# AUTOMORPHIC FORMS ON $\mathrm{GL}_2$

This is an introductory course to modular forms, automorphic forms and automorphic representations.

(1) Modular forms
(2) Representations of $\mathrm{GL}_2(\mathbb{R})$
(3) Automorphic forms on $\mathrm{GL}_2(\mathbb{R})$
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This is a set of notes for my class "Automorphic forms on GL(2)" in the University of Chicago, Spring 2011. There is obviously no originality in the content and presentation of this very classical materials.

## 1. Modular forms

As usual in representation theory, the letter $G$ is overused . In each chapter, $G$ will denote a different group. In this chapter $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}_2(\mathbb{R})$, $\mathbf{H} = G/H$ is the upper half-plane, $\mathbf{D}$ is the open unit disc. $\Gamma$ will denote a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, $\bar{\Gamma}$ its image in $\mathrm{PGL}_2(\mathbb{R})$. In particular, $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ and $\bar{\Gamma}(1)$ is its image in $\mathrm{PGL}_2(\mathbb{R})$.

1.1. **Geometry of the upper half-plane.** The points of projective line are one-dimensional subspaces of a given two-dimensional vector space. The group $\mathrm{GL}_2$ of linear transformations of that two-dimensional vector space thus acts on the corresponding projective line. The action of a $2\times 2$-matrix is given the formula of homographic transformation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d} \tag{1.1.1}$$

if $z$ denotes the standard coordinate of $\mathbf{P}^1$. This formula is valid for any coefficients fields. In particular, $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathbf{P}^1(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{C})$ acts compatibly on $\mathbf{P}^1(\mathbb{C})$. It follows that $\mathrm{GL}_2(\mathbb{R})$ acts on the complement of the real projective line inside the complex projective line

$$\mathbf{P}^1(\mathbb{C}) - \mathbf{P}^1(\mathbb{R}) = \mathbf{H} \cup \mathbf{H}^-$$

where $\mathbf{H}$ (resp. $\mathbf{H}^-$) is the half-plane of complex number with positive (resp. negative) imaginary part. Let $\mathrm{GL}_2^+(\mathbb{R})$ denote the subgroup of $\mathrm{GL}_2(\mathbb{R})$ of matrices with positive determinant; it is also the neutral component of $\mathrm{GL}_2(\mathbb{R})$ with respect to the real topology. Since $\mathrm{GL}_2^+(\mathbb{R})$ is connected, its action on $\mathbf{P}^1(\mathbb{C}) - \mathbf{P}^1(\mathbb{R})$ preserves $\mathbf{H}$ and $\mathbf{H}^-$. Of course, the above assertion is a consequence of the formula

$$\Im\left(\frac{az+b}{cz+d}\right) = \frac{ad-bc}{|cz+d|^2}\Im(z). \tag{1.1.2}$$

which derives from a rather straightforward calculation

$$\begin{aligned} \frac{az+b}{cz+d} &= \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} \\ &= \frac{bd+acz\bar{z}+bc(z+\bar{z})+(ad-bc)z}{|cz+d|^2}. \end{aligned}$$

This equation becomes even simpler when we restrict to the subgroup $\mathrm{SL}_2(\mathbb{R})$ of real coefficients matrix with determinant one

$$\Im\left(\frac{az+b}{cz+d}\right) = \frac{\Im(z)}{|cz+d|^2}. \tag{1.1.3}$$

From now on in this chapter, we will set $G = \mathrm{SL}_2(\mathbb{R})$.

**Lemma 1.1.1.** *The group $G$ acts simply transitively on the upper half-plane $\mathbf{H}$. The isotropy group of the point $i \in \mathbf{H}$ is the subgroup $K = \mathrm{SO}_2(\mathbb{R})$ of rotations :*

$$k_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{1.1.4}$$

*Proof.* The equation

$$\frac{ai+b}{ci+d} = i$$

implies that $a = d$, $b = -c$ in which case the determinant condition $ad - bc = 1$ becomes $a^2 + b^2 = 1$. Thus the matrix is of the form (1.1.4).

Let $z = x+iy$ with $x \in \mathbb{R}$ and $y \in \mathbb{R}^+$. It is enough to prove that there exists $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ such that

$$\frac{ai+b}{ci+d} = z.$$

We set $c = 0$. We check immediately that the system of equations $ad = 1$, $a = yd$, $b = xd$ has real solutions with $d = y^{-1/2}$, $a = y^{1/2}$ and $b = xy^{-1/2}$. We observe that this calculation shows in fact $G = BK$ where $B$ is the subgroup of $G$ consisting of upper triangular matrices. This is a particular instance of the Iwasawa decomposition. □

**Lemma 1.1.2.** *The metric*

$$ds^2 = \frac{dx^2+dy^2}{y^2} \tag{1.1.5}$$

*on $\mathbf{H}$, as well as the density $\mu = dxdy/y^2$ is invariant under the action of $G$.*

*Proof.* With the notations $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $z' = \gamma z$, we have

$$dz' = \frac{(ad-bc)}{(cz+d)^2}dz. \tag{1.1.6}$$

This calculation has the following concrete meaning. The smooth application $g : \mathbf{H} \to \mathbf{H}$ maps $z \mapsto z'$. It induces linear application on tangent spaces $\mathrm{T}_z\mathbf{H} \to \mathrm{T}_{z'}\mathbf{H}$ and its dual linear application $\mathrm{T}^*_{z'}\mathbf{H} \to \mathrm{T}^*_z\mathbf{H}$. The cotangent space $\mathrm{T}^*_{z'}\mathbf{H}$ (resp. $\mathrm{T}^*_z\mathbf{H}$) is a one-dimensional $\mathbb{C}$-vector space generated by $dz'$ (resp. $dz$). The linear application sends $dz$ on $((ad-bc)/(cz+d)^2)dz$.

The element $dz$ induces the canonical quadratic form $dx^2 + dy^2$ on $\mathrm{T}_z\mathbf{H}$ viewed as 2-dimensional real vector space. Similarly, we have the quadratic form $dx'^2 + dy'^2$ on $\mathrm{T}_{z'}\mathbf{H}$. The equation (1.1.6) implies that

$$dx'^2 + dy'^2 = \frac{(cz+d)^2}{|cz+d|^4}(dx^2 + dy^2).$$

It follows that the metric $ds^2 = (dx^2 + dy^2)/y^2$ is invariant under $G$, according to (1.1.2). The same argument applies to the density $\mu = dxdy/y^2$. □

**Lemma 1.1.3.** *The Cayley transform*

$$z \mapsto cz = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} z = \frac{z-i}{z+i}. \tag{1.1.7}$$

*maps isomorphically* $\mathbf{H}$ *onto the unit disk* $\mathbf{D} = \{z \in \mathbb{C} \mid |z| < 1\}$*. The inverse transformation is*

$$w \mapsto c^{-1}w = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} w = \frac{i(1+w)}{1-w}. \tag{1.1.8}$$

*The metric* $ds^2 = (dx^2 + dy^2)/y^2$ *on* $\mathbf{H}$ *transports on the metric*

$$d_{\mathbf{D}}s^2 = \frac{4(du^2 + dv^2)}{(1-|w|^2)^2} \tag{1.1.9}$$

*where* $w = u + iv$*. We also have*

$$dxdy/y^2 = \frac{4dudv}{(1-|w|^2)^2}. \tag{1.1.10}$$

*Proof.* See [5, Lemma 1.1.2] Since $c$ and $c^{-1}$ are inverse functions of each other, it is enough to check that $c(\mathbf{H}) \subset \mathbf{D}$ and $c^{-1}(\mathbf{D}) \subset \mathbf{H}$. For every $z \in \mathbf{H}$, we have $|z - i| < |z + i|$ so that $|c(z)| < 1$. It follows that $c(\mathbf{H}) \subset \mathbf{D}$. For every $w \in \mathbf{D}$, the straightforward calculation

$$\frac{i(1+w)}{1-w} = \frac{-2\Im(w) + i(1-|w|^2)}{|1-w|^2} \tag{1.1.11}$$

shows

$$y = \frac{1-|w|^2}{|1-w|^2} > 0. \tag{1.1.12}$$

if $z = c^{-1}w$ and $y = \Im(z)$. It follows that $c^{-1}(\mathbf{D}) \subset \mathbf{H}$.

By using the chain rule we have

$$dz = \frac{2idw}{(1-w)^2}.$$

If we write $w = u + iv$ in cartesian coordinates, then we have

$$dx^2 + dy^2 = \frac{4(du^2 + dv^2)}{|1-w|^4}.$$

It follows that

$$\frac{dx^2 + dy^2}{y^2} = \frac{4(du^2 + dv^2)}{(1-|w|^2)^2}.$$

The same calculation proves the expression of the measure on the disc (1.1.10). □

**Lemma 1.1.4.** *Any two points of* $\mathbf{H}$ *are joined by a unique geodesic which is a part of a circle orthogonal to the real axis or a line orthogonal to the real axis.*

*Proof.* See [5, Lemma 1.4.1]. Instead of $\mathbf{H}$ we consider the unit disc. We assume that the first point is 0 and the second point is a positive real number $a < 1$. Let $\phi : [0,1] \to D$ with $\phi(t) = (x(t), y(t))$ denote a parametrized joining $0 = (x(0), y(0))$ and $a = (x(1), y(1))$. Its length is

$$\int_0^1 2(1 - |\phi(t)|^2)^{-1} \sqrt{(dx(t)/dt)^2 + (dy(t)/dt)^2} dt$$

that is at least

$$\int_0^1 2(1 - x(t)^2)^{-1} |dx(t)/dt| dt = \int_0^a \frac{2dt}{(1-t^2)}$$

The shortest curve joining 0 and $a$ is thus a part of a radius in the unit disc.

For every two points $x_0, x_1$, there is $g \in \mathrm{SL}_2(\mathbb{R})$ that maps $\mathbf{H}$ on $\mathbf{D}$, $cg(x_0) = 0$ and $cg(x_1) = a$ where $a$ is a positive real number satisfying $a < 1$. Here $c : \mathbf{H} \to \mathbf{D}$ is the Cayley transform. The geodesic joining $x_0$ with $x_1$ is a part of the preimage of the radius from 0 to $a$. That preimage is necessarily part of a circle or a strait line. Moreover as the transformation $cg$ is conformal, that circle or line must be orthogonal with the real line as the radius $[0, a]$ is orthogonal to the unit circle. □

**Exercice 1.1.5.** [2, Ex. 1.2.5] *Let* $\mathrm{SL}(2,\mathbb{C})$ *acts of* $bP^1(\mathbb{C})$ *by the homographic transformation* (1.1.1). *Prove that the subgroup that map the unit disc* $\mathbf{D}$ *onto itself is*

$$\mathrm{SU}(1,1) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \mid |a|^2 - |b|^2 = 1 \right\}. \tag{1.1.13}$$

*Prove that the subgroup* $\mathrm{SU}(1,1)$ *is conjugate to* $\mathrm{SL}(2,\mathbb{R})$ *in* $\mathrm{SL}(2,\mathbb{C})$. *Prove that the subgroup of* $\mathrm{SU}(1,1)$ *that fixes* $0 \in \mathbf{D}$ *is the rotation group*

$$\left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \right\}.$$

1.2. **Fuschian groups.** We will be mainly interested on the quotient of $\mathbf{H}$ by a discrete subgroup of $G$. The most important examples of discrete subgroups are the modular group $\mathrm{SL}_2(\mathbb{Z})$ and its subgroup of finite indices. We will call *Fuchsian group* a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.

**Proposition 1.2.1.** *A Fuchsian group* $\Gamma$ *acts properly on the upper half-plane* $\mathbf{H}$.

*Proof.* Recall that the action $\Gamma$ on $\mathbf{H}$ is proper means that the map $\Gamma \times \mathbf{H} \to \mathbf{H} \times \mathbf{H}$ defined by $(\gamma, x) \mapsto (x, \gamma x)$ is proper i.e. the preimage of a compact is compact. We need to prove that for every compact subsets $U, V \subset \mathbf{H}$, the set $\{\gamma \in \Gamma | \gamma U \cap V \neq \emptyset\}$ is a finite. Because the group $G = \mathrm{SL}_2(\mathbb{R})$ acts on $\mathbf{H}$ with compact stabilizer, the subset $\{g \in G | \gamma U \cap V \neq \emptyset\}$ is compact. Its intersection with the discrete subgroup $\Gamma$ is finite. □

**Corollary 1.2.2.** *For every Fuchsian group* $\Gamma$*, the quotient* $\Gamma\backslash\mathbf{H}$ *is a Haussdorf topological space.*

An element $g \in \mathrm{SL}_2(\mathbb{R})$ is called *elliptic* if it has a fixed point in $\mathbf{H}$. It follows from the above that $g$ is elliptic if and only if it is conjugate to an element of $\mathrm{SO}(2,\mathbb{R})$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Any element $g \in \mathrm{SL}_2(\mathbb{C})$ has at least on fixed point in $\mathbf{P}^1(\mathbb{C})$. If $g \in \mathrm{SL}_2(\mathbb{R})$ is not elliptic, it must have fixed points on $\mathbf{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$. We call $g$ *parabolic* if it has a unique fixed point, and *hyperbolic* if it has two fixed points. A parabolic element is conjugate to a matrix of the form

$$\begin{bmatrix} \epsilon & x \\ 0 & \epsilon \end{bmatrix} \tag{1.2.1}$$

with $\epsilon \in \{\pm 1\}$. A hyperbolic element is conjugate to a diagonal matrix

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \tag{1.2.2}$$

with $a \in \mathbb{R}^\times$.

Let $\Gamma$ be a Fuschian group. For every $x \in \mathbf{H}$, the stabilizer $\Gamma_x$ of $x$ in $\Gamma$ is a finite group because it is the intersection of a compact group with a discrete group. In fact, since $\mathrm{SO}_2(\mathbb{R})$ is isomorphic to the circle, for every $x \in \mathbf{H}$, $\Gamma_x$ is a finite cyclic group. If $\Gamma_x$ is nontrivial, we call $x$ an elliptic point of $\Gamma$. This also implies taht elliptic points are isolated.

**Lemma 1.2.3.** *There is a canonical complex structure on $\Gamma\backslash\mathbf{H}$ so that the quotient map $\mathbf{H} \to \Gamma\backslash\mathbf{H}$ is complex analytic.*

*Proof.* If $[z]$ is the $\Gamma$-orbit of $z \in \mathbf{H}$ such that $\Gamma_z = \Gamma \cap Z$. Then there exists a neighborhood $U$ of $z$ consisting of points with the same property. This neighborhood is homeomorphic with its image in $\Gamma\backslash\mathbf{H}$. Its image in $\Gamma\backslash\mathbf{H}$ is thus equipped with an analytic structure inherited from $U$.

If $z$ is the $\Gamma$-orbit of $z \in \mathbf{H}$ such that $\Gamma_x = \Gamma \cap Z$ such that $\Gamma_z$ is a finite group larger than $\Gamma \cap Z$. $\Gamma_z$ is then a finite cyclic group $\mu_k$. By a homographic transformation we can change the model from $\mathbf{H}$ to $\mathbf{D}$ and maps $z$ to 0. The quotient of a small disc around 0 by the action of $\mu_k$ can be given a complex structure with uniformizing parameter $w^d$ where $w$ is the standard uniformizing parameter of $\mathbf{D}$ around 0. This defines a complex structure on a neighborhood of $[z]$ in $\Gamma\backslash\mathbf{H}$. ☐

**Definition 1.2.4.** *A point $x \in \mathbf{P}^1(\mathbb{R})$ is called a* cusp *for $\Gamma$ if it is the fixed point of a nontrivial parabolic element.*

In that case $\Gamma_x$ is isomorphic to the product of $Z$ with an infinite cyclic group, the first factor $Z = \mu_2$ being the center of $G$. Let $P_\Gamma$ denote the set of cusps of $\Gamma$. We set

$$\mathbf{H}^* = \mathbf{H} \cup P_\Gamma.$$

We consider the topology on $\mathbf{H}^*$ by adding to the real topology on $\mathbf{H}$ a family of neighborhoods of each cusp $x \in P_\Gamma$. If $x = \infty$, we take the family

$$U_l^* = U_l \cup \{\infty\} \text{ with } U_l = \{z \in \mathbf{H} \mid \Im(z) > l\} \tag{1.2.3}$$

The family of neighborhoods of other cusps are constructed from the $U_l$ by conjugation.

**Lemma 1.2.5.** $\Gamma\backslash\mathbf{H}^*$ *is a Haussdorff space.*

*Proof.* See [5, Lemma 1.7.7]. As we already know that $\Gamma\backslash\mathbf{H}$ is Haussdorff, it remains to prove that a cusp and a point of $\mathbf{H}$ are separated and two cusps are separated that can be checked directly upon the definition. □

**Lemma 1.2.6.** *There is a complex analytic structure on* $\Gamma\backslash\mathbf{H}^*$ *that extends the complex analytic structure on* $\Gamma\backslash\mathbf{H}$.

*Proof.* We can restrict ourselves to the case that $\infty$ is a cusp and to define an analytic structure around the image of $\infty$ in $\Gamma\backslash\mathbf{H}^*$. The stabilizer of $\infty$ in $G$ is

$$G_\infty = \{\begin{bmatrix} a & b \\ 0 & a^{-1}\end{bmatrix} | a \in \mathbb{R}^\times, b \in \mathbb{R}\} \tag{1.2.4}$$

By definition, $\infty$ is a cusp of $\Gamma$ if $Z(\Gamma \cap G_\infty)$ is a subgroup of the form

$$\{\begin{bmatrix} \epsilon & mn \\ 0 & \epsilon\end{bmatrix} | \epsilon \in \{\pm 1\}, m \in \mathbb{Z}\} \tag{1.2.5}$$

for some fixed integer $n$. The map $z \mapsto e^{2i\pi z/n}$ defines a homeomorphism from $(G_\infty \cap \Gamma)\backslash U_l^*$ where $U_l^*$ is a standard neighborhood (1.2.3) of $\infty \in \mathbf{H}^*$ on a disc centered at 0. This provides $(G_\infty \cap \Gamma)\backslash\mathbf{H}^*$ with a complex analytic structure. □

**Definition 1.2.7.** *A discrete subgroup* $\Gamma$ *of* $\mathrm{SL}_2(\mathbb{R})$ *is called a Fuchsian group of first kind if* $X_\Gamma = \Gamma\backslash\mathbf{H}^*$ *is compact.*

**Proposition 1.2.8.** *If* $X_\Gamma$ *is compact, then the numbers of elliptic points and cusps of* $\Gamma$ *in* $\Gamma\backslash\mathbf{H}$ *are finite.*

**Theorem 1.2.9** (Siegel)**.** *A discrete subgroup of* $\mathrm{SL}_2(\mathbb{R})$ *is a Fuchsian group of first kind if and only if* $\Gamma\backslash\mathbf{H}$ *has finite area.*

*Proof.* We refer to [5, Theorem 1.9.1] for the proof of this theorem. We will only be interested in the case of arithmetic groups in which the conclusion of the theorem can be established directly by other means. □

A connected domain $F$ of $\mathbf{H}$ is called a fundamental domain of $\Gamma$ if $F$ satisfies the following conditions

(i) $\mathbf{H} = \bigcup_{\gamma\in\Gamma} \gamma F$ ;
(ii) if $U$ is the set of interior points of $F$ then $F = \bar{U}$ ;
(iii) $\gamma U \cap U = \emptyset$ for all $\gamma \in \Gamma$ not belonging to the center $Z$ of $G$.

**Lemma 1.2.10.** *Every Fuchsian group has a fundamental domain.*

*Proof.* An element $\gamma \in \Gamma - Z$ only has finitely many fixed points. Since $\Gamma$ is countable, there exists $z_0 \in \mathbf{H}$ which is not fixed by any element $\gamma \in \Gamma - Z$. For every $\gamma \in \Gamma$, we put

$$\begin{aligned} F_\gamma &= \{z \in \mathbf{H} \mid d(z, z_0) \leq d(z, \gamma z_0)\} \\ U_\gamma &= \{z \in \mathbf{H} \mid d(z, z_0) < d(z, \gamma z_0)\} \\ C_\gamma &= \{z \in \mathbf{H} \mid d(z, z_0) = d(z, \gamma z_0)\} \end{aligned}$$

Here, $d$ indicates the hyperbolic distance on $\mathbf{H}$ defined by the metric $ds^2 = (dx^2 + dy^2)/y^2$. The intersection

$$F = \bigcap_{\gamma\in\Gamma - Z} F_\gamma$$

is a fundamental domain of $\Gamma$. $\square$

We will now review the classification of $2 \times 2$ real matrices up to conjugation. A matrix is said to be :

(1) hyperbolic if its has distinct real eigenvalues;
(2) elliptic if it has distinct complex conjugate eigenvalues;
(3) parabolic if it is not central and and has an eigenvalue of multiplicity two;
(4) central otherwise.

**Lemma 1.2.11.** *Let $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ act on $\mathbf{P}^1(\mathbb{C})$ by homographic transformation. Let $z \in \mathbb{C}$ so that $\gamma z = z$. If $\gamma$ is hyperbolic (resp. elliptic) then $z$ is either rational or generates a real (resp. imaginary) quadratic extension of $\mathbb{Q}$. If $\gamma$ is parabolic then $z \in \mathbb{Q}$.*

Among the Fuschian groups, we are particularly interested in the modular group $\mathrm{SL}_2(\mathbb{Z})$ and its subgroups of congruence

$$\Gamma(N) = \{\gamma \in \mathrm{SL}(2,\mathbb{Z}) \mid \gamma \equiv 1 \mod N\} \tag{1.2.6}$$

$$\Gamma_1(N) = \{\gamma \in \mathrm{SL}(2,\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \mod N\} \tag{1.2.7}$$

$$\Gamma_0(N) = \{\gamma \in \mathrm{SL}(2,\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \mod N\} \tag{1.2.8}$$

In particular, we will use the convenient notation $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ for the full modular group.

We consider the case of the full modular group. Let $F$ denote the domain defined by the conditions $|\Re(z) \leq 1/2|$ and $|z| \geq 1$. Consider the two matrices of $\Gamma(1)$

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{1.2.9}$$

They act on $\mathbf{H}$ by the following rules $Tz = z + 1$ and $Sz = -1/z$.

**Lemma 1.2.12.** *Let $\Gamma'$ denote the subgroup of $\Gamma(1)$ generated by the transformations $S$ and $T$ as above.*

(1) *For every $z \in \mathbf{H}$, there exists $\gamma \in \Gamma'$ such that $\gamma z \in F$.*
(2) *If $z, z' \in F$ and $\gamma \in \Gamma(1)$ non trivial such that $\gamma z = z'$ then $z, z'$ both lies in the boundary of $F$.*
(3) *$F$ is a fundamental domain for $\Gamma(1)$, and $\Gamma(1)$ is generated by the matrices $S$ and $T$.*

*Proof.* For every $z \in \mathbf{H}$, the lattices generated 1 and $z$ have only finitely many members $cz + d$ such that $|cz + d| \leq 1$. It implies that there are only finitely many $z' = \gamma z$ conjugate to $z$ such that $\Im(z') \geq \Im(z)$. We can assume that $\Im(z)$ is maximal among all the conjugates $\gamma z$ with $\gamma \in \Gamma'$. With the help of the translation $T$, we can assume that $z$ belongs the the vertical strip $\Re(z) \leq 1/2$. We only need to prove that under these assumptions, we have $|z| \geq 1$. If $|z| < 1$, we would have $\Im(-1/z) > \Im(z)$ that would contradict the maximality of $\Im(z)$. It follows that $z \in F$.

Let $z, z' \in F$ and

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1) \text{ such that } z' = \gamma z.$$

We can assume that $\Im(z) \leq \Im(z')$. This implies that $|cz+d| \leq 1$. By a careful inspection, this implies in particular that $z$ must lie on the boundary of $F$. More inspection shows that $z'$ lie also on the boundary of $F$. See [6, p.130].

Let $z \in U$ be an element of the interior of $F$. For every $\gamma \in \Gamma$, there exists $\gamma' \in \Gamma'$ such that $\gamma'\gamma z \in F$. The assumption that $z$ lie in the interior of $F$ implies $\gamma'\gamma = 1$, thus $\gamma \in \Gamma'$. We have also checked all the conditions that makes $F$ a fundamental domain of $\Gamma(1)$. ☐

**Proposition 1.2.13.** *For $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$, the set of cusps is $\mathbb{Q} \cup \{\infty\}$. They are all conjugate under the action of $\Gamma(1)$.*

*The quotient $X_{\Gamma(1)} = \Gamma(1)\backslash\mathbf{H}^*$ is isomorphic to $\mathbf{P}^1(\mathbb{C})$ as complex analytic space. Up to equivalence, $\bar{\Gamma}(1)$ has one elliptic point of order 2 that is $i \in \mathbf{H}$ with $b^2 = -1$ and one elliptic point of order 3 that is $j \in \mathbf{H}$ with $j^3 = 1$.*

*Proof.* An element $x \in \mathbb{R}$ which fixed by a parabolic matrix $\gamma \in \Gamma(1)$ must be a rational. It follows from the shape of the fundamental domain $F$ of $\Gamma(1)$ that $\Gamma(1)\backslash\mathbf{H}^*$ is a compact Riemann surface that is homeomorphic to the sphere. It is isomorphic to $\mathbf{P}^1(\mathbb{C})$. ☐

**Corollary 1.2.14.** *Congruence subgroups are Fuchsian groups of first kind who set of cusps is $\mathbb{Q} \cup \{\infty\}$. There are only finitely many cusps up to the action of $\Gamma$.*

*Proof.* Since $\Gamma$ is a subgroup of $\Gamma(1)$ with finite index, they have the same set of cusps $\mathbb{Q} \cup \{\infty\}$. Since $\Gamma(1)$ acts transitively on this set, the number of $\Gamma$-orbits in this set is at most equal to the index of $\Gamma$ in $\Gamma(1)$.

Since $\Gamma(1)\backslash\mathbf{H}^*$ is compact, and $\Gamma$ is a subgroup of $\Gamma(1)$ of finite index, the quotient $\Gamma\backslash\mathbf{H}^*$ is also compact. Let $x_n$ be a sequence of points of $\Gamma\backslash\mathbf{H}^*$. We need to prove that there exists a convergent subsequence. Let $z_n$ be a sequence in $\mathbf{H}^*$ so that $x_n = \Gamma z_n$ is the image of $z_n$ in $\Gamma\backslash\mathbf{H}$. Let $\bar{x}_i$ denote the image of $z_n$ in $\Gamma(1)\backslash\mathbf{H}^*$. Since $\Gamma(1)\backslash\mathbf{H}^*$ is compact, we can assume that $\bar{x}_n$ converges the $\bar{x} \in \Gamma(1)\backslash\mathbf{H}^*$. Let $z \in \mathbf{H}^*$ be a preimage of $\bar{x}$. There exists $\gamma_n \in \Gamma(1)$ so that $\gamma_n z_n$ converges to $z$. Because $\Gamma/\Gamma(1)$ is finite, after extracting a subsequence, we can assume that there exist $\gamma \in \Gamma(1)$ so that $\gamma_n \in \gamma\Gamma(1)$ for all $n$. It follows that $x_n$ converges to $\gamma^{-1}x$ where $x$ is the image of $z$ in $\Gamma\backslash\mathbf{H}^*$. This proves that $\Gamma$ is a Fuschian group of first kind. ☐

**Proposition 1.2.15.** *Let $\bar{\Gamma}$ be a subgroup of $\bar{\Gamma}(1)$ of finite index $\mu$. Let $m_2, m_3$ be the number of $\bar{\Gamma}$-equivalence classes of elliptic points of orders 2 and 3 respectively. Let $m_\infty$ denote the number of $\bar{\Gamma}$-equivalence classes of cusps. Then the genus of $\bar{\Gamma}\backslash\mathbf{H}^*$ is*

$$g = 1 + \frac{\mu}{12} - \frac{m_2}{4} - \frac{m_3}{3} - \frac{m_\infty}{2}.$$

*Proof.* See [7, 1.40]. The map

$$\Gamma\backslash\mathbf{H}^* \to \Gamma(1)\backslash\mathbf{H}^*$$

is a finite proper map of degree $\mu = [\bar{\Gamma}(1) : \bar{\Gamma}]$.

