LECTURE 8: DISCRETENESS FOR CUSPIDAL L^2 FOR REDUCTIVE GROUPS II
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NOVEMBER 29, 2017
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In the notes for Lecture 7, we proved the following fundamental estimate for integral operators acting on $\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})$

Theorem 1. Fix $F \in C_c^\infty(\mathrm{SL}_2(\mathbf{R}))$. For any $\ell > 0$, there exists some $C > 0$ such that for all $y \in \Omega\left(\frac{1}{2}\right)$ and $\varphi \in L^2_{\mathrm{cusp}}$ we have:

$$|T_F\varphi(y)| \leq C|\alpha(m_y)|^{-\ell}\|\varphi\|_{L^2}$$

Here, m_y is the torus coordinate appearing in the NAK decomposition of $\mathrm{SL}_2(\mathbf{R})$ and α is the root $\left(\begin{smallmatrix} t & \\ & t^{-1}\end{smallmatrix}\right) \mapsto t^2$; $\Omega(1/2)$ was a fundamental domain.

Theorem 1 was the key analytic input in proving compactness of T_F. In these notes we will prove the corresponding fundamental estimates for $\mathrm{SL}_2(\mathbf{F}_p(t))\backslash\mathrm{SL}_2(\mathbf{A}_{\mathbf{F}_p(t)})$ and $\mathrm{SL}_3(\mathbf{Z})\backslash\mathrm{SL}_3(\mathbf{R})$; we briefly discuss the general case at the end.

1 $\mathrm{SL}_2(\mathbf{F}_p(t))\backslash\mathrm{SL}_2(\mathbf{A}_{\mathbf{F}_p(t)})$

We will work in the adele ring $\mathbf{A} = \mathbf{A}_{\mathbf{F}_p(t)}$ and consider the space $\mathrm{SL}_2(\mathbf{F}_p(t))\backslash\mathrm{SL}_2(\mathbf{A})$. Let $F \in C_c^\infty(\mathrm{SL}_2(\mathbf{A}))$. We will make repeated use of the following basic result:

Lemma 2. For $F \in C_c^\infty(\mathrm{SL}_2(\mathbf{A}))$, there is a compact open subgroup K_F such that F is bi-invariant under K_F.

Proof. It suffices to find compact open subgroups K_1 and K_2 such that F is left-invariant under K_1 and right-invariant under K_2, since then it is bi-invariant under $K_1 \cap K_2$. The arguments are essentially, the same, so we will show that there is a K such that F is left-invariant under K. Because F is smooth on $\mathrm{SL}_2(\mathbf{A})$, for each $x \in \mathrm{SL}_2(\mathbf{A})$ there is a compact open subgroup K_x such that F is constant on $K_x \cdot x$. If we let x run over a compact open set U such that the support of F is contained in U, then we obtain an open cover of U by $K_x \cdot x$ for $x \in U$. It admits a finite subcover by $K_{x_i} \cdot x_i$ for $x_1, \ldots, x_n$, and we find that F is left-invariant under the compact open subgroup $K = \cap_i K_{x_i}$. □

As we did in the case of $\mathrm{SL}_2(\mathbf{R})$, we define the integral operator $T_F \colon L^2 \to L^2$ by

$$T_F\varphi(y) = \int_{\mathrm{SL}_2(\mathbf{A})} F(x)\varphi(yx)dx = \int_{\mathrm{SL}_2(\mathbf{A})} F(y^{-1}x)\varphi(x)dx$$

To estimate $T_F\varphi(y)$, we once again unwind to obtain:

$$T_F\varphi(y) = \int_{\begin{pmatrix}1 & \mathbf{F}_p(t)\\ 0 & 1\end{pmatrix}\backslash\mathrm{SL}_2(\mathbf{A})} \left(\sum_{a\in\mathbf{F}_p(t)} F\left(y^{-1}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}x\right)\right)\varphi(x)dx$$

As in the $\mathrm{SL}_2(\mathbf{R})$ case, we write $F_{x,y}(s)$ for

$$s \mapsto F\left(y^{-1}m_y \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} m_y^{-1}x\right)$$

so that

$$T_F\varphi(y) = \int_{\begin{pmatrix} 1 & \mathbf{F}_p(t) \\ 0 & 1 \end{pmatrix}\backslash \mathrm{SL}_2(\mathbf{A})} \left(\sum_{a\in\mathbf{F}_p(t)} F_{x,y}(\alpha(m_y)a)\right)\varphi(x)dx$$

and we also note that we may replace the integration over all of $\mathrm{SL}_2(\mathbf{A})$ with integration over a "half plane" $|\alpha(m_x)| \geq c$ for some $c > 0$. Thus, we may assume x and y are both in some Siegel set $\Omega(c)$.

