LECTURE 7: DISCRETENESS FOR CUSPIDAL L^2 FOR REDUCTIVE GROUPS I
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1 Introduction

1.1 References

Our main sources are [Go, 3.1] and [Ga] (cf. the seminar webpage):
* [Ga] *Decomposition and estimates for cuspforms*, notes by Garrett.
* [Go] *Notes on Jacquet-Langlands' theory*, IAS lecture notes by Godement.

1.2 Notation

We start by recalling some notation and definitions. Let k be a global field, G a connected reductive group over k, Z a maximal central torus of G, and $\omega : Z(k)\backslash Z(\mathbf{A}) \to \mathbf{C}^\times$ a unitary character (i.e., valued in S^1). Consider measurable functions $\varphi : G(k)\backslash G(\mathbf{A}) \to \mathbf{C}$ such that for all $z \in Z(\mathbf{A})$ we have $\varphi((\cdot)z) = \omega(z)\varphi$ almost everywhere, so $|\varphi|$ is well-defined on $G(k)\backslash G(\mathbf{A})/Z(\mathbf{A})$ up to change on a measure-0 set.

It makes sense to ask if $\|\varphi\|_2^2 := \int_{G(k)\backslash G(\mathbf{A})/Z(\mathbf{A})} |\varphi|^2$ is finite, in which case its value is insensitive to changing φ on a measure-0 set (upon fixing the unimodular Haar measure on $G(\mathbf{A})$ and a Haar measure of $Z(\mathbf{A})$ underlying the formation of these integrals; Tamagawa measure is a canonical choice). Thus, it makes sense to define $L^2(G(k)\backslash G(\mathbf{A}), \omega)$ to be the space of such φ *up to change on a measure-*0 *set* such that $\|\varphi\|_2 < \infty$. Note that there is a right regular representation of $G(\mathbf{A})$ on this space.

Remark 1.1. Since Z is smooth and geometrically connected, the continuous map $G(\mathbf{A})/Z(\mathbf{A}) \hookrightarrow (G/Z)(\mathbf{A})$ is a homeomorphism onto an open image, so by discreteness of $(G/Z)(k)$ in $(G/Z)(\mathbf{A})$ we see that $G(k)/Z(k)$ is a discrete subgroup of $G(\mathbf{A})/Z(\mathbf{A})$. The integral considered above is really viewed as one on $(G(k)/Z(k))\backslash(G(\mathbf{A})/Z(\mathbf{A}))$, using a choice of Haar measure on $(G/Z)(\mathbf{A})$ or alternatively of such choices on $G(\mathbf{A})$ and $Z(\mathbf{A})$. In practice what matters for us is the finiteness of certain integrals; when exact values matter later in life then one has to be more specific about the choice of measures (such as to use the Tamagawa measure).

We can similarly define $L^p(G(k)\backslash G(\mathbf{A}), \omega)$ for any $1 \le p < \infty$, and the finiteness of the volume of $G(k)\backslash G(\mathbf{A})/Z(\mathbf{A})$ (due to reduction theory) implies $L^p(G(k)\backslash G(\mathbf{A}), \omega) \subset L^1(G(k)\backslash G(\mathbf{A}), \omega)$ for all $1 < p < \infty$ (since $1 \in L^q(G(k)\backslash G(\mathbf{A})/Z(\mathbf{A}))$, where $1/p + 1/q = 1$). We always work with the completion of the Haar measure, so we can disregard any measure-0 set when checking if a function is measureable.

For any φ_1, φ_2 in this space, the product $\varphi_1\overline{\varphi}_2$ is well-defined on $G(k)\backslash G(\mathbf{A})/Z(\mathbf{A})$ by unitarity of ω and the integral

$$\langle \varphi_1, \varphi_2 \rangle = \int_{G(k)\backslash G(\mathbf{A})/Z(\mathbf{A})} \varphi_1 \overline{\varphi_2}$$

converges. This integral is visibly invariant under the right regular representation of $G(\mathbf{A})$ and defines a structure of Hilbert space on $L^2(G(k)\backslash G(\mathbf{A}), \omega)$, so this is a unitary representation of $G(\mathbf{A})$. In Section 2 we will define a closed $G(\mathbf{A})$-invariant subspace $L^2_{\mathrm{cusp}} \subset L^2(G(k)\backslash G(\mathbf{A}), \omega)$ consisting, at least informally, of functions whose integrals vanish along unipotent radicals of proper parabolic subgroups.

1.3 Decomposition of cusp forms

The main result of this lecture and its sequel is:

Theorem 1.2. $L^2_{\mathrm{cusp}} = \widehat{\bigoplus}_i V_i$ with V_i topologically irreducible closed subrepresentations (i.e., an orthogonal Hilbert direct sum), and for any i the set of j with $V_j \simeq V_i$ is finite.