This is an application of Hurwitz' formula. As the morphism $\Gamma\backslash\mathbf{H}^* \to \Gamma(1)\backslash\mathbf{H}^*$ is of degree $\mu$ and $\Gamma(1)\backslash\mathbf{H}^*$ is of genus 0, we have

$$2g - 2 = -2\mu + \sum_P (e_P - 1)$$

with $P$ in the set of ramified points, $e_P$ being the index of ramification. Summing $e_P$ over the ramified points $P$ over $j$ we get $2(\mu - m_3)/3$. The same sum over $i$ is $(\mu - m_2)/2$ and over $\infty$ is $\mu - m_\infty$. By summing altogether, we get the desired formula for the genus of $\bar{\Gamma}\backslash\mathbf{H}^*$. □

**Corollary 1.2.16.** *If $\bar{\Gamma}$ do not have elliptic points then the genus of $\bar{\Gamma}\backslash\mathbf{H}$ is*

$$g = 1 + \frac{\mu}{12} - \frac{m_\infty}{2}.$$

**Exercice 1.2.17.** [2, p.24]

(1) *Prove that a fundamental domain for $\Gamma(2)$ consists of $x+iy$ such that $-1/2 < x < 3/2$, $|z+1/2| > 1/2$, $|z-1/2| > 1/2$ and $|z-3/2| > 1/2$.*
(2) *Prove that $\Gamma(2)$ is generated by the matrices*

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

(3) *Prove that $\Gamma(2)$ has three inequivalent cusps and $\Gamma(2)\backslash\mathbf{H}^*$ is isomoprhic to $\mathbf{P}^1(\mathbb{C})$*
(4) *Prove that if $\phi$ is an entire function such that there exists two distinct complex numbers $a, b$ that don't belong to the image of $\phi$ then $\phi$ is a constant function (Picard's theorem).*

1.3. **Modular forms.** Let $k$ be an even nonnegative integer. A modular form of weight $k$ for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ is a holomorphic function on $\mathbf{H}$ which satisfies the identity

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \tag{1.3.1}$$

for all $z \in \mathbf{H}$ and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

and which is holomorphic at the cusp $\infty$. The last condition requires some discussion. We have define the analytic structure of $\Gamma\backslash\mathbf{H}^*$ near $\infty$ by choosing as the local coordinate the function $q = e^{2\pi i z}$. The equation 1.3.1 implies in particular $f(z+1) = f(z)$, and thus $f$ has a Fourier expansion

$$f(z) = \sum_{n\in\mathbb{Z}} a_n e^{2i\pi n z} = \sum_{n\in\mathbb{Z}} a_n q^n. \tag{1.3.2}$$

The function $f$ is holomorphic at the cusp $\infty$ if in the above expansion $a_n = 0$ for $n < 0$. If furthermore $a_0 = 0$, we say that $f$ is cuspidal at $\infty$.

If $\Gamma$ is a Fuschian group of first kind, we can also modular forms of weight $k$ for $\Gamma$ similarly. The holomorphic function $f$ on $\mathbf{H}$ is required to satisfy the same equation (1.3.1) and to be holomorphic at the cusps of $\Gamma$. If $x$ is a cusp, the stabilizer of $x$ in $\mathrm{PGL}_2(\mathbb{Z})$ is the infinite cyclic group generated by a parabolic element. After conjugation, we can assume that the cusp is the point $\infty$ and its stabilizer in $\Gamma$ is generated by the matrix

$$\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \tag{1.3.3}$$

for some positive real number $m \in \mathbb{R}$. The holomorphicity at this cusp is equivalent to that $f$ admits a Fourier expansion $f =_{n\in\mathbb{Z}} a_n q^n$ with $a_n = 0$ for $n < 0$ with respect to the variable $q = e^{2i\pi z/m}$.

**Lemma 1.3.1.** *Suppose that $\infty$ is a cusp of a Fuschian group $\Gamma$ of first kind with $\Gamma_\infty$ generated by the matrix (1.3.3). Let $f$ be a modular form of weight $k$ and let $\sum a_n q^n$ with $q = e^{2i\pi z/m}$ denote its Taylor expansion near this cusp. Then the series $\sum a_n q^n$ converges absolutely and uniformly on every compact in $\mathbf{H}$.*

*Proof.* The function $z \mapsto q = e^{2i\pi z/m}$ defines an isomorphism between $\Gamma_\infty\backslash\mathbf{H}$ and the punctured disc $\mathbf{D} - \{0\}$. By assumption modular form $f$ defines a holomorphic function on $\mathbf{D} - \{0\}$ that extends holomorphically to $\mathbf{D}$. This implies that the Taylor series $\sum_{n=0}^{\infty} a_n q^n$ converges absolutely uniformly on every compact contained in $\mathbf{D}$. □

**Definition 1.3.2.** *Let $\Gamma$ be a Fuschian group of first kind. We denote $M_k(\Gamma)$ the space of modular forms of weight $k$ for $\Gamma$. We denote by $S_k(\Gamma)$ the space of cusp forms of weight $k$ for $\Gamma$. We also denote $A_k(\Gamma)$ the space of meromorphic functions $f$ of $\mathbf{H}$ satisfying* (1.3.1) *that are meromorphic at the cusps.*

**Proposition 1.3.3.** *If $k = 0$, $A_0(\Gamma)$ is the field $F_\Gamma$ of meromorphic functions on $X_\Gamma$. We have $M_0(\Gamma) = \mathbb{C}$ and $S_0(\Gamma) = 0$.*

We will now determine the dimension of $M_k(\Gamma)$ and $S_k(\Gamma)$ for even integers $k$. We will refer to [7, 2.6] and [5] for the case of odd integers. The case $k = 1$ does not seem to be treated so far.

**Proposition 1.3.4.** *There is a canonical isomorphism between the space $A_{2k}(\Gamma)$ of meromorphic automorphic forms of weight $2k$ and the space $\Omega_{X_\Gamma}^{\otimes k} \otimes_{\mathcal{O}_{X_\Gamma}} F_\Gamma$ of meromorphic $k$-fold differential form on $X_\Gamma$. In particular, $A_{2k}(\Gamma)$ is a one-dimensional $F_\Gamma$-vector space.*

*Proof.* For every $f \in A_k(\Gamma)$, the $k$-fold differential form $f(z)(dz)^{\otimes k}$ is $\Gamma$-invariant. It descends to a meromorphic $k$-fold differential form $\omega_f$ on $\Gamma\backslash\mathbf{H}$. The condition of meromorphicity of $f$ at the cusps impies that $\omega_f$ is a meromorphic form on $X_\Gamma$. The application $f \mapsto \omega_f$ induces an isomorphism $A_{2k}(\Gamma) \to \Omega_{X_\Gamma}^{\otimes k} \otimes_{\mathcal{O}_{X_\Gamma}} F_\Gamma$. □

In order to calculate the dimension of $M_k(\Gamma)$, we will express the condition of holomorphicity of $f$ in terms of the zero divisor of $\omega_f$ on $X_\Gamma$ and then apply the theorem of Riemann-Roch. This calculation will be done separately in three cases : general points, elliptic points and cusps :

- Let $z \in \mathbf{H}$ be a non-elliptic point with image $x \in X_\Gamma$. The function $f$ is holomorphic at $z_0$ if and only if $\omega_f$ is holomorphic at $x$.
- Let $z \in \mathbf{H}$ be an elliptic point of index $e$ and let denote $x \in X_\Gamma$ its image. Let $t$ be a local parameter at $z \in \mathbf{H}$ and $u$ a local parameter at $x \in X_\Gamma$. We have $t^e \sim u$ where the equivalence means equal up to an invertible function on a neighborhood of $z$. By derivation, we have $du \sim z^{e-1}dz$. Raising to the power $k$, we have $(dz)^{\otimes k} \sim z^{-k(e-1)}(du)^{\otimes k}$. Let denote $\nu_x(f)$ the valuation of $f$ with respect to the parameter $u$ i.e. $\nu_x(u) = 1$ and $\nu_x(t) = 1/e$. Let us calculate the order of vanishing of the $k$ -fold differential form
$$\omega_f = f(dz)^{\otimes k} \sim f z^{-k(e-1)}(du)^{\otimes k}$$
at $x$. We have
$$\mathrm{ord}_x(\omega_f) = \nu_x(f) - k(1 - 1/e).$$
The function $f$ is holomorphic at $z$ if and only if $\nu_x(f) \geq 0$ which is equivalent to
$$\mathrm{ord}_x(\omega_f) + k(1 - 1/e) \geq 0.$$

Since $\mathrm{ord}_x(\omega_f)$ is an integer, the above inequality is equivalent to

$$\mathrm{ord}_x(\omega_f) + [k(1 - 1/e)] \geq 0$$

where as usual $[r]$ denotes the largest integer that is not greater than a given real number $r$. In the case of weight two form i.e. $k = 1$, the above condition means simply that $\mathrm{ord}_x(\omega_f) \geq 0$. In the general case the integer $e$ cannot be ignored.

- Let us consider a cusp of $\Gamma$ that we can assume to be $\infty$ without loss of generality. Let us denote $x$ its image in $\Gamma\backslash X_\Gamma$. The development of $f$ at the cusp has the form

$$f(z) = \sum_{n\in\mathbb{Z}} a_n q^n$$

where $q = 2i\pi mz$ for some positive integer $m$. Let $r$ be the least integer such that $a_r \neq 0$. We note $\nu_x(f) = r$. Let us denote $\omega_f = f dz^{\otimes k}$. Since $dz \sim dq/q$ we have $\omega_f \sim f q^{-k} dq^{\otimes k}$. By construction, $q$ is a local parameter of $X_\Gamma$ at $x$. It follows that

$$\nu_x(\omega_f) = \nu_x(f) - k.$$

Thus $f$ is holomorphic at the cusp $\infty$ i.e. $\nu_x(f) \geq 0$ if and only if

$$\nu_x(\omega_f) + k \geq 0$$

and $f$ vanishes at the cusp $\infty$ i.e. $\nu_x(f) \geq 1$ if and only if

$$\nu_x(\omega_f) + k \geq 1.$$

In wight two case $k = 1$, $f$ is holomorphic at $\infty$ if $\omega_f$ is a logarithmic one form and $f$ is a cusp form if and only if $\omega_f$ is a holomorphic one form.

Let denote $x$ its image in $\Gamma\backslash\mathbf{H}$ and let choose a local parameter $u$ of $x \in \Gamma\backslash\mathbf{H}^*$.

**Proposition 1.3.5.** *Let $\Gamma$ be a Fuschian group of first kind. The space $M_2(\Gamma)$ is canonically isomorphic with the space $\mathrm{H}^0(X_\Gamma, \Omega_{X_\Gamma}(cusp))$ of one form with logarithmic singularities at the cusps. The space $S_2(\Gamma)$ is canonically isomorphic with the space of holomorphic one form of $X_\Gamma$*

$$S_2(\Gamma) = \mathrm{H}^0(X_\Gamma, \Omega_{X_\Gamma}).$$

*In particular $\dim S_2(\Gamma) = g$ (calculated in 1.2.15) and $\dim M_2(\Gamma) = g + m - 1$ where $g$ is the genus of $X_\Gamma$ and $m$ is the number of inequivalent cusps.*

**Proposition 1.3.6.** *Let $\Gamma$ be a Fuschian group of first kind and let $2k$ be an even integer greater or equal to 4. We have*

$$\dim M_{2k}(\Gamma) = \dim S_{2k}(\Gamma) + m.$$

*and*

$$\dim S_{2k}(\Gamma) = (2k - 1)(g - 1) + \sum_{i=1}^{s} [k(1 - 1/e_i)] + (k - 1)$$

*where $x_1, \ldots, x_s$ denote the elliptic points, $e_1, \ldots, e_s$ their elliptic index and $m$ is the number of inequivalent cusps. In absence of elliptic points, we have*

$$\dim S_{2k}(\Gamma) = (2k - 1)(g - 1) + (k - 1)m.$$

In the case $\Gamma = \Gamma(1)$, we have $g = 0$, $m = 1$ and two elliptic points with indexes $\{2, 3\}$.

**Corollary 1.3.7.** *We have*

$$\dim S_{2k}(\Gamma(1)) = \begin{cases} 0 & \textit{if } k = 1 \\ [k/6] - 1 & \textit{if } k > 1 \textit{ and } k \equiv 1 \textit{ mod } 6 \\ [k/6] & \textit{otherwise} \end{cases}$$

*and*

$$\dim M_{2k}(\Gamma(1)) = \begin{cases} 0 & \textit{if } k = 2 \\ \dim S_{2k}(\Gamma(1)) + 1 & \textit{otherwise} \end{cases}$$

In particular $\dim M_4(\Gamma(1)) = \dim M_6(\Gamma(1)) = 1$. We can construct an explicit generator for these spaces by Eisenstein series. Let $2k$ be an even integer with $2k \geq 4$. Define

$$E_{2k}(z) = \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2 - (0,0)} (mz + n)^{-2k} \tag{1.3.4}$$

This series is absolutely uniformly convergent on compact domain and defines a holomorphic function on $\mathbf{H}$. This function is a modular form of weight $2k$ for the full modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. The automorphy (1.3.1) dervies from the action of $\mathrm{SL}(2, \mathbb{Z})$ on the set $\mathbb{Z}^2 - (0, 0)$. We will prove in 1.4.1 that Eisenstein are holomorphic at the cusp. In particular, it will be proved that the free coefficient in the Fourier expansion of $E_{2k}$ is $\zeta(2k)$. We will choose a normalization so that the free coefficient be one

$$G_{2k}(z) = \zeta(2k)^{-1} E_{2k}.$$

The space $M_4(\Gamma(1))$ is generated by $G_4$, $M_6(\Gamma(1))$ is generated by $G_6$, $M_8(\Gamma(1))$ is generated by $G_4^2$, $M_{10}(\Gamma(1))$ is generated by $G_4 G_6$. In weight 12 there is the first cusp form

$$\Delta = (G_4^3 - G_6^2)/1728.$$

**Proposition 1.3.8.** *The rational function $j : G_4^3/\Delta$ defines an isomorphism from $X_{\Gamma(1)}$ onto $\mathbf{P}^1_{\mathbb{C}}$*

*Proof.* By the discussion that precedes 1.3.5, there is a line bundle $\mathcal{L}$ on $X_\Gamma(1)$ such that $M_{12}(\Gamma(1)) = \mathrm{H}^0(X_{\Gamma(1)}, \mathcal{L})$. We know $\dim \mathrm{H}^0(X_{\Gamma(1)}, \mathcal{L}) = 2$. Since $X_{\Gamma(1)} = \mathbf{P}^1$, $\mathcal{L} = \mathcal{O}_{\mathbf{P}^1}(1)$. It follows that $G_4^3$ and $\Delta$ as global section of $\mathcal{L}$ vanish exactly at one point. Morever as they are not proportional, their quotient define a morphism

$$j : G_4^3/\Delta : X_{\Gamma(1)} \to \mathbf{P}^1_{\mathbb{C}}$$

that is an isomorphism. □

1.4. **Fourier coefficients of modular forms.** We have an explicit formula for Fourier coeffients of Eiseinstein series

**Proposition 1.4.1.** *The Fourier expansion of $E_k$ has the form*

$$E_k(z) = \zeta(z) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{1.4.1}$$

*where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.*

*Proof.* See [2, p.28] The terms with $m = 0$ in LHS sum up to the term $\zeta(z)$ in RHS. We have

$$\frac{1}{2} \sum_{n \in \mathbb{Z} - \{0\}} n^{-k} = \sum_{n \in \mathbb{N}} n^{-k} = \zeta(k).$$

In order to deal with the other terms, we will need the following lemma.

**Lemma 1.4.2.** *Let $k$ be an integer greater or equal to two. We have the formula*

$$\sum_{n \in \mathbb{Z}} (n - z)^{-k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n \in \mathbb{N}} n^{k-1} e^{2\pi i n z} \tag{1.4.2}$$

*for all $z \in \mathbf{H}$.*

*Proof.* See [2, p.12] for more details. For a fixed $z$, the function $f(x) = (x - z)^{-k}$ is a complex analytic function with a pole at $x = z$. On the real line, it has no pole if $\Im(z) > 0$ and it is $L^1$ if $k \geq 2$. Its Fourier transform is given by the formula

$$\hat{f}(y) = \int_{-\infty}^{\infty} (x - z)^{-k} e^{2i\pi xy} dx.$$

We can evaluate this integral by applying the residue formula to the 1-form $(x-z)^{-k} e^{2i\pi xy} dx$. We get

$$\hat{f}(y) = \begin{cases} 2\pi i \operatorname{res}_{x=z}((x - z)^{-k} e^{2\pi i x y} dx) & \text{if } y > 0 \\ 0 & \text{if } y \leq 0. \end{cases}$$

The calculation of the residue gives

$$\hat{f}(y) = \begin{cases} \frac{(2\pi i)^k}{(k-1)!} y^{k-1} e^{2\pi i y z} & \text{if } y > 0 \\ 0 & \text{if } y \leq 0. \end{cases}$$

We apply now the Poisson summation formula reviewed in B.2.2. □

In (1.3.4), the terms with a fix $m > 0$ and and thoese with its opposite are equal. By taking the factor $1/2$ into account, we only need to consider the terms with $m > 0$. Apply the above lemma to $mz$, we will get

$$\sum_{n \in \mathbb{Z}} (mz + n)^{-k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n \in \mathbb{N}} n^{k-1} e^{2\pi i m n z}. \tag{1.4.3}$$

If we sum the above formula over the positive integers $m$, we will get (1.4.1). □

**Corollary 1.4.3.** *Let $E_k(z) = \sum_{n=0}^{\infty} a_n q^n$ be the Fourier expansion of the Eisenstein series at $\infty$. There exists positive constants $A, B > 0$ such that $An^{k-1} \leq a_n \leq Bn^{k-1}$ for every $n \in \mathbb{N}$.*

*Proof.* By the formula (1.4.1), it is enough to seek such an estimation for $\sigma_{k-1}(n)$. In one side we have the obvious inequality $n^{k-1} \leq \sigma_{k-1}(n)$. On the other hand, we have the inequality

$$\frac{\sigma_{k-1}(n)}{n^{k-1}} = \sum_{d|n} \frac{1}{d^{k-1}} \leq \zeta(k-1)$$

valid for $k > 2$. □

**Proposition 1.4.4.** *If $f = \sum_{n=1}^{\infty} a_n q^n$ is a cusp form of weight $k$, its Fourier coefficients satisfy the inequality $a_n \leq C n^{k/2}$ for some constant $C$ independent of $n$.*

*Proof.* The equations (1.1.2) and (1.3.1) imply that the continuous function $|f(z)y^{k/2}|$ on $\mathbf{H}$ is $\Gamma$-invariant. It so defines a continuous function on the quotient $\Gamma\backslash\mathbf{H}$. Assume now $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Since $|q| = e^{-2\pi y}$, the vanishing of the constant terms of the Fourier expansion of $f$ in the variable $q$ implies that $\lim_{y\to\infty} |f(z)y^{k/2}| = 0$. It follows that the function $|f(z)y^{k/2}|$ is bounded by a constant $C_1$. For every natural integer $n \in \mathbb{N}$, and for every $y > 0$, we have

$$|a_n| e^{-2\pi n y} = |\int_0^1 f(x+iy) e^{-\pi i n x} dx| \leq C_1 y^{-k/2}.$$

Let us pick $y = 1/n$ and derive the inequality

$$|a_n| < C n^{k/2}$$

with $C = e^{2\pi} C_1$. □

This bound can be improved according to the Ramanujan-Peterson conjecture.

**Theorem 1.4.5.** *Let $f = \sum_{n=1}^{\infty} a_n q^n$ is a cusp form of weight $k$ of level $N$. Then for $(n, N) = 1$, we have $a_n = O(n^{\frac{k-1}{2}})$.*

This conjecture was proved by Eichler, Shimura and Igusa in the case $k = 2$. The proof in the $k > 2$ is due to Deligne. It is based on the Eichler-Shimura relation and the Weil conjecture.

Among cusp forms, the eigenvectors with respect to the Hecke operators that we will later introduce, have Fourier coefficients with arithmetic significance. In particular, since the space $S_{12}(\Gamma(1))$ of cusp forms of weight 12 for $\Gamma(1)$ is one dimensional, its generator is automatically an eigenvector. The $\Delta$ function of Ramanujan $\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$ is a normalized cusp form whose Fourier coefficients are integers. Deligne proved the inequality

$$|\tau(p)| \leq 2p^{11/2}$$

that is the original conjecture of Ramanujan.

1.5. ***L*-function attached to modular forms.** If $f \in M_k(\Gamma)$ is a modular form with Fourier expansion $f = \sum_{n=1}^{\infty} a_n q^n$. We call the Dirichlet series

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1.5.1}$$

The bounds on the Fourier coefficients 1.4.3 and 1.4.4 implies that this Dirichlet series converges on a half-plane. We also consider the complete $L$-function

$$\Lambda(s, f) = (2\pi)^{-s} \Gamma(s) L(s, f). \tag{1.5.2}$$

Hecke's theory takes a rather simple form in the case of the full modular group $\Gamma(1)$.

**Proposition 1.5.1.** *Suppose that $f$ is a modular form of weight $k$ for $\Gamma(1)$. If $f$ is a cusp form, $\Lambda(s, f)$ extends to an analytic function of $s$, bounded on vertical strips. If $f$ is not a cusp form, then $\Lambda(s, f)$ extends to a meromorphic function with simple poles $s = 0$ and $s = k$. It satisfies the functional equation*

$$\Lambda(s, f) = i^k \Lambda(k - s, f). \tag{1.5.3}$$

*Proof.* We will restrict ourselves to the case of a cusp form for the full modular group. Because $f$ is cuspidal, $f(iy) \to 0$ vary rapidly as $y \to \infty$. We use the automorphy equation (1.3.1) for the element

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \Gamma(1)$$

and derive the equality

$$f(iy) = i^k y^{-k} f(i/y) \tag{1.5.4}$$

It follows that $f(iy) \to 0$ very rapidly as $y \to 0$ too. It follows that the integral

$$\int_0^\infty f(iy) y^s \frac{dy}{y} \tag{1.5.5}$$

is convergent for all $s$ and defines an analytic function of $s$.

The following Mellin integral is absolutely convergent for $\Re(s) > \nu + 1$

$$\begin{aligned}
\int_0^\infty f(iy) y^s \frac{dy}{y} &= \int_0^\infty \sum_1^\infty a_n e^{-2n\pi y} y^s \frac{dy}{y} && (1.5.6)\\
&= \sum_1^\infty a_n (2n\pi)^{-s} \int_0^\infty e^{-y} y^s \frac{dy}{y} && (1.5.7)\\
&= (2\pi)^{-s} \Gamma(s) \sum_1^\infty a_n n^{-s} && (1.5.8)\\
&= \Lambda(s, f) && (1.5.9)
\end{aligned}$$

The exchange of the integration and infinite series is licit because the series $\sum_1^\infty a_n e^{-2n\pi y}$ is absolutely convergent as well as $\sum_1^\infty a_n n^{-s}$. It follows that the expression 1.5.5 defines an analytic continuation of $\Lambda(s, f)$.

The functional equation (1.5.3) derives from the substitution of $y$ by $1/y$ in (1.5.5). □

The converse theorem is also easy in the case of $\Gamma(1)$.

**Proposition 1.5.2.** *Let $a_1, a_2 \ldots$ be a sequence of complex numbers which is $O(n^\nu)$ for some positive real number $\nu$. Let $L(s, f)$ be defined by the series* (1.5.1)*, convergent for $\Re(s) > \nu + 1$. Let $\Lambda(s, f)$ be the function defined by* (1.5.2) *and suppose it has an analytic continuation for the complex plan of $s$ which is bounded in any vertical strips. Assume that $\Lambda(s, f)$ satisfies the functional equation* (1.5.3) *for some positive integer $k$.*

*Then $f(z) = \sum_{n=1}^\infty a_n q^n$ is a cusp form of weight $k$ for $\Gamma(1)$.*

*Proof.* The assumption $a_n = O(n^\nu)$ implies that the series $\sum_{n=1}^\infty a_n q^n$ is absolutely for $|q| < 1$ or over the half-plane $z = \mathbf{H}$ if $q = e^{2\pi i z}$. The Mellin integral

$$\begin{aligned}
\int_0^\infty f(iy) y^s \frac{dy}{y} &= \int_0^\infty \sum_1^\infty a_n e^{-2n\pi y} y^s \frac{dy}{y} && (1.5.10)\\
&= \sum_1^\infty a_n (2n\pi)^{-s} \int_0^\infty e^{-y} y^s \frac{dy}{y} && (1.5.11)\\
&= \Lambda(s, f) && (1.5.12)
\end{aligned}$$

is absolutely convergent for $\Re(s) > \nu+1$. Let $\sigma$ be a positive real number such that $\sigma > \nu+1$. The inverse Mellin transform will permit us to recover the value of $f$ on the imaginary axis $i\mathbb{R}$ from $\Lambda(s, f)$

$$f(iy) = \frac{1}{2\pi i}\int_{t=-\infty}^{\infty} \Lambda(\sigma + it, f) y^{-\sigma-it} dt. \tag{1.5.13}$$

The convergence of this integral follows from the absolute convergence of the Dirichlet series and the Stirling formula for the Gamma function:

$$\Gamma(s) \sim \sqrt{2\pi} e^{-s} s^{s-1/2} \tag{1.5.14}$$

as $|s| \to \infty$ and $\Re(s) \geq \delta > 0$. On the vertical line $s = \sigma + it$ for fixed $\sigma > 0$, we have

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}|t|^{\sigma-1/2} e^{-\pi|t|/2}$$

as $|t| \to \infty$.

With the functional equation (1.5.3), we can transform (1.5.13) as follows

$$f(iy) = i^k \frac{1}{2\pi i}\int_{t=-\infty}^{\infty} \Lambda(k - \sigma - it, f) y^{-\sigma-it} dt \tag{1.5.15}$$

$$= i^k y^{-k} \frac{1}{2\pi i}\int_{t'=-\infty}^{\infty} \Lambda((k - \sigma) + it', f) y^{k-\sigma+it'} dt' \tag{1.5.16}$$

after the change of variable $t' = -t$. We note that the line $\Re(s) = k - \sigma$ is out of the convergence domain on the Dirichlet series and we would like to move back the line of integration to the domain of convergence of the Dirichlet series. In order to apply Cauchy theorem, we need to prove that $\Lambda(x + iy, f) \to 0$ as $y \to \pm\infty$ uniformly when $x$ varies in a compact set. For a fixed $x > \sigma$, this follows again from the absolute convergence of the Dirichlet series and the Stirling formula for the Gamma function.For a fixed $x << 0$, we obtain the same convergence $\Lambda(x + iy, f) \to 0$ as $y \to \pm\infty$ by using the functional equation (1.5.3). The uniform convergence follows from an application of the Phragmén-Lindelof principle.

**Proposition 1.5.3.** *Let $f(s)$ be a holomorphic function on the upper part of a strip defined by the inequalities*

$$\sigma_1 \leq \Re(s) \leq \sigma_2 \text{ and } \Im(s) > c.$$

*Suppose that $f(\sigma + it) = O(e^{t^\alpha})$ for some real number $\alpha > 0$. Suppose that for some $M \in \mathbb{R}$, $f(\sigma + it) = O(t^M)$ for $\sigma = \sigma_1$ or $\sigma = \sigma_2$. Then $f(\sigma + it) = O(t^M)$ uniformly in $\sigma \in [\sigma_1, \sigma_2]$.*

*Proof.* See [5, p.118] and [10, p.124]. □

We can now apply the Cauchy theorem and move back the integration line to $\Re(s) = \sigma$

$$f(iy) = i^k y^{-k} \frac{1}{2\pi i}\int_{t=-\infty}^{\infty} \Lambda(\sigma + it, f) y^{\sigma+it} dt \tag{1.5.17}$$

and apply again the inverse Mellin transform

$$f(iy) = i^k y^{-k} f(iy^{-1}). \tag{1.5.18}$$

This is the automorphy relation (1.3.1) with respect to the element $S$ of (1.2.9). The automorphy relation for $T$ requires no proof since $f$ is defined as a Fourier series. As $S$ and $T$ generates the full modular group $\Gamma(1)$, $f$ is a cusp form of weight $k$ for $\Gamma(1)$. □

The case of a general congruence group $\Gamma \subset \Gamma(1)$ is considerably more complicated, mainly because the matrix $S$ fails to belong to $\Gamma$. As we will see later 1.7, it it enough to restrict to congruence subgroup of the form $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N)$ with

$$\Gamma(N) = \{\gamma \in \mathrm{SL}(2,\mathbb{Z}) \mid \gamma \equiv 1 \mod N\} \tag{1.5.19}$$

$$\Gamma_1(N) = \{\gamma \in \mathrm{SL}(2,\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \mod N\} \tag{1.5.20}$$

$$\Gamma_0(N) = \{\gamma \in \mathrm{SL}(2,\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \mod N\} \tag{1.5.21}$$

For every $N$, there is an exact sequence

$$1 \to \Gamma_1(N) \to \Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times \to 1.$$

It follows that the group $(\mathbb{Z}/N\mathbb{Z})^\times$ acts on the space of modular forms $M_k(\Gamma_1(N))$ as well as the space of cusp forms $S_k(\Gamma_1(N))$. For every primitive character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, we will denote

$$M_k(N,\chi) = \{f \in M_k(\Gamma_1, N) \mid cf = \chi(c)f \, \forall c \in (\mathbb{Z}/N\mathbb{Z})^\times\}. \tag{1.5.22}$$

Similar notation $S_k(N,\chi)$ for cusp forms will also prevail. We will call these forms modular forms of level $N$ of nebentypus $\chi$.