As a result, we obtain the following uniform regularity for the functions $F_{x,y}$:

Lemma 3. Given a Siegel set $\Omega(c)$, there exists $U \in \mathbf{A}$ compact open such that for all $x, y \in \Omega(c)$, $F_{x,y}$ is right invariant under U.

Proof. We note that for

$$g \in \mathbf{A} = \begin{pmatrix} 1 & \mathbf{A} \\ 0 & 1 \end{pmatrix},$$

$$F_{x,y}(sg) = F(y'sgx') = F(y'sx'x'^{-1}gx').$$

Thus it suffices to show we can find a compact open U subgroup such that $x'^{-1}gx'$ is contained in K_F for all x'. But, we know x' lies in a compact set K' (because we have fixed a Siegel set – cf. the argument in the $\mathrm{SL}_2(\mathbf{R})$ case), thus for a sufficiently small compact open subgroup W of $SL_2(\mathbf{A})$, $x'^{-1}gx'$ is in K_F for all g in W and $x' \in K'$, and we may take $U = W \cap bA$ (where $\mathbf{A}$ is thought of as upper unipotent matrices). □

Now, Poisson summation says:

$$\sum_{a\in\mathbf{F}_p(t)} F_{x,y}\big(\alpha(m_y)^{-1}a\big) = |\alpha(m_y)| \sum_{a\in\mathbf{F}_p(t)} \widehat{F_{x,y}}(\alpha(m_y)a)$$

The term $a = 0$ on the right-hand side integrates to 0 due to cuspidality. Now, because the $F_{x,y}$ are all invariant under a compact open subgroup U, there is a compact open U' such that for all x, y, $\widehat{F_{x,y}}$ has support in U'. Moreover, there is a uniform bound on the values of $\widehat{F_{x,y}}$: the values are bounded by

$$\int_{\mathbf{A}} |F_{x,y}|$$

which is itself bounded by a constant C: this integral is a continuous function of x' and y', which vary in a compact set.

Putting everything together, we find that for $|\alpha(m_y)|$ sufficiently large, the sum vanishes, and for $|\alpha(m_y)|$ smaller (but keeping in mind $|\alpha(m_y)| > c$), there is a bound on the number of non-zero terms, each of which has bounded absolute value, and thus

$$|\sum_{a\in\mathbf{F}_p(t)} F_{x,y}\big(\alpha(m_y)^{-1}a\big)| \leq C'$$

for some constant C'.

Introducing this estimate for the kernel back into our computation of $T_F\varphi(y)$, we conclude

Theorem 4. For $y \in \Omega(c)$ and $\varphi \in L^2_{\text{cusp}}$,

$$|T_F\varphi(y)| \leq C|\varphi|_{L^2}.$$

Remark 5. The analytic inputs here were simpler than the analytic inputs over $\mathbf{R}$: in the non-archimedean setting, the space of smooth (locally constant) compactly supported functions are a good analogue for Schwarz space, and the exchange of regularity for support in the Fourier transform can be understood in elementary terms.

Remark 6. We note that the deduction of compactness from the fundamental estimate is also simplified in the absence of an archimedean place: Theorem 4 says that T_F factors as a continous map from L^2_{cusp} to L^∞_{cusp} combined with the inclusion L^∞_{cusp} into L^2_{cusp}. On the other hand, we claim that $T_F\varphi$ is always right-invariant under K_F: Indeed, for $k \in K_F$ and any y,

$$T_F\varphi(yk) = \int F(k^{-1}y^{-1}x)\,\varphi(x)\,dx = \int F\big(y^{-1}x\big)\,\varphi(x)\,dx = T_F\varphi(y)$$

Thus, T_F factors as composition of continuous maps

$$L^2_{\text{cusp}}(\mathrm{SL}_2(\mathbf{F}_p(t))\backslash SL_2(\mathbf{A})) \xrightarrow{T_F} L^\infty_{\text{cusp}}(\mathrm{SL}_2(\mathbf{F}_p(t))\backslash SL_2(\mathbf{A})/K_F) \to L^2_{\text{cusp}}(\mathrm{SL}_2(\mathbf{F}_p(t))\backslash SL_2(\mathbf{A})/K_F) \hookrightarrow L^2_{\text{cusp}}(\mathrm{SL}_2(\mathbf{F}_p(t))\backslash SL_2(\mathbf{A}))$$

Because $\mathrm{SL}_2(\mathbf{F}_p(t))\backslash\mathrm{SL}_2(\mathbf{A})/K_F$ is a discrete space of finite volume, it is easy to check that the middle map is compact, and we conclude that the composition is also compact.