Remark 1.3. Instead of working with the adelic double coset space, we could fix an arithmetic lattice $\Gamma \subset G(\mathbf{R})$ and consider the space

$$L^2_{\mathrm{cusp}}(\Gamma\backslash G(\mathbf{R}), \omega)$$

as defined in Lecture 5 with the action of the real points $G(\mathbf{R})$. The same techniques can be used to show that this decomposes as a Hilbert direct sum of topologically irreducible closed subrepresentations of $G(\mathbf{R})$, each appearing with finite multiplicity.

One might guess that this is implied by the corresponding adelic result, but this is not quite true. For example, for $k = \mathbf{Q}$, $G = \mathrm{SL}_2$, the statement about real points for a given level Γ includes the following two statements (at least, after invoking some representation theory of $\mathrm{SL}_2(\mathbf{R})$):

- The space of cuspidal modular forms of any fixed weight and level Γ is finite-dimensional.
- The 2-dimensional Riemannian orbifold $\Gamma\backslash\mathbf{H}$ admits finitely many cuspidal eigenfunctions for the Laplacian of any fixed eigenvalue.

The corresponding adelic result does not imply either of these statements without bringing in some more refined information concerning "admissibility" of the adelic representation.

Given a topologically irreducible closed subrepresentation of a unitary (Hilbert) representation of $G(\mathbf{A})$, using the inner product it always splits off as a direct summand. Thus, the only difficulty in proving Theorem 1.2 is that it is not a priori clear that there are *any* topologically irreducible closed subrepresentations of a given Hilbert space representation – the spaces involved are infinite-dimensional, so there is no formal reason that a descending chain of closed subrepresentations needs to stabilize[1].

We will prove Theorem 1.2 by showing that certain natural integral operators on L^2_{cusp} are compact, then using the spectral theorem for compact self-adjoint operators to find finite dimensional eigenspaces, which can be exploited to produce topologically irreducible closed subrepresentations. The integral operators we define on L^2_{cusp} will extend naturally to all of L^2, but working in L^2_{cusp} will be necessary to prove the key estimates that allow us to show they are compact.

In this lecture, we will define L^2_{cusp} and the integral operators, then deduce Theorem 1.2 assuming compactness of these operators. We will also prove the analogous compactness statement in the

[1] Think of the shift operator on $\ell^2(\mathbf{N})$, $f \mapsto (n \mapsto f(n+1))$.

simplest case of $\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})$, before treating the general (adelic) case in the next lecture. Thus, whereas this lecture can be mostly understood independently of the previous lectures, in the next lecture we will need the full strength of the reduction theory discussed in previous lectures.

2 Cusp forms

We now define the Hilbert space of cusp forms $L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbf{A}),\omega) \subseteq L^2(G(k)\backslash G(\mathbf{A}),\omega)$ (to then be denoted L^2_{cusp} for convenience). We consider the unipotent radicals U of parabolic k-subgroups $P \subseteq G$. One could define L^2_{cusp} as the set of all $\varphi \in L^2(G(k)\backslash G(\mathbf{A}),\omega)$ such that for each such U and for almost all g,

$$\int_{U(k)\backslash U(\mathbf{A})} \varphi(ug)\, du = 0. \tag{1}$$

However, to avoid a subtlety later on in applying Fubini's theorem (cf. Remark 2.4 below), we will instead adopt a distributional definition. Before making this definition, we introduce some notation: for any affine k-group scheme H of finite type we define

$$H_\infty := H(\prod_{v|\infty} k_v), \quad H_{\mathrm{fin}} := H(\mathbf{A}_{\mathrm{f}})$$

(so in the function field case we have $H_\infty = 1$ and $H_{\mathrm{fin}} = H(\mathbf{A})$).

Definition 2.1. For $\varphi \in L^2(G(k)\backslash G(\mathbf{A}),\omega)$, we say that φ is *cuspidal at* P if for all

$$f \in C_c^\infty(U(\mathbf{A})\backslash G(\mathbf{A})) := C_c^\infty(U_\infty\backslash G_\infty) \otimes C_c^\infty(U_{\mathrm{fin}}\backslash G_{\mathrm{fin}})$$

(with $C_c^\infty(U_{\mathrm{fin}}\backslash G_{\mathrm{fin}})$ the space of compactly supported locally constant functions $U_{\mathrm{fin}}\backslash G_{\mathrm{fin}} \to \mathbf{C}$) the "constant term"

$$(f\varphi)_P := \int_{U(k)\backslash G(\mathbf{A})} f\varphi \tag{2}$$

vanishes. (Note that the measureable $f\varphi$ is integrable on $U(k)\backslash G(\mathbf{A})$ because f is continuous and compactly supported on $U(k)\backslash G(\mathbf{A})$ and $f\varphi$ inherits local integrability from the integrability of $|\varphi|$ on $G(k)\backslash G(\mathbf{A})/Z(\mathbf{A})$ due to the discreteness of $G(k)$ in $G(\mathbf{A})$ and the map $G(\mathbf{A}) \to G(\mathbf{A})/Z(\mathbf{A})$ being a topological $Z(\mathbf{A})$-fibration.)