We will need the following lemma that replaces the role of the element $S \in \Gamma(1)$.

**Lemma 1.5.4.** *The matrix*

$$S_N = \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix} \tag{1.5.23}$$

*normalizes $\Gamma_0(N)$. Furthermore, it transforms the space $S_k(N,\chi)$ into $S_k(N,\bar\chi)$ where $\bar\chi$ is the opposite Dirichlet character of $\chi$.*

**Theorem 1.5.5** (Hecke)**.** *Let $f(z) = \sum_{n=0}^\infty a_n q^n$ and $g(z) = \sum_{n=0}^\infty b_n q^n$ where $q = e^{2\pi i z}$ and $a_n, b_n$ are $O(n^\nu)$ for some real number $\nu$. For positive integers $k$ and $N$, the following conditions are equivalent*

(A) $g(z) = (-i)^k N^{k/2} z^{-k} f(-1/Nz)$.

(B) *Both $\Lambda_N(s,f) = (2\pi/\sqrt{N})^{-s}\Gamma(s)L(s,f)$ and $\Lambda_N(s,g) = (2\pi/\sqrt{N})^{-s}\Gamma(s)L(s,g)$ can be analytically continued to the whole $s$-plane, satisfy the functional equation*

$$\Lambda(s,f) = \Lambda(k-s,g) \tag{1.5.24}$$

*and*

$$\Lambda_N(s,f) + \frac{a_0}{s} + \frac{b_0}{k-s}$$

*is holomorphic on the $s$-plane and bounded on any vertical strip.*

See [5, 4.3.5] for proof.

**Theorem 1.5.6** (Weil)**.** *Let $N$ be a positive integer and $\chi$ be a Dirichlet character modulo $N$. Suppose $a_n, b_n$ are sequences of complex numbers such that $a_n, b_n = O(n^\nu)$ for some real number $\nu$. If $D$ is positive integer number, relatively prime to $N$, and if $\mu$ is a primitive Dirichlet character modulo $D$, we consider the Dirichlet series $L_a(s,\mu) = \sum_{n=0}^\infty \mu(n)a_n n^{-s}$ and $L_b(s,\bar\mu) = \sum_{n=0}^\infty \bar\mu(n) b_n n^{-s}$. Let $\Lambda_a(s,\mu) = (2\pi)^{-s}\Gamma(s)L_a(s,\mu)$ and $\Lambda_b(s,\bar\mu) = (2\pi)^{-s}\Gamma(s)L_b(s,\bar\mu)$.*

*Let $S$ be a set of primes including those dividing $N$. Assuming that the conductor $D$ of $\mu$ is either 1 or a prime $D \notin S$, $\Lambda_a(s,\mu)$ and $\Lambda_b(s,\bar{\mu})$ have analytic continuation to the whole s-plane, are bounded in every vertical strips, and satisfy the functional equation*

$$\Lambda_a(s,\mu) = i^k \mu(N)\chi(D)\frac{\tau(\mu)^2}{D}(D^2N)^{-s+\frac{k}{2}}\Lambda_b(k-s,\bar{\mu}) \tag{1.5.25}$$

*Then $f(z) = \sum_{n=0}^{\infty} a_n q^n$ is a modular form of level $N$ with nebentypus $\chi$ i.e. $f \in M_k(N,\chi)$.*

1.6. **Hecke operators and Euler product.** It is sometimes convenient to express the automorphy condition (1.3.1) in terms of group action. For every matrix with positive determinant

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})^+,$$

we define the right action of $\gamma$ on a function $f : \mathbf{H} \to \mathbb{C}$ by the transformation rule

$$(f \mid_k \gamma)(z) = \det(\gamma)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) \tag{1.6.1}$$

A straightforward calculation shows that $(f \mid_k \alpha) \mid_k \beta = f \mid_k (\alpha\beta)$. Thus this transformation rule is indeed a right action of $\mathrm{GL}_2^+(\mathbb{R})$ on the space of holomorphic functions on $\mathbf{H}$.

With this definition, a modular form of weight $k$ with respect to a Fuschian group $\Gamma$ is a holomorphic function $f$ on $\mathbf{H}$ such that $f \mid_k \gamma = f$ for all $\gamma \in \Gamma$ and which satisfies a growth condition near the cusps. It also follows from this definition that the algebra of double cosets of $\Gamma$ in $\mathrm{GL}_2^+(\mathbb{Q})$ acts naturally on the space $M_k(\Gamma)$ as well as $S_k(\Gamma)$.

The construction of the algebra of double cosets of a congruence subgroup $\Gamma$ in $\Sigma = \mathrm{GL}_2^+(\mathbb{Q})$ relies on the following property.

**Lemma 1.6.1.** *Let $\Gamma$ be a congruence subgroup and $\alpha \in \Sigma$. Then $\Gamma \cap \alpha\Gamma\alpha^{-1}$ is a subgroup of $\Gamma$ with finite index.*

This lemma can be reformulated in the following way : Each double coset $\Gamma\alpha\Gamma$ in $\Sigma$ is a finite union of right cosets or left cosets

$$\Gamma\alpha\Gamma = \bigsqcup_i \Gamma\alpha_i \tag{1.6.2}$$

where $i$ runs over a finite set of indexes. It is convenient to see double cosets as notion of relative position between two let cosets. We will say that the two cosets $\Gamma\alpha_1$ and $\Gamma\alpha_2$ are in position $\alpha$ if $\alpha_1\alpha_2^{-1} \in \Gamma\alpha\Gamma$. Lemma 1.6.1 can be reformulated in an yet another way :

**Lemma 1.6.2.** *For each left coset $\Gamma\alpha_1$ and each double coset $\Gamma\alpha\Gamma$, there are only finitely many left cosets $\Gamma\alpha_2$ so that $\Gamma\alpha_1$ and $\Gamma\alpha_2$ are in position $\Gamma\alpha\Gamma$.*

Let $\mathcal{H}_\Gamma$ denote the free abelian group generated by a basis indexed by the set of double cosets of $\Gamma$ in $\Sigma$. We simple write $T_\alpha \in \mathcal{H}_\Gamma$ for the element in this basis indexed by the doubl coset $\Gamma\alpha\Gamma$. We define a multiplication on $\mathcal{H}_\Gamma$ by the following rule

$$T_{\alpha_1}T_{\alpha_2} = \sum_\alpha c^\alpha_{\alpha_1,\alpha_2} T_\alpha \tag{1.6.3}$$

where $c^\alpha_{\alpha_1,\alpha_2}$ is the number of left coset $\Gamma\beta$ such that $\Gamma\beta_1$ and $\Gamma\beta$ are in position $\alpha_1$ and $\Gamma\beta$ and $\Gamma\beta_2$ are in position $\alpha_2$, here $\Gamma\beta_1$ and $\Gamma\beta_2$ are fixed cosets in position $\alpha$. The finiteness of the numbers $c^\alpha_{\alpha_1,\alpha_2}$ follows immediately from 1.6.2.

We define the action of $\Gamma\alpha\Gamma = \bigsqcup_i \Gamma\alpha_i$ on $M_k(\Gamma)$ by the formula

$$f \mid_k T_\alpha = \sum_i f \mid_k \alpha_i. \tag{1.6.4}$$

This formula defines an action of the Hecke algebra $\mathcal{H}_\Gamma$ on the space of modular forms $M_k(\Gamma)$. This action preserves the subspace of cusp forms $S_k(\Gamma)$.

We consider the Hecke algebra $\mathcal{H}_{\Gamma(1)}$ with respect to the full modular group $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ as subgroup of $\Sigma = \mathrm{GL}_2(\mathbb{Q})^+$. The double cosets of $\Gamma(1)$ be described explicitly by the theory of elementary divisors.

**Proposition 1.6.3.** *Each double coset of $\Gamma(1)$ in $\mathrm{GL}_2(\mathbb{Q})^+$ contains a unique diagonal matrix of the form*

$$\alpha(d_1, d_2) = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \tag{1.6.5}$$

*with $d_1, d_2 \in \mathbb{Q}^+$ such that $d_1 d_2^{-1}$ is an integer.*

*Proof.* This can be best explained in term of relative position between lattices. A lattice of $\mathbb{Q}^2$ is a free abelian group of rank two contained in $\mathbb{Q}^2$. The map $\alpha \mapsto \alpha^{-1}(\mathbb{Z}^2)$ defines a bijection between the set of left coset $\Gamma(1)\alpha$ in $\Sigma = \mathrm{GL}_2^+(\mathbb{Q})$ into the set of lattices of $\mathbb{Q}^2$. If $L, L' \in \mathbb{Q}^2$ are two lattices, the theorem of elementary divisors assert that there exist a basis $\{x_1, x_2\}$ of $L$ such that $\{d_1 x_1, d_2 x_2\}$ is a basis of $L'$ where $d_1, d_2$ are well defined positive rational numbers such that $d_1 d_2^{-1}$ is an integer. □

**Proposition 1.6.4.** *The algebra $\mathcal{H}_{\Gamma(1)}$ is commutative.*

*Proof.* The transposed matrix $g \mapsto g^\top$ being an anti-homomorphism of $\Sigma$, it induces an anti-homomorphism on $\mathcal{H}_{\Gamma(1)}$. On the other hand, as it fixed the diagonal matrices $\alpha(d_1, d_2)$, it induces identity on $\mathcal{H}_{\Gamma(1)}$. This means that $\mathcal{H}_{\Gamma(1)}$ is a commutative algebra. □

The Hecke operators are self-adjoint with respect to the Peterson inner product on $S_K(\Gamma(1))$. Recall that this inner product is defined by the integral

$$\langle f, g \rangle = \int_{\Gamma(1)\backslash \mathbf{H}} f(z)\bar{g}(z) y^k \frac{dxdy}{y^2}. \tag{1.6.6}$$

Here the invariant $dxdy/y^2$ on $\mathbf{H}$ and defines a measure on the quotient $\Gamma(1)\backslash\mathbf{H}$, the expression $f(z)\bar{g}(z)y^k$ is also invariant under $\Gamma(1)$ and defines a function on $\Gamma\backslash\mathbf{H}$. Since $f, g$ are cusp forms, $f(z)\bar{g}(z)y^k$ tends to zero near the cusps and thus is bounded function on $\Gamma\backslash\mathbf{H}$. Now the Peterson inner product is well defined since $\Gamma(1)\backslash\mathbf{H}$ has finite measure with respect to $dxdy/y^2$.

**Theorem 1.6.5.** *The action of $\mathcal{H}_{\Gamma(1)}$ on $S_k(\Gamma(1))$ can be simultaneously diagonalized.*

*Proof.* The Hecke operators are self-adjoint with respect to to the Peterson inner product. A commutative algebra of self-adjoint operators on a finite dimensional vector space can be simultaneously diagonalized. □

We will next single out a particular family of Hecke operators that are relevant to $L$-functions. For every $n \in \mathbb{N}$, we define the Hecke operator $T(n) \in \mathcal{H}_{\Gamma(1)}$ by the following formula

$$T(n) = \sum_{\substack{d_1,d_2\in\mathsf{N}, d_2|d_1\\ d_1d_2=N}} T_{\alpha(d_1,d_2)}. \tag{1.6.7}$$

The corresponding union of double cosets

$$\mathcal{T}(n) = \bigsqcup_{\substack{d_1,d_2\in\mathsf{N}, d_2|d_1\\ d_1d_2=N}} \Gamma(1)\alpha(d_1,d_2)\Gamma(1) \tag{1.6.8}$$

is the set of integral matrices of determinant $n$

$$\mathcal{T}(n) = \{\alpha \in \mathrm{Mat}_2(\mathbb{Z}) \mid \det(\alpha) = n\}. \tag{1.6.9}$$

**Proposition 1.6.6.** *Let $f = \sum_{m=1}^{\infty} a_m q^m$ be a cusp form of weight $k$ for $\Gamma(1)$. Let $f \mid_k T(n) = \sum_{m=1}^{\infty} b_m q^m$ be the Fourier development of $T(n)f$. We have*

$$b_m = \sum_{ad=n, a|m} n^{\frac{k}{2}} d^{-k+1} a_{\frac{md}{a}}. \tag{1.6.10}$$

*In particular*

$$b_1 = n^{-\frac{k}{2}+1} a_n. \tag{1.6.11}$$

*Proof.* We can make explicit the action of $T(n)$ on modular forms by decomposing $\mathcal{T}(n)$ in left cosets of $\Gamma(1)$

$$\mathcal{T}(n) = \bigsqcup_{\substack{a,b,d\in\mathbb{N}\\ ad=n, 0\le b<d}} \Gamma(1) \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}. \tag{1.6.12}$$

Thus

$$f \mid_k T(n) = \sum_{ad=n} \sum_{0\le b<d} n^{k/2} d^k f\left(\frac{az+b}{d}\right) \tag{1.6.13}$$

$$= \sum_{ad=n} n^{k/2} d^k \sum_{m=1}^{\infty} a_m\, e^{\frac{2\pi i m a z}{d}} \sum_{0\le b<d} e^{\frac{2\pi i m b}{d}} \tag{1.6.14}$$

The term $\sum_{0\le b<d} e^{\frac{2\pi i m b}{d}}$ vanishes unless $d|m$ in which case it is equal to $d$. By replacing $m$ by $dm$ in the above formula, we get the following expression

$$f \mid_k T(n) = \sum_{ad=n} n^{k/2} d^{k+1} \sum_{m=1}^{\infty} a_{md}\, e^{2\pi i m a z} \tag{1.6.15}$$

$$= \sum_{m=1}^{\infty} \sum_{ad=n, a|m} n^{k/2} d^{k+1} a_{\frac{md}{a}} e^m \tag{1.6.16}$$

from which we derive (1.6.10) □

**Corollary 1.6.7.** *The Fourier coefficient $a_n$ of a normalized eigenform $f$ of weight $k$ with respect to $\Gamma(1)$ is the eigenvalue of the operator $n^{\frac{k}{2}-1}T(n)$.*

**Theorem 1.6.8.** *Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a cuspidal eigenform of weight $k$ for $\Gamma(1)$ normalized so that $a_1 = 1$. Then we have*

$$T(n)f = n^{-\frac{k}{2}+1} a_n f. \tag{1.6.17}$$

*In particular, we have the relation*

$$a_{mn} = a_m a_n \tag{1.6.18}$$

*for all relatively prime integers $m, n \in \mathbb{N}$. Moreover, the Dirichlet series $L(s, f)$ admits a development in Euler product*

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}. \tag{1.6.19}$$

*Proof.* Let $c(n)$ denote the eigenvalue of $T(n)$ with respect to the eigenvector $f$. We have $b_1 = c(n)a_1 = c(n)$ since $a_1 = 1$ with our normalization. Now (1.6.17) follows from (1.6.11).

The multiplicative relation (1.6.18) follows from the relation in the Hecke algebra $T(mn) = T(m)T(n)$ that holds for $(m, n) = 1$.

The multiplicative relation implies a development in product of the Dirichlet series

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p (\sum_{r=0}^{\infty} a_{p^r} p^{-rs}). \tag{1.6.20}$$

The formula

$$\sum_{r=0}^{\infty} a_{p^r} p^{-rs} = (1 - a_p p^{-s} + p^{k-1-2s})^{-1} \tag{1.6.21}$$

follows from similar formula in the Hecke algebra. □

1.7. **Old and new forms.** The theory of Hecke operators and expansion $L$-function as an Euler product can be generalized to any congruence subgroup. It will take however a much more complicated form. As the theory of Hecke operators can be significantly streamlined with the introduction of the adèles and the interpretation of modular form as automorphic forms on an adèlic group, we will postpone discussion after the adèles being introduced.

For the record, we will just state the result of the theory of new forms, due to Atkin and Lehner. The proof will be postponed.

Let $M, N \in \mathbb{N}$ such that $M|N$. For any character $\chi : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}$, we have a character $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ also denoted by $\chi$ obtained by composing with $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/M\mathbb{Z})^\times$. For any integer $d$ such that $dM|N$, there is a map

$$[d] : S_k(M, \chi) \to S_k(N, \chi)$$

that associate to a form $f \in S_k(M, \chi)$ the form $z \mapsto f(dz)$ that belongs to $S_k(N, \chi)$. Let denote $S_k(N, \chi)_{\mathrm{old}}$ the subspace generated by the image of $[d]$ for all integers $d, M$ so that $dM|N$. The orthogonal complement of $S_k(N, \chi)_{\mathrm{old}}$ is denoted $S_k(N, \chi)_{\mathrm{new}}$.

**Theorem 1.7.1.**
1. *The space of new forms $S_k(N, \chi)_{\mathrm{new}}$ admits a basis consisting of normalized eigenform for the operators $T_p$ associated to the primes $p$ not dividing $N$.*
2. *The $L$-function associated to such a normalized eigenform has a complete expansion as a Euler product.*

(3) *If* $f, f' \in S_k(N, \chi)_{\rm new}$ *such that for every prime* $p$ *not dividing* $N$*, there exists* $a_p$ *such that* $T_p f = a_p f$ *and* $T_p f' = a_p f'$ *then* $f$ *and* $f'$ *are proportional.*

## 2. Representations of $\mathrm{GL}(2,\mathbb{R})^+$

2.1. **Representations of locally compact groups.** Let $G$ be a topological group that is locally compact. We will be interested in representations of $G$ on Hilbert spaces.

**Definition 2.1.1.** *A representation of* $G$ *on a Hilbert space* $\mathsf{H}$ *is a homomorphism* $\pi$ *from* $G$ *to the group of continuous linear transformations of* $H$ *so that for evry* $v \in \mathsf{H}$*, the map* $g \to \pi(g)v$ *is continuous. If moreover* $\pi$ *preserves the inner product on* $\mathsf{H}$*, we will say that the representation* $\pi$ *is unitary.*

**Lemma 2.1.2.** *Every representation is locally bounded. In other words, for every compact* $K$ *of* $G$*, there exists a positive real number* $C$ *such that* $|\pi(g)| < C$ *for every* $g \in K$*.*

*Proof.* For every $v$, the vectors $\pi(g)v$ with $g \in K$ form a compact set of $\mathsf{H}$ which is necessarily bounded. The lemma follows from the uniform boundedness principle. □

Let $C_c(G)$ denote the space of complex valued continuous functions on $G$ with compact support. Assume that $G$ is unimodular. The Haar measure $dg$ defines a positive linear form $\mathbb{C}_c(G) \to C$

$$\phi \mapsto \int_G \phi(g)dg.$$

The Haar measure also provides $C_c(G)$ with a structure of algebra under the convolution product

$$\phi * \psi(x) = \int_G \phi(xy^{-1})\psi(y)dy. \tag{2.1.1}$$

Let $\pi$ be a representation of a locally compact topological group $G$ on a Hilbert space $\mathfrak{g}$. We consider the integral

$$\pi(\phi)v = \int_G \phi(g)\pi(g)vdg$$

for every $\phi \in C_c(G)$ and $v \in \mathsf{H}$. The above integral can be defined as follows. We first consider the continous linear form

$$v' \mapsto \int_G \phi(g)\langle \pi(g)v, v'\rangle dg.$$

By Riesz representation theorem there exists a unique vector $\pi(\phi)v \in \mathsf{H}$ so that

$$\langle \pi(\phi)v, v'\rangle = \int_G \phi(g)\langle \pi(g)v, v'\rangle dg.$$

This defines an action of $C_c(G)$ on $\mathsf{H}$.

**Definition 2.1.3.** *A sequence of positive functions* $\phi_n$ *is said to be approximating the delta function of the identity of* $G$ *in the following sense*

(1) $\phi_n \in C_c(G)$ *are supported in a certain compact* $K$ *of* $G$
(2) $\int_G \phi_n(g)dg = 1$ *for every* $n$
(3) *for every neighborhood* $U$ *of* $1_G$ *we have* $\lim_{n\to\infty} \int_{G-U} \phi_n(g)dg = 0$.

**Lemma 2.1.4.** *Let $\pi$ be a representation of $G$ on a Hilbert space $\mathsf{H}$. If $\phi_n$ is a sequence of function approximating the delta function of $1_G$, then we have*

$$\lim_{n\to\infty} \pi(\phi_n)v = v$$

*for all $v \in \mathsf{H}$.*

*Proof.* Let $v \in V$. For every $\epsilon > 0$, there exists a neighborhood $U$ of $1_G$ so that for every $g \in U$, $||\pi(g)v - v|| < \epsilon$. For an integer $n$ large enough, we have $\lim_{n\to\infty} \int_{G-U} ||\phi_n(g)v||dg < \epsilon$ since the family $|\phi_n|$ is bounded and $\lim_{n\to\infty} \int_{G-U} \phi_n(g)dg = 0$. We can also check that $\int_U ||\pi(\phi_n)v - v|| < \epsilon$ for $n$ large enough. □

2.2. **Representations of the circle group.** The theory of unitary representations of the circle group is very much a reformulation of the series of Fourier series associated to square integrable functions on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The characters of is of the forms

$$\chi_n(x) = e^{2\pi i n x}.$$

Let $\mathsf{H} = L^2(\mathbb{T})$ denote the space of square integrable functions on $\mathbb{T}$. The translation by $\mathbb{T}$ defines a unitary representation of $\mathbb{T}$ on $\mathsf{H}$.

**Theorem 2.2.1.** *Let $\pi$ be a unitary representation of $\mathbb{T}$ on a Hilbert space $\mathsf{H}$. For every $v \in \mathsf{H}$ and $n \in \mathbb{Z}$, we consider*

$$v_n = \pi(\chi_{-n})v = \int_{\mathbb{T}} e^{-2\pi i n x}\pi(x)v dx.$$

*Let $\mathsf{H}_n$ denote the image $p_n : v \mapsto v_n = \pi(\chi_{-n})v$.*

1. *For $n \neq m$, the subspaces $\mathsf{H}_n$ and $\mathsf{H}_m$ are orthorgonal.*
2. *The space $\mathsf{H}$ is the Hilbert direct sum of its subspace $\mathsf{H}_n$.*

*Proof.* Only the last statement is non trivial. For the last statement, it suffices to prove that for all $v \in \mathsf{H}$, $v = \lim_{N\to\infty} \sum_{n=-N}^{N} p_n(v)$. Since $v - \sum_{n=-N}^{n=N} p_n(v)$ is orthogonal to $\bigoplus_{n=-N}^{N} \mathsf{H}_n$, for every $v' \in \bigoplus_{n=-N}^{N} \mathsf{H}_n$, we have

$$|v - v'| \geq |v - \sum_{n=-N}^{n=N} p_n(v)|.$$

It follows that we only need to construct a sequence $v'_N \in \bigoplus_{n=-N}^{N} \mathsf{H}_n$ so that

$$\lim_{N\to\infty} |v - v'_N| = 0.$$

This can be achieved by constructing a delta sequence of functions $\phi_N$ on $\mathbb{T}$ so that for all $N$, $\phi_N$ is a linear combination of $e^{2\pi i n x}$ with $|n| \leq N$. Classical examples of this is Fejer's kernel

$$K_N(x) = \sum_{n=-N}^{N} (1 - \frac{|n|}{N+1})e^{2\pi i n x} \tag{2.2.1}$$

$$= \frac{1}{N+1}\frac{\sin^2(\pi(N+1)x)}{\sin^2(\pi x)} \tag{2.2.2}$$

On the complement of neighborhood of $0 \in \mathbb{T}$, $\sin^2(\pi x) > \alpha$ for some given positive number $\alpha$ which implies that $K_N(x) < e^{-1}/(N+1)$ for all $N$. This implies that Fejer's sequence approximate the delta function. ☐

A vector $v$ of some finite direct sum $\bigoplus_{n=-N}^{N} \mathsf{H}_n$ is called a finite vector i.e. its transforms $\pi(x)v$ with $x \in \mathbb{T}$ generates a finite dimensional vector space. It follows from the above theorem that there are non zero finite vectors in any unitary representation of the circle groups, and moreover, the finite vectors form a dense subspace of the Hilbert space. This statement can be generalized to arbitrary compact group.

**Lemma 2.2.2.** *Let $(\pi, \mathsf{H})$ be a Hilbert representation of a compact group $K$. Then there exists a hermitian inner product on $\mathsf{H}$ inducing the same topology as the original one so that $\pi$ is unitary i.e. $\langle \pi(g)v, \pi(g)v' \rangle = \langle v, v' \rangle$ for all $g \in K$ and $v, v' \in \mathsf{H}$.*

*Proof.* [2, 2.4.3] Let $\langle v, v' \rangle_1$ denote the given inner product on $\mathsf{H}$. For all $v$, the function $k \mapsto \langle \pi(k)v, \pi(k)v' \rangle_1$ is a continuous function onthe compact group $K$ which is then necessarily bounded. By the uniform boundedness principle, there exist a real number $C > 0$ such that $|\pi(k)v| < C|v|$ for all non zero vector $v \in \mathsf{H}$. This also implies that $|\pi(k)v| > C^{-1}|v|$.

The inner form

$$\langle v, v' \rangle = \int_K \langle \pi(k)v, \pi(k)v' \rangle_1 dk$$

is obviously positively definite. The inequalities $C^{-1}|v| < |\pi(k)v| < C|v|$ imply that it defines the same topology on $\mathsf{H}$ as $\langle v, v' \rangle_1$. ☐

2.3. **Lie groups.** Let $G$ be a real Lie group. Let $\mathfrak{g}$ denote its Lie algebra.

**Proposition 2.3.1.** *There exists natural isomorphism between*

(1) *The tangent space of $G$ at the origin $\mathfrak{g}$.*
(2) *The space of left invariant vector fields on $G$.*
(3) *The space of left invariant derivations of $C^\infty(G)$.*

For every vector $X \in \mathfrak{g}$, there is a unique invariant vector field $\mathcal{L}_X$ having $X$ as the vector $X$ at the origin. For every $g \in G$, the left translation map $l_g : G \to G$ induces an isomorphism $dl_g : \mathfrak{g} \to T_g G$ and we have $\mathcal{L}_{X,g} = dl_g(X)$. Left invariant vector fields are equivalent to left invariant derivations. We have the following formula for the bracket

$$\mathcal{L}_{[X,Y]} f = (\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X) f.$$

**Proposition 2.3.2.** *There exists a map $\exp : \mathfrak{g} \to G$ that is a local homeomorphism in a neighborhood of the origin in $\mathfrak{g}$ such that, for any $X \in \mathfrak{g}$, $t \mapsto \exp(tX)$ in an integral curve for the left invariant vector field $\mathcal{L}_X$. Moreover $\exp((t+u)X) = \exp(tX)\exp(uX)$.*

For every smooth function $f \in C^\infty(G)$, we have the following formula for the left invariant derivation

$$(\mathcal{L}_X f)(g) = \frac{d}{dt} f(g \exp(tX))|_{t=0}.$$

**Definition 2.3.3.** *A representation of the Lie algebra is a linear application $\pi : \mathfrak{g} \to \mathrm{End}(V)$ from $\mathfrak{g}$ to the space of endomorphisms of a vector space $V$ such that for all $X, Y \in \mathfrak{g}$ , we have*

$$\pi[X, Y] = \pi(X)\pi(Y) - \pi(Y)\pi(X).$$

Let us consider examples of finite dimensional representation of Lie algebras. Let $G = \mathrm{SL}_2(\mathbb{R})$ and $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ its Lie algebra. Let $V$ be the standard 2-dimensional vector space on which $G$ acts. For every integer $k$, $\mathrm{Sym}^{k-1}(V)$ is a vector space of dimension $k$ equipped with an induced action of $G$. This representation is algebraic and its derivation is a representation of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(\mathrm{Sym}^{k-1}(V))$. But we are mostly interested in infinite dimensional representations.

We define the universal enveloping algebra $U(\mathfrak{g})$ as the quotient of the tensorial algebra $\bigotimes \mathfrak{g} = \bigoplus_N \bigotimes^n \mathfrak{g}$ by the ideal generated by $[x, y] - x \otimes y + y \otimes x$. Every representation $\pi : \mathfrak{g} \to \mathrm{End}(V)$ can be extended in a unique way into a homomorphism of algebras $U(\mathfrak{g}) \to \mathrm{End}(V)$.

