LECTURE 9: ADELIZATION OF MODULAR FORMS PART II
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1 Hecke Operators

1.1 Classical Setting

First, we will recall the classical theory of Hecke operators. Let Γ be a congruence subgroup of $\mathrm{GL}_2^+(\mathbf{R})$ and $f \in M_k(N)$. Recall the operation of Γ on $M_k(N)$ is given by:

$$(f|_k\gamma)(z) = f\left(\frac{az+b}{cz+d}\right)\left(\frac{\sqrt{\det\gamma}}{cz+d}\right)^k$$

The space of Hecke Operators $\mathcal{H}_\Gamma$ is defined to be the free abelian group generated by the double cosets $\{T_\alpha := \Gamma\alpha\Gamma\colon \alpha \in \mathrm{GL}_2^+(\mathbf{Q})\}$. In the case $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, this group is generated by $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$, so the double cosets are essentially the same thing as matrices up to determinant-preserving elementary row and column operations.

We will depend on the following group theoretic fact for many of our calculations:

Fact 1.1. Each double coset decomposes as $\Gamma\alpha\Gamma = \bigsqcup_i \Gamma\alpha_i$ for *finitely many* $\alpha_i \in \Gamma\alpha\Gamma$.

We can use this to define a multiplication on $\mathcal{H}_\Gamma$. Suppose $\Gamma\alpha\Gamma = \bigsqcup_i \Gamma\alpha_i$ and $\Gamma\beta\Gamma = \bigsqcup_i \Gamma\beta_i$; then define

$$T_\alpha T_\beta := \sum_\gamma c_{\alpha\beta\gamma} T_\gamma$$

where $c_{\alpha\beta\gamma}$ is the number of pairs i, j such that $\alpha_i\beta_j \in \Gamma\gamma$. We will show below that this operation is well-defined and makes $\mathcal{H}_\Gamma$ into an associative $\mathbf{Z}$-algebra.

Observe that $\mathcal{H}_\Gamma$ can be naturally associated with the set of functions $f : \Gamma\backslash\mathrm{GL}_2^+(\mathbf{Q})/\Gamma \to \mathbf{Z}$ with finite support; namely, for any $T \in \mathcal{H}_\Gamma$, the value of the corresponding f_T on $\Gamma\alpha\Gamma$ is the coefficient of $\Gamma\alpha\Gamma$ in T.

Proposition 1.2. The set of functions $f : \Gamma\backslash\mathrm{GL}_2^+(\mathbf{Q})/\Gamma \to \mathbf{Z}$ with finite support has a convolution structure

$$(f_1 \star f_2)(g) = \sum_{\Gamma h} f_1(gh^{-1})f_2(h),$$

which agrees with the multiplication on $\mathcal{H}_\Gamma$ (and hence makes $\mathcal{H}_\Gamma$ an associative $\mathbf{Z}$-algebra).

Proof. First note that the sum is a finite sum by Fact 1.1: since f_2 has finite support, there are only finitely many Γh such that $f_2(h)$ is nonzero. The sum is independent of choice of coset representatives because

$$f_1(gh^{-1}\gamma^{-1})f_2(\gamma h) = f_1(gh^{-1})f_2(h)$$

by right Γ-invariance of f_1 and left Γ-invariance of f_2.

Now $(f_1 \star f_2)(g)$ is left Γ-invariant because f_1 is left Γ-invariant, and it is right Γ-invariant because

$$(f_1 \star f_2)(g\gamma) = \sum_{\Gamma h} f_1(g\gamma h^{-1}) f_2(h) = \sum_{\Gamma h} f_1(gh^{-1}) f_2(h\gamma)$$

by the change of variables $h \mapsto h\gamma$, and this equals $(f_1 \star f_2)(g)$ by right Γ-invariance of f_2. This proves that $f_1 \star f_2$ is well-defined on the double-coset space.

If $(f_1 \star f_2)(g) \neq 0$, this would mean that there exists $h \in \text{supp}(f_2)$ such that $gh^{-1} \in \text{supp}(f_1)$; in other words, $g \in \text{supp}(f_1)\text{supp}(f_2)$. But since each of $\text{supp}(f_1)$ and $\text{supp}(f_2)$ is a finite union of double cosets, and hence by Fact 1.1 a finite union of right cosets of Γ, $\text{supp}(f_1)\text{supp}(f_2)$ is a finite union of sets of the form $(\Gamma\alpha)(\Gamma\beta)$. Since each $\Gamma\alpha\Gamma$ is itself a finite union of right cosets of Γ, and hence so is $(\Gamma\alpha)(\Gamma\beta)$, this means $\text{supp}(f_1)\text{supp}(f_2)$ is a finite union of right cosets of Γ. This proves that $f_1 \star f_2$ has finite support.

