimedean place ∞ and locally constant at the finite-adelic part.

Remark 4.4. As in the classical setting, one can define the space of *automorphic forms* on $G(\mathbf{A})$ that are not necessarily cuspidal. As before, one must then separately insist on an adelic moderate growth condition similar to that of Remark 3.7. We record this here briefly; see [4, §4] for more. Also, see [5] for relevant sanity-checks about the adelic topology (for example one can use the results therein to verify that the moderate growth condition doesn't depend on the choice of norm we make below).

Because G is not Zariski-closed in Mat_2, we cannot simply use norms of entries of $g \in G(\mathbf{A})$ to define an adelic norm. Instead, if a, b, c, d are the entries of g, we can let

$$\|g\| = \sup_v \left(\max \|a_v\|_v, \|b\|_v, \|c\|_v, \|d\|_v, \|ad - bc\|_v^{-1}\right).$$

With this definition, we say $\phi : G(\mathbf{A}) \to \mathbf{C}$ is of *moderate growth* if there exists an $A > 0$ such that $|\phi(g)| \ll \|g\|^A$ for all $g \in G(\mathbf{A})$. This may look ad hoc; in general, the idea is to control the size using a finite generating set of the coordinate ring of the algebraic group (the choice of which turns out not to matter).

5 L-functions

Recall that in his proof of the analytic continuation and functional equation for the zeta function, Riemann defines:

$$Z(2s) := \pi^{-s}\Gamma(s)\zeta(2s) = \int_0^\infty \frac{\theta(s) - 1}{2} t^s \, d^\times t$$

where $\theta(s)$ is the theta function $\theta(s) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}$. The Poisson summation formula implies that $\theta(s^{-1}) = s^{1/2}\theta(s)$, and this in turn implies the functional equation $Z(s) = Z(1 - s)$, or in other words:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

as meromorphic functions with poles at 0 and 1. Also, we have the Euler product formula:

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}$$

for s with $\mathrm{Re}(s) > 1$.

Tate's thesis provides an adelic interpretation of these facts. He shows that:

$$Z(s) = \int_{\mathbf{A}^\times} f(a)|a|^s \, d^\times a \tag{2}$$

where $|\cdot|$ is the idelic norm and $f(a) = \prod_p f_p(a) \cdot \left(e^{-\pi t^2}\right)_\infty$, with f_p the indicator function of $\mathbf{Z}_p$ on $\mathbf{Q}_p$. Then, we can manipulate (2) when $\mathrm{Re}(s) > 1$ as:

$$Z(s) = \left(\prod_p \int_{\mathbf{Z}_p} |a|^s \, d^\times a\right) \int_{-\infty}^{\infty} e^{-\pi t^2} t^s \, d^\times t = \left(\prod_p (1 - p^{-s})^{-1}\right) \pi^{s/2}\Gamma(s/2)$$

Then, we get the functional equation for $Z(s)$ by adelic Poisson summation.

We can also do this with a Dirichlet character $\chi\colon (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$. In the classical version, we define:

$$L_{\text{fin}}(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Note that this definition only "sees" primes $p \nmid N$, i.e. that $\chi(m) = 0$ whenever $(m, N) \neq 1$.

For the adelic version of this, we pull back χ from $(\mathbf{Z}/N\mathbf{Z})^\times$ via the quotient map $\mathbf{Q}^\times\backslash\mathbf{A}^\times \simeq \mathbf{R}^\times \times \widehat{\mathbf{Z}}^\times \to (\widehat{\mathbf{Z}}/N\widehat{\mathbf{Z}})^\times$ to an idèle character $\psi\colon \mathbf{Q}^\times/\mathbf{A}^\times \to \mathbf{C}^\times$. Then we define the adelic L-function:

$$L(s,\psi) = \int_{\mathbf{A}^\times} f(a)\psi(a)|a|^s \, d^\times a$$

$L(s,\psi)$ will be the product of $L_{\text{fin}}(s,\chi)$ with local factors at $p \mid N$ and ∞. Here, we have to choose f more carefully than before, according to the conductor of χ. Then, applying adelic Poisson summation, we get a functional equation for this L-function. We can get a functional equation for $L_{\text{fin}}(s,\chi)$ from this by using local functional equations for the extra local factors and dividing through by these factors.

We want to repeat this story for modular forms. In the classical version, we can define L-functions of modular forms via the Fourier expansion. Let $f = \sum_{n\geq 1} a_n q^n \in S_k^{\text{new}}(N,\chi)$. We assume that f is a normalized eigenform, meaning that it is a simultaneous eigenfunction for the Hecke operators and that $a_1 = 1$ (in which case the eigenvalue for the Hecke operator T_p is a_p). The subset $S_k^{\text{new}}(N,\chi)$ is the orthogonal complement (with respect to the Petersson inner product) to the space of "oldforms" defined by $f(z) = g(Mz)$ where g is a cusp form with level dividing N/M with $M > 1$.