2 $\mathrm{SL}_2(\mathbf{Q})\backslash\mathrm{SL}_2(\mathbf{A_Q})$

Combining the arguments for $\mathrm{SL}_2(\mathbf{F}_p(t))\backslash\mathrm{SL}_2(\mathbf{A}_{\mathbf{F}_p(t)})$ above with the arguments for $\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})$, we can handle the case $\mathrm{SL}_2(\mathbf{Q})\backslash\mathrm{SL}_2(\mathbf{A})$: we can assume F on $\mathrm{SL}_2(\mathbf{A})$ is of the form $F = F_f \cdot F_\infty$ where F_∞ is a compactly supported C^∞ function on $\mathrm{SL}_2(\mathbf{R})$ and F_f is a compactly supported smooth function on $\mathbf{A}_f$. Then the bound for F_f is handled just as in the function field case, and the bound for F_∞ is essentially what was already proved for $\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})$.

Remark 7. In the last lecture, we noted that the decomposition and finite-multiplicity results for L^2_{cusp} for $\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})$ *does not* follow from the analogous result for $\mathrm{SL}_2(\mathbf{Q})\backslash\mathrm{SL}_2(\mathbf{A})$. However, the compactness of the operators T_F for $\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})$ can be easily deduced from the compactness of the operators T_F for $\mathrm{SL}_2(\mathbf{Q})\backslash\mathrm{SL}_2(\mathbf{A})$, and thus the proof in the latter case encompasses the former.

3 SL_3

How do things change when we go beyond the case $G = \mathrm{SL}_2$? Let's examine the case of $\mathrm{SL}_3(\mathbf{Z})\backslash\mathrm{SL}_3(\mathbf{R})$, and the case of general split groups over $\mathbf{Q}$ will follow similarly. For the rank-two group SL_3, we consider the basis of simple roots α_1, α_2 for the positive roots, where α_1 corresponds to the root group $\begin{pmatrix} 1 & \mathbf{Z} & \\ & 1 & \\ & & 1 \end{pmatrix}$ and α_2 appears in the root group $\begin{pmatrix} 1 & & \\ & 1 & \mathbf{Z} \\ & & 1 \end{pmatrix}$. The sum $\alpha_{12} = \alpha_1 + \alpha_2$ corresponds to the root group $\begin{pmatrix} 1 & & \mathbf{Z} \\ & 1 & \\ & & 1 \end{pmatrix}$. Now, as in the case of SL_2, we can rewrite the integral for $F \cdot e(y)$ as a sum, indexed by $a \in \mathbf{Z}^3 = \begin{pmatrix} 1 & \mathbf{Z} & \mathbf{Z} \\ & 1 & \mathbf{Z} \\ & & 1 \end{pmatrix}$ of integrals over $\begin{pmatrix} 1 & \mathbf{Z} & \mathbf{Z} \\ & 1 & \mathbf{Z} \\ & & 1 \end{pmatrix} \backslash \mathrm{SL}_3(\mathbf{R})$.

Thus, we end up with a sum, which we would like analyze via Poisson summation again:

$$\sum_{a = \begin{pmatrix} 1 & a_1 & a_{12} \\ & 1 & a_2 \\ & & 1 \end{pmatrix}} g\big(\alpha_1(m_y)^{-1} a_1, \alpha_2(m_y)^{-1} a_2, \alpha_{12}(m_y)^{-1} a_{12}\big)$$

We will denote this upper-triangular subgroup of $\mathrm{SL}_3(\mathbf{Z})$ as $\mathbf{Z}^3$. The natural group structure is not the ordinary one on $\mathbf{Z}^3$, but this is immaterial for our purposes. Poisson summation says that the sum may be rewritten as:

$$\delta(m_y) \sum_{a \in \mathbf{Z}^3} \widehat{g}\big(\alpha_1(m_y) a_1, \alpha_2(m_y) a_2, \alpha_1(m_y)\alpha_2(m_y) a_{12}\big)$$

We will use the bound $|\widehat{g}(t_1, t_2, t_3)| \leq C_\ell \left(\sup_i |t_i|\right)^{-\ell}$, but it is no longer as clear that when $\delta(m_y)$ is large, this number will be small: for example, for $(a_1, a_2, a_3) = (1, 0, 0)$, $\delta(m_y)$ could be made very large by making $\alpha_2(m_y)$ very large while keeping $\alpha_1(m_y)$ small, whereas our estimate for $\widehat{g}$ would only involve a negative power of $\alpha_1(m_y)$. However, the cuspidality condition rules out precisely this possibility: if the a_j are zero for all of the indices where a simple root α_i contributes, then this term will vanish when integrated against a cusp form (indeed, this set of positive roots is exactly the set of positive roots in the unipotent radical of the maximal proper parabolic corresponding to the simple root α_i).

4 And beyond

The main technical difficulty in carrying out the general case seems to be showing that there is a suitable parameterization of the $\mathbf{A}$-points of the nilpotent radical N of a minimal parabolic by $\mathbf{G}_a(\mathbf{A})^{\dim N}$. Hopefully this will be added either to these notes or as an addendum at a later date.