The Killing form is the symmetric bilinear form on $\mathfrak{g}$ defined by

$$B(x, y) = \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y)).$$

Since it is invariant under the adjoint action of $G$ on $\mathfrak{g}$, after derivation we get

$$B([z, x], y) + B(x, [z, y]) = 0.$$

**Proposition 2.3.4.** *Assume that the Killing form is non degenerate. Let $x_1, \ldots, x_d$ be a basis of $\mathfrak{g}$ and $y_1, \ldots, y_d$ be the dual basis so that $B(x_i, y_j) = \delta_{ij}$. Then the element $\Omega = \sum_i x_i y_i$ does not depend on the choice of the basis. Moreover, it belongs to the center of $U(\mathfrak{g})$.*

*Proof.* The adjoint action of $G$ on $\mathfrak{g}$ induces an action of $G$ on $\mathfrak{g}$. Its derivation is an action of $\mathfrak{g}$ on $U(\mathfrak{g})$. We can check that this action is $X(A) = XA - AX$. Sice $\Omega$ is fixed under the action of $G$, it follows that it commutes with $\mathfrak{g}$. Since $\mathfrak{g}$ generates $U(\mathfrak{g})$, $\Omega$ belongs to the center of $U(\mathfrak{g})$. ☐

Let us consider the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$. Its standard base

(2.3.1) $$H_+ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, R_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, L_+ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

satisfies the commutation rule $[H_+, R_+] = 2R_+$, $[H_+, L_+] = -2L_+$ and $[R_+, L_+] = H_+$. The dual basis with respect to the Killing form is $H_+, 2L_+, 2R_+$. The Casimir element is

(2.3.2) $$\Omega = H_+^2 + 2R_+L_+ + 2L_+R_+$$

It belongs to the center of the universal enveloping algebras. In fact the Casimir element is a generator of the center of the universal algebra of $\mathfrak{sl}_2$.

**Proposition 2.3.5.** *The Casimir element $\Omega$ acts on the irreducible $n$-dimensional representation of $\mathfrak{g}$ by the scalar $n^2 - 1$.*

2.4. **Smooth, analytic and $K$-finite vectors in a Hilbert representation.** A representation of a Lie group $G$ on a topological vector space $\mathsf{H}$ is a continuous linear action $\pi : G \times \mathsf{H} \to \mathsf{H}$. If $\mathsf{H}$ is a Hilbert space, we call $\pi$ a Hilbert representation. Observe that we do not require that the action of $G$ preserve the inner product of $\mathsf{H}$. If it does, we say that $\pi$ is a unitary representation.

A vector $v \in \mathsf{H}$ is $C^1$ if for every $X \in \mathfrak{g}$, the limit

$$\pi(X)v := \lim_{t\to 0} \frac{1}{t}(\pi(\exp(tX))v - v)$$

exists. Recursively, we can define $C^k$-vectors for all $k \in \mathbb{N}$. A smooth or $C^\infty$-vector $v$ is a vector that is $C^k$ for all $k$. The space of smooth vectors is a representation of the Lie algebra $\mathfrak{g}$.

**Proposition 2.4.1.** *Let $\pi$ be a Hilbert representation of a Lie group $G$ in a Hilbert space $\mathsf{H}$. Then the subspace of smooth vectors $\mathsf{H}^{\mathrm{sm}}$ is dense in $\mathsf{H}$.*

*Proof.* For a continuous function with compact support $\phi \in C_c(G)$, we define

$$\pi(\phi)v = \int_G \phi(g)\pi(g)v\, dg.$$

If $\phi$ is a smooth function, $\pi(\phi)v$ is a smooth vector. The proposition follows from the existence of a delta sequence of smooth compactly supported function. □

**Proposition 2.4.2.** *Let $(\pi, \mathsf{H})$ be an irreducible unitary representation of $G$ and $\mathsf{H}^{\mathrm{sm}}$ the dense subspace of smooth vectors. Then all elements $\Omega \in Z(U(\mathfrak{g}))$ acts on $\mathsf{H}^{\mathrm{sm}}$ as a scalar.*

*Proof.* This is an elaborate version of the Schur lemma. See [11, 1.6.5]. □

We can assume that the restriction of $\mathsf{H}$ to $K$ is unitary. Then we have a decomposition of $\mathsf{H}$ as Hilbert direct sum

$$\mathsf{H} = \hat{\bigoplus}_{\ell \in \mathbb{Z}} \mathsf{H}(\ell).$$

The algebraic direct sum $\bigoplus_{\ell \in \mathbb{Z}} \mathsf{H}(\ell)$ is the subspace of $K$-finite vectors.

**Proposition 2.4.3.** *For every $\ell \in \mathbb{Z}$, the subspace $\mathsf{H}^{\mathrm{sm}} \cap \mathsf{H}(\ell)$ is dense in $\mathsf{H}(\ell)$. The subspace of smooth, $K$-finite vectors is dense in $\mathsf{H}$.*

*Proof.* [11, 3.3.5]. □

We will also consider the subspace $\mathsf{H}_{\mathrm{an}}$ of analytic vectors. A vector $v \in \mathsf{H}$ is analytic if the function $g \mapsto \pi(g)v$ is analytic. It is equivalent to the apparently weaker condition that for all $w \in \mathsf{H}$, the function $g \mapsto \langle \pi(g)v, w\rangle$ is analytic [12, p.278]. An analytic vector is obviously a smooth vector. The space $\mathsf{H}^{\mathrm{an}}$ is a subspace of $\mathsf{H}^{\mathrm{sm}}$ which is table under the action of $\mathfrak{g}$.

**Proposition 2.4.4.** *Smooth, $K$-finite vectors are analytic.*

*Proof.* In order to prove that a function is analytic, it is enough to prove that it is annihilated by an elliptic differential operator. Recall that the Casimir element

$$\Omega = H_+^2 + 2R_+L_+ + L{+}R_+ = H^2 + (R_+ + L_+)^2 - (R_+ - L_+)^2$$

acts as scalar on smooth vectors. If $v$ is $K$-eigenvector then it is also an eigenvector of $(R_+ - L_+)$. It follows that it is an eigenvector of the elliptic operator $\Omega + 2(R_+ - L_+)^2$. □

**Lemma 2.4.5.** *Let $G$ be a connected Lie group. If $V$ is a $\mathfrak{g}$-invariant subspace of $\mathsf{H}^{\mathrm{an}}$, then its closure $\bar{V}$ is a $G$-invariant subspace of $\mathsf{H}$.*

*Proof.* See [11, 1.6.6]. If $V^\perp$ denote the orthogonal complement of $V$, then $\bar{V} = (V^\perp)^\perp$. Let $X \in \mathfrak{g}$, $v \in \bar{V}$ and $w \in V^\perp$. There exists $\epsilon > 0$ such that if $|t| < \epsilon$ then

$$\langle \pi(\exp(tX))v, w\rangle = \sum_n \frac{t^n}{n!}\langle \pi(X)^n v, w\rangle$$

and the series converges absolutely.

Since $\pi(X)^n v \in V$ and $w \in V^\bullet$ it follows that $\langle \pi(\exp(tX))v, w\rangle = 0$ for $|t| < \epsilon$. The real analiticity of the function $t \mapsto \langle \pi(\exp(tX))v, w\rangle$ implies that $\langle \pi(\exp(X))v, w\rangle = 0$ for all $v \in \bar{V}$, $w \in W$ and $X \in \mathfrak{g}$. It follows that $\bar{V}$ is stable under $\exp(X)$ for all $X \in \mathfrak{g}$ thus under $G$ since $\exp(\mathfrak{g})$ generates $G$. □

Let $G$ be a real reductive Lie group, $K$ a maximal compact subgroup of $G$ anf $\mathfrak{g}$ its Lie algebra. By a $(\mathfrak{g}, K)$-module, we mean a vector space $V$ together with representation $\pi$ of $K$ and of $\mathfrak{g}$ subject to the following conditions

(1) $V$ is a direct sum of finite dimensional irreducible representations of $K$
(2) The actions of $K$ and of $\mathfrak{g}$ are compatible i.e. for every $X$ in the Lie algebra of $K$ and for every vector $v \in V$, we have the relation

$$\pi(X)v = \frac{d}{dt}(\exp(tX))f|_{t=0}. \tag{2.4.1}$$

(3) For every $g \in K$ and $X \in \mathfrak{g}$, we have the relation

$$\pi(g)\pi(X)\pi(g^{-1})v = \pi(\mathrm{Ad}(g)X)v. \tag{2.4.2}$$

The module is said to be admissible if each irreducible representations of $K$ occurs with finite multiplicity.

**Proposition 2.4.6.** *Let $(\pi, \mathsf{H})$ be an irreducible Hilbert representation of $G$. Then the subspace of smooth $K$-finite vectors $V$ is a $(\mathfrak{g}, K)$-module.*

*Proof.* The commutation relation (2.4.1) and (2.4.2) can be checked upon the definition of $\pi(X)$. If $v$ is a $K$-finite vector, then so is $\pi(X)v$ because the finite dimensional vector space generated by $\pi(Y)\pi(k)v$ with $Y \in \mathfrak{g}$ and $k \in K$ is stable under the action of $K$. As in 2.2.2, the Hermitian form can be made $K$-invariant without changing the underlying topology of $\mathsf{H}$. □

**Proposition 2.4.7.** *Let $V$ be a finitely generated $(\mathfrak{g}, K)$-module on which $\Omega$ acts as a scalar. Then $V$ is admissible.*

**Proposition 2.4.8.** *Let $V$ be an irreducible $(\mathfrak{g}, K)$-module. Then the Casimir element $\Omega$ acts on it as a scalar.*

*Proof.* We first prove that $V$ as a countable basis. For every $v \in V$, the vectors $kv$ with $k \in K$ span a finite dimensional subspace $\mathrm{span}(Kv)$. The space $U(\mathfrak{g})\mathrm{span}(Kv)$ is then a nonzero $(\mathfrak{g}, K)$-submodule of $V$. Since $V$ is irreducible, we have $V = U(\mathfrak{g})\mathrm{span}(Kv)$ and in particular, $V$ has a countable basis.

The following version of Schur lemma, due to Dixmier, implies that $\Omega$ acts as a scalar. We will see later that from this fact, we can classify completely all irreducible $(\mathfrak{g}, K)$-modules and then derive their admissibility. □

**Lemma 2.4.9.** *Suppose that $V$ is countable dimensional and that $S \subset \mathrm{End}(V)$ act irreducibly. If $T \in \mathrm{End}(V)$ commutes with all elements of $S$ then $T$ is a scalar multiple of identity.*

*Proof.* It is enough to prove that there exists $x \in \mathbb{C}$ so that $T - c1_V$ is not invertible. If it is the case, either its kernel or its image is a proper subspace which is stable under $S$. This would contradict the irreducibility.

If for all $c \in \mathbb{C}$, $T - c1_v$ is irreducible. It follows that for all polynomial $q(t) \in \mathbb{C}[t]$, $Q[T]$ is invertible and therefore we can define the operator $R[T]$ for all rational fraction $R(t) \in \mathbb{C}(t)$. We derive a linear injective map $\mathbb{C}(t) \to V$ defined by $r(t) \mapsto r[T]v$. This implies $\mathbb{C}(t)$ has a countable basis which is a contradiction. ☐

**Proposition 2.4.10.** *Let $(\pi, \mathsf{H})$ be a irreducible unitary representation of $G$. Then $V = \mathsf{H}^{\mathrm{sm}} \cap \mathsf{H}^{\mathrm{fin}}$ is an irreducible admissible $(\mathfrak{g}, K)$-module.*

*Proof.* Let $v \in V$ a non zero vector. Then $V' = U(\mathfrak{g})\mathrm{span}(Kv)$ is a $(\mathfrak{g}, K)$-module of finite type on which $\Omega$ acts as a scalar. It follows that its is admissible i.e for every $\ell \in \mathbb{Z}$, $V'(\ell)$ is finite dimensional. But we know that $V'$ is a $\mathfrak{g}$-invariant subspace of $\mathsf{H}^{\mathrm{an}}$, then $\bar{V}'$ is $G$-stable thus $\bar{V}' = \mathsf{H}$. It follows that $\mathsf{H} = \hat{\bigoplus}_{\ell\in\mathbb{Z}} V'(\ell)$. It follows that $V(\ell) = V'(\ell)$ is finite dimensional and $\mathsf{H}$ is admissible. ☐

Two Hilbert representations $(\pi, \mathsf{H})$ and $(\pi', \mathsf{H}')$ of $G$ are said to be infinitesimally equivalent if their associated $(\mathfrak{g}, K)$-modules are isomorphic. A priori there exist non isomorphic irreducible Hilbert representations that are infinitesimally equivalent. But as we will see in the case of $\mathrm{GL}_2(\mathbb{R})$, irreducible unitary representations that are infinitesimally equivalent, have to be isomorphic.

2.5. **$(\mathfrak{g}, K)$-modules for $\mathrm{GL}_2(\mathbb{R})$.** Let $G = \mathrm{GL}_2(\mathbb{R})^+$ and $K = \mathrm{SO}_2(\mathbb{R})$ its maximal compact group.

$$K = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \mid \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}.$$

Since $K$ is isomorphic to the circle group $\mathbb{R}/\mathbb{Z}$, all irreducible unitary representations of $K$ are one dimensional and classified by the integers. For an integer $\ell \in \mathbb{Z}$, the $\ell$-th representation of $K$ is given by the $\ell$-th power in the circle group.

Let $V$ be admissible $(\mathfrak{g}, K)$-modules. We have a decomposition of $V$ into algebraic direct sum

$$V = \bigoplus_{\ell\in\mathbb{Z}} V(\ell)$$

where $K$ acts on $V(\ell)$ by its $\ell$-th power.

In $\mathfrak{sl}_2(\mathbb{C})$, we consider the following triple

$$H = -i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ R = \frac{1}{2} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix},\ L = \frac{1}{2} \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \tag{2.5.1}$$

which is obtained from the triple $(H_+, R_+, L_+)$ by conjugation by the Cayley matrix

$$c = -\frac{i+1}{2} \begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix} \tag{2.5.2}$$

i.e. $c^{-1}(H_+, R_+, L_+)c = (H, R, L)$. We also have

$$c^{-1} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} c = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{2.5.3}$$

Thus $H$ is a generator of the complexified Lie algebra of $K$. In fact, the Lie algebra of circle group $K$ is generated by the vector

$$W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.5.4}$$

and we have $H = -iW$.

Since $\Omega$ is invariant under the adjoint action, we have

$$\Omega = H^2 + 2RL + 2LR. \tag{2.5.5}$$

By Dixmier's lemma 2.4.9, $\Omega$ acts on an irreducible $(\mathfrak{g}, K)$-module by a scalar.

**Proposition 2.5.1.** *Let $V$ be admissible $(\mathfrak{g}, K)$-modules and $V = \bigoplus_\ell V(\ell)$ be is decomposition in direct sum of isotypical $K$-components.*

1. *$V(\ell)$ is the space of vectors $v \in V$ such that $Hv = kv$.*
2. *If $v \in V(\ell)$, then $Rv \in V(\ell+2)$ and $Lv \in V(\ell-2)$. Moreover, if $v \neq 0$, then $R^n v$ is a generator of $V(\ell+2n)$ and $L^n v$ is a generator of $V(\ell-2n)$.*
3. *The dimension of $V(\ell)$ is at most one. If $V(\ell) \neq 0$ and $V(k) \neq 0$ then $k-l$ is an even integer. If $V(\ell) \neq 0$ for some even (resp. odd) integer, we say that $V$ is an even (resp. odd) module.*
4. *Let $\Omega$ act on $V$ by the scalar $\omega$. For $v \in V(\ell)$, we have*

$$4LRv = (\omega - \ell(\ell+2))v \ , \ 4RLv = (\omega + \ell(2-\ell))v. \tag{2.5.6}$$

*If $v \neq 0$ and $Rv = 0$ then $\omega = (\ell+1)^2 - 1$. If $v \neq 0$ and $Lv = 0$ then $\omega = (\ell-1)^2 - 1$.*

*Proof.* For $v \in V(\ell)$, we have

$$Wv = \frac{d}{dt} e^{i\ell t} v|_{t=0} = i\ell v.$$

It follows that $Hv = kv$.

By using the commutation rule $[H, R] = 2R$, we have

$$HRv = RHv + [H, R]v = (\ell+2)Rv$$

that implies $Rv \in V(\ell+2)$. We have similarly $HLv = (\ell-2)Lv$.

By using the commutation rule $[R, L] = 2H$, we have

$$4LRv = \Omega v - H^2 v - 2[R, L]v = (\omega - \ell(\ell+2))v.$$

The other affirmations follow from these computations. □

**Definition 2.5.2.** *Let $V$ be an irreducible admissible $(\mathfrak{g}, K)$-module. The $K$-type of $V$ is the set $\Sigma(V)$ of integers $k \in \mathbb{Z}$ so that $V(\ell) \neq 0$. The central character of $V$ is given by the eigenvalue $\alpha$ of $Z$. The infinitesimal character is given be the eigenvalue $\omega$ of $\Omega$.*

**Corollary 2.5.3.** *The isomorphism class of an irreducible admissible $(\mathfrak{g}, K)$-module is determined by its central character, its infinitesimal character and its $K$-type $\Sigma(V)$. Let $\omega$ be the eigenvalue of $\Omega$ on $V$ (the infinitesimal character). Let us write $\omega = s^2 - 1$ with $s \in \mathbb{C}$ well determined up to a sign. Let $\epsilon \in \{0, 1\}$ be the parity of $V$:*

1. *If $s$ in not an integer, or an integer congruent to $\epsilon$ modulo 2 then the $K$-type of $V$ is*

$$\Sigma = \{\ell \in \mathbb{Z} \mid \ell \equiv \epsilon \mod 2\}.$$

2. *If $s$ is an integer not congruent to $\epsilon$ modulo 2, then the $K$-type of $V$ can be either*

$$\begin{aligned} \Sigma_+ &= \{\ell \geq |s| + 1 \mid \ell \equiv \epsilon \mod 2\} \\ \Sigma_0 &= \{-|s| - 1 < l < |s| + 1 \mid \ell \equiv \epsilon \mod 2\} \\ \Sigma_- &= \{l \leq -|s| - 1 \mid \ell \equiv \epsilon \mod 2\} \end{aligned}$$

*Note that the case $s = 0$ is exceptional because in that case $\Sigma_0 = \emptyset$.*

*In every case, there exists nonzero vector $v_\ell \in V(l)$ with $l$ in the set of $K$-type a a good choice of $s$ so that*

$$\begin{aligned} Hv_\ell &= lv_\ell \\ Rv_\ell &= \frac{s+1+\ell}{2} v_{\ell+2} \\ Lv_\ell &= \frac{s+1-\ell}{2} v_{\ell-2} \end{aligned}$$

*In the first case, the choice of $s$ between the two solutions of teh equation $\omega = s^2 - 1$ does not matter. In the second case, we have to choose positive $s$ in the cases $\Sigma_\pm$ and negative $s$ in the case $\Sigma_0$.*

2.6. **Unitary $(\mathfrak{g}, K)$-modules.** Let $(\pi, \mathsf{H})$ be a unitary representation of $G$. The space of $K$-finite vectors $V = \mathsf{H}^{\text{fin}}$ is then a dense subspace of $G$ which is equipped with a structure of $(\mathfrak{g}, K)$-module.

**Proposition 2.6.1.** *If $V = \mathsf{H}^{\text{fin}}$ is the $(\mathfrak{g}, K)$-module associated with a unitary representation then the action of $K$ is unitary and the action of $fg$ is anti-self-adjoint i.e. that for all $X \in \mathfrak{g}$, $v, w \in V$, we have*

$$\langle Xv, w \rangle = -\langle v, Xw \rangle.$$

*Proof.* We derive $\langle Xv, w \rangle + \langle v, Xw \rangle = 0$ from the relation

$$\langle \exp(tX)v, \exp(tX)w \rangle = \langle v, w \rangle$$

by Leibnitz rule. □

Recall that $R = \frac{1}{2}(H_+ + iX)$ and $L = \frac{1}{2}(H_+ - iX)$ where $H_+, X \in \mathfrak{g}$ are real matrices. In other words, $R, L$ are complex conjugate matrices. It follows that

$$\langle Rv, w \rangle = -\langle v, Lw \rangle$$

for all $v, w \in V$.

**Proposition 2.6.2.** *If $(\pi_1, \mathsf{H}_1)$ and $(\pi_2, \mathsf{H}_2)$ are irreducible unitary representations of $G$, they are unitarily equivalent if and only if they are infinitesimally equivalent i.e. if $V_1 = \mathsf{H}_1^{\text{fin}}$ and $V_2 = \mathsf{H}_2^{\text{fin}}$ are isomorphic as $(\mathfrak{g}, K)$-modules.*

*Proof.* Let $k \in \mathbb{Z}$ so that $V_1(\ell) \neq 0$. Then $V_2(\ell) \neq 0$. Choose non zero vector $x_1 \in V_1(\ell)$ and $x_2 \in V_2(\ell)$ so that $\langle x_1, x_1 \rangle = \langle x_2, x_2 \rangle$. It is enough to prove $\langle Lx_1, Lx_1 \rangle = \langle Lx_2, Lx_2 \rangle$ and the same for $R$ because $L^n x_1$ and $R^n x_1$ form a basis for $V_1$. But this follows from the calculation

$$\langle Lx_1, Lx_1 \rangle = -\langle RLx_1, x_1 \rangle \tag{2.6.1}$$

$$= \frac{1}{4}(\omega - \ell(\ell+2))\langle x_1, x_1 \rangle \tag{2.6.2}$$

and the identical calculation for $x_2$. □

**Proposition 2.6.3.** *Let $V$ be an infinite dimensional irreducible unitary $(\mathfrak{g}, K)$-module. Then the central element $Z$ act on $V$ by a purely imaginary scalar $\mu$ and the Casimir element $\Omega$ act by a real scalar $\omega$. Moreover, if we write $\omega$ in the form $\omega = s^2 - 1$ then there are only four possibilities*

(1) *$s$ is purely imaginary (unitary principal series),*

(2) *$s$ is real and $-1 < s < 1$ (complementary series),*
(3) *$s \in \mathbb{Z}$ (discrete series and limit of discrete series),*

*Proof.* Since $V$ is irreducible $Z$ and $\Omega$ ought act by scalars. Since $Z$ is skew-symmetric, the scalar $\mu$ has to be a purely imaginary number. Since $\Omega = H_+^2 + 2R_+L_+ + 2L_+R_+$ is symmetric, the scalar $\omega$ has to be a real number.

Assume that $\omega$ is not of the form $s^2 - 1$ for an integer $s$, then $\omega < 0$ (resp. $\omega < -1$) if $V$ is an even (resp. odd) module. In fact, if $\omega$ is not of the form $s^2 - 1$ with integer $s$ then $V(\ell) \neq 0$ for all even (resp. odd) integer $k$ depending on the parity of $V$. Moreover, if $v \in V(\ell)$ is a non zero vector, then $Rv \neq 0$. It follows from

$$(\omega - \ell(\ell+2))\langle v, v\rangle = \langle v, 4LRv\rangle = -4\langle Rv, Rv\rangle < 0$$

that $\omega - \ell(\ell+2) < 0$ for all even (resp. odd) integer $k$ if $V$ is even (resp. odd). In the first case, we have $\omega < 0$ by taking $\ell = 0$. In the second case, we have $\omega < -1$ by taking $\ell = -1$.

- If $\omega < -1$, then $s^2$ is a real negative integer i.e. $s$ is purely imaginary.
- If $-1 \leq \omega < 0$, then $s$ is a real integer in the interval $(-1, 1)$. In this case $V$ must be an even module.
- The remaining case $\omega = s^2 - 1$ with $s \in \mathbb{N}$. If $s \geq 1$, $V$ is called a discrete series. If $s = 0$ then $V$ is called limit of discrete series.

In the subsequent paragraphs, we will construct unitary representations of $G$ who $(\mathfrak{g}, K)$-modules fit with the three cases enumerated above. □

In order to complete the picture, we will have to construct unitary representations in three classes enumerated in Proposition 2.6.3 : unitary principal series, complementary series and discrete series. For later references, we we will name

- $\mathcal{P}_s$ the $(\mathfrak{g}, K)$-module in the unitary principal series, associated to a purely imaginary number $\pm s$,
- $\mathcal{C}_s$ the $(\mathfrak{g}, K)$-module in the complementary principal series, associated to a purely real number $-1 < \pm s < 1$,
- $\mathcal{D}_{\pm k}$ the $(\mathfrak{g}, K)$-module in the discrete series. A $(\mathfrak{g}, K)$-module with a lowest weight $k \in \mathbb{N}$ is denoted $\mathcal{D}_k$, in this case $s = \pm(k-1)$. A $(\mathfrak{g}, K)$-module with a lowest weight $-k \in -\mathbb{N}$ is denoted $\mathcal{D}_{-k}$, in this case $s = \pm(k-1)$.

2.7. **Unitary principal series of $\mathrm{GL}_2(\mathbb{R})^+$.** Let $B$ denote the subgroup of triangular matrices of $\mathrm{GL}_2(\mathbb{R})$

$$b = \begin{bmatrix} y_1 & x \\ 0 & y_2 \end{bmatrix} = \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} \begin{bmatrix} 1 & y_1^{-1}x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & y_2^{-1}x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} \tag{2.7.1}$$

with $y_1, y_2 \in \mathbb{R}^\times, x \in \mathbb{R}$. On $B$, the left and right invariant measures can be expressed as follows

$$d_{l(B)}b = \frac{dy_1 dy_2 dx}{|y_1|^2|y_2|} \text{ and } d_{r(B)}b = \frac{dy_1 dy_2 dx}{|y_1||y_2|^2}.$$

As $\delta_B(b) d_{l(B)}b = d_{r(B)}b$, the modulus character $\delta$ is given by the expression

$$\delta_B(b) = |y_1||y_2|^{-1}.$$

For all $s_1, s_2 \in \mathbb{C}$ and $\epsilon : \{\pm 1\} \to \mathbb{C}^\times$, we have a character $\chi : B \to \mathbb{C}^\times$

$$\chi(b) = \epsilon(\mathrm{sgn}(y_1))|y_1|^{s_1}|y_2|^{s_2}$$

which is unitary if and only if both $s_1, s_2$ are purely imaginary. For every character $\chi$ of $B$, let $\mathsf{H}_0(\chi)$ be the space of continuous functions $f : G \to \mathbb{C}$ so that

$$f(bg) = \delta_B(b)^{1/2}\chi(b)f(g)$$

We have a representation of $G$ on the Banach space $\mathsf{H}_0(\chi)$ equipped with the $\infty$-norm.

**Lemma 2.7.1.** *Let $\mathsf{H}_0(\epsilon)$ denote the space of continuous functions $f : K \to \mathbb{C}$ so that $f(-k) = \epsilon(-1)f(k)$. If $\chi(b) = \epsilon(\mathrm{sgn}(y_1))|y_1|^{s_1}|y_2|^{s_2}$ as above, the restriction to $K$ defines a linear bijection $\mathsf{H}_0(\chi) \to \mathsf{H}_0$. The subspace of $K$-finite vectors of $\mathsf{H}(\chi)$ is $\bigoplus_{n \text{ even}} \mathsf{H}_n$ if $\epsilon$ is trivial and $\bigoplus_{n \text{ odd}} \mathsf{H}_n$ if $\epsilon$ is non trivial.*

*Proof.* This follows from the Iwasawa decomposition $G = BK$ and $B \cap K = \{\pm 1\}$. □

**Proposition 2.7.2.** *Let $\mathsf{H}(\epsilon)$ be the space of $L^2$-functions on $K$ satisfying $f(-k) = \epsilon(-1)f(k)$. For every character $\chi(b) = \epsilon(\mathrm{sgn}(y_1))|y_1|^{s_1}|y_2|^{s_2}$, let $\mathsf{H}(\chi)$ be the space of functions on $G$ satisfying $f(bg) = \delta_B(b)^{1/2}\chi(b)f(g)$ so that the restriction to $K$ is square integrable. Then the restriction to $K$ defines an isomorphism $\mathsf{H}(\chi) \to \mathsf{H}(\epsilon)$. Moreover the right action of $G$ on $\mathsf{H}(\chi)$ defines Hilbert representation $\mathrm{Ind}_B^G(\chi)$ on the Hilbert space $\mathsf{H}(\epsilon)$.*

The inner product for every $f_1, f_2 \in \mathsf{H}(\chi)$ is given by

$$\int_K f_1\bar{f}_2(k)dk \tag{2.7.2}$$

so that the restriction of $\mathrm{Ind}_B^G(\chi)$ to $K$ is unitary. But the representation $\mathrm{Ind}_B^G(\chi)$ of $G$ is not in general unitary.