Then we define:

$$L_{\text{fin}}(s,f) := \sum_{n\geq 1} \frac{a_n}{n^s} = \prod_{p\nmid N} \left(1 - a_p p^{-s} + \chi(p)p^{k-1+2s}\right) \prod_{p\mid N} \left(1 - a_p p^{-s}\right)^{-1}$$

Here, the second equality holds for $\operatorname{Re}(s)$ sufficiently large.

A slightly better version is:

$$L(s,f) := \int_0^\infty f(iy)y^s \, d^\times y = \int_0^\infty \left(\sum_{n\geq 1} a_n e^{-2\pi n y}\right) y^{s-1} \, dy = (2\pi)^{-s}\Gamma(s)\cdot L_{\text{fin}}(s,f)$$

Observe that $\phi_f\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)_\infty\right) = f(iy)\cdot y^{k/2}$. Then we have:

$$L(s+k/2,f) = \int_{\mathbf{Q}^\times\backslash\mathbf{A}^\times} \phi_f\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)_\infty\right)\psi^{-1}(y_0)|y|^s \, d^\times y \tag{3}$$

where $y = y_{\mathbf{Q}} \cdot y_\infty \cdot y_0$ with $y_{\mathbf{Q}} \in \mathbf{Q}^\times$, $y_\infty \in \mathbf{R}_{>0}$, $y_0 \in \widehat{\mathbf{Z}}^\times$.

Now, this gives an L-function for f which is written in terms of adelic information, but it is unsatisfying in a number of ways: it does not clearly generalize to settings other than $\mathrm{GL}_2(\mathbf{Q})$, and it does not obviously lead to a functional equation or product formula.

The idea is that we can find a "canonical function" which depends on a representation associated to ϕ_f (e.g. a matrix coefficient) and take an appropriate adelic version of its "Mellin transform."

Depending on how we phrase this, we might need to also include a well-chosen Bruhat–Schwartz function to ensure convergence (e.g. $f(a)$ in (2), which is compactly-supported at the finite places and Schwartz at the infinite place).[5]

Recall that $\mathbf{A}$ is its own Pontryagin dual. This comes from compatible versions of self-duality at each place: at the real place, this is the familiar statement that $\mathbf{R} \simeq \widehat{\mathbf{R}}$, normalized such that $\xi \longleftrightarrow (\chi_\xi : x \mapsto e^{-2\pi i \xi x})$. At each place $p < \infty$, we have an isomorphism $\mathbf{Q}_p \simeq \widehat{\mathbf{Q}_p}$ having the normalization $\xi \longleftrightarrow (\chi_\xi : x \mapsto e^{2\pi i \xi x})$.[6]

Note these isomorphisms respect the natural actions of $\mathbf{Q}_v^\times$ (including $v = \infty$) on each side (therefore, we have the same for the global self-duality statement). Moreover, under the induced isomorphism $\mathbf{A} \simeq \widehat{\mathbf{A}}$ (denoted $a \longleftrightarrow \chi_a$), we see that χ_a vanishes on $\mathbf{Q}$ if and only if $a \in \mathbf{Q} \subset \mathbf{A}$. In particular, the Pontryagin dual $\widehat{(\mathbf{A}/\mathbf{Q})}$ is isomorphic to $\mathbf{Q}$, and this respects the multiplication action of $\mathbf{Q}^\times$ on each side. We call $\lambda_1 \in \widehat{(\mathbf{A}/\mathbf{Q})}$ the character corresponding to $1 \in \mathbf{Q}$. Concretely, $\lambda_1(a_v) = e^{-2\pi i a_\infty} \prod_{v<\infty} e^{2\pi i a_v}$. Alternatively, one can use strong approximation for $\mathbf{A}$ to (make sense of) and write $\lambda_1(a_v) = e^{-2\pi i a_\infty} \prod_{v<\infty} \mathbb{1}_{\mathscr{O}_v}(a_v)$.