Now suppose f_α, f_β are the characteristic functions of $\Gamma\alpha\Gamma$, $\Gamma\beta\Gamma$ respectively (corresponding to $T_\alpha, T_\beta \in \mathcal{H}_\Gamma$). Then

$$(f_\alpha \star f_\beta)(g) = \sum_{\Gamma h} f_\alpha(gh^{-1}) f_\beta(h).$$

Letting $\Gamma\beta\Gamma = \bigsqcup_j \Gamma\beta_j$, this can be written as

$$(f_\alpha \star f_\beta)(g) = \sum_j f_\alpha(g\beta_j^{-1}).$$

This counts the number of j such that $g\beta_j^{-1} \in \Gamma\alpha\Gamma$, or equivalently, by writing $\Gamma\alpha\Gamma = \bigsqcup_i \Gamma\alpha_i$, the number of j such that $g\beta_j^{-1} \in \Gamma\alpha_i$ for some (necessarily unique) i. Thus $(f_\alpha \star f_\beta)(g)$ is the number of pairs (i, j) such that $g \in \Gamma\alpha_i\beta_j$, proving that $f_\alpha \star f_\beta$ is the function corresponding to $T_\alpha T_\beta$. Since convolution is bilinear, it agrees with the multiplication defined on $\mathcal{H}_\Gamma$ by extending linearly. $\square$

We may define an action of $\mathcal{H}_\gamma$ on $M_k(\Gamma)$ as follows: using Fact 1.1, we can write $\Gamma\alpha\Gamma = \bigsqcup_i \Gamma\alpha_i$, and define

$$f|_k T_\alpha = \sum_i f|_k \alpha_i.$$

This may profitably be thought of as convolution of f with T_α (though ths doesn't currently make sense because f and T_α don't have the same domain, this intuition that will be made rigorous in the adelic setting; see Definition 1.9).

Example 1.3. If $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, then every double coset contains a unique element of the form $\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ with $d_1, d_2 \in \mathbf{Q}^+$ and d_1 a (positive) integer multiple of d_2: this can be proved via row reduction and the theory of elementary divisors.

Now, we may define, for general Γ:

Definition 1.4.

$$T(n) = \sum_{\substack{d_1, d_2 \in \mathbf{Z} \\ d_2 | d_1, d_1 d_2 = n}} T_{\begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix}}$$

In the case of $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, this corresponds to the sum of T_α for all $\Gamma\alpha\Gamma$ with α of determinant n. This definition satisfies the property that if $(a, b) = 1$, then $T(ab) = T(a)T(b)$.

Proposition 1.5. If $f = \sum_{m=1}^{\infty} \alpha_m q^m \in S_k(\Gamma)$ is the Fourier expansion at the cusp ∞, then we can compute the action of $T(m)$ as:

$$f|_k T(m) = \sum_{m=1}^{\infty} \beta_m q^m$$

with $\beta_m = \displaystyle\sum_{\substack{ad=n \\ a|m}} n^{k/2} d^{-k+1} \alpha_{md/a}$.

Proof. The sum defining $T(n)$ corresponds to a disjoint union of double cosets, each of which is a disjoint union of right cosets; in particular, the collection of right cosets defining $T(n)$ is given by

$$\bigsqcup_{\substack{a,b,d \in \mathbf{N} \\ ad=n \\ 0 \le b < d}} \Gamma \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$$

Plugging this into the definition of $T(n)$ tells us that

$$f|_k T(n) = \sum_{\substack{a,b,d \in \mathbf{N} \\ ad=n \\ 0 \le b < d}} f|_k \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right),$$

from which we may evaluate the Fourier coefficients by direct computation. $\square$

Remark 1.6. The fact that the Hecke operators have such nice formulas in terms of the q-expansion at the cusp ∞ is a special fact about GL_2 and ∞. We may determine such formulas for general cusps, but they get nastier and often are not essential.

The formula in the proposition giving β_m as a sum over divisors $ad = n$ looks reminiscent of a Dirichlet convolution, i.e. a multiplicative convolution of f with "something encoding divisibility by n". This is the perspective we will take on the adelic side (see Definition 1.9).

Corollary 1.7. Notice that $\beta_1 = n^{k/2} n^{-k+1} \alpha_n$, so $T(n)$ lets you isolate the n-th Fourier coefficient of f.

Now suppose $f \in S_k(N, \chi)$, $\Gamma = \Gamma_1(N)$, and let p be a prime. In this case, $T(p)$ is defined by a unique double coset; hence we seek to understand the double coset $\Gamma_1(N) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma_1(N)$[1]. If $p \mid N$, we have:

$$\Gamma_1(N) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma_1(N) = \bigsqcup_{i=0}^{p-1} \Gamma_1(N) \left(\begin{smallmatrix} 1 & i \\ 0 & p \end{smallmatrix}\right).$$

One way to see this is by noting that if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$ (so that $a, d \equiv 1 \mod N$, implying they are both $1 \mod p$), we have for any $k \in \mathbf{Z}$ that

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ pc & pd \end{smallmatrix}\right) = \begin{pmatrix} a & \frac{1-a}{p} b + ka \\ pc & d + c(kp - b) \end{pmatrix} \begin{pmatrix} 1 & b - kp \\ 0 & p \end{pmatrix},$$

[1]Since $T(p) = T_{\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)}$, it seems as though we should really be studying the double coset $\Gamma_1(N) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \Gamma_1(N)$. The analysis will obviously be very similar in each case, but importantly, it is actually $\Gamma_1(N) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma_1(N)$ that will have a direct relation to the corresponding adelic concept for $T(p)$ that uses $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$

so that any element of $\Gamma_1(N)\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\Gamma_1(N)$ is in $\Gamma_1(N)\left(\begin{smallmatrix}1&i\\0&p\end{smallmatrix}\right)$ for some $0\le i\le p-1$. These cosets are all disjoint because $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\left(\begin{smallmatrix}1&i\\0&p\end{smallmatrix}\right)=\left(\begin{smallmatrix}a&ai+bp\\c&ci+dp\end{smallmatrix}\right)$ satisfies $ai+bp\equiv i \mod p$, and so $\left(\begin{smallmatrix}1&i\\0&p\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&j\\0&p\end{smallmatrix}\right)$ will be in distinct cosets if $i\not\equiv j \mod p$.