**Proposition 2.7.3.** $\mathrm{Ind}_B^G(\chi)$ *is unitary if $\chi$ is unitary.*

*Proof.* For every $f_1, f_2$ the function $f_1\bar{f}_2(g) = f_1(g)\overline{f_2(g)}$ satisfies the equation $f_1\bar{f}_2(bg) = \delta_B(b)f_1\bar{f}_2(g)$. As in C.2.1, there exists a canonical linear map

$$C_c(B\backslash G, \delta_B) \to \mathbb{C}$$

that is $G$ invariant. Moreover, this linear form is proportional to the integration over $K$ by C.2.2 so that the integral is $G$-invariant i.e. $\mathrm{Ind}_B^G(\chi)$ is unitary. □

By Iwasawa decomposition $G = BK$, any matrix $g \in \mathrm{GL}_2(\mathbb{R})^+$ can be written uniquely under the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} uy^{-1/2} & 0 \\ 0 & uy^{-1/2} \end{bmatrix} \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2.7.3}$$

with $z, y \in \mathbb{R}^+$, $x \in \mathbb{R}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Note that we have written the central matrix in a complicated way to make sure that $\det(g) = u^2$.

**Proposition 2.7.4.** *For every $\chi = (\epsilon, s_1, s_2)$, the space of $K$-finite vectors $V(\chi)$ in $\mathsf{H}(\chi)$ is generated by the vectors $f_\ell$*

$$f_\ell\left(\begin{bmatrix} uy^{-1/2} & 0 \\ 0 & uy^{-1/2} \end{bmatrix} \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right) = u^{s_1+s_2}y^{\frac{s+1}{2}}e^{il\ell\theta} \tag{2.7.4}$$

*indexed by even (resp. odd) integers $\ell$ if $\epsilon$ is trivial (resp. non trivial). Here we have introduced the notation $s = s_1 - s_2$.*

*The action of* $\mathfrak{g}$ *on* $\mathsf{H}^{\text{fin}}$ *is given by the formulas*

$$\mathcal{L}_H f_\ell = l f_\ell,\ \mathcal{L}_R f_\ell = \frac{s+1+\ell}{2} f_{\ell+2},\ \mathcal{L}_L f_\ell = \frac{s+1-\ell}{2} f_{\ell-2} \tag{2.7.5}$$

*and* $\mathcal{L}_Z f_\ell = (s_1+s_2) f_\ell$. *In particular, the Casimir element* (2.5.5) *acts as follows*

$$\Omega f_\ell = (s^2-1) f_\ell.$$

The proposition from direct calculations of Lie derivatives. The following tabulations can be found in [2, p.155] and [4, p.116] with slightly different coordinates. We follow Lang's calculations.

**Proposition 2.7.5.** *We have the following formula for the Lie derivatives* $\mathcal{L}_H, \mathcal{L}_R, \mathcal{L}_L$ *on* $C^\infty(G)$

$$\mathcal{L}_H = -i\frac{\partial}{\partial\theta} \tag{2.7.6}$$

$$\mathcal{L}_R = e^{2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{i}{2}\frac{\partial}{\partial\theta}\right) \tag{2.7.7}$$

$$\mathcal{L}_L = e^{-2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{i}{2}\frac{\partial}{\partial\theta}\right) \tag{2.7.8}$$

*Proof.* To all $X \in \mathfrak{g}$ is attached a left invariant vector field whose associated flow is $g_X(t) = g\exp(tX)$ on G. We have

$$\phi(g_X(t)) = \phi(u_X(t), x_X(t), y_X(t), \theta_X(t))$$

and

$$\mathcal{L}_X\phi(g) = \frac{du_X(t)}{dt}|_{t=0}\frac{\partial\phi}{\partial u} + \frac{dx_X(t)}{dt}|_{t=0}\frac{\partial\phi}{\partial x} + \frac{dy_X(t)}{dt}|_{t=0}\frac{\partial\phi}{\partial y} + \frac{d\theta_X(t)}{dt}|_{t=0}\frac{\partial\phi}{\partial\theta}. \tag{2.7.9}$$

We are thus about to calculate the derivatives $\frac{dz_X(t)}{dt}|_{t=0}$, $\frac{dx_X(t)}{dt}|_{t=0}$, $\frac{dy_X(t)}{dt}|_{t=0}$ and $\frac{d\theta_X(t)}{dt}|_{t=0}$ for any $X \in \mathfrak{g}$.

We need to remember the formula relating the coordinates $(z, x, y, \theta)$ with the matrix entries $a, b, c, d$ in the formula (2.7.3). Apply $g$ to $i \in \mathbf{H}$ we get the equality

$$\frac{ai+b}{ci+d} = yi + x$$

that allows us to express $x, y$ as functions of $a, b, c, d$. We can also derive from the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = uy^{-1/2}\begin{bmatrix} y\cos\theta - x\sin\theta & y\sin\theta + x\cos\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2.7.10}$$

the equality $d - ic = te^{i\theta}$ that allows us to calculate $t$ as the modulus of $d - ic$ and $\theta$ as its argument. In particular we have

$$\frac{d-ic}{|d-ic|} = e^{i\theta}. \tag{2.7.11}$$

Note that we also have the formula

$$y = \frac{ad-bc}{|ci+d|^2}. \tag{2.7.12}$$

For $R_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, we have

$$\exp(tR_+) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

It follows that

$$\begin{aligned} g_{R_+}(t) &= \begin{bmatrix} a & at+b \\ c & ct+d \end{bmatrix} \\ g_{R_+}(t)i &= \frac{ai+at+b}{ci+ct+d} \\ &= x_{R_+}(t) + iy_{R_+}(t) \\ &= z_{R_+}(t) \end{aligned}$$

We go on to calculate the derivative of $x, y$ with respect to $t$

$$\begin{aligned} \frac{dz_{R_+}}{dt}|_{t=0} &= \frac{ad-bc}{(ci+d)^2} \\ &= \frac{ad-bc}{|ci+d|^2}\frac{(d-ci)^2}{|d-ci|^2} \\ &= ye^{2i\theta} \end{aligned}$$

after (2.7.11) and (2.7.12). In particular, we have the formulae

$$\begin{aligned} x'_{R_+}(0) &= y\cos 2\theta \\ y'_{R_+}(0) &= y\sin 2\theta \end{aligned}$$

We calculate the derivative of $\theta$

$$\begin{aligned} i\theta_{R_+}(t) &= \log\frac{ct+d-ic}{((ct+d)^2+c^2)^{1/2}} \\ i\theta'_{R_+}(0) &= \frac{(c^2+d^2)^{1/2}}{d-ic}\frac{c(c^2+d^2)^{1/2} - cd(d-ic)(c^2+d^2)^{-1/2}}{c^2+d^2} \\ &= \frac{c(d+ic)}{c^2+d^2} - \frac{cd}{c^2+d^2} \\ &= \frac{ic^2}{c^2+d^2} \\ \theta'_{R_+}(t) &= \frac{c^2}{c^2+d^2} \\ &= \sin^2\theta \end{aligned}$$

The derivative of $u$ vanishes because $\det(g) = u^2$ and $\det\exp(tR_+) = 1$. From these calculations we obtain

(2.7.13) $$\mathcal{L}_{R_+} = y\cos 2\theta\frac{\partial}{\partial x} + y\sin 2\theta\frac{\partial}{\partial y} + \sin^2\theta\frac{\partial}{\partial\theta}.$$

For $W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, we have

$$
\begin{aligned}
\exp(tW) &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \\
g\exp(tW) &= \begin{bmatrix} uy^{-1/2} & 0 \\ 0 & uy^{-1/2} \end{bmatrix} \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta+t) & \sin(\theta+t) \\ -\sin(\theta+t) & \cos(\theta+t) \end{bmatrix}
\end{aligned}
$$

and we get

$$
\mathcal{L}_W = \frac{\partial}{\partial \theta}. \tag{2.7.14}
$$

For $H_+ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we have

$$
\begin{aligned}
\exp(tH_+) &= \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \\
g\exp(tH_+) &= \begin{bmatrix} ae^t & be^{-t} \\ ce^t & de^{-t} \end{bmatrix} \\
g\exp(tH_+)i &= \frac{iae^t + be^{-t}}{ice^t + de^{-t}} \\
&= x_{H_+}(t) + iy_{H_+}(t) \\
i\theta_{H_+}(t) &= \log \frac{de^{-t} - ice^t}{(c^2e^{2t} + d^2e^{-2t})^{1/2}}.
\end{aligned}
$$

From this we obtain the Lie derivative

$$
\mathcal{L}_{H_+} = -2y\sin 2\theta \frac{\partial}{\partial x} + 2y\cos 2\theta \frac{\partial}{\partial y} + \sin 2\theta \frac{\partial}{\partial \theta}. \tag{2.7.15}
$$

We can now calculate the Lie derivatives $\mathcal{L}_H, \mathcal{L}_R, \mathcal{L}_L$ as linear combination with complex coefficients of $\mathcal{L}_{R_+}, \mathcal{L}_W, \mathcal{L}_{H_+}$. We have

$$
\begin{aligned}
\mathcal{L}_H &= -i\mathcal{L}_W \\
&= -i\frac{\partial}{\partial \theta} \\
\mathcal{L}_R &= i\mathcal{L}_{R_+} - \frac{i}{2}\mathcal{L}_W + \frac{1}{2}\mathcal{L}_{H_+} \\
&= e^{2i\theta}\left( iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{i}{2}\frac{\partial}{\partial \theta} \right) \\
\mathcal{L}_L &= -i\mathcal{L}_{R_+} + \frac{i}{2}\mathcal{L}_W + \frac{1}{2}\mathcal{L}_{H_+} \\
&= e^{-2i\theta}\left( -iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{i}{2}\frac{\partial}{\partial \theta} \right)
\end{aligned}
$$

according to (2.5.1). □

2.8. **Complementary series of $\mathrm{GL}_2(\mathbb{R})^+$.** The construction of the unitary principal series was based on the fact that for purely imaginary numbers $s_1, s_2$, the induced representation $\mathsf{H}(s_1, s_2)$ affords a positively definite Hermitian inner form. In fact, for all copmlex numbers $s_1, s_2 \in \mathbb{C}$ there is a Hermitian pairing

$$\mathsf{H}^\infty(s_1, s_2) \times \mathsf{H}^\infty(-\bar{s}_1, -\bar{s}_2) \to \mathbb{C}.$$

The construction of complementary series is based on the existence of an intertwining operator.

$$\mathsf{H}^\infty(s_1, s_2) \to \mathsf{H}^\infty(s_1, s_2)$$

so that $\mathsf{H}^\infty(s_1, s_2)$ affords a Hermitian inner form if $s_1 = -\bar{s}_2$ and $s_2 = \bar{s}_1$. The latter condition is equivalent with $s_1 + s_2 \in i\mathbb{R}$ and $s = s_1 - s_2 \in \mathbb{R}$. We will construct the intertwining operator and prove that the induced inner form is positively definite is $s = s_1 - s_2 \in (-1, 1)$.

For every smooth vector $f \in \mathsf{H}(\chi)$ with $\chi = (\epsilon, s_1, s_2)$, we consider the integral

$$(M(s)f)(g) = \int_{-\infty}^{\infty} f\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) dx. \tag{2.8.1}$$

**Proposition 2.8.1.** *Suppose that $\Re(s) > 0$. For every $K$-finite vector $f \in V(\epsilon, s_1, s_2)$, the integral* (2.8.1) *is convergent, and $M(s)f$ is a $K$-finite vector in $\mathsf{H}(\epsilon, s_2, s_1)$. If $f$ is $K$-finite, then so is $M(s)f$, and $f \mapsto M(s)f$ defines a homomorphism of $(\mathfrak{g}, K)$-modules $V(\epsilon, s_1, s_2) \to V(\epsilon, s_2, s_1)$.*

*Proof.* Assume more generally that $f$ is a smooth vector. Let us evaluate the integral $M(s)f$ at $g = 1$. For that, we use Iwasawa decomposition

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = (1+x^2)^{1/2}\begin{bmatrix} (1+x^2)^{-1} & -x(1+x^2)^{-1} \\ 0 & 1 \end{bmatrix} k_{\theta(x)} \tag{2.8.2}$$

where $k \in K$. It follows that

$$M(s)f(1) = \int_{-\infty}^{\infty} (1+x^2)^{-\frac{1+s}{2}} f(k_{\theta(x)}) dx. \tag{2.8.3}$$

If $f$ is smooth, its restriction to $K$ is bounded so that the above integral is absolutely convergent as long as $\Re(s) > 0$.

For every $g \in G$, the integral $M(s)f(g)$ is still absolutely convergent since one can replace $f$ by its right translated by $g$. In fact, the operator $M(s)$ commutes with the right translation by $G$ as long as the integrals converge. In particular, $M(s)$ commutes with the right action of $K$, and therefore it transforms a $K$-finite vector onto a $K$-finite vector.

We derive from the commutation rule

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} = \begin{bmatrix} y_2 & 0 \\ 0 & y_1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & y_1^{-1}y_2 x \\ 0 & 1 \end{bmatrix} \tag{2.8.4}$$

that if $f \in V(\epsilon, s_1, s_2)$ then $M(s)f \in V(\epsilon, s_2, s_1)$. □

**Proposition 2.8.2.** *Assume that $\Re(s) > 0$, then we have*

$$M(s)f_{\ell,s} = (-i)^\ell \sqrt{\pi}\frac{\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s+1+\ell}{2})\Gamma(\frac{s+1-\ell}{2})} f_{\ell,-s}. \tag{2.8.5}$$

*Proof.* We already know that $M(s)f_{\ell,s}$ is a multiple of $f_{\ell,-s}$ so that it is enough to calculate the constant $M(s)f_{\ell,s}(1)$. See [2, p.230] for the calculation of the constant. □

**Theorem 2.8.3.** *The intertwining operator $M(s) : V(\epsilon, s_1, s_2) \to V(\epsilon, s_2, s_1)$ for $\Re(s) > 0$ by the integral* (2.8.1) *can be meromorphically continued to the whole complex plane. It induces a positively definite inner form on $\mathsf{H}(\epsilon, s_1, s_2)$ for all real numbers $s = s_1 - s_2$ satisfying $-1 < s < 1$.*

*Proof.* It follows from 2.8.2 that $M(s)$ can be meromorphically continued to the whole complex plane.

For every complex numbers $s_1, s_2$, there is a Hermitian pairing

$$\mathsf{H}^\infty(s_1, s_2) \times \mathsf{H}^\infty(-\bar{s}_1, -\bar{s}_2) \to \mathbb{C}$$

defined by

$$(f_1, f_2) \mapsto \int_K f_1(k)\bar{f}_2(k)dk.$$

By C.2.2, this pairing is $G$-invariant.

Now we have a meromorphic family of intertwining operators

$$M(s) : V(s_1, s_2) \to M(s_2, s_1).$$

If $s_1 + s_2$ is purely imaginary and $s = s_1 - s_2$ is real, then $s_1 = -\bar{s}_2$ and $s_2 = -\bar{s}_1$ so that we have a Hermitian inner product on $\mathsf{H}^\infty(\epsilon, s_1, s_2)$.

It remains to check that for all even integers $\ell$ the constants

$$(-1)^{\ell/2}\sqrt{\pi}\frac{\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s+1+\ell}{2})\Gamma(\frac{s+1-\ell}{2})}$$

are positive for all real numbers $0 < s < 1$. Suppose $\ell \geq 0$. It suffices now to observe that in this product $\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})$ and $\Gamma(\frac{s+1+\ell}{2})$ are positive, and the sign of $\Gamma(\frac{s+1-\ell}{2})$ is $(-1)^{\ell/2}$.

Without this explicit formula we can check that the inner form is positive definite as follows. If $v$ is the generator of $V(\ell)$ then we have

$$\langle Rv, Rv\rangle = (\ell(\ell+2) - \omega)\langle v, v\rangle$$

Here since $\ell$ is even integer $\ell(\ell+2) \geq 0$. We also know that $\omega < 0$. It follows that $\langle Rv, Rv\rangle$ and $\langle v, v\rangle$ have the same sign. □

2.9. **Discrete series of $\mathrm{GL}_2(\mathbb{R})^+$.** We recall that on the upper half plan $\mathbf{H}$, we have an invariant measure $\mu = dxdy/y^2$. For every integer $k \geq 2$, let $\mu_k = y^k\frac{dxdy}{y^2}$. We consider the vector space

$$\mathsf{H}_k = L^2_{\mathrm{hol}}(\mathbf{H}, \mu_k) \tag{2.9.1}$$

of holomorphic functions on $\mathbf{H}$ which are square integrable with respect to the measure $\mu_k$. The following proposition shows that $\mathsf{H}_k$ is complete with repect to the topology defined by the $L_2$-norm, and thus is a Hilbert space.

**Lemma 2.9.1.** *Let $\{f_n\}$ be a sequence of holomorphic functions on the open disc $\mathbf{D}$ which is $L^2$-convergent. Then $f_n$ converges uniformly to a holomorphic function on any compact set.*

*Proof.* [4, p.182] Recall the Cauchy formula for a holomorphic function $f$ the unit disc $\mathbf{D}$

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta}} d\theta$$

thus

$$|f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(re^{i\theta})|}{r} d\theta.$$

It follows that

$$|f(0)|\frac{\delta^3}{3} \leq \frac{1}{2\pi}\int_0^{\delta}\int_0^{2\pi} \frac{|f(re^{i\theta})|}{r} d\theta.$$

It follows that $|f(0)|$ and more generally the uniform norm is dominated by the local $L^1$-norm and therefore the local $L^2$-norm.

Since the uniform norm is dominated by the $L_2$-norm, a $L_2$-convergent sequence $f_n$ is also for the uniform topology. The limit is necessarily a holomorphic function again by application of the Cauchy formula. □

**Proposition 2.9.2.** *For every*

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$$

*let us define*

$$(\pi_k(g^{-1})f)(z) = f|_k g = (ad-bc)^{k/2}(cz+d)^{-k} f(gz).$$

*Then $\pi_k$ is an irreducible unitary representation on $\mathsf{H}_d$.*

*Proof.* We have already seen that $f \mapsto f|_k g$ defines a representation. We now verify that $\pi_k(g^{-1})$ preserves the $L^2$-norm. If $z' = gz$ with real coordinates $z' = x' + iy'$, then we have

$$y' = \frac{ad-bc}{|cz+d|^2} y$$

according to (1.1.2) and $\mu'_z = \mu_z$ according to 1.1.2. Now, for every $f \in \mathsf{H}_k$ we have

$$\begin{aligned} ||\pi_k(g^{-1})f||^2 &= \int_{\mathbf{H}} |f(z')|^2 \frac{(ad-bc)^k}{|cz+d|^{2k}} y^k \mu_z \\ &= \int_{\mathbf{H}} |f(z')|^2 {y'}^k \mu_{z'} \\ &= ||f||^2 \end{aligned}$$

which proves the unitarity of the representation $\pi_k$. □

With help of the Cayley transform, $\mathsf{H}_d$ is equipped with a convenient orthogonal basis. Recall that the Cayley map 1.1.3

$$z \mapsto w = \frac{z-i}{z+i}$$

defines an analytic isomorphism $C$ between the upper half plane $\mathbf{H}$ and the open unit disc $\mathbf{D}$. Let us write $w = u + iv$ in cartesian coordinates and $w = re^{i\theta}$ in polar coordinates.

**Proposition 2.9.3.** *The isomorphism maps the density $\mu_k$ of $\mathbf{H}$ on the density $\nu_k$ of $\mathbf{H}$ given by the formula*

$$\nu_k = \frac{4(1-|w|^2)^{k-2}dudv}{|1-w|^{2k}}.$$

*In particular, the Cayley map induces an isomorphism of Hilbert spaces*

$$L^2(\mathbf{H}, \mu_k) = L^2(\mathbf{D}, \nu_k).$$

*Proof.* This is a direct consequence of (1.1.10) and (1.1.12). □

**Proposition 2.9.4.** *The functions $\{w^n(1-w)^k\}_{n=0}^\infty$ form an orthonormal basis for $L^2(\mathbf{D}, \nu_k)$. The functions*

$$\left\{\phi_n = \left(\frac{z-i}{z+i}\right)^n \frac{(2i)^k}{(z+i)^k} \mid n = 0, 1, \dots\right\}$$

*form an orthonormal basis for $L^2(\mathbf{H}, \mu_k)$.*

*Proof.* Since the functions $w^n(1-w)^k$ on $\mathbf{D}$ and $(2i)^k(z-i)^n/(z+i)^{n+k}$ correspond to each other via the Cayley transform, it is enough to prove that $\{w^n(1-w)^k\}_{n=0}^\infty$ form an orthonormal basis for $L^2(\mathbf{D}, \nu_k)$.

We first verify that $\psi_n = w^n(1-w)^k$ are square integrable on $\mathbf{D}$ with respect to the measure $\nu_k$. In polar coordinate $w = re^{i\theta}$, we have $dudv = rdrd\theta$ so that

$$\nu_k = \frac{4(1-r^2)^{k-2}rdrd\theta}{|1-w|^{2k}}.$$

For every $n$, the integral

$$||\psi_n||^2 \leq 4\int_0^{2\pi}\int_0^1 r^{2n}(1-r^2)^{k-2}drd\theta$$

the latter integral being obviously convergent.

We verify now that $\psi_m$ and $\psi_n$ are orthogonal with respect to the measure $\nu_k$ if $m \neq n$. The integral

$$\begin{aligned}\int_D \psi_m(w)\bar{\psi}_n(w)dw &= 4\int_D w^m\bar{w}^n(1-r^2)^{k-2}drd\theta \\ &= 4\int_0^1 r^{m+n}(1-r^2)^{k-2}\int_0^{2\pi} e^{i(m-n)\theta}dq\end{aligned}$$

vanishes if $m \neq n$. □

**Proposition 2.9.5.** *The infinitesimal class of $\pi_k$ is the $(\mathfrak{g}, K)$-module $\mathcal{D}_k^-$ of highest weight $k$.*

*Proof.* According to the classification of irreducible $(\mathfrak{g}, K)$-modules, it is enough to show that the $K$-type of $\pi_k$ is the set of integers $\{-k-2n | n \in \mathbb{N}\}$.

It is enough to show that $k_\theta \in K$ acts on the vector

$$\phi_n = \left(\frac{z-i}{z+i}\right)^n \frac{(2i)^k}{(z+i)^k}$$

by $\pi_k(k_\theta^{-1})\phi_n = e^{2i\pi(k+2n)\theta}\phi_n$. This can be done by a direct calculation.

Let us do this calculation by writting down explicitly the action of $G$ on $L^2(D, \nu_k)$. Conjugate by the Cayley matrix, we get

$$\frac{1}{2}\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix} = \begin{bmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{bmatrix} \tag{2.9.2}$$

where $\alpha = (a+bi-ci+d)/2$ and $\beta = (a-bi-ci-d)/2$. We also have

$$\begin{aligned} cz+d &= \frac{(ci-d)w+(ci+d)}{1-w} \\ &= \frac{(-\alpha+\bar{\beta})w+(\bar{\alpha}-\beta)}{1-w} \end{aligned}$$

If $g = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then we have

$$\pi_k(g^{-1})\psi(w) = \psi\left(\frac{\alpha w+\beta}{\bar{\beta}w+\bar{\alpha}}\right)\frac{(|\alpha|^2-|\beta|^2)^{k/2}(1-w)^k}{((-\alpha+\bar{\beta})w+(\bar{\alpha}-\beta))^k} \tag{2.9.3}$$

for every $\psi \in L^2(\mathbf{D}, \nu_k)$. If

$$\begin{bmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{bmatrix} = \begin{bmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{bmatrix}$$

then we have

$$\pi_k(g^{-1})\psi(w) = \psi(e^{2i\theta}w)\frac{e^{ik\theta}(1-w)^k}{(1-e^{2i\theta}w)^k}.$$

For $\psi_n = w^n(1-w)^k$ we have

$$\pi_n(g^{-1})\psi_n(w) = e^{i(2n+k)\theta}\psi_n(w).$$

This implies that the $K$-type of $\pi_k$ is the set of integers $\{-k-2n | n \in \mathbb{N}\}$. □

We attach to each holomorphic function $f$ on $\mathbf{H}$ the smooth function $\phi$ on $G$

$$\phi(g) = f(gi)j(g,z)^{-k} \tag{2.9.4}$$

with the automorphy factor

$$j(g,z) = \det(g)^{-1/2}(cz+d). \tag{2.9.5}$$

if $g = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$.

We will make the formula for $\phi$ more explicit with help of the Iwasawa decomposition. Every matrix $g \in G$ can be written uniquely in the form

$$g = \begin{bmatrix}uy^{-1/2} & 0\\ 0 & uy^{-1/2}\end{bmatrix}\begin{bmatrix}y & x\\ 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} \tag{2.9.6}$$

where $x, y, u \in \mathbb{R}$ with $y, u > 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Apply $g$ to $i \in \mathbf{H}$, we get $gi = x+iy$. The automorphy factor can be calculated to be $j(g,i) = y^{-1/2}e^{-i\theta}$ so that we get

$$\phi(g) = y^{k/2}e^{ik\theta}f(x+iy). \tag{2.9.7}$$

The above construction $f \mapsto \phi$ defines a bijection between smooth functions $f$ on $\mathbf{H}$ and smooth functions $\phi$ on $G$ that satisfy the equation

$$\phi(gk_\theta) = e^{ik\theta}\phi(g) \tag{2.9.8}$$

The opposite mapping $\phi \mapsto f$ associates to a smooth function $\phi$ on $G$ the function on $\mathbf{H}$

$$f(z) = y^{-k/2}\phi\left(y^{-1/2}\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}\right) \tag{2.9.9}$$

for every $z = x + iy$ with $y > 0$.

**Proposition 2.9.6.** *The above construction $f \mapsto \phi$ defines a bijection between holomorphic functions $f$ on $\mathbf{H}$ and smooth functions $\phi$ on $G$ that satisfy*

$$\phi(gk_\theta) = e^{ik\theta}\phi(g) \tag{2.9.10}$$

*and*

$$\mathcal{L}_L\phi = 0 \tag{2.9.11}$$

*where $\mathcal{L}_L$ is the left differential derivation attached to the nilpotent matrix $L$ of* (2.5.1).

*Proof.* We only to prove that $f$ is holomorphic if and only if $\phi$ is annihilated by $\mathcal{L}_L$. If $f$ is holomorphic, it is annihilated by the operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right). \tag{2.9.12}$$

According to 2.7.5, we have

$$\mathcal{L}_L = e^{-2i\theta}\left(-iy\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + \frac{i}{2}\frac{\partial}{\partial \theta}\right)$$

In order to prove that $\mathcal{L}_L$ annihilates $\phi(g) = y^{k/2}e^{ik\theta}f(x+iy)$, it is enough to verify that it annihilates $y^{k/2}e^{ik\theta}$ and $f(x+iy)$. One can check

$$\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{i}{2}\frac{\partial}{\partial \theta}\right)(y^{k/2}e^{ik\theta}) = 0$$

by direct calculation. The factor $f(x+iy)$ is also annihilated because of the Cauchy-Riemann equation. □

**Proposition 2.9.7.** *The map $f \mapsto \phi$ defined in* (2.9.4) *defines an intertwining operator from $L^2(\mathbf{H}, \mu_k)$ into $L^2(G)$. In particular, $L^2(\mathbf{H}, \mu_k)$ is a square integrable representation.*

## 3. Automoprhic forms on $\mathrm{SL}_2(\mathbb{R})$

In this chapter, we denote $G = \mathrm{SL}_2(\mathbb{R})$ and $\mathbf{H}$ the upper half plane. The main reference is [1].

3.1. **Siegel domains.** Let $\Gamma$ be a Fuschian group of first kind. Let us suppose that $\infty$ is a cusp for $\Gamma$. The group $B$ of triangular matrices is the stabilizer of $\infty$ for the homographic action of $G$ on $\mathsf{P}^1(\mathbb{C})$. We have $B = NA$ where $N$ is the group of unipotent triangular matrices and $A$ the group of diagonal matrices. The action of $A$ on the Lie algebra of $N$ defines a character $\alpha : A \to \mathbb{R}^\times$. For each compact subset $\Omega_N$ of $N$ and $t > 0$, we define the Siegel domain

$$\mathfrak{S} = \omega A_t K$$

where $A_t = \{y \in A \mid \alpha(y) > t\}$. If $\omega = \{|x| \leq h\}$, then the image of $\mathfrak{S}$ is the upper half band

$$\{x + iy \mid |x| \leq h \text{ and } y > t\}.$$

Assume that $\omega$ contains an interval in $N = \mathbb{R}$ which is a fundamental domain for the discrete subgroup $\Gamma_\infty$. Then $\Gamma_\infty \mathfrak{S}$ is an upper half plane $\{x + iy \mid y > t\}$.

We define by conjugation the notion of Siegel sets for other cusps.