When this holds for all P, we say φ is a *cusp form*.

Remark 2.2. Using the action of $G(k)$ on $G(\mathbf{A})$ by conjugation and the unimodularity of $G(\mathbf{A})$, it follows that for any $\gamma \in G(k)$ we have $(f\varphi)_P = (f\varphi)_{\gamma P\gamma^{-1}}$. Thus, to check cuspidality it suffices to consider one P from each $G(k)$-conjugacy class of parabolic k-subgroups; by the structure theory of connected reductive groups over fields, if P_0 is a minimal parabolic k-subgroup then the set of parabolic k-subgroups of G containing P_0 is finite and meets each $G(k)$-conjugacy class of parabolic k-subgroups in a single member, called its "standard" representative relative to P_0.

For each $f \in C_c^\infty(N(\mathbf{A})\backslash G(\mathbf{A}))$, the integral condition (2) defines a closed subspace in $L^2(G(k)\backslash G(\mathbf{A}),\omega)$, and the set of these conditions for all f is invariant under the $G(\mathbf{A})$-action. Thus, we obtain:

Lemma 2.3. $L^2_{\text{cusp}}(G(k)\backslash G(\mathbf{A}),\omega) \subseteq L^2(G(k)\backslash G(\mathbf{A}),\omega)$ is a closed $G(\mathbf{A})$-stable subspace.

Remark 2.4. By the Fubini theorem, which can be shown to apply due to volume-finiteness of $G(k)\backslash G(\mathbf{A})/Z(\mathbf{A})$, the integral (2) is equal to:

$$\int_{U(\mathbf{A})\backslash G(\mathbf{A})} \int_{U(k)\backslash U(\mathbf{A})} f(ug)\varphi(ug)\, du\, dg = \int_{U(\mathbf{A})\backslash G(\mathbf{A})} f(g) \int_{U(k)\backslash U(\mathbf{A})} \varphi(ug)\, du\, dg$$

where we have used that f is left-invariant under $U(\mathbf{A})$. Thus, (1) for almost all g implies this integral is zero, and so the definition of cuspidality in terms of the integrals (1) implies cuspidality in the distributional sense we have adopted in Definition 2.1.

3 The integral action

Consider the algebra $C_c^\infty(G(\mathbf{A}))$ with convolution product

$$(f_1 \star f_2)(g) = \int_{G(\mathbf{A})} f_1(g')f_2({g'}^{-1}g)dg'.$$

This algebra acts on $L^2(G(k)\backslash G(\mathbf{A}),\omega)$ via integral operators:

$$(f \cdot \varphi)(y) = \int_{G(\mathbf{A})} f(x)\varphi(yx)dx.$$

(It is easy to check via Fubini's Theorem that $f\cdot\varphi$ is measureable and belongs to $L^2(G(k)\backslash G(\mathbf{A}),\omega)$). To explain why the definition of $f\cdot\varphi$ is appropriate, observe that if we let ρ denote the right regular representation of $G(\mathbf{A})$ on the unitary Hilbert representation $L^2(G(k)\backslash G(\mathbf{A}),\omega)$ then

$$f \cdot \varphi = \int_{G(\mathbf{A})} f(g)\rho(g)(\varphi)dg$$

as an integral valued in the Hilbert space $L^2(G(k)\backslash G(\mathbf{A}),\omega)$; i.e., this is an instance of the standard procedure (called a *Gelfand-Pettis integral*) by which $C_c^\infty(G(\mathbf{A}))$ acts on *any* unitary Hilbert representation (V,ρ) of $G(\mathbf{A})$ (namely, $\pi(f)(v) = \int f(g)\rho(g)(v)dg$).