When $p\nmid N$, then each $\Gamma_1(N)\left(\begin{smallmatrix}1&i\\0&p\end{smallmatrix}\right)$ is still a subset of $\Gamma_1(N)\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\Gamma_1(N)$, but there are also elements in the double coset which are not in any of the $\Gamma_1(N)\left(\begin{smallmatrix}1&i\\0&p\end{smallmatrix}\right)$; for example, if $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_1(N)$ has $p\mid a, p\nmid b$, then a similar computation as above shows that $\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ is not in any $\Gamma_1(N)\left(\begin{smallmatrix}1&i\\0&p\end{smallmatrix}\right)$. If $p\equiv 1 \mod N$, it is possible to show that these extra elements are in $\Gamma_1(N)\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)$; unfortunately, for other values of p, $\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)$ is not even in the double coset space! To fix this, we shall apply a correction factor W that will take $\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)$ into the double coset space; the corresponing right coset will then capture all the remaining elements.

Proposition 1.8. Suppose $p\nmid N$. There exists $W\in\mathrm{SL}_2(\mathbf{Z})$, depending only on p and N such that

$$\Gamma_1(N)\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\Gamma_1(N)=\Gamma_1(N)(W\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right))\sqcup\bigsqcup_{i=0}^{p-1}\Gamma_1(N)\left(\begin{smallmatrix}1&i\\0&p\end{smallmatrix}\right)$$

and $f|_kW=\chi(p)f$.

(Note: This proof is essentially just a lot of modular arithmetic and may be safely skipped without losing much content.)

Proof. By the Chinese remainder theorem, there exists α,β satisfying $\alpha\equiv 1 \mod N$, $\alpha\equiv 0 \mod p^2$, $\beta\equiv 0 \mod N$, and $\beta\equiv 1 \mod p^2$. Under these conditions, $\alpha^2+\beta^2\equiv 1 \mod p^2N$, so $\left(\begin{smallmatrix}\alpha&-\beta\\\beta&\alpha\end{smallmatrix}\right)$ has determinant $1 \mod p^2N$. Since the natural map $\mathrm{SL}_2(\mathbf{Z})\to\mathrm{SL}_2(\mathbf{Z}/p^2N\mathbf{Z})$ is surjective, this lifts to a matrix $M\in\mathrm{SL}_2(\mathbf{Z})$ which is the identity mod N and $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ mod p^2.

Set $W=\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)M\left(\begin{smallmatrix}1/p&0\\0&1\end{smallmatrix}\right)$. This is an integer matrix because $\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)M\equiv\left(\begin{smallmatrix}0&-1\\p&0\end{smallmatrix}\right) \mod p^2$, so the left column can be divided by p. It has determinant 1, and so lies in $\mathrm{SL}_2(\mathbf{Z})$. We have $W\equiv\left(\begin{smallmatrix}p^{-1}&0\\0&p\end{smallmatrix}\right) \mod N$, so $f|_kW=\chi(p)f$. We have $W\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)=\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)M\in\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\Gamma_1(N)$, so $W\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)$ is in the double coset space.

Now suppose $W\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\left(\begin{smallmatrix}1&i\\0&p\end{smallmatrix}\right)$ for some $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_1(N)$. On one hand, $W\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)\equiv\left(\begin{smallmatrix}0&-1\\p&0\end{smallmatrix}\right) \mod p^2$, but on the other hand $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\left(\begin{smallmatrix}1&i\\0&p\end{smallmatrix}\right)=\left(\begin{smallmatrix}a&ai+bp\\c&ci+dp\end{smallmatrix}\right)$; since $a\equiv 0 \mod p^2$ would imply $ai+bp\not\equiv -1 \mod p^2$, this is a contradiction. Hence $\Gamma_1(N)(W\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right))$ is disjoint from all $\Gamma_1(N)\left(\begin{smallmatrix}1&i\\0&p\end{smallmatrix}\right)$.

Finally, we will construct a list of all possible right cosets of $\Gamma_1(N)$ in the desired double coset. Take $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_1(N)$. If $p\nmid a$, let $t\equiv a^{-1} \mod p$; if $p\mid a$ and $p\mid b$, let $t=1$. In either case, we have for any $k\in\mathbf{Z}$ that

$$\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\left(\begin{smallmatrix}a&b\\pc&pd\end{smallmatrix}\right)=\begin{pmatrix}a&\frac{b(1-at)}{p}+ka\\pc&d+c(kp-tb)\end{pmatrix}\begin{pmatrix}1&tb-kp\\0&p\end{pmatrix},$$

so $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_1(N)\left(\begin{smallmatrix}1&i\\0&p\end{smallmatrix}\right)$ for some $1\le i\le p-1$.