**Proposition 3.1.1.** *Let $\Gamma$ be a Fuschian group of first kind and let $u_1, \ldots, u_m$ be a set of representatives of $\Gamma$-equivalence of cusps. There exists a compact subset $C$ of $\mathbf{H}$ and a Siegel domain $\mathfrak{S}_i$ for each cusp such that $C \cup \bigcup_{i=1}^m \mathfrak{S}_i$ contain a fundamental domain of $\Gamma$.*

*Proof.* For each cusp $x_i$, we choose a Siegel domain $\mathfrak{S}_i$ whose image in $\Gamma\backslash\mathbf{H}^*$ is a neighborhood of $x_i$. Since $\Gamma\backslash\mathbf{H}^*$ is compact by assumption, the complement of the union of images of $\mathfrak{S}_i$ is relatively compact. We can choose a compact $C \subset \mathbf{H}$ whose image contain it. □

3.2. **Growth condition.** Assume that $\infty$ is a cusp for $\Gamma$ and let $\Gamma_\infty$ is the stabilizer of $\infty$ in $\Gamma$. Let $\phi$ be a function on $G$ that is $\Gamma_\infty$-invariant on the left and has $K$-type $k$ for the action of $K$ on the right i.e.

$$\phi(\gamma g k_\theta) = f(g) e^{ik\theta} \text{ for all } \gamma \in \Gamma_\infty, k_\theta \in K.$$

Then $\phi$ is said to be of *moderate growth* at $\infty$ if there exists $\lambda > 0$ so that

$$|\phi(g)| < y^\lambda$$

if $g(i) = x + iy$ with $x, y \in \mathbb{R}$ and $y > t$ for some fixed $t$. In other words, this inequality is valid on a Siegel domain. The function $\phi$ is said to be of rapid decay if this inequality is valid for all $\lambda$.

**Definition 3.2.1.** *A function $\phi : \Gamma\backslash G \to \mathbb{C}$ is said to be of moderate growth if it is of moderate growth at every cusps of $\Gamma$.*

**Lemma 3.2.2.** *If $f$ has moderate growth of exponent $\lambda$ then so does $f * \alpha$ for all $\alpha \in C_c^\infty(G)$.*

**Lemma 3.2.3.** *Suppose that $\phi$ has moderate growth of exponent $\alpha$ on a Siegel domain $\mathfrak{S}$. Assume that $\phi_1 = \phi * \alpha$ for some $\alpha \in C_c^\infty(G)$ then $|D\phi(g)| < y^\alpha$ for any $D \in U(\mathfrak{g})$ and $g \in \mathfrak{S}$. We will say that $\phi_1$ as above has uniform moderate growth.*

*Proof.* Since

$$D\phi_1(g) = D(\phi * \alpha)(g) = (\phi * D\alpha)(g).$$

this lemma follows from the previous one. □

There is another way to define the condition of moderate growth. Let

$$||g||^2 = a^2 + b^2 + c^2 + d^2.$$

**Proposition 3.2.4.** *A function $\phi : \Gamma\backslash G$ has moderate at the cusp of and only if there exists $m > 0$ such that*

$$|\phi(g)| \leq ||g||^m.$$

*The function $\phi$ is of rapid decay if this inequality is valid for all $m$.*

3.3. **From modular forms to automorphic forms.** Let $f \in M_k(\Gamma)$ be a modular form of weight $k$ with respect to $\Gamma$. The formula (2.9.4) defines a function $\phi$ on $G$ that satisfies the equation and is annihilated by $\mathcal{L}_L$. Since the map $f \mapsto \phi$ is $G$-equivariant, and $f$ is $\Gamma$-invariant with respect to the action (1.6.1), $\phi$ is also invariant $\Gamma$-invariant. Thus $\phi$ defines a function $\Gamma\backslash G \to \mathbb{C}$ that satisfies (2.9.4) and is annihilated by $\mathcal{L}_L$.

**Proposition 3.3.1.** *Let $f$ be a modular form of weight $k$ and $\phi$ the smooth function on $G$ defined by* (2.9.4)*. Then $\phi$ satisfies the differential equation $\Omega\phi = k(k-2)\phi$.*

*Proof.* It is clear for (2.7.16) that $\mathcal{L}_H\phi = k\phi$. In combining with $\mathcal{L}_{E_-}\phi = 0$, the calculation

$$\begin{aligned}
\Omega\phi &= (H^2 + 2E_+E_- + 2E_-E_+)\phi && (3.3.1)\\
&= k^2\phi + 2[E_-, E_+]\phi && (3.3.2)\\
&= k^2\phi - 2H\phi && (3.3.3)\\
&= k(k-2)\phi. && (3.3.4)
\end{aligned}$$

proves our proposition. □

Modular forms satisfy a condition of holomorphicity at the cusp, the associated function $\phi$ will satisfy a growth condition at the cusps that we are about to describe.

**Proposition 3.3.2.** *Let $f$ be a modular form of weight $k$ with respect to $\Gamma$. Then the function $\phi$ on $G$ defined as in* (2.9.4) *has moderate growth. If $f$ is a cusp form, then $\phi$ has rapid decay.*

*Proof.* The function $\phi$ is given by the formula (2.9.7) $phi(g) = y^{k/2}e^{ik\theta}f(x+iy)$ where $f$ admits the Fourier expansion

$$f(x+iy) = \sum_{n=0}^{\infty} a_n e^{2ni\pi x} e^{-2n\pi y}.$$

As $y \to \infty$, $\phi(g)$ has moderate growth. If $a_0 = 0$, $\phi(g)$ has rapid decay as $y \to \infty$. □

**Definition 3.3.3.** *A smooth function $\phi : G \to \mathbb{C}$ is an automorphic function for a Fuschian group of first kind $\Gamma$ if*

1. *$\phi$ has a unitary central character*
2. *$\phi(\gamma g) = \phi(g)$ for all $\gamma \in G$ and $g \in G$,*
3. *$\phi$ is $K$-finite on the right,*
4. *$\phi$ is $Z(U(\mathfrak{g}))$-finite,*
5. *$\phi$ has moderate growth at the cusps,*

**Proposition 3.3.4.** *Automorphic form are real analytic.*

*Proof.* Because $\phi$ is both $Z(U(\mathfrak{g}))$-finite and $K$-finite, it satisfies an elliptic differential equation. It follows that $\phi$ is a real analytic function. □

**Proposition 3.3.5** (Harish-Chandra)**.** *Let $\phi$ be an automorphic form. For every neighborhood $U$ of identity there exists $\alpha \in C_c^\infty(G)$ with support contained in $U$ so that $\phi = \phi * \alpha$. In particular, $\phi$ has uniform moderate growth.*

*Proof.* Let $V$ be the smallest closed $G$-invariant subspace containing $\phi$. Since $\phi$ is $K$-finite and $Z(U(\mathfrak{g}))$-finite, $V$ is admissible i.e. for all all integer $\ell$, the eigenspace space $V(\ell)$ for the $\ell$-th power character of $K$ is finite dimensional.

Let $I_c^\infty(G)$ denote the space of smooth function with compact support that are invariant with respect to the conjugation by $K$. We will prove that there exists a $\alpha \in I_c^\infty(G)$ with arbitrarily small support such that $\phi = \phi * \alpha$.

By assumption $\phi$ is a finite sum $\sum_{\ell \in L} \phi_\ell$ with $\phi_\ell \in V(\ell)$ with $L$ a finite subset of $\mathbb{Z}$. The convolution $\phi \mapsto \phi * \alpha$ preserves $L$ and defines a continuous linear map $I_c^\infty(U) \to \mathrm{End}(L)$. Its image is vector subspace is a finite-dimensional vector space hence closed. Let $\alpha_n \in I_c^\infty(U)$ be a delta sequence whose image in $\mathrm{End}(L)$ tends to identity. It follows that identity can be represented in the form $\phi \mapsto \phi * \alpha$ for some $\alpha \in I_c^\infty(U)$. ☐

3.4. **$L^2$-automorphic forms.** The Peterson inner product (1.6.6) for cusp forms has a natural transposition to the framework work of automorphic forms. Let $f, f' \in S_k(\Gamma)$ be cusp forms of weight $k$ with respect to a Fuschian group of first kind $\Gamma$. Let $\phi$ and $\phi'$ be functions on $G$ associated with $f$ and $f'$ as (2.9.7). Then

$$\int_{\Gamma\backslash \mathbf{H}} f(z)\bar{f}'(z)y^k \frac{dxdy}{y^2} = \int_{Z\Gamma\backslash G} \phi(g)\bar{\phi}'(g)dg.$$

where $dg$ is the Haar measure of $G$.

**Definition 3.4.1.** *Let $\omega$ be a unitary character of $Z$. A $L^2$-automorphic form with central character $\omega$ is a function $\phi : \Gamma\backslash G \to \mathbb{C}$ which transforms as $\omega$ under the action of the center so that $|\phi(z)|$ is a square integrable function on $Z\Gamma\backslash G$.*

**Proposition 3.4.2.** *Suppose that $\infty$ is a cusp of $\Gamma$. Let $\alpha \in C_c(G)$. Then there exists a constant $c$ so that*

$$|(\phi * \alpha)(g)| \leq cy||\phi||_2$$

*for all $f \in L^2(\Gamma\backslash G)$, all $g \in NA_tK$ and $g(i) = x + iy$.*

*Proof.* See [1, 5.7]. Assume that $\alpha$ is supported by $C^{-1}$ where $C$ is a compact subset of $G$. By definition

$$(\phi * \alpha)(g) = \int_G \phi(gx^{-1})\alpha(x)dx \tag{3.4.1}$$

from which we derive the estimate

$$|(\phi * \alpha)(g)| \leq ||\alpha||_\infty \int_{gC} |\phi(h)|dh. \tag{3.4.2}$$

Let $C_N$ and $C_A$ be compact subsets of $N$ and $A$ such that $KC \subset C_N C_A K$. After writing $g$ in Iwasawa's form

$$g = y^{-1/2} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{3.4.3}$$

we have $gC \subset gKC$

$$y^{-1/2} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} (\mathrm{Ad} \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix} C_N)(\begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix} C_A)K. \tag{3.4.4}$$

Assume that $C_N \in \mathbb{R}$ and $C_A \in \mathbb{R}^+$ are closed finite intervals. The image of this compact in $\mathbf{H}$ is a rectangle. The question is how many fundamental domains we need to cover this rectangle. Since $y > t$ for some fixed positive $t$ the fundamental domain we need to "cover in vertical direction" is bounded. The adjoint action of $y$ on $C_N$ dilates in length with rate $y$. Therefore the number of fundamental domain we need to cover the rectangle is $O(y)$. We get the estimate

$$\int_{gC} |\phi(h)|dh \leq C_1 y||\phi||_1 \tag{3.4.5}$$

for some constant $C_1$. Using Cauchy-Schwarz inequality and the finiteness of the volume of $\Gamma\backslash G$, we get

$$\int_{gC} |\phi(h)|dh \leq C_2 y||\phi||_2. \tag{3.4.6}$$

THe proposition derives from this and (3.4.2). □

**Corollary 3.4.3.** *If $\phi \in L^2(\Gamma\backslash G)$ satisfying the condition $(1,2,3,4)$ of 3.3.3, then it also satisfies the moderate growth condition $(5)$.*

*Proof.* By 3.3.4 and 3.3.5, if $\phi$ satisfies $(1,2,3,4)$ of 3.3.3, then $\phi$ is real analytic end there exists $\phi \in C_c^\infty(G)$ with support arbitrarily close identity such that $\phi = \phi * \alpha$. We have the estimate

$$|\phi(g)| \leq cy||\phi||_2$$

on the Siegel domain $\Omega_N A_t K$ of the cusp $\infty$. We have also similar estimates for other cusps. It remains now to apply 3.1.1. □

3.5. **Constant terms.** Suppose that $\infty$ is a cusp of $\Gamma$. Let $B = NA$ is the subgroup of triangular matrices in $G$, $N$ its unipotent radical and $\Gamma_N = \Gamma \cap N$. Let $\phi$ be a $\Gamma_N$-invariant function on $G$ that is locally integrable. The constant term $\phi_B$ of $f$ is, by definition the function

$$\phi_B(g) = \int_{\Gamma_N\backslash N} \phi(ng)dn. \tag{3.5.1}$$

where the invariant measure is normalized so that the quotient $\Gamma_N\backslash N$ has volume one.

Since the constant term is defined by integration on the left, this operation commutes with left-invariant differential operators as well as the convolution on the right

$$D(\phi_B) = (D\phi)_B \text{ and } (\phi * \alpha)_B = \phi_B * \alpha$$

for all $D \in U(\mathfrak{g})$ and $\alpha \in C_c(G)$.

**Lemma 3.5.1.** *Let $f$ be a modular form of weigh $k$ and $f = \sum_{n=0}^\infty a_n q^n$ its Fourier expansion at $\infty$. Let $\phi$ denote the associate function $\phi(g) = y^k e^{ik\theta} f(x+iy)$. Then the constant term of $\phi$ is given by the formula $\phi_B(g) = y^k e^{ik\theta} a_0$.*

**Proposition 3.5.2.** *Let $L^1_{\text{cusp}}(\Gamma\backslash G)$ be the subspace of $L^1(\Gamma\backslash G)$ of integrable functions that have vanishing constant terms at every cusp. Then $L^1_{\text{cusp}}(\Gamma\backslash G)$ is a closed subspace, stable under the convolution on the right of $C_c(G)$.*

*Proof.* □

Since $\Gamma\backslash G$ has finite volume, by applying the Cauchy-Schwarz inequality

$$\int_{\Gamma\backslash G} |\phi(x)|dx \le \left(\int_{\Gamma\backslash G} |\phi(x)|^2 dx\right)^{1/2} \left(\int_{\Gamma\backslash G} dx\right)^{1/2}$$

we have an inclusion $L^2(\Gamma\backslash G) \subset L^1(\Gamma\backslash G)$, and the injection is continuous. We define

$$L^2_{\text{cusp}}(\Gamma\backslash G) = L^2(\Gamma\backslash G) \cap L^1_{\text{cusp}}(\Gamma\backslash G).$$

It follows that

**Proposition 3.5.3.** *$L^2_{\text{cusp}}(\Gamma\backslash G)$ is a closed subspace of $L^2(\Gamma\backslash G)$, stable under the convolution on the right of $C_c(G)$.*

**Proposition 3.5.4.** *Let $X_1, X_2, X_3$ be a basis of the Lie algebra of $\mathfrak{g}$. Let $\phi \in C^1(\Gamma_N\backslash G)$. Then there exists a constant $c > 0$, independent of $f$, such that*

$$|(\phi - \phi_B)(g)| \le cy^{-1}\left(\sum_{i=1}^{3} |\mathcal{L}_{X_i}\phi|_B(g)\right)$$

*where $g(i) = x + iy$.*

*Proof.* We have denoted

$$R_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{3.5.2}$$

which is a generator of the Lie algebra of $N$. We have

$$(\phi_B - \phi)(g) = \int_0^1 (\phi(e^{tR_+}g) - \phi(g))dt. \tag{3.5.3}$$

But clearly,

$$\phi(e^{tR_+}g) - \phi(g) = \int_0^t (\mathcal{R}_{R_+}\phi)(e^{uR_+}g)du \tag{3.5.4}$$

where $\mathcal{R}_{R_+}$ is the right invariant derivation attached to $R_+$. We want to convert the above expression to left invariant derivation.

$$(\mathcal{R}_{R_+}\phi)(e^{uR_+}g) = \frac{d}{dt}\phi(e^{uR_+}e^{tR_+}g)|_{t=0} = \frac{d}{dt}\phi(e^{uR_+}ge^{t\mathrm{Ad}g^{-1}R_+})|_{t=0}$$

We will use again the Iwasawa decomposition

$$g = y^{-1/2}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

we have

$$e^{uR_+}ge^{t\mathrm{Ad}g^{-1}R_+} = e^{uR_+}ge^{ty^{-1}\mathrm{Ad}k_\theta^{-1}R_+}.$$

There are smooth functions $c_1, c_2, c_3$ on $K$ so that

$$k_\theta^{-1}R_+ = \sum_{i=1}^{3} c_i(k_\theta)X_i$$

We have

$$(\mathcal{R}_{R_+}\phi)(e^{uR_+}g) = y^{-1}\sum_{i=1}^{3} c_i(k_\theta)(\mathcal{L}_{X_i}\phi)(e^{uR_+}g).$$

The continuous functions on the compact set $K$ are bounded by some constant $c$ so that

$$|(\mathcal{R}_{-R_+}\phi)(e^{uR_+}g)| \le cy^{-1}\sum_{i=1}^{3} |(\mathcal{L}_{X_i}\phi)(e^{uR_+}g)|. \tag{3.5.5}$$

It follows that

$$\begin{aligned} |\int_0^t (\mathcal{R}_{R_+}\phi)(e^{uR_+}g)du| &\le cy^{-1}\sum_{i=1}^{3}\int_0^1 |(\mathcal{L}_{X_i}\phi)(e^{uR_+}g)|du \\ &= cy^{-1}\sum_{i=1}^{3} |\mathcal{L}_{X_i}\phi|_B(g) \end{aligned}$$

from which derives our desired estimate. □

### 3.6. **Convolution on the cuspidal spectrum.**

**Proposition 3.6.1.** *Let $\alpha \in C_c^1(G)$. There exists a constant $c(\alpha)$ so that*

$$|(\phi * \alpha)(g)| \le c(\alpha)||\phi||_2$$

*for all $\phi \in L^2_{\mathrm{cusp}}(\Gamma\backslash G)$.*

*Proof.* Let $P$ be a cuspidal subgroup of $\Gamma$ and $\mathfrak{S}$ s Siegel set relative to $P$. By 3.4.2, there exists a constant $c_1(\alpha)$, depending only on $\alpha$ such that

$$|(\phi * \alpha)(g)| \le c_1(\alpha)y||\phi||_2 \tag{3.6.1}$$

for $g \in \mathfrak{S}$ and $g(i) = x + iy$. If $X \in \mathfrak{g}$ and $\mathcal{L}_X$ is the associated left invariant derivation, we have $\mathcal{L}_X(\phi * \alpha) = \phi * \mathcal{L}_X(\alpha)$. Thus there exists a constant $c(\alpha, X)$ such that

$$|(\mathcal{L}_X(\phi * \alpha))(g)| \le c(\alpha, X)y||\phi||_2. \tag{3.6.2}$$

for $g \in \mathfrak{S}$. Since the translation on the left by the unipotent radical $N$ of $B$ keeps the coordinate $y$ of $g$ invariant, the integration over $N$ gives

$$|(\mathcal{L}_X(\phi * \alpha))_B(g)| \le c(\alpha, X)y||\phi||_2. \tag{3.6.3}$$

We are now applying the inequality to the elements $X_1, X_2, X_3$ of a basis of $\mathfrak{g}$. By assumption $\phi_B = 0$ so that $(\phi * \alpha)_B = 0$, we have the estimate

$$|(\phi * \alpha)(g)| \le c(\alpha)||\phi||_2 \tag{3.6.4}$$

for some constant $c(\alpha)$ that depends only on $\alpha$. □

**Proposition 3.6.2.** *For all $\alpha \in C_c^2(G)$, the convolution $\phi \mapsto \phi*\alpha$ defines a compact operator on $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$.*

*Proof.* Let $\phi_n$ be a sequence of function on $\Gamma\backslash G$ such that $||\phi_n||_2 \leq 1$. The estimate 3.6.1 implies that there exists a contant $c(\alpha)$ so that $|(\phi_n * \alpha)(g)| < c(\alpha)$ for all $g$ and all $n$. Thus the sequence $\phi_n * \alpha$ is bounded with respect to the uniform norm. The same estimate applies to $\mathcal{L}_X(\phi_n * \alpha)$ for all left invariant derivation $\mathcal{L}_X$. It implies that this family is equicontinuous. Now by application of the Arzela-Ascoli theorem and we can extract a subsequence $\phi_{n'} * \alpha$ that converges locally uniformly to a function $\phi'$ that is continuous and bounded. Since $\Gamma\backslash G$ has finite volume, $\phi'$ is square integrable. It remains only to recall that $L^2_{\text{cusp}}(\Gamma\backslash G)$ is a closed subspace of $L^2(G)$ so that $\phi' \in L^2_{\text{cusp}}(\Gamma\backslash G)$. □

The following general statement is due to Gelfand, Graev and PS. The following proof is due to Langlands. See [11].

**Proposition 3.6.3.** *Let $(\pi, \mathsf{H})$ be a unitary representation of $G$. If there exists a delta sequence $\phi_n$ on $G$ such that $\pi(\phi_n)$ is compact operator on $\mathsf{H}$ then $\mathsf{H}$ is a Hilbert direct sum of irreducible unitary representations which appear with finite multiplicity.*

*Proof.* The idea is to use the spectral theory of compact self-adjoint operators. Recall that a compact self-adjoint operators have eigenvalues, the set of eigenvalues has no accumulation point except 0, and for each non zero eigenvalue, the corresponding eigenspace is finite dimensional.

Let $(\pi_1, \mathsf{H}_1)$ be an irreducible unitary representation of $G$. Since $\phi_n$ is a delta sequence, $\pi_1(\phi_n) \neq 0$ for some $n$. There exists $\lambda \neq 0$ so that the $\lambda$-eigenspace of $\pi_1(\phi_n)$ is non-zero. Since the $\lambda$-eigenspace of the compact self-adjoint operator $\pi(\phi_n)$ is finite dimensional, $(\pi_1, \mathsf{H}_1)$ appears in $(\pi, \mathsf{H})$ with finite multiplicity.

We will now prove that $\mathsf{H}$ has at least one irreducible subspace with the help of the compact self-adoint operators $\pi(\phi_n)$. For $n$ large enough $\pi(\phi_n) \neq 0$ has a nonzero eigenvalue $\lambda$. Let $V$ denote the $\lambda$-eigenspace of $\pi(\phi_n)$ which is finite dimensional. Consider the non zero subspace $V' \subset V$ so that $V' = V \cap \mathsf{H}'$ where $\mathsf{H}'$ is a closed invariant subspace of $\mathsf{H}$. Since $V$ is finite dimensional, this family has minimal element for the inclusion order. Let $V_1$ is a minimal element of this family.

Let $\mathsf{H}_1$ denote the intersection of all closed invariant subspaces $\mathsf{H}' \subset \mathsf{H}$ such that $\mathsf{H}' \cap V = V_1$. We will prove that $\mathsf{H}_1$ is irreducible. If it is not, we have a direct decomposition $\mathsf{H}_1 = A \oplus B$ where $A, B$ are closed invariant subspaces. Since $A$ and $B$ are stable under $\pi(\phi_n)$, if we decompose a vector $v \in V_1$ as $v = a \oplus b$ with $a \in A$ and $b \in B$, then $a, b$ are also $\lambda$-eigenvectors. It follows that

$$V_1 = (V_1 \cap A) \oplus (V_1 \cap B).$$

The minimality of $V_1$ implies that either $V_1 \cap A = V_1$ or $V_1 \cap B = V_1$. But this contradicts with the minimality of $\mathsf{H}_1$.

Let $\mathsf{H}'$ be the closure of the sum of all irreducible subspaces in $\mathsf{H}$ and let $\mathsf{H}''$ be the orthogonal complement of $V$. If $\mathsf{H}'' \neq 0$, it contains an irreducible subspace. This would contradict with the contradict with the definition of $\mathsf{H}'$. Therefore $\mathsf{H}'' = 0$ and $\mathsf{H} = \mathsf{H}'$. □

**Theorem 3.6.4.** *The Hilbert space $L^2_{\text{cusp}}(\Gamma\backslash G)$ decomposes as a direct sum of irreducible unitary representations of $G$, each occurs with finite multiplicity.*

3.7. **Duality theorem.**

**Proposition 3.7.1** (Gelfand, Graev, PS)**.** *There is an isomorphism*

$$S_k(\Gamma) = \mathrm{Hom}_G(\mathcal{D}_k^+, L^2_{\mathrm{cusp}}(\Gamma\backslash G)).$$

*Proof.* Let fix a lowest weight vector $v_k$ of $\mathcal{D}_k^+$. It is of $K$-weight $k$ and annihilated by $\mathcal{L}_L$. Let $\alpha : \mathcal{D}_k^+ \to L^2_{\mathrm{cusp}}(\Gamma\backslash G)$. Then $\alpha(v_k)$ is an analytic $L^2$-function that is of $K$-weight $K$, annihilated by $\mathcal{L}_L$. By 2.9.6, $\phi$ comes from a cuspidal modular form $f$ of weight $k$. □

Let $s$ be a purely imaginary number and let $\mathcal{P}_s$ be the unitary principal series. Let $v_0$ be a vector of weight 0 in $\mathcal{P}_s$. For any intertwining operator

$$\alpha \in \mathrm{Hom}_G(\mathcal{P}_s, L^2_{\mathrm{cusp}}(\Gamma\backslash G))$$

the function $\alpha(v_0)$ is a cuspidal analytic function on $\Gamma\backslash G$ that is right $K$-invariant. It satisfies the differential equation $\Omega\phi = (s^2-1)\phi$. It defines an analytic function $f : \Gamma\backslash \mathbf{H} \to \mathbb{C}$ which is an eigenvalue of the Laplacian

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \text{ then } \Delta f = \frac{1-s^2}{4} f. \tag{3.7.1}$$

Those are Maass modular forms, eigenvector for the Laplacians with eigenvalues $\lambda \geq 1/4$.

**Conjecture 3.7.2** (Selberg)**.** *For all $s \in [-1,1]$, and $\mathcal{C}_s$ the complementary series, we have*

$$\mathrm{Hom}_G(\mathcal{C}_s, L^2_{\mathrm{cusp}}(\Gamma\backslash G)) = 0$$

*for all congruence subgroup.*

This is actually false for some non congruence subgroup.

## 4. Smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

In this chapter, $G = \mathrm{GL}_2(F)$ where $F$ is a local nonarchimedean field. Let $\mathcal{O}_F$ denote the ring of integers of $F$.

4.1. **Cartan and Iwasawa decompositions.** As in the case of real Lie groups, Cartan and Iwasawa decomposition plays an important role in their representation theory. In the case of $\mathrm{GL}_2$, these decompositions can be described with linear algebras over the $p$-adic fields. Let $B = AN$ denote the subgroup of upper triangular matrices, $A$ is the group of diagonal matrices and $N$ the group of unipotent upper triangular matrices. The group $K(1) = \mathrm{GL}_2(\mathcal{O}_F)$ is a maximal compact open subgroup of $G$. Every compact subgroup subgroup of $G$ is conjugate to a subgroup of $K(1)$ of finite index.

**Proposition 4.1.1** (Cartan decomposition)**.**

$$G = \bigsqcup_{d_1 \leq d_2, d_i \in \mathbb{Z}} K(1) \begin{bmatrix} p^{d_1} & 0 \\ 0 & p_2^d \end{bmatrix} K(1).$$

*Proof.* Let $X$ denote the set of all lattices $\mathcal{L} \subset F^2$. The group $G$ acts transitively on $X$ and its stabilizer at the standard lattice $\mathcal{L}_0 = \mathcal{O}_F^2$ is $K$. It follows that $X = G/K(1)$. The Cartan decomposition now follows from the theory of elementary divisors : for every lattice $\mathcal{L} \in X$, there exist a basis $x_1, x_2$ of $\mathcal{L}_0$ so that $\mathcal{L} = p^{d_1} x_1 \mathcal{O}_F \oplus p^{d_2} x_2 \mathcal{O}_F$ where $d_1, d_2$ are integers satisfying $d_1 \leq d_2$. □

**Proposition 4.1.2** (Iwasawa decomposition)**.**

$$G = BK(1) = \bigsqcup_{d_1,d_2\in\mathbb{Z}} N \begin{bmatrix} p^{d_1} & 0 \\ 0 & p_2^d \end{bmatrix} K(1).$$

*Proof.* Let $x_1, x_2$ denote the standard basis of $F$. The group $B$ upper triangular matrices is the stabilizer of the line generated by $x_1$. For every lattice $\mathcal{L} \in X$, we can define two integers $d_1, d_2 \in \mathbb{Z}$ as follows : $\mathcal{L}_1 = \mathcal{L} \cap x_1 F$ being a lattice of $x_1 F$, must be of the form $p^{d_1} x_1 \mathcal{O}_F$ for some integer $d_1$; the quotient $\mathcal{L}/\mathcal{L}_1$ being naturally a lattice in $x_2 F$, must be of the form $p^{d_2} x_2 \mathcal{O}_F$. It is not hard to check that there exists a triangular matrix $n \in N$ so that

$$\mathcal{L} = n \begin{bmatrix} p^{d_1} & 0 \\ 0 & p_2^d \end{bmatrix} \mathcal{L}_0.$$

The Iwasawa decomposition follows. □

4.2. **Hilbert representations.** Let $C_c^\infty(G)$ denote the space of locally constant function with compact support. With a choice of Haar measure on $G$, $C_c^\infty(G)$ is an algebra with respect to the convolution product. Be aware that this algebra is not equipped wit ha unit. We observe that every function $f \in C_c^\infty(G)$ is biinvariant with respect to some compact open subgroup $K$. In other words, if $\mathcal{H}_K$ denotes the algebra, with unit, of $K$-biinvariant functions with compact support, then

$$C_c^\infty(G) = \bigcup_K \mathcal{H}_K$$

the union being taken over all compact open subgroups of $G$.