The operator $T_f : \varphi \mapsto f \cdot \varphi$ is compatible with the $G(\mathbf{A})$-action on V and on $C_c^\infty(G(\mathbf{A}))$ via $(g.f)(x) = f(xg^{-1})$:

$$\begin{aligned}(f \cdot \rho(g_0)(v)) = \int_{G(\mathbf{A})} f(g)\rho(g)(\rho(g_0)(v))dg &= \int_{G(\mathbf{A})} f(g)\rho(gg_0)(v)dg \\ &= \int_{G(\mathbf{A})} f(gg_0^{-1})\rho(g)(v)dg \\ &= (g_0.f)\cdot v.\end{aligned}$$

Likewise, $\rho(g_0)(f\cdot v) = (\ell_{g_0}.f)\cdot v$ for the left regular representation of $G(\mathbf{A})$ on $C_c^\infty(G(\mathbf{A}))$ defined by $(\ell_g.f)(x) = f(g^{-1}x)$. Observe as well that the operator T_f is bounded:

$$\|T_f v\|^2 \le \int |f(g)\overline{f(g')}||\langle gv, g'v\rangle|dgdg' \le \int |f(g)f(g')|\|gv\|\|g'v\|dgdg' = \|f\|_1^2\|v\|^2,$$

so $\|T_f\| \leq \|f\|_1$.

Although for our purposes we will not need anything more than the individual operators, we verify that this defines an algebra action:

$$\begin{aligned}(f_1 \cdot (f_2 \cdot v)) &= \int f_1(g)\rho(g)(f_2 \cdot v)dg \\ &= \int f_1(g)((\ell_g.f_2) \cdot v)dg \\ &= \int f_1(g) \int (\ell_g.f_2)(g')\rho(g')(v)dg'dg \\ &= \int (\int f_1(g)(\ell_g.f_2(g'))dg)\rho(g')(v)dg' \\ &= \int (f_1 \star f_2)(g')\rho(g')(v)dg \\ &= (f_1 \star f_2) \cdot v.\end{aligned}$$

Moreover, the operator T_f has output that is continuous since unimodularity allows us to move the input variable inside of f in the integral:

$$(T_f(\varphi))(y) = \int_{G(\mathbf{A})} f(y^{-1}g)\varphi(g)dg$$

(so the uniform continuity of f on its compact support and the locally-L^1 property of φ do the job). In fact, writing f as a finite sum of elementary tensors in $C_c^\infty(G(\mathbf{A})) = C_c^\infty(G_\infty) \otimes C_c^\infty(G_{\text{fin}})$ shows that $T_f(\varphi)$ is right-invariant by a compact open subgroup of G_{fin} and smooth in G_∞ (with fixed non-archimedean part); the theorem on differentiation through the integral implies that for smooth vector fields X on G_∞ we have $X(T_f) = T_{Xf}$.

Lemma 3.1. The action of $C_c^\infty(G(\mathbf{A}))$ clearly preserves any closed G-stable subspace. Furthermore, if two closed G-stable subspaces are isomorphic as unitary representations of G, then any such isomorphism respects the induced actions of $C_c^\infty(G(\mathbf{A}))$.

Proof. From the definition, $f \cdot \varphi$ can be approximated by finite sums of the form $\sum c_i(\rho(g_i)(\varphi))$ for some $c_i \in \mathbf{C}$ and $g_i \in G(\mathbf{A})$, and thus is contained in any G-invariant subspace containing φ. Passage to limits preserves a closed subspace, so this gives the first claim. The second claim is obvious since our description of the effect of $C_c^\infty(G(\mathbf{A}))$ in terms of a Hilbert-valued integral only relied on the underlying unitary $G(\mathbf{A})$-representation space. $\square$

4 Decomposition and compact operators

In this section we will show that Theorem 1.2 follows from:

Theorem 4.1. For $f \in C_c^\infty(G(\mathbf{A}))$, the bounded operator $T_f : \varphi \mapsto f \cdot \varphi$ on $L^2(G(k)\backslash G(\mathbf{A}), \omega)$ restricts to a compact operator on the closed subspace $L^2_{\text{cusp}}(G(k)\backslash G(\mathbf{A}), \omega)$.

Theorem 4.1 will be proven in the next lecture; an analog for $\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})$ is proven in the next section.

4.1 Analytic Preliminaries

We will use the notation H for a Hilbert space over $\mathbf{C}$ with inner product $\langle \cdot, \cdot \rangle$, and T for a bounded operator from H to H. [2] We have the following basic definitions:

Definition 4.2.

- T is *compact* if $\overline{T(B_1)}$ is compact, where B_1 is the unit ball in H.
- T is *self-adjoint* if $T^* = T$, where T^* is the operator defined uniquely by the requirement that $\langle Tx, y \rangle = \langle x, T^* y \rangle$.

We define an anti-involution $f \mapsto f^*$ on $C_c^\infty(G(\mathbf{A}))$ by

$$f^*(x) = \overline{f(x^{-1})}$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbf{C}$.