If instead $p \mid a$ and $p \nmid b$, Let $A = \left(\begin{smallmatrix} a & b \\ pc & pd \end{smallmatrix}\right) \left(\begin{smallmatrix} 1/p & 0 \\ 0 & 1 \end{smallmatrix}\right) W^{-1}$; this is an integer matrix since a and pc are both divisible by p. Since $W \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \mod N$, we have $A \equiv \left(\begin{smallmatrix} a & b \\ pc & pd \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} a & bp^{-1} \\ pc & d \end{smallmatrix}\right)$, which is in $\Gamma_1(N)$ by assumption on a, c, d. Hence

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ pc & pd \end{smallmatrix}\right) = A(W \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)) \in \Gamma_1(N)(W \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)),$$

completing the classification. □

1.2 Adelic Setting

Consider an automorphic form ϕ with nebentypus ψ, according to definition 4.3 in Lecture 5. In particular, it will be a smooth cuspidal function on $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$, it will satisfy $\phi(gz) = \phi(g)\psi(z)$ for $g \in G(\mathbf{A})$, $z \in Z(\mathbf{A})$, and it will be invariant under some compact open subgroup of $G(\mathbf{A}_f)$. In particular, by openness, we can find N such that ϕ is invariant under

$$K_1(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{\mathbf{Z}})\colon a \equiv d \equiv 1, c \equiv 0 \mod N\}.$$

Taking a multiple of N if necessary (which preserves invariance), we can guarantee that ψ factors through $(\mathbf{Z}/N\mathbf{Z})^\times$; then ψ extends to a character on $K_0(N)$ by $\psi \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \psi(d)$.

Consider $H_p = K_1(N) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_1(N)$, where $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is concentrated at p (i.e. it is the identity at all other places). Just as in the classical setting (though here we use left cosets, in order to play well the right action of $G(\mathbf{A})$ on ϕ), we may check that if $p \mid N$, we have:

$$H_p = \bigsqcup_{i=0}^{p-1} \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right) K_1(N)$$

where, as above, the given matrix is concentrated at p. When $p \nmid N$, we have:

$$H_p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) K_1(N) \sqcup \bigsqcup_{i=0}^{p-1} \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right) K_1(N).$$

Note that in this case we don't need the elaborate construction involving W, because $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ is already in $K_1(N) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_1(N)$! Namely, the element w, which is $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ at p and the identity elsewhere, is in $K_1(N)$ since $p \nmid N$, and $-w \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) w = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$[2].

In either case, let $\{h_i\}$ be the set of coset representatives given in one of the above decompositions, so for any p we have $H_p = \bigsqcup_i h_i K_1(N)$ (where of course the h_i depend on p and N).

Now, we define the adelic Hecke operators, first in a way that appears to depend heavily on N.

Definition 1.9.

$$(\widetilde{T}_N(p) \cdot \phi)(g) = \frac{1}{\mathrm{vol}(K_1(N))} \int \phi(gh) \mathbb{1}_{H_p}(h)\, dh$$

where $\mathbb{1}_{H_p}$ is the characteristic function of H_p.

[2] This w takes the role of W in the classical case – note that $W \equiv w \bmod p$

Notice that if $h = h'k$ for some $k \in K_1(N)$, we have $\phi(gh) = \phi(gh')$ by right-invariance. Hence the integrand is constant on left-cosets of $K_1(N)$, and so the integral is equal to:

$$\left(\widetilde{T}_N(p) \cdot \phi\right)(g) = \sum_i \phi(gh_i).$$

Proposition 1.10. The definition above depends only on whether $p \mid N$. That is, suppose ϕ is invariant under $K_1(N)$ and $K_1(M)$, the nebentypus ψ of ϕ factors through both $(\mathbf{Z}/M\mathbf{Z})^\times$ and $(\mathbf{Z}/N\mathbf{Z})^\times$, and $p \mid M$ iff $p \mid N$. Then $\widetilde{T}_N(p)\phi = \widetilde{T}_M(p)\phi$.

Proof. If p divides both M and N or neither M and N, then the same coset representatives can be used in both cases; that is,

$$K_1(N) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_1(N) = \bigsqcup_i h_i K_1(N),$$

$$K_1(M) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_1(M) = \bigsqcup_i h_i K_1(M)$$

for the same collection of representatives $\{h_i\}$ in each line. Thus,

$$\left(\widetilde{T}_N(p) \cdot \phi\right)(g) = \sum_i \phi(gh_i) = \left(\widetilde{T}_M(p) \cdot \phi\right)(g) \qquad \square.$$

This allows us to define $\widetilde{T}(p)$ independently of ϕ; namely, if there exists an N with $p \nmid N$, ϕ invariant under $K_1(N)$, and ψ factoring through $(\mathbf{Z}/N\mathbf{Z})^\times$, then we will have $\widetilde{T}(p) = \phi(g \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)) + \sum_{i=0}^{p-1} \phi(g \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right))$; otherwise we have $\widetilde{T}(p) = \sum_{i=0}^{p-1} \phi(g \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right))$.