We set $W_\phi(g) := \phi_1(g) = \int_{\mathbf{A}} \phi\big(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\big)\overline{\lambda_1(x)}\, dx$. Then Fourier inversion gives us a new version of (3) to get a canonical definition of the L-function. In particular, we have

$$L(s + k/2, f) = \int_{\mathbf{Q}^\times \backslash \mathbf{A}^\times} \sum_{\xi \in \mathbf{Q}^\times} W_\phi\Big(\left(\begin{smallmatrix} y\xi & 0 \\ 0 & 1 \end{smallmatrix}\right)_\infty\Big) \psi^{-1}(y_0)|y|^s \, d^\times y \tag{4}$$

$$= \int_{\mathbf{A}^\times} W_\phi\Big(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)_\infty\Big) \psi^{-1}(y_0)|y|^s \, d^\times y. \tag{5}$$

The upshot, as we will see after the talk on Kirillov models, is that the *Whittaker functional* W_ϕ arises locally. Moreover, we can characterize it uniquely using the local representation at each place. This leads to a canonical definition of the completed L-function with a built-in product formula. We finish by listing some tasks that still remain (some of these will be addressed in "Adelization of Modular Forms, Part II"; others will be major themes of the seminar at large):

1. Use Whittaker models to define the global L-function purely from the perspective of representation theory, and deduce its product formula and functional equation (depending on a version of the $|_k w_N$ operator), plus functional equations for the local factors.
2. Explain the action of the Hecke operators T_p in terms of corresponding Hecke operators acting on automorphic forms.
3. Identify the space $S_k^{\text{new}}(N, \psi)$ of newforms inside $L^2_{\text{cusp}}(G(\mathbf{Q})\backslash G(\mathbf{A}), \psi)$ using an adelic version of the Petersson inner product.
4. Understand how the representation π_f generated by ϕ_f behaves (especially locally):
   - (i) Show that π_f is a completed tensor product of local representations at each place.
   - (ii) Show that π_f is irreducible if f is a cuspidal eigenform.

[5] This turns out not to be necessary for (4).

[6] This is well-defined as a character on $\mathbf{Q}$ and continuous for the p-adic topology, so it extends to $\mathbf{Q}_p$.

(iii) Prove representation-theoretic multiplicity-one statements and the Converse Theorem. For example, we would like to deduce the following fact from representation-theoretic knowledge: "if f_1, f_2 are normalized Hecke eigenforms in the new space for which the eigenvalues a_p are equal for all but finitely many p then $f_1 = f_2$."

A Proof of Proposition 3.3

Recall that $\phi : \Gamma\backslash\mathrm{SL}_2(\mathbf{R}) \to \mathbf{C}$ is a smooth function with $\mathrm{SO}_2(\mathbf{R})$-weight $k \in \mathbf{Z}$ such that $\Delta\phi = -(k/2)(k/2-1)\phi$. Our aim is to show that $f(x+iy) := \phi\left(\begin{smallmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{smallmatrix}\right) y^{-k/2}$ is holomorphic if ϕ is cuspidal.

The key point is going to be that the $K = \mathrm{SO}_2(\mathbf{R})$-weight k and the Δ eigenvalue $-\frac{k}{2}(\frac{k}{2}-1)$ of ϕ are compatible. This allows us to show that ϕ is the lowest-weight vector in an irreducible $(\mathfrak{g}, K)$-module (a copy of the so-called $\mathcal{D}_k^+$). In particular, the lowering operator in $\mathfrak{g}$ kills ϕ; a straightforward calculation shows that $L\phi$ vanishes exactly when $\frac{\partial f}{\partial \bar{z}}$ does; see Lemma A.3. Our proof will be phrased, however, in the language of differential operators.

Lemma A.1 encodes the compatibility of the K-weight and the Δ-eigenvalue of ϕ in terms of the action of the raising and lowering operators. Lemma A.3 shows the connection between ϕ being annihilated by the lowering operator and f being holomorphic. The key step is Lemma A.2, whose representation-theoretic interpretation is that the $(\mathfrak{g}, K)$-module generated by ϕ is unitary, allowing us to deduce that the lowering operator indeed kills ϕ. For more representation-theoretic context, including the theory of $(\mathfrak{g}, K)$-modules and the definition of $\mathcal{D}_k^+$, see Ch. 2 of [3].[7]

We have the usual basis for $\mathfrak{g}$

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfying the familiar commutation relations:

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Similarly, a basis for $\mathfrak{g}_{\mathbf{C}}$ is given by the elements

$$R := \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad L := \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \qquad H = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that R and L are complex conjugates, we have "equivalent" commutation relations:

$$[H, R] = 2R, \qquad [H, L] = -2L, \qquad [R, L] = H,$$

and $H = ir'(0)$ for the parameterization $r(\theta) = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ that we have been using for $K = \mathrm{SO}_2(\mathbf{R}) \subset \mathrm{SL}_2(\mathbf{R})$.

[7] Recall that for us, ϕ having "weight k" means that $\phi(gr(\theta)) = \phi(g)e^{-ik\theta}$, where we use the usual "counterclockwise" parametrization $r(\theta)$ of $\mathrm{SO}_2(\mathbf{R})$. Both of these are the opposite conventions to those in [3], where the clockwise parametrization and the opposite definition of weights are used. The net effect is that an eigenfunction ϕ has the same weight k in our setting as it does in [3].