Lemma 4.3. For $f \in C_c^\infty(G(\mathbf{A}))$, $T_{f^*} = (T_f)^*$ as operators on $L^2(G(k)\backslash G(\mathbf{A}), \omega)$.

Proof. Compute from the definition. □

We will use the following weak version on the spectral theorem for compact self-adjoint operators:

Theorem 4.4 (Spectral Theorem)**.** If $T \colon H \to H$ is a non-zero compact self-adjoint operator, then T admits a non-zero eigenvalue λ, and for any non-zero $\lambda \in \mathbf{C}$, the eigenspace $H_\lambda := \ker(T - \lambda)$ is finite-dimensional.

Remark 4.5. Because the action of $C_c^\infty(G(\mathbf{A}))$ preserves any closed $G(\mathbf{A})$-stable subspace, and the restriction of a compact self-adjoint operator to a closed $G(\mathbf{A})$-stable subspace is compact self-adjoint, we see that if $f \in C_c^\infty(G(\mathbf{A}))$ satisfies $f = f^*$ then T_f is a compact self-adjoint operator on any closed $G(\mathbf{A})$-stable subspace of $L^2_{\text{cusp}}(G(k)\backslash G(\mathbf{A}), \omega)$.

We will also need the following non-degeneracy result for the action of $C_c^\infty(G(\mathbf{A}))$.

Lemma 4.6. There exists a sequence $f_i \in C_c^\infty(G(\mathbf{A}))$ such that for all $v \in L^2(G(k)\backslash(G(\mathbf{A}), \omega)$, $f_i \cdot v \to v$ as $i \to \infty$.

A sequence as in Lemma 4.6 is called an *approximate identity*. To construct it, one can take a sequence of functions approaching the delta function at the identity (as a distribution); the existence of such a sequence holds very generally, and such a sequence will satisfy the conclusion of Lemma 4.6 for the Gelfand-Pettis action on any Hilbert space.

[2] All operators considered between Hilbert spaces will be bounded, even if not claimed explicitly.

4.2 Compactness implies decomposition

Now, the fact that Theorem 4.1 implies Theorem 1.2 follows essentially from:

Lemma 4.7. Assuming Theorem 4.1, every non-zero closed $G(\mathbf{A})$-stable subspace $V \subset L^2_{\text{cusp}}$ contains a non-zero topologically irreducible closed $G(\mathbf{A})$-stable subspace.

Proof. Choose $v \in V - \{0\}$. Then, using an approximate identity as in Lemma 4.6, there is some $f_i \in C_c^\infty(G(\mathbf{A}))$ with $f_i \cdot v \neq 0$, so the compact operator $T_{f_i}|_V$ is nonzero. Now, either

$$T_{f_i} + T_{f_i}^* = T_{f_i + f_i^*}$$

or

$$\frac{1}{\sqrt{-1}}\left(T_{f_i} - T_{f_i}^*\right) = \frac{1}{\sqrt{-1}}\left(T_{f_i - f_i^*}\right)$$

is non-zero on V since f_i is a linear combination of the two. Both operators are self-adjoint and compact, so there exists $f \in C_c^\infty(G(\mathbf{A}))$ such that $T_f|_V$ is a nonzero compact self-adjoint operator. Theorem 4.4 gives a non-zero eigenvalue λ of T_f such that the eigenspace V_λ is a (non-zero) finite-dimensional subspace of V.

We now chose $w \in V_\lambda - \{0\}$ which minimizes the dimension of the intersection of V_λ with the closed $G(\mathbf{A})$-stable subspace $\overline{\langle G(\mathbf{A}) \cdot w \rangle}$ topologically generated by w (i.e., the closure of the span of its $G(\mathbf{A})$-orbit). This gives the desired topologically irreducible closed $G(\mathbf{A})$-stable subspace:

Claim 4.8. The closure $\overline{\langle G(\mathbf{A}) \cdot w \rangle}$ is topologically irreducible.