Proposition 1.11. If $\phi = \phi_f$ for $f \in S_k(N, \chi)$, then we have

$$p^{k/2-1}\widetilde{T}(p)\phi_f = \phi_{f|_k T(p)}.$$

Proof. Write $g = \gamma\kappa u$ for $\gamma \in G(\mathbf{Q})$, $\kappa \in K_1(N)$, and $u \in G^+(\mathbf{R})$, and let $z = u \cdot i \in \mathbf{C}$. Then for each representative h_i, $gh_i = \gamma\kappa h_i u = \gamma h_j \kappa' u$ for some other h_j and $\kappa' \in K_1(N)$ by properties of the double coset. Now observe that we can interpret h_j as an element in $G(\mathbf{Q})$ (i.e. diagonally embedded rather than concentrated at p) as long as we multiply by its inverse at each place besides p. Thus, we replace u with $h_j^{-1}u$ at ∞, and replace κ' with $\widetilde{h_j}^{-1}\kappa'$ (where $\widetilde{h_j}$ is trivial at p and equals the matrix defining h_j at all other finite places). Then

$$\begin{aligned}
p^{k/2-1}(\widetilde{T}(p)\phi)(g) &= p^{k/2-1} \sum_j \chi(\widetilde{h_j}^{-1})^{-1} (f|_k h_j^{-1} u)(i) \\
&= p^{k/2-1} \sum_j \chi(\widetilde{h_j})(f|_k p^{-1}(ph_j^{-1})u)(i) \\
&= \sum_j \chi(\widetilde{h_j})(f|_k(ph_j^{-1})u)(i).
\end{aligned}$$

Note that the appearance of h_j^{-1} is why we considered different cosets for the classical and adelic cases; in particular, the map $h_j \mapsto ph_j^{-1}$ provides a bijection between adelic representatives and classical representatives.

Now observe that $\chi(\widetilde{h_j}) = \chi(p)$ for $h_j = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$, and $\chi(\widetilde{h_j}) = 1$ for all other representatives. Hence, if $p \nmid N$,

$$\begin{aligned} p^{k/2-1}(\widetilde{T}(p)\phi)(g) &= \left(\sum_{b=0}^{p-1} (f \mid_k \begin{pmatrix} 1 & -b \\ 0 & p \end{pmatrix} u)(i) + \chi(p)(f \mid_k \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) u)(i) \right) \\ &= \left(\sum_{b=0}^{p-1} (f \mid_k \begin{pmatrix} 1 & -b \\ 0 & p \end{pmatrix} u)(i) + (f \mid_k W \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) u)(i) \right) \\ &= ((f \mid_k T(p)) \mid_k u)(i) \\ &= \phi_{f \mid_k T(p)}(g). \end{aligned}$$

If $p \mid N$, on the other hand,

$$\begin{aligned} p^{k/2-1}(\widetilde{T}(p)\phi)(g) &= \sum_{b=0}^{p-1} (f \mid_k \begin{pmatrix} 1 & -j \\ 0 & p \end{pmatrix} u)(i) \\ &= ((f \mid_k T(p)) \mid_k u)(i) \\ &= \phi_{f \mid_k T(p)}(g). \end{aligned}$$

□

So this definition does agree with the classical definition of Hecke operators.

Fact 1.12. For *any* p, $\widetilde{T}(p)$ is self-adjoint with respect to the Petersson inner product.

Proof. Suppose ϕ_1 and ϕ_2 share a central character. Then

$$\begin{aligned} \langle \widetilde{T}(p)\phi_1, \phi_2 \rangle &= \int_{Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} (\widetilde{T}(p)\phi_1)(g)\overline{\phi_2(g)}\, dg \\ &= \frac{1}{\mathrm{vol}(K_1(N))} \int_{Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} \left(\int_{H_p} \phi_1(gh)\, dh \right) \overline{\phi_2(g)}\, dg \\ &= \frac{1}{\mathrm{vol}(K_1(N))} \int_{H_p} \int_{Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} \phi_1(gh)\overline{\phi_2(g)}\, dg\, dh \end{aligned}$$

substitute g with gh^{-1}:

$$\begin{aligned} &= \frac{1}{\mathrm{vol}(K_1(N))} \int_{H_p} \int_{Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} \phi_1(g)\overline{\phi_2(gh^{-1})}\, dg\, dh \\ &= \frac{1}{\mathrm{vol}(K_1(N))} \int_{Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} \phi_1(g) \int_{H_p} \overline{\phi_2(gh^{-1})}\, dh\, dg \end{aligned}$$

substitute h with h^{-1}:

$$\begin{aligned} &= \frac{1}{\mathrm{vol}(K_1(N))} \int_{Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} \phi_1(g)\overline{\left(\int_{H_p} \phi_2(gh)\, dh \right)}\, dg \\ &= \int_{Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} \phi_1(g)\overline{(\widetilde{T}(p)\phi_2)(g)}\, dg \\ &= \langle \phi_1, \widetilde{T}(p)\phi_2 \rangle. \end{aligned}$$

Observe that both of the substitutions were group operations, and hence both measure-preserving and domain-preserving. This proves that $\widetilde{T}(p)$ is self-adjoint. □

2 Newforms and Oldforms

In the second part of this lecture, we will enhance our dictionary between the classical theory of modular forms and the adelic theory. In particular, a number of results in the classical setting are only valid for Hecke eigenforms or for newforms, so we need to find a way to interpret these conditions. An example of such a result is the classical multiplicity one theorem.

Now, let $\phi \in L^2_{\text{cusp}}(\mathrm{GL}_2(\mathbf{Q})\backslash\mathrm{GL}_2(\mathbf{A}), \psi)$. We could ask if it is of the form ϕ_f for $f \in S_k(N, \psi)$, but unlike in the classical setting, the level is not specified as part of the data of ϕ. Asking ϕ to be a cusp form requires that it have a finite-dimensional orbit under the right regular action of $K = K_{\text{fin}}K_\infty$, as well as be "$\mathfrak{z}$-finite,"[3] where $\mathfrak{z}$ is the center of the universal enveloping algebra of $\mathfrak{gl}_2$. But in the case that $\phi = \phi_f$ for $f \in S_k(N, \psi)$, we have that ϕ is actually invariant for K_0^N.