Using the commutation relations, it is a classical exercise to check that the Killing form $\mathfrak{sl}_2$ in characteristic 0 is $(X, Y) \mapsto 4 \cdot \mathrm{tr}(XY)$ and from this that the Casimir operator on $\mathfrak{g} = \mathfrak{sl}_2(\mathbf{R})$ and $\mathfrak{g}_{\mathbf{C}}$ is given by

$$\frac{1}{8}(H^2 + 2RL + 2LR) = C = \frac{1}{8}(h^2 + 2ef + 2fe).$$

Returning to the action on $C^\infty(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$, one can check (via differentiation)[8] that the $\mathfrak{sl}_2$-triple $\{R, L, H\}$ giving a basis of $\mathfrak{g}_{\mathbf{C}}$ acts via the following operators:

$$R := e^{-2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta}\right)$$

$$L := e^{2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial\theta}\right)$$

$$H := i\frac{\partial}{\partial\theta}.$$

One can then compute by hand [9] that

$$\begin{aligned} -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - y\frac{\partial^2}{\partial x\,\partial\theta} &= -\frac{1}{4}(H^2 + 2RL + 2LR) \\ &= -2C \\ &= -RL + \frac{1}{4}\frac{\partial^2}{\partial\theta^2} - \frac{1}{2i}\frac{\partial}{\partial\theta}. \end{aligned}$$

The effect of the left side on $C^\infty(\mathrm{SL}_2(\mathbf{R})/K)$ makes the ∂_θ-term disappear, yielding the classical formula $y^2(\partial_x^2 + \partial_y^2)$ for the Laplacian associated to the left-invariant hyperbolic Riemannian metric

$$\mathrm{d}s^2 = y^{-2}(\mathrm{d}x^{\otimes 2} + \mathrm{d}y^{\otimes 2})$$

on $\mathrm{SL}_2(\mathbf{R})/K = \mathbf{H}$.

Before proceeding with the proof of Proposition 3.3, we make one more observation: with these operators in place, the weight-k eigenfunction relation $R(r(\theta)) : \phi \mapsto e^{-ik\theta}\phi$ for ϕ under the $\mathrm{SO}_2(\mathbf{R})$-action implies at the level of the action of $\mathfrak{sl}_2(\mathbf{R})$ via differential operators that $r'(0) : \phi \mapsto -ik\phi$. Since $H = ir'(0)$ in $\mathfrak{sl}_2(\mathbf{C})$, we conclude that ϕ is an eigenfunction for H with eigenvalue k; i.e. $H\phi = k\phi$.

Lemma A.1. Let $\phi : \Gamma\backslash\mathrm{SL}_2(\mathbf{R}) \to \mathbf{C}$ be a smooth function satisfying the first two conditions of Proposition 3.3. That is, ϕ is an eigenfunction for $\mathrm{SO}_2(\mathbf{R})$ with weight k and an eigenfunction for Δ with eigenvalue $-(k/2)(k/2 - 1)$. Then $RL\phi = 0$.

Proof. Using the Iwasawa (i.e., NAK) decomposition, the function

$$\widetilde{\phi}(x, y, \theta) := e^{ik\theta}\phi(x, y, \theta)$$

[8] see [3, Prop. 2.2.5] for this computation, keeping in mind that the opposite parametrization of $\mathrm{SO}_2(\mathbf{R})$ is used there
[9] again, see [3, Prop. 2.2.5]

is independent of θ due to ϕ being an $\mathrm{SO}_2(\mathbf{R})$-eigenfunction of weight k. Therefore,

$$\left(\frac{1}{4}\frac{\partial^2}{\partial\theta^2} - \frac{1}{2i}\frac{\partial}{\partial\theta}\right)\phi = \left(\frac{1}{4}\frac{\partial^2}{\partial\theta^2} - \frac{1}{2i}\frac{\partial}{\partial\theta}\right)(e^{-ik\theta}\widetilde{\phi}) = -\frac{k^2}{4}\phi + \frac{k}{2}\phi = -(k/2)(k/2-1)\phi,$$

yet

$$RL = (1/4)\partial_\theta^2 - (1/2i)\partial_\theta - \Delta,$$

so the Δ-eigenfunction hypothesis on ϕ implies that $(RL)(\phi) = 0$ as desired. □

Lemma A.2. Let $\phi : \Gamma\backslash\mathrm{SL}_2(\mathbf{R}) \to \mathbf{C}$ be a smooth, square-integrable eigenfunction for $\mathrm{SO}_2(\mathbf{R})$ of weight k such that $RL\phi = 0$. If ϕ is cuspidal then $L\phi = 0$.