Let $(\pi, \mathsf{H})$ be a Hilbert representation of $G$. The algebra $C_c^\infty(G)$ acts on $\mathsf{H}$ by

$$\pi(\phi)v = \int_G \phi(g)\pi(g)v dg.$$

If $\phi \in \mathcal{H}_K$, then $\pi(\phi)v \in \mathbf{H}^K$.

A vector $v \in \mathsf{H}$ is said to be smooth if is stabilized by a compact open subgroup of $G$. Let $V$ be the subspace of smooth vectors in $\mathsf{H}$. The group $G$ acts continuously on $V$ equipped with the discrete topology. For all compact open subgroup $K$ of $G$, the subspace $V^K$ of $K$-invariant vectors is a module of $\mathcal{H}_K$.

**Proposition 4.2.1.** *Let $(\pi, \mathsf{H})$ be a Hilbert representation of $G$ and let $V$ denote its subspace of smooth vectors. Then $\mathsf{H}$ is irreducible if and only if $V$ an irreducible smooth representation of $G$, if and only if for every compact open subgroup $K$ of $G$, $V^K$ is an irreducible $\mathcal{H}_K$-module.*

Smooth representations of $p$-adic groups play a similar roles to $(\mathfrak{g}, K)$-modules of real Lie groups.

4.3. **Unramified representations.**

**Definition 4.3.1.** *An irreducible smooth representation of $G$ is said to be unramified if $V^{K(1)} \neq 0$ where $K(1)$ is the maximal compact subgroup of $G$.*

Unramified representations are just irreducible $\mathcal{H}$-modules. We have a complete description of the algebra $\mathcal{H}$ that allows us to describe its irreducible modules. By using Gelfand's trick, one sees that $\mathcal{H}$ is commutative. In fact, we have a much more precise description of this algebra using Harish-Chandra's constant terms.

**Theorem 4.3.2** (Satake isomorphism)**.** *For every $\phi \in \mathcal{H}$, the constant term of $\phi$ along $B$ is the function $\phi_B : A \to \mathbb{C}$ given by the formula*

$$\phi_B(a) = \delta_B(a)^{1/2} \int_N \phi(an)dn$$

*the Haar measure $dn$ on $N$ is choosen so that $K \cap N$ has measure one. The constant term map defines an isomorphism from $\mathcal{H}$ onto the algebra of compactly supporte functions on $A$ invariant under $A \cap K$ and under the action of the Weyl group $W$.*

*Proof.* Orbital integral of a diagonal elements can be expressed as constant terms. ☐

**Corollary 4.3.3.** *Let $\hat{A}$ denote the complex dual torus of $A$. There is an isomorphism of algebras between $\mathcal{H}$ and $\mathbb{C}[\hat{A}]^W$. In particular, the unramified representations of $G$ are in natural bijection with $\hat{A}/W$.*

**Corollary 4.3.4.** *Let $\hat{G} = \mathrm{GL}_2(\mathbb{C})$ denote the Langlands dual group of $G = \mathrm{GL}_2(F)$. The unramified representations of $G$ are classified by the semisimple conjugacy classes of $\hat{G}$.*

The trivial representation of $G$ is certainly unramified. It corresponds to the homomorphism $\mathcal{H} \to \mathbb{C}$ given by

$$\phi \mapsto \int_G \phi(g)dg.$$

The associated conjugacy semisimple class in $\mathrm{GL}_2(\mathbb{C})$ is the class of the matrix

$$\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \tag{4.3.1}$$

**Proposition 4.3.5.** *If $V$ is an unramified representation then its contragredient $V'$ is also an unramified.*

*Proof.* The linear form $v \mapsto \int_{K(1)} \pi(k)v$ is $K(1)$-invariant. ☐

**Proposition 4.3.6.** *Let $V$ be an unramified representation of $V$ and $v \in V$ be a $K(1)$-invariant vector.*

4.4. **Smooth admissible representations.** A representation of $G$ on a complex vector space $V$ is called smooth if every vector $v \in V$ is stabilized by a compact open subgroup $K$.

**Lemma 4.4.1.** *Let $V'$ be a subspace of $V$ that stable under the action of $K$ and irreducible as representation of $K$. Then $V'$ is finite-dimensional.*

*Proof.* A vector $v' \in V'$ is stabilized by a finite index subgroup $K'$ of $K$. The subspace of $V'$ generated by $gv$ for a set of representatives $K/K'$ is $K$-stable. Since $V'$ is irreducible, $V'$ is generated by those vectors and hence finite-dimensional. ☐

Let $K$ be a compact open subgroup of $G$. Let $V$ be a smooth representation of $G$. For every finite dimensional irreducible representation $(\rho, V_\rho)$ of $K$ be the sum of all $K$-invariant subspaces of $V$ which are isomorphic to $V_\rho$. The $V$ can be decomposed as an algebraic direct sum

$$V = \bigoplus_{\rho \in \hat{K}} V(\rho).$$

$V$ is said to admissible if and only if for every $\rho \in \hat{K}$, $V(\rho)$ is finite dimensional. A smooth representation of $G$ is admissible if and only if $\dim(V^K)$ is finite for all compact open subgroups $K$ of $G$.

**Theorem 4.4.2** (Harish-Chandra)**.** *All irreducible smooth representations of $G$ are admissible.*

Let $V$ be a smooth representation of $G$. The contragredient representation $V'$ is the space of smooth vectors in the vector space of linear forms on $V$.

**Proposition 4.4.3.** *Let $V$ be a smooth admissible representation of $G$. Let $V = \bigoplus_\rho V(\rho)$ denote the decomposition of $V$ into isotypical components with respect to a compact open subgroup $K$ of $G$. Then the contragredient of $G$ is an algebraic direct sum*

$$V' = \bigoplus V(\rho)'$$

*where $V(\rho)' = \mathrm{Hom}(V(\rho), \mathbb{C})$. In particular, $V'$ is admissible if $V$ is. In that case, we have $V = (V')'$.*

**Theorem 4.4.4** (Gelfand-Kazhdan)**.** *The group $G = \mathrm{GL}_2(\mathbb{Q}_p)$ is equipped with the outer automorphism*

$$g \mapsto {}^\top g^{-1}.$$

*Let $(\pi, V)$ be a irreducible admissible representation of $G$. Then the contragredient $(\pi', V')$ is isomorphic with $(\pi'', V)$ where $\pi''(g) = \pi({}^\top g^{-1})$.*

This theorem is surprisingly difficult to prove. We will need in particular to introduce the notion of character of an admissible representation. For every $\phi \in C_c^\infty(G)$, there exists open compact subgroup $G$ so that $\phi \in \mathcal{H}_K$. The operator $\pi(\phi) : V \to V$ has image contained in the finite-dimensional space $\mathcal{H}^K$. The number

$$\mathrm{Tr}_\pi(\phi) = \mathrm{Tr}(\pi(\phi)|_{V^K})$$

does not depend on the choice of $K$. We thus define a linear map

$$\mathrm{Tr}_\pi : C_c^\infty(G) \to \mathbb{C}$$

hence a distribution on $G$.

**Proposition 4.4.5.** *Two irreducible admissible representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ having the same character $\mathrm{Tr}_{\pi_1} = \mathrm{Tr}_{\pi_2}$, are isomorphic.*

*Proof.* Let $K$ be a compact open subgroup of $G$ small enough so that $V_1$ and $V_2$ have nonzero $K$-invariant vectors. It follows that $V_1^K$ and $V_2^K$ are irreducible $\mathcal{H}_K$-modules having the same function trace. It follows that they are isomorphic as $\mathcal{H}_K$-modules (Bourbaki, see Lang's discussion on the Jacobson density theorem). Hence $V_1$ and $V_2$ are isomorphic as smooth representations of $G$. □

In order to prove the Gelfand-Kazhdan theorem, we need to prove that $(\pi', V')$ and $(\pi'', V)$ have the same character. Let $\phi \in C_c^\infty(G)$. Let denote $\phi'(g) = \phi(g^{-1})$ and $\phi''(g) = \phi({}^\top g^{-1})$. By definition, $\pi'(\phi)$ and $\pi(\phi')$ are adjoint operators so that they have the same trace. Also by definition, we have $\pi''(\phi) = \pi(\phi'')$. It is thus enough to prove that

$$\mathrm{Tr}_\pi(\phi') = \mathrm{Tr}_\pi(\phi'')$$

or in other words that the distribution $\mathrm{Tr}_\pi$ is transposition invariant.

**Proposition 4.4.6.** *Over* $\mathrm{GL}(2,F)$*, a conjugation invariant distribution is also transposition invariant.*

4.5. **Analysis on totally disconnected spaces.** A td-space is a separated topological space in which every point has a basis of neighborhood given by compact open subsets.

**Lemma 4.5.1.** *Let* $X$ *be a td-space. Let* $Y$ *be a compact subset of* $X$*. From every covering of* $X$ *by a open subsets of* $X$*, we can extract a finite family that, after refinement, forms a covering of* $Y$ *by disjoint open subsets of* $X$*.*

We will denote $S(X) = C_c^\infty(X)$ the space of complex valued locally constant functions with compact support on $X$. For every compact open subset $U$ of $X$, we denote $1_U$ the characteristic function of $U$. All function $\phi \in S(X)$ is a finite linear combination of such functions $1_U$.

The vector space $S(X)$ has a structure of algebra with respect to the pointwise multiplication. For every $x \in X$, let $\mathfrak{m}_x$ the ideal of $S(X)$ of functions vanishing at $x$. Every element $\phi \in \mathfrak{m}_x$ is a finite linear combination of $1_U$ where $U$ is a compact open not containing $x$. In fact, we have $\mathfrak{m}_x = S(X - x)$. We have an exact sequence of $\mathbb{C}$-vector spaces

$$0 \to \mathfrak{m}_x \to S(X) \to \mathbb{C} \to 0.$$

In fact, all morphism in this sequence are ring morphism. We have more generally the following proposition.

**Lemma 4.5.2.** *Let* $Y$ *be a closed subset of* $X$ *and* $U = X - Y$*. The we have an exact sequence*

$$0 \to S(U) \to S(X) \to S(Y) \to 0.$$

*Proof.* Only the surjectivity $S(X) \to S(Y)$ is not totally obvious. It follows in fact from 4.5.1. □

**Definition 4.5.3.** *A* $S(X)$*-module* $M$ *is called smooth if for every* $m \in M$*, there exists a compact open subset* $U$ *of* $X$ *such that* $1_U m = m$*.*

Let $M$ be a smooth $S(X)$-module. We can define the fiber of $M$ at a point $x$ as follows

$$M_x = M/\mathfrak{m}_x M$$

where $\mathfrak{m}_x M$ is the submodule of $M$ generated by the elements $\phi m$ with $\phi \in \mathfrak{m}_x$ and $m \in M$.

**Lemma 4.5.4.** *Let* $M$ *be a smooth* $S(X)$*-module and* $x \in X$*. An element* $m \in \mathfrak{m}_x M$ *if and only if for all small enough compact open* $U$ *containing* $x$ *so that* $1_U m = 0$*.*

*Proof.* If $m = 1_V m'$ where $V$ is a compact open subset not containing $x$, then for all compact open $U$ containing $x$ but disjoint from $V$ we have $1_U m = 1_U 1_V m' = 0$. Let $m \in M$ such that $1_U m = 0$ for all small enough compact open $U$ containing $x$. By smoothness hypothesis there exists a compact open subset $V$ such that $1_V m = m$. If $x \notin V$, we are done. If $x \in V$, there exists a compact open $U$ with $x \in U \subset V$ such that $1_U m = 0$. Then $1_{V-U} m = m$. □

**Lemma 4.5.5.** *Let* $M$ *be a smooth* $S(X)$*-module. Suppose that* $M_x = 0$ *for all* $x \in X$*. Then* $M = 0$*.*

*Proof.* Let $m \in M$ be an arbitrary element. There exists a compact open subset $V$ so that $1_V m = m$. For every $x \in V$, since $M_x = 0$ there exists a compact open subset $U_x$ containing $x$ and contained in $V$ such that $1_{U_x} x = 0$. We can even require that $1_{U'_x} m = 0$ for all compact open $U'_x$ contained in $U_x$ containing $x$. It follows that $1_{U'_x} = 0$ for all compact open subset $U'_x$ of $U_x$ that may contain $x$ or not because if $x \notin U'_x$, we can write $1_{U'_x} = 1_{U_x} - 1_{U_x - U'_x}$. By applying 4.5.1, we can write $1_V$ as finite sum of $1_{U'_x}$ which implies $m = 0$. □

**Lemma 4.5.6.** *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of smooth $S(X)$-modules. Then for every $x \in X$, we have an exact sequence*

$$0 \to M'_x \to M_x \to M''_x \to 0.$$

*Proof.* The right exactness is a general property of tensor product. Here, we also have the left exactness because $M_x$ is not only the residual fiber but the fiber itself. More concretely, let $m' \in M'$ whose image $m \in M$ satisfies $m_x = 0$. There exists a compact open $U$ containing $x$ such that $1_U m = 0$. This implies that $1_U m' = 0$ and of course $m'_x = 0$. □

There are plenty of natural smooth $S(X)$-module.

**Proposition 4.5.7.** *Let $Y \to X$ be a continuous map of td-spaces. Then $S(Y)$ is naturally a smooth $S(X)$-module. Moreover, for every $x \in X$, we have $S(Y)_x = S(Y_x)$.*

*Proof.* Let $\phi$ be a locally constant with compact support $U$ in $Y$. The image of $U$ in $X$ is a compact subset of $X$ that can be covered by a compact open subset $U$ of $X$. Then we have $1_U \phi = \phi$.

We have an exact sequence

$$0 \to S(Y - Y_x) \to S(Y) \to S(Y_x) \to 0$$

where $S(Y - Y_x) = \mathfrak{m}_x S(Y)$. It follows that $S(Y_x) = S(Y)_x$. □

**Proposition 4.5.8.** *Let $Y \to X$ be a continuous map between td-spaces. Let $G$ be a td-group acting on $Y$, preserving the map $Y \to X$ and acting transitively on the fibers $Y_x$. Let $S(Y)_G$ be the quotient of $S(Y)$ by the subspace $S(Y)(G)$ generated by the functions of the form $g\phi - \phi$. Then $S(Y)_G$ is a smooth $S(X)$-module whose fibers $S(Y)_{G,x}$ is of dimension at most one.*

*Proof.* Since the action of $G$ on $S(Y)$ commute with multiplication by $S(X)$, the subspace $S(Y)(G)$ is a $S(X)$-submodule. The module quotient is automatically smooth because $S(Y)$ is a smooth $S(X)$-module.

The fiber $S(Y)_{G,x}$ is the quotient of $S(Y)$ by the subspace generated by $\mathfrak{m}_x S(Y)$ and $S(Y)(G)$. It is thus the quotient of $S(Y_x)$ by the image of $S(Y)(G)$ in $S(Y_x)$ which is $S(Y_x)(G)$.

Now, since $G$ acts simply transitively on $S(Y_x)$, the space of $G$-invariant distribution on $Y_x$ is of dimension at most one. □

*Proof.* (of 4.4.6) Let $Y$ denote $G - Z$ the space of non central $2 \times 2$-matrices. Let $Y \to X = F \times F^\times$ denote the characteristic polynomial $g \mapsto (\mathrm{Tr}(g), \det(g))$. The group $G$ acts transitively on the fibers of $Y \to X$ by conjugation. For every $y \in Y$, the centralizer $G_y$ is commutative, in particular unimodular, hence the quotient $G/G_y$ has a $G$-invariant measure. The $S(X)$-module $S(Y)_G$ has fibers of dimension one. The transposition defines an involution on this space that acts as identity on every fiber. It follows from 4.5.5 that it acts trivially on the whole module $S(Y)_G$. In other words, a conjugation invariant distribution on $Y$ in transposition invariant. Since the similar statement is trivial for $Z$, the statement follows from the exact sequence 4.5.2. $\square$

The same technique of proving 4.4.6 will permit us to prove an important property of Bessel distributions. Let us fix a non trivial additive charater $\psi : F \to \mathbb{C}^\times$. We call Bessel functions the locally constant functions $b : G \to \mathbb{C}$ satisfying the transformation property

$$b(n_1 g n_2) = \psi(n_1)\psi(n_2)b(g).$$

We will be interested in the more general notion of Bessel distributions $b : S(G) \to \mathbb{C}$ satisfying

$$b(l_{n_1^{-1}} r_{n_2} \phi) = \psi(n_1)\psi(n_2)b(\phi).$$

Recall that by definition $l_{n_1^{-1}} r_{n_2}\phi(g) = \phi(n_1 g n_2)$.

Let $\theta : G \to G$ be the anti-involution defined by $\theta(g) = w_0 {}^\top g w_0$ where $w_0$ is the permutation matrix. For every $n \in N$ we have $\theta(n) = n$. The involution $\theta$ induces an involution on $S(G)$ and on the space of distribution.

**Proposition 4.5.9.** *If $b$ is a Bessel distribution, then we have $\theta(b) = b$.*

4.6. **Uniqueness of the Whittaker model.** Let $N$ be the subgroup of unipotent upper triangular matrices. Let $\psi : F \to \mathbb{C}^\times$ be a non trivial character of $F$. A Whittaker function with respect to $\psi$ is a locally constant function $\phi : G \to \mathbb{C}$ such that $\phi(ng) = \psi(n)\phi(g)$ for all $n \in N$ and $g \in G$ and there exists a compact open subset $C \subset G$ so that the support of $\phi$ is contained in $NC$. The space of all Whittaker functions is

$$\mathrm{cInd}_N^G(\psi)$$

the compact induction of $\psi$. This space affords with a representation of $G$ by translation on the right.

Let $(\pi, V)$ be an irreducible admissible representation of $G$. A Whittaker model of $G$ is an intertwining operator $\alpha : V \to \mathrm{cInd}_N^G(\psi)$. For all such $\alpha$, we have a linear form $l_\alpha : V \to \mathbb{C}$ given by $l_\alpha(v) = \alpha(v)(1_G)$ which satisfies $l_a(\pi(n)v) = \psi(n)l_a(v)$. In general, $l_a$ is not fixed by any compact open subgroup so that $l_\alpha$ does not belong to the contragredient representation $V'$.

We have in fact a bijection

$$\mathrm{Hom}_G(V, \mathrm{cInd}_N^G(\psi)) = \mathrm{Hom}_N(V, \psi)$$

by application of the Frobenius reciprocity. Let $V_\psi(N)$ be the subspace of $V$ spanned by the vectors of the form $\pi(n)v - \psi(n)v$ with $n \in N$ and $v \in V$. Let $V_{N,\psi} = V/V_\psi(N)$. Every linear functional $l \in \mathrm{Hom}_N(V, \psi)$ must factorize though $V_{N,\psi}$ and in fact we have

$$\mathrm{Hom}_N(V, \psi) = \mathrm{Hom}(V_{N,\psi}, \mathbb{C}).$$

If these vector spaces are nonzero then we say that $V$ admits a Whittaker model.

**Lemma 4.6.1.** *If $V$ admits a Whittaker model with for a non trivial additive character $\psi : F \to \mathbb{C}^\times$, then it has Whittaker model for any non trivial additive character.*

*Proof.* The group $A$ of diagonal matrices acts transitively on the set of non trivial characters of $N$. $\square$

**Proposition 4.6.2.** *Let $V$ be an irreducible admissible representation. If $V$ admits a model of Whittaker then so does its contragredient $V'$.*

*Proof.* We know by Gelfand and Kazhdan that the contragredient representation $(\pi', V')$ is isomorphic to $(p'', V)$ where $\pi''(g) = \pi({}^{\top}g^{-1})$. If $l : V \to \mathbb{C}$ is a Whittaker functional of $\pi$ with respect to $(N, \psi)$ then it is a Whittaker functional for $\pi''$ with respect to ${}^{\top}(N, \psi)^{-1}$. After conjugation, we will get a Whittaker functionalof $\pi''$ with respect to $(N, \psi)$. □

**Theorem 4.6.3.** *An irreducible admissible representation admits at most one Whittaker model. In other words*

$$\dim_G(V, \mathrm{cInd}_N^G(\psi)) \leq 1.$$

The theorem follows from 4.6.2 and the following proposition.

**Proposition 4.6.4.** *Let $V$ be an irreducible admissible representation. Then we have*

$$\dim_G(V, \mathrm{cInd}_N^G(\psi)) \dim_G(V', \mathrm{cInd}_N^G(\psi)) \leq 1.$$

*Proof.* Let $l' : V \to \mathbb{C}$ and $l : V' \to \mathbb{C}$ be nonzero Whittaker functionals of $V$ and $V'$. If $l, l'$ are smooth linear form i.e. $l' \in V'$ and $l \in V$ we can define the matrix coefficient function $G \to \mathbb{C}$

$$c(g) = \langle l, \pi'(g)l' \rangle$$

that satisfies

$$c(n_1 g n_2) = \psi(n_1)\psi(n_2)c(g).$$

But in fact $l, l'$ are not smooth vectors, we will have to define $c$ as a Bessel distribution.

For every $\phi \in S(G)$, we consider the linear form $\pi'(\phi)l' : V \to \mathbb{C}$ defined by $v \mapsto l'(\pi(\phi')v)$ where $\phi'$ is the function $\phi(g) = \phi'(g^{-1})$. This is a smooth linear form on $V$ so that $\pi'(\phi)l' \in V'$. In the same way, for every $\phi \in S(G)$, we have a smooth vector $\pi(\phi)l \in V$. One can check that

$$\pi(r_n\phi)l = \psi(n)\pi(\phi)l \text{ and } \pi'(r_n\phi)l' = \psi(n')\pi'(\phi)l' \tag{4.6.1}$$

We also have

$$\pi(l_g\phi)l = \pi(g)\pi(\phi)l \text{ and } \pi'(l_g\phi)l' = \pi'(g)\pi'(\phi)l$$

so that the map $\phi \mapsto \pi(\phi)l$ and $\phi \mapsto \pi'(\phi)l'$ are intertwining operators $S(G) \to V$ and $S(G) \to V'$ respectively.

We have a bilinear form on $S(G)$

$$b(\phi_1, \phi_2) = \langle \pi(\phi_1)l, \pi'(\phi_2)l' \rangle$$

which satisfies

$$b(l_g\phi_1, l_g\phi_2) = b(\phi_1, \phi_2).$$

Note that $S(G) \otimes S(G) = S(G \times G)$. Consider the diagonal action of $G$ acting by left translation on $G \times G$ given by the formula $g(g_1, g_2) = (gg_1, gg_2)$. The quotient $G\backslash(G \times G)$ can be identified with $G$ by the map $G \times G \to G$ given by $(g_1, g_2) \mapsto g_1^{-1}g_2$. The integration along the fibers of this map defines a linear map

$$S(G) \otimes S(G) \to S(G)$$

which is $(\phi_1, \phi_2) \mapsto \phi_1' * \phi_2$ where $\phi_1'(g) = \phi_1(g^{-1})$. Because the linear form $\phi_1 \otimes \phi_2 \mapsto b(\phi_1, \phi_2)$ is $G$-invariant with respect to the diagonal action of $G$ by lest translation on $G \times G$, there exists a unique linear form

$$c : S(G) \to \mathbb{C}$$

such that $c(\phi_1' * \phi_2) = b(\phi_1, \phi_2)$.

Because of (4.6.1), $c$ is a Bessel distribution

$$c(l_{n_1} r_{n_2} \phi) = \psi^{-1}(n_1)\psi(n_2)c(\phi).$$

By applying 4.5.9, we know that $\theta(c) = c$ where $\theta$ is the induced operaotr on the space of distribution of $\theta(g) = w_0\, {}^{\top}g w_0$. Thus

$$c(\phi_1' * \phi_2) = c(\theta(\phi_1' * \phi_2)) = c(\theta(\phi_2) * \theta(\phi_1'))$$

thus

$$b(\phi_1, \phi_2) = b(\theta(\phi_2'), \theta(\phi_1')).$$

If $\phi_2$ is in the kernel of the intertwining operator $l' : S(G) \to V'$ the $\theta(\phi_2')$ is in the kernel of $l : S(G) \to V$. It follows that $l'$ determines the kernel of $l$. By Schur lemma, as $V$ is irreducible, $l'$ determines $l$ up to a scalar. The proposition follows. □

**Lemma 4.6.5.** *For every compact subgroup $N_0 \subset N$, we set*

$$V_\psi(N_0) = \{v \in V \mid \int_{N_0} \psi^{-1}(n)\pi(n)v dn = 0\}$$

*Then we have $V_\psi(N) = \bigcup_{N_0} V_\psi(N_0)$.*

*Proof.* An element $v \in V_\psi(N)$ is a linear combination of vectors of the form $\pi(n_0)v_0 - \psi(n_0)v_0$. Every $n_0 \in N$, there exists a compact subgroup $N_0$ of $N$ such that $n_0 \in N_0$. Then we have

$$\int_{N_0} \psi^{-1}(n)\pi(n)\pi(n_0)v_0 - \psi(n_0)v_0 dn = 0 \tag{4.6.2}$$

thus $\pi(n_0)v_0 - \psi(n_0)v_0 \in V_\psi(N_0)$. It follows that $V_\psi(N) \subset \bigcup_{N_0} V_\psi(N_0)$.

Let $v \in V(N_0)$. There exists a compact open subgroup $N_1 \subset N_0$ that stabilizes $v$. By shrinking $N_1$ we can also assume that $\psi$ is trivial on $N_1$. Then we have

$$\int_{N_0} \psi^{-1}(n)\pi(n)v dn = \sum_{n \in N_0/N_1} \psi^{-1}(n)\pi(n)v = 0 \tag{4.6.3}$$

Let $r$ denote the cardinal $N_0/N_1$. Then we have

$$v = \frac{1}{r} \sum_{n \in N_0/N_1} \psi^{-1}(n)(\psi(n)v - \pi(n)v)$$

and thus $v \in V_\psi(N)$. □

Let $V$ be an admissible representation of $G$. The above lemma suggest that Whittaker functionals are fibers of $V$ with respect to a structure of $S(F)$-module that we are going to define. Choose a non trivial additive character $\psi : F \to \mathbb{C}^\times$. The Fourier transform

$$\hat{\phi}(x) = \int_F \phi(xy)\psi(x)dx$$

defines an isomorphism $S(F) \to S(F)$. There exists a unique choice of Haar measure $dx$ so that $\hat{\hat{\phi}}(x) = \phi(-x)$. The Fourier transform turns pointwise multiplication into convolution product. It follows that if we let $\phi \in S(F)$ act on $V$ by

$$\iota(\phi)v := \int_N \hat{\phi}(n)\pi(n)v dn \tag{4.6.4}$$

then $V$ has a structure of $S(F)$-module.

**Lemma 4.6.6.** *$V$ is a smooth $S(F)$-module.*

*Proof.* Since $V$ is smooth as representation of $G$, there exists a compact open subgroup $N_0$ that stabilizes $v$. The Fourier transform of $1_{N_0}$ is of the form $1_U$ for certain compact open subgroup $U$. Then we have $\iota(1_U)v = \pi(1_{N_0})v = v$. □

**Lemma 4.6.7.** *For every $a \in F^\times$, $V_a$ and $V_{N,\psi_a}$, where $\psi_a$ is the additive character $x \mapsto \psi(ax)$ are the same quotient of $V$.*

*Proof.* The description of $V_{\psi_a}(N)$ given in 4.6.5 identifies this submodule with $V(a)$. It follows an identification with quotient modules $V_a = V_{N,\psi_a}$. □

**Proposition 4.6.8.** *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequences of admissible representations. Then we have an exact sequence*

$$0 \to M'_{N,\psi} \to M_{N,\psi} \to M''_{N,\psi} \to 0.$$

**Proposition 4.6.9.** *Let $V$ be an irreducible admissible representation. Then $G$ has Whittaker model.*

*Proof.* Assume that $V$ does not have Whittaker model. Then $V_a = V_{N,\psi_a} = 0$. It follows that $V = V_0$. The fiber $V_0$ is the Jacquet module of $V$ that is finite dimensional as we will later show. □

4.7. **Jacquet module.** Let $(\pi, V)$ be a smooth representation of $G$. The space $V(N)$ is the subspace of $V$ spanned by the vectors of the form $\pi(n)v - v$ with $n \in N$ and $v \in V$. For a compact subgroup $N_0 \subset N$ define $V(N_0)$ to be

$$V(N_0) = \{v \in V \mid \int_{N_0} \pi(n)v dn = 0\}.$$

As in 4.6.5, we have $V(N)$ to be $V(N) = \bigcup_{N_0} V(N_0)$. Let define the Jacquet module $V_N = V/V(N)$ which is also the fiber $V_0$ of $V$ as $S(G)$-module defined in (4.6.4).