Our solution will be to define by hand the subspace of $L^2_{\text{cusp}}(\mathrm{GL}_2(\mathbf{Q})\backslash\mathrm{GL}_2(\mathbf{A}))$ corresponding to $S_k(N, \psi)$ and then prove that it's the right thing. More generally, let σ be some irreducible representation of K_∞^0 (the connected component of the identity of K_∞), let N be such that $\psi\colon \mathbf{Q}^\times\backslash\mathbf{A}^\times \to \mathbf{C}^\times$ factors through the projection of $\mathbf{Q}^\times\backslash\mathbf{A}^\times$ to $(\mathbf{Z}/N\mathbf{Z})^\times$, and let $\lambda \in \mathbf{C}$. Then we define:

$$H(\psi, \lambda, N, \sigma) = \Big\{\phi \in L^2_{\text{cusp}}(\mathrm{GL}_2(\mathbf{Q})\backslash\mathrm{GL}_2(\mathbf{A}), \psi) \mid \Delta\phi = \lambda\phi, \phi((\cdot)k_\infty k_f) = \sigma(k_\infty)\psi(k_0)\phi\Big\}$$

Here, as in Lecture 5, we think of ϕ as a left-$\mathrm{GL}_2(\mathbf{Q})$-invariant function on $\mathrm{GL}_2(\mathbf{A})$. Then Δ is the Laplacian operator acting on the archimedean place - this spans the center of the universal enveloping algebra of $\mathfrak{sl}_2 \subset \mathfrak{gl}_2$ and we regard it as acting on the infinite place (again in the distributional sense).

Then, we can see by construction that the image of the map $f \mapsto \phi_f$ lands inside

$$H_k(N, \psi) := H(\psi, -\frac{k}{2}(\frac{k}{2} - 1), N, \theta \mapsto e^{-ik\theta})$$

Indeed, as verified in Lecture 5, elements of $H_k(N, \psi)$ are actually *smooth functions* (see the comments on elliptic regularity in Proposition 3.3 and immediately thereafter), and this map is an *isomorphism* onto this subspace. The inverse map is given by defining a function f on the upper half-plane:

$$\phi\bigg(\Big(\begin{smallmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{smallmatrix}\Big)_\infty\bigg) = f(x + iy)y^{k/2}$$

In Lecture 5, we verified that if $\phi \in H_k(N, \psi)$, then $f \in S_k(N, \psi)$: the hard part was to verify that f is holomorphic.

Now, suppose that $dm \mid N$ and ψ factors through $(\mathbf{Z}/m\mathbf{Z})^\times$. Classically, we define an inclusion from $S_k(m, \psi)$ to $S_k(N, \psi)$ by sending $f(z)$ to $f(dz)$. Adelically, we will define $H_k(m, \psi) \to H_k(N, \psi)$ by:

$$\phi(g) \mapsto d^{-k/2}\phi\Big(\big(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\big)_\infty \cdot g\Big)$$

[3] Of course, the universal enveloping algebra only honestly acts on smooth functions, not on L^2 "functions," but one can still define $\mathfrak{z}$-finiteness in a distributional sense. For more details, see the Remark at the end of §4.1 in [6].

This allows us to define the space of *oldforms* by:

Definition 2.1. $H_k^{\text{old}}(N,\psi)$ is intersection inside $L^2_{\text{cusp}}(\text{GL}_2(\mathbf{Q})\backslash\text{GL}_2(\mathbf{A}),\psi)$ of $H_k(N,\psi)$ and the span of all right-translates (by $\text{GL}_2(\mathbf{A})$) of images of all $H_k(m,\psi)$ for all proper divisors $m \mid N$ under the maps defined above for all $d|(N/m)$.[4]

In the classical theory, the newforms are the orthogonal complement of the oldforms with respect to the Petersson inner product, so in order to define the appropriate adelic version of newforms, we need an analogue of the Petersson inner product on L^2_{cusp}. The inner product on L^2_{cusp} defining its structure as a Hilbert space recovers the Petersson inner product. In other words, if we write

$$\langle \phi_1, \phi_2 \rangle = \int_{Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} \phi_1(g)\overline{\phi_2(g)}\, dg$$

then for $\phi_i = \phi_{f_i}$, we have (see (5.12) and (5.13) of [1], as well as the Haar measure calculation in Lecture 3)

$$\langle \phi_1, \phi_2 \rangle = \int_{\Gamma_0(N)\backslash\mathfrak{h}^+} f_1(z)\overline{f_2(z)}y^{\frac{k_1+k_2}{2}}\,\frac{dx\,dy}{y^2}$$

We define $H_k^{\text{new}}(N,\psi) := (H_k^{\text{old}}(N,\psi))^{\perp}$ inside $H_k(N,\psi)$. In order to check that this recovers the classical notion of newforms, we need to check that if f is an oldform, then $\phi_f \in H_k^{\text{old}}(N,\psi)$.

There is likely a direct proof of this fact, but one can certainly deduce it from the action of the Hecke operators: for each prime p, we have the operator $T(p)$ on $S_k(N,\psi)$ and its adjoint $T(p)^*$ with respect to the Petersson inner product. These correspond to operators $\widetilde{T}(p)$ and $\widetilde{T}(p)^*$ on $H_k(N,\psi)$.