Granting this lemma for a moment, we connect the lowering operator to holomorphicity of our candidate modular form f:

Lemma A.3. Let $\phi : \Gamma\backslash\mathrm{SL}_2(\mathbf{R}) \to \mathbf{C}$ be a smooth eigenfunction of $\mathrm{SO}_2(\mathbf{R})$-weight k. Define $f(x+iy) := \phi\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2}\end{pmatrix} y^{-k/2}$. Then

$$2\frac{\partial f}{\partial \overline{z}} = iy^{-k/2-1}e^{i(k-2)\theta}L\phi.$$

In particular, $L\phi = 0$ if and only if f is holomorphic.

Proof. We can rewrite $f = y^{-k/2}e^{ik\theta}\phi(x,y,\theta)$ since ϕ is a weight-k eigenfunction for $K = \mathrm{SO}_2(\mathbf{R})$. Then

$$\begin{aligned}
2\frac{\partial}{\partial\overline{z}}f = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f &= y^{-k/2}e^{ik\theta}\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right) - \frac{ik}{2}y^{-k/2-1}e^{ik\theta}\phi \\
&= iy^{-k/2-1}e^{ik\theta}\left(-iy\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y}\right) - \frac{ik}{2y}f \\
&= iy^{-k/2-1}e^{ik\theta}\left(-\frac{1}{2i}\frac{\partial\phi}{\partial\theta}\right) - \frac{ik}{2y}f + iy^{-k/2-1}e^{i(k-2)\theta}L\phi,
\end{aligned}$$

where the final equality uses the description of L as a first-order differential operator. By "differentiating" the condition that ϕ is a weight-k eigenfunction for K, we have $\partial\phi/\partial\theta = -ik\phi$. Plugging this into the first term on the right side at the end, we obtain that

$$\begin{aligned}
2\frac{\partial}{\partial\overline{z}}f &= iy^{-k/2-1}e^{ik\theta}\left(\frac{k}{2}\phi\right) - \frac{ik}{2y}f + iy^{-k/2-1}e^{i(k-2)\theta}L\phi \\
&= \frac{ik}{2y}f - \frac{ik}{2y}f + iy^{-k/2-1}e^{i(k-2)\theta}L\phi \\
&= iy^{-k/2-1}e^{i(k-2)\theta}L\phi
\end{aligned}$$

□

It remains to prove Lemma A.2:

Proof. Using the NAK decomposition for $\mathrm{SL}_2(\mathbf{R})$ and the identification of $\mathrm{SL}_2(\mathbf{R})/K$ with $\mathbf{H}$, the Haar measure on $\mathrm{SL}_2(\mathbf{R})$ in such "(x, y, θ)" coordinates is $(y^{-2}dx\,dy)d\theta$. We will show that $\int_{\Gamma\backslash \mathrm{SL}_2(\mathbf{R})} |L\phi|^2 \frac{dxdy}{y^2} d\theta$ vanishes.

The proof strategy is to show that R and L roughly behave as adjoints with respect to the inner product for certain K-eigenfunctions in $L^2_{\mathrm{cusp}}(\Gamma\backslash \mathrm{SL}_2(\mathbf{R}))$ (depending on the K-weight of those functions). More specifically, we will find that our integrand is exact, and more specifically is equal to $d\omega$ for a 2-form ω on $\Gamma\backslash \mathrm{SL}_2(\mathbf{R})$ such that $\omega(x, y, \theta)$ decays rapidly as (x, y) approaches any cusp of Γ (ultimately due to cuspidality of ϕ). A careful limiting application of Stokes' Theorem to compact "cutoffs" of the K-bundle $\Gamma'\backslash \mathrm{SL}_2(\mathbf{R}) \to \Gamma'\backslash\mathbf{H}$ near the cusps will then give the result, where Γ' is a suitable finite-index subgroup of Γ.

By the Leibniz rule for R, together with the fact that R and L are complex conjugates, we have

$$|L\phi|^2 = (L\phi)\left(\overline{L\phi}\right) = R\left(L\phi(\overline{\phi})\right) - (RL\phi)\left(\overline{\phi}\right) = R\left(L\phi(\overline{\phi})\right).$$

We claim this function multiplied against the oriented volume form $(y^{-2}dx\,dy)d\theta$ is exact. Indeed, consider the 2-form

$$\omega = -e^{-2i\theta}(L\phi)\overline{\phi}\left(\frac{d\overline{z}d\theta}{y} + i\frac{dxdy}{2y^2}\right).$$

Note that

$$\frac{\partial\overline{\phi}}{\partial\theta} = ik\overline{\phi} \qquad \text{and} \qquad \frac{\partial}{\partial\theta}(L\phi) = -iHL\phi = -i(LH - 2L)\phi = -i(k-2)(L\phi)$$