Suppose the claim is false. Then there exists a proper non-zero closed $G(\mathbf{A})$-stable subspace $V' \subsetneq \overline{\langle G(\mathbf{A}) \cdot w \rangle}$. We may write $\overline{\langle G(\mathbf{A}) \cdot w \rangle} = V' \oplus V'^{\perp}$, and $w = (v', x)$ with respect to this decomposition with both v' and x non-zero. Since $w \in V_\lambda$, and V' and $V'^{\perp}$ are stable under T_f, v' and v are both in V_λ. In particular, we find

$$\dim \overline{\langle G(\mathbf{A}) \cdot v' \rangle} \cap V_\lambda \leq \dim V'_\lambda < \dim V'_\lambda + \dim (V'_\lambda)^\perp = \dim \overline{\langle G(\mathbf{A}) \cdot w \rangle} \cap V_\lambda$$

and thus v' contradicts the minimality assumption for w, and we conclude. □

Now, we will prove that Theorem 4.1 implies Theorem 1.2:

Proof. First, we will show the decomposability statement. By Zorn's lemma, there exists a maximal collection $\{V_i\}$ of topologically irreducible mutually orthogonal closed $G(\mathbf{A})$-stable subspaces. Then Lemma 4.7 implies that $L^2_{\text{cusp}} = \widehat{\bigoplus} V_i$ (otherwise, apply the lemma to the *nonzero* orthogonal complement of the Hilbert direct sum of these subspaces to find a subspace that can be added to the collection $\{V_i\}$).

Now, we will show the finite multiplicity statement. Suppose that V is a non-zero unitary representation of $G(\mathbf{A})$ such that

$$\widehat{\bigoplus}_{i=1}^{\infty} V \hookrightarrow L^2_{\text{cusp}} \tag{3}$$

as unitary representations. Looking at a single copy of $V \subset L^2_{\text{cusp}}$ and arguing as in the proof of Lemma 4.7, we find that there is $f \in C_c^\infty(G(\mathbf{A}))$ such that $T_f|_V$ is a non-zero self-adjoint compact operator. In particular, by Theorem 4.4, it admits a non-zero eigenvalue λ. But, for a closed $G(\mathbf{A})$-invariant subspace W, by Lemma 3.1 the restriction $T_f|_W$ depends only on the structure of

W as a unitary representation of $G(\mathbf{A})$, and thus (3) implies that the λ-eigenspace of T_f on L^2_{cusp} is infinite-dimensional. This contradicts Theorem 4.4 since T_f is a compact self-adjoint operator on L^2_{cusp}. $\square$

5 Compactness of integral operators for $\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})$

As a warm-up to Theorem 4.1 (to be proved in the sequel lecture), we now prove an analogue in which the adelic coset space $G(k)\backslash G(\mathbf{A})$ is replaced with $\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})$. In the statement of the theorem below, the integral operators are defined by essentially the same formulas as in the adelic setting.

Theorem 5.1. For any $f \in C_c^\infty(\mathrm{SL}_2(\mathbf{R}))$, the integral operator T_f on $L^2_{\text{cusp}}(\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R}))$ is compact.

We give an outline of the proof, which will be similar in the adelic setting for a general G:

1. Our goal is to show that $f \cdot B_1 \subset L^2_{\text{cusp}}$ has compact closure (with B_1 the closed unit ball).

2. This compactness will follow by a theorem reminiscent of the Arzelà-Ascoli theorem once we prove that the functions in $f \cdot B_1$ are uniformly bounded and equicontinuous (i.e. the functions in this set are each uniformly continuous with the constants for uniform continuity the same across all such functions, and they all have a common bound on their L^2-norm as well).

3. To prove the uniform boundedness and equicontinuity, we will use a fundamental estimate which uniformly bounds $(f \cdot \varphi)(y)$ for all $y \in \mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})$ and *cuspidal* φ in the unit ball of L^2. This will give equicontinuity when applied to $X(f \cdot \varphi) = (Xf) \cdot \varphi$ for X any smooth vector field on $G(\mathbf{R})$ – the functions $f \cdot \varphi$ are smooth, so to control the constants for their continuity it will suffice to control these derivatives.

We will focus on establishing the fundamental estimate, and will not give further details on the deduction of Theorem 5.1 from it.

5.1 Siegel sets

For $c > 0$, we consider *Siegel set* $\Omega(c) \subset \mathrm{SL}_2(\mathbf{R})$,

$$\Omega(c) := \begin{pmatrix} 1 & [-\frac{1}{2}, \frac{1}{2}] \\ 0 & 1 \end{pmatrix} \cdot \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t > 0, t^2 \geq c \right\} \cdot K$$

where $K = \mathrm{SO}_2(\mathbf{R})$. Note that t^2 is the value of the "upper-right" root α when applied to the indicated diagonal torus element, so that the condition on the torus element m can be written as $\alpha(m) \geq c$ (and the condition of positivity on t says that m belongs to the identity component A of the group of $\mathbf{R}$-points of the diagonal torus).

The image of this set under the orbit map for $i = \sqrt{-1} \in \mathbf{H}$ is equal the closed half-strip of points $x + iy \in \mathbf{H}$ satisfying $x \in [-\frac{1}{2}, \frac{1}{2}], y \geq c$.