The key characterizing property to check in each case (classical and adelic) is that our candidate space of newforms ($S_k^{\text{new}}(N,\psi)$ or $H_k^{\text{new}}(N,\psi)$) is the following: if L (respectively $\widetilde{L}$, which is isomorphic) is the algebra generated by these operators for all p, then one can show $S_k^{\text{new}}(N,\psi)$ (resp. $H_k^{\text{new}}(N,\psi)$) is the largest L-invariant (resp. $\widetilde{L}$-invariant) subspace of $S_k(N,\psi)$ (resp. $H_k(N,\psi)$) on which L (resp. $\widetilde{L}$) is commutative and semisimple. Therefore, the isomorphism of $S_k(N,\psi)$ and $H_k(N,\psi)$ takes the subspace $S_k^{\text{new}}(N,\psi)$ to $H_k^{\text{new}}(N,\psi)$.

For this characterizing property in the adelic case, see Theorem A of [4]. This property in the classical case follows from standard facts about modular forms. For example, Corollary 4.6.21 of Miyake's book [5] implies that any simultaneous eigenfunction $f \in S_k(N,\psi)$ of all of L necessarily lies in $S_k^{\text{new}}(N,\psi)$. Hence, any subspace of $S_k(N,\psi)$ on which L is commutative and semisimple must be contained in $S_k^{\text{new}}(N,\psi)$. On the other hand, Theorem 4.6.13(2) *op. cit.* means that L indeed acts commutatively and semisimply on $S_k^{\text{new}}(N,\psi)$.[5]

We note that §5.B of [1] also contains an account of this material, but doesn't contain proofs.

3 L-functions

In this section, we continue the adelic/representation-theoretic description of the L-function of $f \in S_k^{\text{new}}(N,\psi)$. Rather than proving anything, we'll refer to representation-theoretic facts that will

[4]Membership in $H_k(N,\psi)$ does not interact well with the action of $\text{GL}_2(\mathbf{A})$, so this definition aims to include any right-translates of elements of $H_k(N,\psi)$ that unexpectedly also belong to $H_k(N,\psi)$.

[5]The adjective *primitive* in [5] means "normalized eigenform."

be proved later in this seminar. Conditional on these facts, we will have a representation-theoretic interpretation of classical properties of the L-function (e.g. its Euler product and functional equation). Towards the end of the seminar, our converse theorem (i.e. that of Jacquet–Langlands, as presented in [2]) will be stated and proved in representation-theoretic terms (then we'll be able to deduce classical consequences using this classical $\longleftrightarrow$ adelic dictionary). Last time (Lecture 5), we saw that

$$L(s+k/2, f) = \int_{\mathbf{A}^\times} W_{\phi_f}\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\psi^{-1}(y_0)|y|^s \, d^\times y$$

where $W_{\phi_f}(g) := \phi_1(g) = \int_{\mathbf{A}} \phi_f\big(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\big)\overline{\lambda_1(x)}\, dx$ (since ϕ_f is smooth and defines an element of L^2_{cusp}, the function W_{ϕ_f} is smooth), and $y = y_{\mathbf{Q}} y_\infty y_0$ for $y_{\mathbf{Q}} \in \mathbf{Q}^\times, y_\infty \in \mathbf{R}^\times, y_0 \in \mathbf{A}_f^\times$. Here, λ_1 is a particular character of $\mathbf{A}$ which is defined in Lecture 5.

If Π is the representation of $\mathrm{GL}_2(\mathbf{A})$ which is generated by $\phi_f \in L^2_{\text{cusp}}$ (under the right regular representation), then W is a "Whittaker functional" for Π. Indeed, we have:

(1) For $u = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in U$ where U is the unipotent radical of the upper triangular Borel subgroup, $W_\phi(u(\cdot)) = \lambda_1(t) W_\phi$. In the language of Lecture 6, this says that $W_\phi \in \mathrm{Ind}_U^G \mathbf{C}_{\lambda_1}$.

(2) $W_{g\cdot\phi} = g \cdot W_\phi$, where g acts on both sides by the right regular representation.

What does this $\mathrm{Ind}_U^G \mathbf{C}_\lambda$ mean in this setting? We define:

$$\mathscr{W} = \{W \colon \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C} \mid \substack{W\left(\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)(\cdot)\right) = \lambda_1(t) W \\ W(z(\cdot)) = \psi(z) W \\ \text{finiteness, smoothness, and growth conditions}} \}$$

Here, the extra conditions on W are the similar ones to those we impose on automorphic forms: we require finite-dimensional span under the right regular representation of $K = K_{\text{fin}} K_\infty$, smoothness at the archimedean place, and that $W\big(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)\big) = O(|x|^{-N})$ for all N as $|x| \to +\infty$ (note that W_{ϕ_f} satisfies this last condition by the decay properties of ϕ_f).

In Lecture 6, we saw "local multiplicity one", i.e. that irreducible representations have multiplicity one in the $\mathrm{Ind}_U^G \mathbf{C}_\lambda$. In fact, we will have a global version, saying roughly that global Whittaker models[6] are unique (up to scaling) and implying "global multiplicity one," i.e. each Hilbert-irreducible representation of $\mathrm{GL}_2(\mathbf{A})$ appears at most once in $L^2_{\text{cusp}}(\mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}), \psi)$. We will not prove these facts until the spring term; they appear in 3.4–5 of Godement's notes [2].