In particular, this implies that

$$\frac{\partial}{\partial\theta}\left((L\phi)\overline{\phi}\right) = 2i(L\phi)\overline{\phi},$$

so

$$\frac{\partial}{\partial\theta}\left(e^{-2i\theta}(L\phi)\overline{\phi}\right) = 0.$$

Hence, we have

$$\begin{aligned}
d\omega &= d\left(-e^{-2i\theta}(L\phi)\overline{\phi}\left(y^{-1}\,dxd\theta - iy^{-1}\,dyd\theta + \frac{i}{2}y^{-2}\,dxdy\right)\right)\\
&= e^{-2i\theta}\frac{\partial}{\partial y}\left(y^{-1}(L\phi)\overline{\phi}\right)\,dxdyd\theta + ie^{-2i\theta}y^{-1}\frac{\partial}{\partial x}\left(L(\phi)\overline{\phi}\right)\,dxdyd\theta\\
&\qquad - \frac{i}{2y^2}\frac{\partial}{\partial\theta}\left(e^{-2i\theta}(L\phi)\overline{\phi}\right)\,dxdyd\theta.
\end{aligned}$$

The final term being subtracted vanishes, so

$$\begin{aligned}
d\omega &= \left(e^{-2i\theta}(-y^{-2})(L\phi)\overline{\phi} + e^{-2i\theta}y^{-1}\frac{\partial}{\partial y}\left((L\phi)\overline{\phi}\right) + ie^{-2i\theta}y^{-1}\frac{\partial}{\partial x}\left(L(\phi)\overline{\phi}\right)\right)\,dxdyd\theta\\
&= \left(-e^{-2i\theta}(L\phi)\overline{\phi}\right)\,\frac{dxdyd\theta}{y^2} + \left(e^{-2i\theta}y\frac{\partial}{\partial y}\left((L\phi)\overline{\phi}\right) + e^{-2i\theta}iy\frac{\partial}{\partial x}\left((L\phi)\overline{\phi}\right)\right)\,\frac{dxdyd\theta}{y^2}\\
&= \left(-e^{-2i\theta}(L\phi)\overline{\phi}\right)\,\frac{dxdyd\theta}{y^2} + \left(R\left((L\phi)\overline{\phi}\right) + e^{-2i\theta}\frac{1}{2i}\frac{\partial}{\partial\theta}\left((L\phi)\overline{\phi}\right)\right)\,\frac{dxdyd\theta}{y^2},
\end{aligned}$$

the final equality by the determination of R as a first-order differential operator. Rearranging the order of summation, we get

$$\begin{aligned} d\omega &= \left(R\left((L\phi)\overline{\phi}\right) + \left(-e^{-2i\theta} + e^{-2i\theta}\frac{1}{2i}\frac{\partial}{\partial\theta}\right)\left((L\phi)\overline{\phi}\right)\right)\frac{dxdyd\theta}{y^2} \\ &= \left(R\left((L\phi)\overline{\phi}\right) + \left(-e^{-2i\theta} + e^{-2i\theta}\right)\left((L\phi)\overline{\phi}\right)\right)\frac{dxdyd\theta}{y^2} \\ &= R\left(L\phi(\overline{\phi})\right)\frac{dxdyd\theta}{y^2} \end{aligned}$$

(the second equality due to the identity $\partial_\theta((L\phi)\overline{\phi}) = 2i(L\phi)\overline{\phi}$.

We conclude that

$$\int_{\Gamma\backslash \mathrm{SL}_2(\mathbf{R})} |L\phi|^2 \frac{dxdy}{y^2} d\theta = \int_{\Gamma\backslash \mathrm{SL}_2(\mathbf{R})} d\omega$$

as oriented integrals, provided that the right side is at least *finite*. In particular, it is harmless to replace $d\omega$ with the corresponding density $|d\omega|$ (so all convergence issues involve manifestly non-negative quantities).

The NAK-decomposition realizes $\mathrm{SL}_2(\mathbf{R})$ as an S^1-bundle over $\mathbf{H}$. As long as Γ acts freely on $\mathbf{H}$, the coset space $\Gamma\backslash\mathrm{SL}_2(\mathbf{R})$ is also an S^1-bundle over $\Gamma\backslash\mathbf{H}$. For convenience, we will denote the structure map in that case as $\pi : \Gamma\backslash\mathrm{SL}_2(\mathbf{R}) \to Y(\Gamma)$. Suppose $\Gamma' \subset \Gamma$ is a finite-index subgroup. Then $L\phi = 0$ if and only if $L\phi' = 0$, where ϕ' is the pullback of ϕ to the finite cover $\Gamma'\backslash\mathrm{SL}_2(\mathbf{R})$.[10] Since Γ is arithmetic, it has a finite-index subgroup $\Gamma' \subset \mathrm{SL}_2(\mathbf{Z})$ such that $\Gamma' \cap K = \{1\}$. Then Γ' acts freely on $\mathbf{H}$ and we may use it in place of Γ for the remainder of the argument.