5.2 The fundamental estimate

For any $y \in \mathrm{SL}_2(\mathbf{R})$, we use the NAK decomposition to write $y = n_y m_y k_y$ with n_y in the upper unipotent subgroup of $\mathrm{SL}_2(\mathbf{R})$, m_y in the identity component of the diagonal torus, and k_y in the maximal compact $K := \mathrm{SO}_2(\mathbf{R})$.

Theorem 5.2. Fix $f \in C_c^\infty(\mathrm{SL}_2(\mathbf{R}))$. For any $\ell > 0$, there exists $C = C_{f,\ell} > 0$ such that for all $y \in \Omega\left(\frac{1}{2}\right)$ and all $\varphi \in L^2_{\mathrm{cusp}}(\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R}))$, we have:

$$\left|(f \cdot \varphi)(y)\right| \le C \left|\alpha(m_y)\right|^{-\ell} \|\varphi\|_{L^2}$$

Proof. We have

$$(f \cdot \varphi)(y) = \int_{\mathrm{SL}_2(\mathbf{R})} f(x)\varphi(yx)\,dx = \int_{\mathrm{SL}_2(\mathbf{R})} f(y^{-1}x)\varphi(x)\,dx. \tag{4}$$

Our basic strategy will be to bound the term coming from f (with an important modification) in order to estimate the value of this integral using the integral of φ on $\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})$. In order to do so, we must first cut down our domain of integration uniformly in y to one that can be covered by finitely many fundamental domains.

To do so, note that $y^{-1}x \in \operatorname{Supp} f$ if and only if $x \in y \cdot \operatorname{Supp} f$, in which case

$$x \in n_y m_y K \cdot \operatorname{Supp} f.$$

Since $K \cdot \operatorname{Supp} f$ is compact, the value of $\alpha(m_g)$ is bounded for $g \in K \cdot \operatorname{Supp} f$. Thus there exists an interval $[c_1, c_2]$ such that

$$\alpha(m_x) \subseteq [\alpha(m_y)c_1, \alpha(m_y)c_2] \ge \frac{1}{2}c_1.$$

So, taking $c = \frac{1}{2}c_1$, we may rewrite the integral (4) as

$$\int_{\{\alpha(m_x)\ge c\}} f\left(y^{-1}x\right)\varphi(x)\,dx = \int_{\left(\begin{smallmatrix}1 & \mathbf{Z}\\ 0 & 1\end{smallmatrix}\right)\backslash\{\alpha(m_x)\ge c\}} \left(\sum_{j\in\mathbf{Z}} f\left(y^{-1}\left(\begin{smallmatrix}1 & j\\ 0 & 1\end{smallmatrix}\right)x\right)\right)\varphi(x)\,dx \tag{5}$$

We now analyze the sum

$$\sum_{j\in\mathbf{Z}} f\left(y^{-1}\left(\begin{smallmatrix}1 & j\\ 0 & 1\end{smallmatrix}\right)x\right) \tag{6}$$

appearing in the integrand. It is helpful to consider this sum geometrically: we have a parameterized curve

$$\gamma : \mathbf{R} \mapsto \mathrm{SL}_2(\mathbf{R}),\ t \mapsto y^{-1}\left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)x$$

and we are summing up the values of f at the points $\gamma(j)$ for $j \in \mathbf{Z}$. The image of γ in $\mathbf{H}$ is a horocycle around $y^{-1}\cdot\infty$ (i.e. a circle in the upper half plane tangent to $y^{-1}\cdot\infty$). In particular, different choices of y and x can lead to different parameterization of the same horocycle (or, rather, the same lift of a horocycle to the unit tangent bundle $\mathrm{SL}_2(\mathbf{R})$ of $\mathbf{H}$), and it will be helpful for some estimates later on in the proof to account for this by fixing some standard parameterization.

To accomplish this, note that we may rewrite $\gamma(t)$ as

$$\gamma(t) = y^{-1} m_y m_y^{-1} \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) m_y m_y^{-1} x = y^{-1} m_y \begin{pmatrix} 1 & \alpha(m_y)^{-1} t \\ 0 & 1 \end{pmatrix} m_y^{-1} x,$$

so that the parameterization coming from x and y is the same as the one coming from $m_y^{-1}x$ and $m_y^{-1}y$ up to a change in speed. Thus, it is natural to define a function $f_{x,y} \in C_c^\infty(\mathbf{R})$ by pulling back f via this standard parameterization,

$$f_{x,y}(t) := f(y^{-1} m_y \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) m_y^{-1} x).$$