The upshot is that this formula for the L-function is "canonical". If f is a *cuspidal, new eigenform*, then it will turn out that $\Pi_f = \overline{\mathrm{GL}_2(\mathbf{A}) \cdot \phi_f}$ is irreducible (in the usual Hilbert space sense) and $\mathbf{C}\phi_f$ is distinguished inside as the unique line consisting of simultaneous eigenfunctions for $\widetilde{L}$ (in consideration of the eigenvalues of ϕ_f; see again Corollary 4.6.21 of [5]). Then by global multiplicity one, the line spanned by W_{ϕ_f} is also distinguished inside $\mathscr{W}$. Thus, up to a choice of scaling, $L(s, f)$ only depends on Π_f. In addition, if Π is any irreducible Hilbert representation of $\mathrm{GL}_2(\mathbf{A})$ inside of L^2_{cusp} (and we have a distinguished $v \in \Pi$), we may define $L(s, \Pi)$ by the same formula as above via a Whittaker model for Π.

Later on in the seminar, we will prove that a Hilbert-irreducible representation of $\mathrm{GL}_2(\mathbf{A})$ decomposes as a completed tensor product of irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ (including

[6]Using the global Hecke algebra $\mathscr{H}_{\mathbf{A}}$ (also to be defined later), the official definition of a *global Whittaker model* is a Hecke-equivariant inclusion of a smooth representation of $\mathscr{H}_{\mathbf{A}}$ into the space $\mathscr{W}$.

$p = \infty$). Granting this fact, it follows that such a decomposition usually does not hold for Hilbert representations of $\mathrm{GL}_2(\mathbf{A})$ that are not Hilbert-irreducible.

We now want to make some comments on the functional equation. Recall (from Lecture 5) that for $f \in S_k^{\text{new}}(N, \psi)$ we defined $L(s, f) = \int_0^\infty f(iy) y^s \, d^\times y$ (when $\mathrm{Re}(s) \gg 0$), and proved that:

$$L(s + k/2, f) = \int_{\mathbf{A}^\times} W_{\phi_f}\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \psi^{-1}(y_0) |y|^s \, d^\times y$$

From the definition of L in terms of the Mellin transform of f, it is clear that the integral defining $L(s, f)$ makes sense for s in some right half-plane of $\mathbf{C}$. However, it is *not obvious* where the functional equation comes from via this adelic perspective. Remember the original functional equation relating $L(s, f) = \epsilon L(k - s, f|_k W_N)$, where $W_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ and $\epsilon \in \mathbf{C}^\times$. As we will see much later (Lecture 23), the representation-theoretic functional equation takes the form

$$L(s, \Pi) = \epsilon_\pi(s) L(1 - s, \Pi^\vee),$$

where $\Pi^\vee$ is the contragredient of Π (also to-be-understood later as a completed tensor product of contragredients of the local factors). In [2], Godement arrives at this functional equation (along with the similar functional equations for the twists of Π by Grossencharacters) via multiplying together local functional equations at each place and dividing out the functional equation for an L-integral which is a variant of our adelic formula for the L-function. Again, this will all be discussed in detail later in the seminar, but see Theorem 4 in 3.7 (and its proof) of [2].

We will not explain how the epsilon factors arise in these local functional equations (the whole story, including this point, will be discussed fully later in the seminar). Instead, we briefly record the functional equation for the variant, just to illustrate the relevance of W_N from this perspective.

Let $L(g, s, \phi, \chi) := \int_{\mathbf{Q}^\times \backslash \mathbf{A}^\times} W_\phi\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) g\right) \chi^{-1}(y_0) |y|^{s - \frac{1}{2}} \, d^\times y$, again for s in some right-half plane. Then we get:

$$L(W_N g, 1 - s, \phi, \psi\chi^{-1}) = L(g, s, \phi, \chi)$$

Here, $\phi \in L^2_{\text{cusp}}$ with central character ψ and $\chi : \mathbf{Q}^\times \backslash \mathbf{A}^\times \to \mathbf{C}^\times$ is a character. Then the above equality follows from the identity of matrices $W_N \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} y & 0 \\ 0 & y \end{smallmatrix}\right) \left(\begin{smallmatrix} y^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) W_N$.

Remark 3.1. We mention in passing that in [3], a more general recipe for constructing zeta and L functions is given. Let Φ be a Schwartz-Bruhat function on $\mathrm{Mat}_n(\mathbf{A})$, and let $\varphi, \widetilde{\varphi}$ be in $\Pi, \Pi^\vee$ respectively. Then we may define a zeta integral (for $\mathrm{Re}(s) \gg 0$) via

$$Z(s, \Phi, \varphi, \widetilde{\varphi}) = \int_{\mathrm{GL}_n(\mathbf{A})} \Phi(g) \left(\int_{G(\mathbf{Q}) Z(\mathbf{A}) \backslash \mathrm{GL}_n(\mathbf{A})} \int \widetilde{\phi}(h) \phi(gh) \, dh \right) |\det g|^s \, dg.$$

These zeta integrals each obey a functional equation coming from an adelic Poisson summation formula. One can then deduce the functional equation for the L-function itself in a manner more in the spirit of Tate's thesis (and of Riemann); this theory is all developed in [3].

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