Let $\mathcal{M}$ be the 2-manifold with boundary obtained from $X(\Gamma)$ by removing an (open) ϵ-neighborhood of each cusp.[11] Then $\mathcal{E} := \pi^{-1}(\mathcal{M}) \subset \Gamma\backslash\mathrm{SL}_2(\mathbf{R})$ is a compact oriented Riemannian 3-manifold with boundary ($\partial\mathcal{E}$ is a disjoint union of S^1-bundles over circles of radius ϵ, indexed by the cusps of Γ). Then

$$\begin{aligned} \int_{\Gamma\backslash \mathrm{SL}_2(\mathbf{R})} |L\phi|^2 \frac{dxdy}{y^2} d\theta &= \int_{\Gamma\backslash \mathrm{SL}_2(\mathbf{R})} |d\omega| \\ &= \int_{\mathcal{E}} |d\omega| + \int_{(\Gamma\backslash \mathrm{SL}_2(\mathbf{R}))\ -\ \mathcal{E}} |d\omega| \\ &\leq \int_{\partial\mathcal{E}} |\omega| + \int_{(\Gamma\backslash \mathrm{SL}_2(\mathbf{R}))\ -\ \mathcal{E}} |d\omega|, \end{aligned}$$

the final inequality by Stokes' Theorem for the oriented compact manifold with boundary $\mathcal{E}$. We claim that both of these final integrals decay to 0 as $\epsilon \to 0$. Since ϕ is automorphic and L is $\mathrm{SL}_2(\mathbf{R})$-invariant, the formula defining ω shows that the decay (in y) of the first integral is reduced to the decay of $(L\phi)\overline{\phi}$ near each cusp. Likewise, our computation of $d\omega$ reduces the decay of the second integral to the decay (in y) of $R((L\phi)(\overline{\phi}))$ near each cusp. We explain how to deduce these decay properties from results proved in [8].

[10]The L operator on $C^\infty(\Gamma'\backslash\mathrm{SL}_2(\mathbf{R}))$ takes the same form as the L operator on $C^\infty(\Gamma'\backslash\mathrm{SL}_2(\mathbf{R}))$.

[11]This corresponds to cutting away the "large-y" portion of a Siegel domain in $Y(\Gamma)$ for each cusp.

In [8, Cor. 3.6.1], it is shown that for each smooth compactly-supported function $\alpha \in C_c^\infty(\mathrm{SL}_2(\mathbf{R}))$, there is a constant $c_0(\alpha) > 0$ such that

$$|(\psi * \alpha)(g)| \leq c_0(\alpha)\|\psi\|_2 \tag{6}$$

for any $\psi \in L^2_{\mathrm{cusp}}(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$.[12] But the proof is robust: we will show via the same method (plus induction) that for each $n \geq 0$, there is a constant $c_{-n}(\alpha) > 0$ such that for any $\psi \in L^2_{\mathrm{cusp}}(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$ and cusp s,

$$|(\psi * \alpha)(g_s g)| \leq c_{-n}(\alpha) y^{-n}\|\psi\|_2, \tag{7}$$

where $g_s \in \mathrm{SL}_2(\mathbf{Z})$ such that $g_s(\infty) = s$ and $g \in \mathrm{SL}_2(\mathbf{R})$ such that $g(i) = x + iy$. Note that Equation (6) is the $n = 0$ case. Suppose now that we have Equation (7) for some $n \geq 0$.

In view of the identity $\mathcal{L}_X(\psi * \alpha) = \psi * \mathcal{L}_X(\alpha)$ for any left-invariant differential operator $\mathcal{L}_X$, one can apply Equation (7) to the functions $\mathcal{L}_X(\alpha) \in C_c^\infty(\mathrm{SL}_2(\mathbf{R}))$ to conclude that

$$|\mathcal{L}_X(\psi * \alpha)(g_s g)| \leq c_{-n}(\mathcal{L}_X(\alpha)) y^{-n}\|\psi\|_2.$$

Note that left-translation by $U_\infty(\mathbf{R}) = g_s^{-1} U_s(\mathbf{R}) g_s$ preserves the quantity y^{-n}. Thus, by integration over $\Gamma_s\backslash U_s(\mathbf{R})$ (for s a cusp), we get a similar estimate for the "constant term" at s:

$$|\mathcal{L}_X(\psi * \alpha)_{B_s}(g_s g)| \leq \mathrm{vol}(\Gamma_s\backslash U_s(\mathbf{R})) \cdot c_{-n}(\mathcal{L}_X(\alpha)) y^{-n}\|\psi\|_2,$$

where $F_{B_s}(g) := \int_{\Gamma_s\backslash U_s(\mathbf{R})} F(ug)\, du$.