We can then express the sum (6) as

$$\sum_{j \in \mathbf{Z}} f_{x,y}(\alpha(m_y)^{-1} j)$$

which, by Poisson summation, is equal to

$$\alpha(m_y) \sum_{j \in \mathbf{Z}} \widehat{f_{x,y}}(\alpha(m_y) j).$$

where $\widehat{f_{x,y}}$ is the Fourier transform of $f_{x,y}$. Combining with (5), we obtain

$$(f \cdot \varphi)(y) = \int_{\left(\begin{smallmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{smallmatrix}\right) \setminus \{\alpha(m_x) \geq c\}} \left(\alpha(m_y) \sum_{j \in \mathbf{Z}} \widehat{f_{x,y}}(\alpha(m_y) j) \right) \varphi(x)\, dx \tag{7}$$

Note that

$$\widehat{f_{x,y}}(0) = \int_{\mathbf{R}} f(y^{-1} \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) x) dt$$

is invariant under left translation of x by upper unipotent matrices (as this just correspond to a shift in the variable t). Thus, the integral

$$\int_{\mathbf{R}} \widehat{f_{x,y}}(0) \varphi(x) dx$$

vanishes by cuspidality of φ, and as we are free to exchange the order of summation and integration in (7), we find that we may remove the term $j = 0$ to obtain

$$(f \cdot \varphi)(y) = \int_{\left(\begin{smallmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{smallmatrix}\right) \setminus \{\alpha(m_x) \geq c\}} \left(\alpha(m_y) \sum_{j \in \mathbf{Z} - \{0\}} \widehat{f_{x,y}}(\alpha(m_y) j) \right) \varphi(x)\, dx \tag{8}$$

Taking absolute values and using the fundamental domain $\Omega(c)$ for the domain of integration, we find

$$|(f \cdot \varphi)(y)| \leq \sup_{x \in \Omega(c)} \left| \alpha(m_y) \sum_{j \in \mathbf{Z} - \{0\}} \widehat{f_{x,y}}(\alpha(m_y) j) \right| \cdot \int_{\Omega(c)} |\varphi(x)| dx.$$

Because $\Omega(c)$ can be covered by finitely many copies of a fundamental domain for $\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})$[3], there is a constant C_1 such that

$$\int_{\Omega(c)} |\varphi(x)| \leq C_1 \int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})} |f| \leq C_1 \|f\|_{L^2}$$

where the latter inequality holds because $\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})$ has finite volume. Thus,

$$|(f\cdot\varphi)(y)| \leq C_1 \cdot \sup_{x\in\Omega(c)} \left|\alpha(m_y) \sum_{j\in\mathbf{Z}-\{0\}} \widehat{f_{x,y}}(\alpha(m_y)j)\right| \|f\|_{L^2},$$

and it remains only to control the supremum appearing.

By a calculation similar to the one we used at the beginning of the proof to show the domain of integration could be restricted, we find that there is a compact subset $K' \subset \mathrm{SL}_2(\mathbf{R})$ such that $y^{-1}m_y$ and $m_y^{-1}x$ are both contained in K' for any $y \in \Omega(1/2)$, $x \in \Omega(c)$. Using this, it can be shown that the functions $f_{x,y}$ for all such $y,\ x$ are contained in a compact subset of $C_c^\infty(\mathbf{R})$. Thus, by continuity properties of the Fourier transform on the Schwartz space (for its natural topology), the Fourier transforms of these functions are contained in a compact subset of Schwartz space, which implies that there is a constant C_2 such that for all $y,\ x$ as above and any t,

$$\widehat{f_{x,y}}(t) \leq C_2(1+|t|)^{-\ell-1}.$$

Thus,

$$\left|\alpha(m_y) \sum_{j\in\mathbf{Z}-\{0\}} \widehat{f_{x,y}}(\alpha(m_y)j)\right| \leq C_2|\alpha(m_y)| \sum_{j\in\mathbf{Z}-\{0\}} (1+|\alpha(m_y)j|^{-\ell-1}) \leq C_2|\alpha(m_y)|^{-\ell} \sum_{j\in\mathbf{Z}-\{0\}} |j|^{-\ell-1}.$$

The sum converges to a value C_3, and combining everything we conclude

$$|(f\cdot\varphi)(y)| \leq C_1C_2C_3\|f\|_{L^2}.$$

$\square$

Remark 5.3. We will explain more of the analytic details used in bounding the Fourier transforms when we tackle the general case of Theorem 4.1 in the next lecture.

[3]This point is not handled carefully in [Ga], and the argument given there for bounding the integral in terms of the L^2 norm of φ and the supremum of the terms involving f **uniformly for all** y **in** $\Omega(1/2)$ seems to be insufficient.