On the other hand, note that $(F * \alpha)_{B_s} = F_{B_s} * \alpha$. Hence, cuspidality of ψ (i.e. vanishing of ψ_{B_s} for each cusp s) implies cuspidality of $\psi * \alpha$. Then [8, Prop. 3.5.4] implies that

$$|(\psi * \alpha)(g_s g)| \leq c y^{-1} \left(\sum_{i=1}^{3} |\mathcal{L}_{X_i}(\psi * \alpha)_{B_s}(g_s g)| \right),$$

where $\{X_1, X_2, X_3\}$ is a basis of $\mathfrak{g}$ and $c > 0$ is a constant depending on the choice of g_s and $\{X_1, X_2, X_3\}$.[13] For convenience, fix $\{X_1, X_2, X_3\}$ to be $\{e, f, h\}$. Combining these two inequalities, we see that there is a constant $c_{-n-1}(\alpha, g_s) > 0$ depending on α and g_s such that

$$|(\psi * \alpha)(g_s g)| \leq c_{-n-1}(\alpha, g_s) y^{-n-1}\|\psi\|_2$$

for any $\psi \in L^2_{\mathrm{cusp}}(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$ and cusp s. But there are only finitely many cusps s and at most two choices of $g_s \in \mathrm{SL}_2(\mathbf{Z})$ for each s, so we may set $c_{-n-1}(\alpha) := \max_{g_s} c_{-n-1}(\alpha, g_s) > 0$. Then

$$|(\psi * \alpha)(g_s g)| \leq c_{-n-1}(\alpha) y^{-n-1}\|\psi\|_2$$

for any $\psi \in L^2_{\mathrm{cusp}}(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$ and cusp s, which is Equation (7) in the $n + 1$ case. Thus, we have established Equation (7) for all $n \geq 0$.

By [8, Prop. 3.3.5], there is an $\alpha_0 \in C_c^\infty(\mathrm{SL}_2(\mathbf{R}))$ such that $\phi = \phi * \alpha_0$. Recalling that the operators L and R are $\mathrm{SL}_2(\mathbf{R})$-invariant, we apply (7) with $\psi = \phi$ and $\alpha = \alpha_0$ (or α equal to one of $L\alpha_0, RL\alpha_0, R\alpha_0$, as needed) to see that $|(L\phi)\overline{\phi}|$ and $|R((L\phi)\overline{\phi})|$ decay faster than any power of y^{-n} near the cusp s. Therefore, both of the integrals $\int_{\partial\mathcal{E}} |\omega|$ and $\int_{(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))\,-\,\mathcal{E}} |d\omega|$ decay to 0 as $\epsilon \to 0$, proving that $L\phi = 0$. $\square$

[12] Recall that we assumed the cuspidal ϕ belongs to $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$, since this is the situation of Proposition 3.3.

[13] For example, if $s = \infty$, $g_s = 1$, and we use the basis $\{e, f, h\}$, then $c = 1$ suffices.

References

[1] L. Bers, F. John, M. Schechter, *Partial Differential Equations*, Lectures in Applied Math (vol. 3A), AMS, Providence, 1971.

[2] N. Bourbaki, *Lie Groups and Lie Algebras* (Ch. I-III), Springer-Verlag, New York, 1989.

[3] D. Bump, *Automorphic forms and representations,* Cambridge Univ. Press, Cambridge, 1998.

[4] A. Borel, H. Jacquet, "Automorphic forms and automorphic representations" in *Automorphic forms, representations and L-functions.* Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977, Part 1. Amer. Math. Soc., Providence, 1979.

[5] B. Conrad, *Weil and Grothendieck approaches to adelic points*, Enseignement Math., vol. 58 (2012), pp. 61-97.

[6] S. Helgason, *Differential Operators on Homogeneous Spaces*, Acta Mathematica, 102 (1959), pp. 239–299.

[7] G. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, New York, 1991.

[8] Ngo, Bao Chau. *Automorphic forms on* GL_2, complete course notes at `http://math.stanford.edu/~conrad/conversesem/refs/NgoGL2.pdf`

[9] D. Trotabas, *Modular forms and automorphic representations*, online notes at `http://math.stanford.edu/~conrad/conversesem/refs/trotabas.pdf`

[10] F. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, GTM 94, Springer-Verlag, New York, 1983.