The Jacquet module $V_N$ is equipped with an action $\pi_N$ of $A = B/N$. Observe that if a vector $v \in V$ is fixed by a compact open subgroup $K$ of $G$, its image $\bar{v}$ in $V_N$ is necessarily fixed by $K \cap A$. Hence, $(\pi_N, V_N)$ is a smooth representation of $A$ provided that $(\pi, V)$ to be a smooth representation of $G$.

**Proposition 4.7.1.** *$V_N$ is an admissible representation of $A$.*

The proof of the theorem is based on the notion of Iwahori decomposition. This decomposition is valid for an important family of compact open subgroups of $G$. Let denote

$$
\begin{aligned}
K(p^r) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cong \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mod p^n \right\} \\
K_1(p^r) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cong \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \mod p^n \right\} \\
K_0(p^r) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cong \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \mod p^n \right\}
\end{aligned}
$$

For $r \geq 1$, all these groups satisfy the following properties

**Lemma 4.7.2.** *Let $K$ be one of the above compact open subgroups for an integer $r \geq 1$. Then every $k \in K$ can be written uniquely in the form $r = nan^-$ where $n \in N_K = N \cap K$, $a \in A_K = A \cap K$ and $n^- \in N_K^- = N^- \cap K$ is a unipotent lower triangular matrix.*

*Proof.* The existence of the Iwahori decomposition in the above example is $d \in \mathcal{O}_F^\times$ in all cases. □

**Proposition 4.7.3.** *Let $K$ be a compact open subgroup that admit an Iwahori decomposition as above. Then the canonical map*

$$V^K \to V_N^{A_K}$$

*is surjective.*

*Proof.* Let $E$ be a finite dimensional subspace of $V_N^{A_K}$. Since $A_K$ is a compact subset, there exists a finite dimensional subspace $E'$ of $V^{A_K}$ that maps onto $E$. Since $E'$ is finite dimensinoal there exists a compact subgroup $N_0^-$ of $N^-$ that fixes $N^-$. There exists $a \in A$ such that $\pi(a)N_0\pi^{-1}(a) \subset N_K$ hence $A_K N_K^-$ fixes $\pi(a)E'$. In other words, for every $v \in \pi(a)E'$ we have

$$\int_{A_K} \int_{N_K} \pi(a)\pi(n^-)v dn^- da = v.$$

It follows that

$$\int_{N_K} \int_{N_K} \pi(n)\pi(a)\pi(n^-)v dn^- da dn = \int_{N_K} \pi(n) v dn.$$

The application $v \mapsto \pi(1_{N_K})v$ send $\pi(a)E'$ into $V^K$. It follows that the image of $V^K$ in $V_N$ contains $\pi_N(a)E$. But since $\pi_N(a)$ is invertible in $V_N$, it follows that

$$\dim(E) = \dim(\pi_N(a)E) \leq \dim(V^K).$$

Since $E$ have been chosen as an arbitrary finite dimensional subspace of $V_N^{A_K}$, it follows that $V_N^{A_K}$ is finite and bounded by $\dim(V^K)$. We can thus take $E = V_N^{A_K}$ hence $V^K \to V_N^{A_K}$ is surjective. □

In fact, the structure of the map $V^K \mapsto V_N^{A_K}$ can be made very precise because it is linear over a rather large algebra. Let $A^-$ denote the sub semigroup of $A$ of diagonal matrices $(a_1, a_2)$ such that $a_1 a_2^{-1} \in \mathcal{O}_F$ in other words $A^- = \{a \in A \mid |\alpha(\alpha)| \leq 1\}$ where $\alpha$ is the positive root. For $a \in A$, the conjugation by $A$ contracts $N$ and dilates $N^-$. More precisely, we have

(4.7.1) $$aN_K a^{-1} \subset N_K \text{ and } a^{-1} N_K^- a \subset N_K^-$$

Let $\mathbb{C}[A^-/A_K]$ be the algebra generated by the monoid $A^-/A_K$.

**Lemma 4.7.4.** *Let $K$ be as above. The application $a \mapsto 1_{KaK}$ defines a ring morphism $\mathbb{C}[A^-/A_K] \to \mathcal{H}_K$.*

*Proof.* Using (4.7.1), we have

$$a_1Ka_2 \subset Ka_1a_2K$$

for all $a_1, a_2 \in A^-$. The lemma follows. $\square$

**Proposition 4.7.5.** *The map $V^K \to V_N^{A_K}$ is $\mathbb{C}[A^-/A_K]$-linear. There is a decomposition $V^K = E_0 \oplus E_1$ where $\pi(a) = 0$ on $E_0$ and is bijective on $E_1$. The map $V^K \to V_N^{A_K}$ is null on $E_0$ and induces a bijection $E_1 \to V_N^{A_K}$.*

**Corollary 4.7.6.** *The map $V^I \to V_N^{A_I}$ is a bijection if $I = K_0(p)$ is the Iwahori subgroup.*

*Proof.* In this case $1_{IaI}$ is an invertible element of the Hecke-Iwahori algebra $\mathcal{H}_I$. $\square$

In our case $G = \mathrm{GL}_2(F)$, a much stronger statement is true.

**Proposition 4.7.7.** *If $V$ is irreducible then the Jacquet module $V_N$ is finite dimensional. In fact we have* $\dim(V_N) \leq 2$.

*Proof.* We first prove that $V_N$ contains a one-dimensional representation of $A$. We already know that $V_N$ is a smooth admissible as representation of $A$. For some compact open subgroup $A_K$ of $A$, $V_N^{A_K}$ is non zero and finite dimensional. The commutative group of finite type $A/A_K$ acting on this finite dimensional vector space admits at least one eigenvector. Thus there exists a non zero vector $v \in V_N$ and a character $\chi : A \to \mathbb{C}^\times$ so that for every $a \in A$, $\pi_N(a)v = \chi(a)v$.

Now we have a non zero $B$-map $V \to \chi$. We derive from the Frobenius reciprocity a non zero map $V \to \mathrm{Ind}_B^G(\chi)$ which is necessarily injective since $V$ is irreducible. Since the Jacquet functor is exact, we have an injective map $V_N \to \mathrm{Ind}_B^G(\chi)_N$. It is thus enough to prove the following statement. $\square$

**Proposition 4.7.8.** $\dim \mathrm{Ind}_B^G(\chi)_N = 2$ *for all character $\chi$ of $A$.*

*Proof.* It amounts to prove that the space of linear functional $L : \mathrm{Ind}_B^G(\chi) \to \mathbb{C}$ such that $L(\pi_\chi(n)v) = L(v)$ for all $v \in \mathrm{Ind}_B^G(\chi)$, $n \in N$ and $\pi_\chi$ is the representation of $G$ in $\mathrm{Ind}_B^G(\chi)$. Let $L$ be such a functional which is nonzero. The space of the induced representation consists of function $f : G \to C$ so that for all $b \in B$, we have $f(bg) = \chi(b)f(g)$. We have a linear map

$$\Lambda : S(G) \to \mathrm{Ind}_B^G(\chi)$$

defined by

$$\Lambda(\phi)(g) = \int_B \phi(bg)\delta^{-1}(b)\chi^{-1}(b)db$$

where $db$ is the left invariant measure on $B$. We are now considering the distribution

$$\phi \mapsto L(\Lambda(\phi)).$$

It satisfies the property

$$l_b r_n(L \circ \Lambda) = \delta(b)\chi(b)(L \circ \Lambda) \tag{4.7.2}$$

Recall the Bruhat decomposition $G = Bw_1B \sqcup B$ where $Bw_1B$ is an open subset and $B$ is closed. In applying 4.5.2, it is enough to prove that on $Bw_1B$ and on $B$ there is exactly on distribution satisfying (4.7.2). This is easy to see because $B \times N$, $B$ acting on the left and $N$ on the right, acts transitively on each Bruhat cell. □

One important consequence of the finiteness of the Jacquet module is the existence of Whittaker model for infinite dimensional representations.

**Corollary 4.7.9.** *Infinite dimensional smooth admissible representations of $G$ have Whittaker model.*

*Proof.* Let $V$ be an infinite dimensional irreducible admissible representation of $G$. Assume that $V$ has no Whittaker model for some non trivial additive character $\psi$. Using the transitive action of $A$ on the set of non trivial characters of $N$, we conclude that $V$ has no Whittaker model for any non trivial additive character of $N$. As $S(F)$-module 4.6.4, we have $V_x = 0$ for all $x \neq 0$. It follows that map $V \to V_0$ is an isomorphism. We know that the fiber of $V$ over $x = 0$ is the Jacquet module $V_N$ which is finite dimensional. It follows that $V$ is finite dimensional. □

4.8. **Principal series.** A smooth character $\chi : F^\times \to \mathbb{C}^\times$ is a character that is trivial on a compact open subgroup of $F^\times$. Recall that $F^\times = \mathcal{O}_F^\times \times p^{\mathbb{Z}}$. The character $\chi$ consists in a finite order character $\chi^0 : \mathcal{O}_F^\times \to \mathbb{C}^\times$ and a complex number $s$ well determined modulo $2i\pi \log(p)$ such that $\chi(p) = p^{-s}$. The character $\chi$ is unitary if $|z| = 1$. The character is said to be unramified if $\chi^0 = 1$.

Let $\chi : A \to \mathbb{C}^\times$ be a smooth character. Since $A = F^\times \times F^\times$, $\chi = (\chi_1, \chi_2)$ where $\chi_1, \chi_2$ are characters of $F^\times$ as above. We say $\chi$ is unramified if $\chi$ is trivial on $A \cap K$ which amounts to saying that both $\chi_1$ and $\chi_2$ are unramified. We define the principal series representation $i_B^G(\chi)$ as the induces representation

$$i_B^G(\chi) = \mathrm{Ind}_B^G(\chi \otimes \delta^{1/2}).$$

Recall that this is the space of locally constant functions $f : G \to \mathbb{C}$ such that

(1) for every $b \in B$, $f(bg) = \delta_B^{1/2}(b)\chi(b)f(g)$,
(2) there exists a compact open subgroup $K$ of $G$ so that $f(gk) = f(g)$ for all $k \in K$.

The translation of $G$ on the right defines an action of $G$ on $i_B^G(\chi)$. The second condition makes sure that $i_B^G(\chi)$ is a smooth representation.

**Proposition 4.8.1.** *If $\chi$ is unitary the $i_B^G(\chi)$ is a unitary representation. More generally, the contragredient of $i_B^G(\chi)$ is $i_B^G(\chi^{-1})$ where $\chi^{-1} = (\chi_1^{-1}, \chi_2^{-1})$.*

*Proof.* Let $f_1 \in i_B^G(\chi_1, \chi_2)$ and $f_2 \in i_B^G(\chi_1, \chi_2)$ then $f_1 f_2 \in \mathrm{Ind}_B^G(\delta)$. There is now a canonical $G$-invariant linear form $\mathrm{Ind}_B^G(\delta) \to \mathbb{C}$. This defines a nonzero $G$-invariant pairing between $i_B^G(\chi_1, \chi_2)$ and $i_B^G(\chi_1^{-1}, \chi_2^{-1})$. □

**Proposition 4.8.2.** *Let $\chi = (\chi_1, \chi_2)$ be a character of $A$. Except if $\chi' = (\chi_1, \chi_2)$ or $\chi' = (\chi_2, \chi_1)$, there is no nonzero intertwining operators from $i_B^G(\chi)$ to $i_B^G(\chi')$.*

*Proof.* Assume that there is a nonzero intertwining operator $i_B^G(\chi) \to i_B^G(\chi')$. It induces a non zero $G$-invariant bilinear form $i_B^G(\chi) \times i_B^G(\chi'^{-1}) \to \mathbb{C}$. We have canonical projections $\Lambda_\chi : S(G) \to i_B^G(\chi)$ defined by

$$\Lambda_\chi(\phi)(g) = \int_B \phi(bg)\delta^{-1/2}(b)\chi(b)db$$

where $db$ is the left invariant measure on $B$. Similarly, there is a $G$-equivariant canonical maps

$$\Lambda_{\chi'^{-1}} : S(G) \to i_B^G({\chi'}^{-1}).$$

It follows that there is a canonical linear form

$$c : S(G \times G) \to C$$

defined by the formula

$$c(\phi_1 \otimes \phi_2) = \langle \Lambda_\chi(\phi_1), \Lambda_{\chi'^{-1}}(\phi_2) \rangle$$

using the pairing between $i_B^G(\chi)$ and $i_B^G({\chi'}^{-1})$. This is a $G$-invariant with respect to the diagonal action of $G$ by right translation of $G$ on $G \times G$. There exists a unique linear form $b : S(G) \to \mathbb{C}$ such that

$$c(\phi_1 \otimes \phi_2) = b(\phi_1 * \phi_2')$$

where $\phi_2'(g) = \phi_2(g^{-1})$. Now from the construction we have

$$\Lambda_\chi(l_{b_1}\phi_1) = \delta^{-1/2}(b_1)\chi(b_1)\Lambda_\chi(\phi_1).$$

Similarly, we have

$$\Lambda_{\chi'^{-1}}(l_{b_2}\phi_2) = \delta^{-1/2}(b_2){\chi'}^{-1}(b_2)\Lambda_{\chi'^{-1}}(\phi_2).$$

We have

$$(l_{b_1}\phi_1) * (l_{b_2}\phi_2)' = l_{b_1} r_{b_2}(\phi_1 * \phi_2'). \tag{4.8.1}$$

It follows taht for all $\phi \in S(G)$, we have

$$b(l_{b_1} r_{b_2}\phi) = \delta^{-1/2}(b_1)\chi(b_1)\delta^{-1/2}(b_2){\chi'}^{-1}(b_2)\Lambda(\phi).$$

We use again the Bruhat decomposition $G = Bw_1B \cup B$. On each Bruhat double coset there is at most one distribution satisfying the transformation property (4.8.1).

On the big cell $Bw_1B$, there is a non zero distribution satisfying (4.8.1) if and only if the restriction of the character

$$(b_1, b_2) \mapsto \delta^{-1/2}(b_1)\chi(b_1)\delta^{-1/2}(b_2){\chi'}^{-1}(b_2) \tag{4.8.2}$$

to the stabilizer $\{(b_1, b_2) \mid l_{b_1} r_{b_2} w_1 = w_1\}$. The stabilizer consists in pairs $(a_1, a_2)$ with $a_1, a_2 \in A$ such that $a_1 = w_1 a_2 w_1$. In that case $\delta^{-1/2}(a_1)\delta^{-1/2}(a_2) = 1$ so that the restriction of the above character to the stabilizer is trivial if and only if $\chi' = w_1\chi$ i.e if $\chi = (\chi_1, \chi_2)$ then $\chi' = (\chi_2, \chi_1)$.

On the small cell $B$, there is a non zero distribution satisfying (4.8.1) if and only if $\chi = \chi'$. Here we have to take the modulus character $\delta$ of $B$ in to account. □

**Proposition 4.8.3.** *The representation $i_B^G(\chi)$ admits an invariant one-dimensional subspace if and only if $\chi_1\chi_2^{-1}(y) = |y|^{-1}$ for all $y \in F^\times$. It admits an invariant one-dimensional quotient if and only if $\chi_1\chi_2^{-1}(y) = |y|$ for all $y \in F^\times$.*

*Proof.* Consider one-dimensional representation of $G$ which is a subspace of $i_B^G(\chi)$. The commutator group $\mathrm{SL}(2, F)$ acts trivially on the one dimensional subspace. It implies that the restriction of the character $\delta^{1/2}\chi$ to the one dimensional torus

$$\begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix}$$

is trivial. In other words, we have $\chi_1\chi_2^{-1}(y) = |y|^{-1}$.

Now $i_B^G(\chi)$ admits a one-dimensional quotient if and only if its contragredient $i_B^G(\chi^{-1})$ admits a one-dimensional subspace. $\square$

If $i_B^G(G)$ admits a one-dimensional representation as subspace, then the quotient by this subspace is called special representation. For every character $\chi_2 : F^\times \to \mathbb{C}^\times$, we denote $\sigma(\chi_2)$ the special representation given as the irreducible quotient of $i_B^G(\chi_1, \chi_2)$ where $\chi_1\chi_2^{-1}(y) = |y|^{-1}$.

**Proposition 4.8.4.** *Except the above mentioned cases $\chi_1\chi_2^{-1}(y) = |y|^{\pm 1}$, the representation $i_B^G(\chi)$ is irreducible. Special representations are irreducible.*

*Proof.* Except in the cases $\chi_1\chi_2^{-1}(y) = |y|^{\pm 1}$, $i_B^G(\chi)$ do not have one-dimensional (so finite dimensional) subspace or quotient. If it is not irreducible, there exists an exact sequence

$$0 \to V' \to i_B^G(\chi) \to V'' \to 0$$

where both $V'$ and $V''$ are infinite dimensional. It follows that $V'$ and $V''$ admits at least one Whittaker functional after 4.7.9 and therefore $i_B^G(\chi)$ admits at least two linearly independent Whittaker functionals which would contradict the following statement $\square$

**Proposition 4.8.5.** *For every nontrivial additive character $\psi$,* $\dim(i_B^G(\chi))_{N,\psi} = 1$.

*Proof.* The proof is similar to 4.8.2 and is based on the Bruhat decomposition. $\square$

**Theorem 4.8.6.** *Let $V$ be an irreducible admissible representation of $G$. If $V$ is a principal series representation $i_B^G(\chi)$ then* $\dim(V_N) = 2$ *and $A$ acts on $V_N$ through $\delta^{1/2}\chi$ and $\delta^{1/2}w_1(\chi)$. If $V$ is a special representation the* $\dim(V_N) = 1$. *Otherwise $V_N = 0$ i.e. $V$ is supercuspidal representation.*

*Proof.* If the irreducible representation $V \simeq i_B^G(\chi)$ is a principal series representation then $V \simeq i_B^G(w_1\chi)$. The isomorphism $V \simeq i_B^G(\chi)$ induces a nonzero $A$-map $V_N \to \delta^{1/2}\chi$. Similarly, we have a nonzero $A$-map $V_N \to \delta^{1/2}w_1(\chi)$. This implies that $\dim(V_N) \geq 2$. If $\dim(V_N > 2)$ then a character $\chi' \notin \{\chi, w_1(\chi)\}$ such that there exists a nonzero intertwining operator $V \simeq i_B^G(\chi) \to i_B^G(\chi')$ which would contradict 4.8.2. $\square$

**Proposition 4.8.7.** *The space of $K(1)$-fixed vectors $i_B^G(\chi)$ is of dimension no greater than one. It is of dimension one if and only if $\chi_1$ and $\chi_2$ are unramified characters.*

*Proof.* It follows from the Iwasawa decomposition 4.1.2, a function which is $K(1)$-right invariant and which transform on the left by the character $\chi$ is completely determined by its value at the origin. At the origin, the two conditions can reconciled if and only if $\chi$ is trivial on $B \cap K$ in other words if $\chi$ is an unramified character. In that case $\dim i_B^G(\chi)^{K(1)} = 1$. $\square$

4.9. **Kirillov model.** Let $\psi : F \to \mathbb{C}^\times$ denote a non trivial additive character. Then all characters of $F$ must be of the form $\psi_a : x \mapsto \psi(ax)$ for some $a \in F^\times$.

Let $(\pi, V)$ a smooth irreducible representation of $G$ that admits a $\psi$-Whittaker functional $\Lambda_\psi : V \to \mathbb{C}$. For every $v \in V$, the corresponding Whittaker function.$W_v \in \mathrm{Ind}_N^G(\psi)$ is given by $g \mapsto \Lambda(\pi(g)v)$. The construction of the Kirillov model consist in restricting Whittaker function to the subgroup $F^\times$ of matrices of the form

$$a = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

**Proposition 4.9.1.** *Assume that $V$ is infinite dimensional. If $v \neq 0$, then there exists $a \in F^\times$ so that $W_v \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \neq 0$.*

*Proof.* Assume that $W_v(a) = 0$ for all $a \in F^\times$. This implies that $\Lambda_{\psi_a}(v) = 0$ because

$$v \mapsto \Lambda_\psi(\pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} v)$$

is a $\psi_a$-Whittaker functional. In other words, the image of $v$ in $V_a$ vanishes for every $a \in F^\times$. Here we are considering $V$ as a smooth $S(F)$-module as in 4.6.4. For every $x \in F$, we have

$$\pi \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} v = v. \tag{4.9.1}$$

In fact, if $v'$ denote the difference of these two vectors, then the image of $v'$ in $V_a$ vanishes for every $a \in F$ so that $v' = 0$.

The equation (4.9.1) being established for all $x \in F$, $v$ is a $N$-invariant vector. By assumption, $v$ is also invariant under an open compact subgroup $K$ and in particular by a $K \cap N^-$ where $N^-$ is the group of unipotent lower matrices. Now $v$ is stablized by the $\mathrm{SL}(2, F)$ because this group is generated by any two non trivial unipotent matrices, one upper and the other lower triangular. This implies that $V$ is finite dimensional. ☐

For every $v$, let $\phi_v : F^\times \to \mathbb{C}$ denote the function

$$\phi_v(a) = W_v \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}.$$

We denote $\mathrm{Kr}(V)$ the subspace of complex valued functions of $F^\times$ image of the application $v \mapsto \phi_v$. Since this application is injective, $V = \mathrm{Kr}(V)$ so that $\mathrm{Kr}(V)$ also affords a smooth irreducible representation of $G$. This is the Kirillov model of $V$. It is easy to write down the action of upper triangular matrices in $\mathrm{Kr}(V)$

$$\pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \phi_v(y) = \phi_v(ay) \quad \text{and} \quad \pi \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \phi_v(y) = \psi(by)\phi_v(y) \tag{4.9.2}$$

The equations (4.9.2) impose restrictions on $\phi_v$ associated to a vector $v$. Since $v$ is fixed by some compact subgroup $K$, $\phi_v$ must be invariant under the translation by a compact subgroup of $F^\times$ by the first equation. Since $v$ is invariant under $N \cap K$, the second equation implies $\phi_v(y) = 0$ if the character $b \mapsto \psi(by)$ is non trivial on $N \cap K$, in other words there exists $C > 0$ such that $\phi(y) = 0$ for all $|y| > C$.

**Proposition 4.9.2.** *For every $v \in V$, the function $\phi_v : F^\times \to \mathbb{C}$ is locally constant and its support is contained in a compact subset of $F$. Moreover, if $v$ has zero image in the Jacquet module $V_N$, then the function $\phi_v$ has compact support*

*Proof.* The first statement has just been proved. For the second statement, the kernel $V(N)$ of $V \mapsto V_N$ is generated by vectors of the form

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} v - v.$$

The associated function $\phi(y)$ vanishes for $|y|$ small enough so that $\psi(by) = 1$. ☐

**Theorem 4.9.3.** *Let $(\pi, V)$ be an infinite-dimensional irreducible smooth representation of $G$. Let identify $V$ with its Kirillov model. Then the kernel $V(N)$ of the projection from $V$ to its Jacquet module $V_N$ is exactly $C_c^\infty(F^\times)$.*

*Proof.* $V(N)$ is a non-zero $B_1$-invariant subspace of $C_c^\infty(F^\times)$. It is enough to prove that $C_c^\infty(F^\times)$ as representation of $B_1$ is irreducible. □

From the description of Jacquet module of $V$. we deduce the following proposition.

**Proposition 4.9.4.** *Let us denote by $\pi_{\mathrm{Kr}}$ the action of the group*

$$B_1 = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \mid a \in F^\times, b \in F \right\} \tag{4.9.3}$$

*acts on $C_c^\infty(F^\times)$ by the equation (4.9.2). Then $(\pi_{\mathrm{Kr}}, C_c^\infty)$ is an irreducible representation.*

*Proof.* Let $\phi \in C_c^\infty(F^\times)$. The action of $F$ on $\phi$ according to (4.9.2) gives rise to an action of the algebra $C_c^\infty(F)$. For every $f \in C_c^\infty(F)$, we have

$$\pi_{\mathrm{Kr}}(f)(\phi)(y) = \hat{f}(y)\phi(y).$$

The irreducibility of $\pi_{\mathrm{Kr}}$ can be easily deduced from this equation. □

**Theorem 4.9.5.** *Let $(\pi, V) = i_B^G(\chi_1, \chi_2)$ be an irreducible principal series i.e. $\chi_1\chi_2^{-1}(t)$ is not $|y|$ or $|y|^{-1}$.*

(1) *Assume taht $\chi_1 \neq \chi_2$. Then the space of Kirillov model of $V$ consists of the function on $F^\times$ that are locally constant, that vanish for large values of $|y|$ and*

$$\phi(y) = c_1|y|^{1/2}\chi_1(y) + c_2|y|^{1/2}\chi_2(y) \tag{4.9.4}$$

*for for all $|y|$ small and for some constants $c_1, c_2$.*

(2) *Assume that $\chi_1 = \chi_2 = \chi$. The space of Kirillov model of $V$ consists of the function on $F^\times$ that are locally constant, that vanish for large values of $|y|$ and*

$$\phi(y) = c_1|y|^{1/2}\chi(y) + c_2\mathrm{val}(y)|y|^{1/2}\chi(y) \tag{4.9.5}$$

*for for all $|y|$ small and for some constants $c_1, c_2$.*

4.10. **Local functional equation.** We define local $L$ factor of an irreducible representation as follows. If $\pi = i_B^G(\chi_1, \chi_2)$ is a principal series representation, we put

$$L(s, \pi) = (1 - \alpha_1 q^{-s})^{-1}(1 - \alpha_2 q^{-s})^{-1}$$

where $\alpha_i = \chi_i(\varpi)$ if $\chi_i$ is unramified and $\alpha_i = 0$ if $\chi_i$ is ramified. If $\pi = \sigma(\chi_2)$ is a special representation then $L(s, \pi) = (1 - \alpha_2 q^{-s})^{-1}$ where again $\alpha_2 = \chi_2(\varpi)$ if $\chi_i$ is unramified and $\alpha_2 = 0$ if $\chi_2$ is ramified.

**Proposition 4.10.1.** *Let $(\pi, V)$ be an infinite dimensional irreducible smooth representation of $G$. If $f$ is an element in the space of its Kirillov model, consider the interal*

$$Z(s, f) = \int_{F^\times} f(y)|y|^{s-1/2} d^\times y \tag{4.10.1}$$

*This integral is convergent for $\Re(s) >> 0$ has meromoprphic continuation to all $s \in \mathbb{C}$. There exists a unique polynomial $L(\pi, s)^{-1} \in \mathbb{C}[q^{-s}]$ in the variable $q^{-s}$ with free coefficient one such that for every $f \in V$, there exists $p(s, f)$ such that*

$$Z(s, f) = p(s, f)L(s, \pi). \tag{4.10.2}$$

*Moreover, one can choose $f_0$ so that $Z(s,f_0) = L(s,\pi)$ i.e. $p(s,f_0) = 1$.*

*Proof.* Suppose that $(\pi, V) = i_B^G(\chi)$ is a principal series with $\chi = (\chi_1, \chi_2)$. For every function $f : F^\times \to \mathbb{C}$ in the Kirillov model of $\pi$, there exist $c_1, c_2 \in C$ such that

$$f(y) = c_1|y|^{1/2}\chi_1(y) + c_2|y|^{1/2}\chi_2(y)$$

for all $t \in F^\times$ with $|y|$ small.

Let $f_0$ denote the locally constant on $F^\times$ supported by $\mathcal{O}_F$ defined by

$$f_0(y) = c_1 t^{1/2}\chi_1(y) + c_2 t^{1/2}\chi_2(y).$$

Then $f - f_0$ is a locally constant function on $F^\times$ with compact support. The zeta integral $Z(s, f - f_0)$ is then a polynomial function on $q^{\pm s}$.

The zeta integral

$$Z(s,f_0) = c_1 \int_{\mathcal{O}_F} \chi_1(y)|y|^s d^\times y + c_2 \int_{\mathcal{O}_F} \chi_2(y)|y|^s d^\times y.$$

Both summands are absolutely convergent for $\Re(s) >> 0$. Assume $\Re(s) >> 0$. If $\chi_1 : F^\times \to \mathbb{C}$ is a ramified character then

$$\int_{\varpi^r \mathcal{O}_F} \chi_1(y)|y|^s d^\times y = 0$$

for all integer $r$. It follows that the integral

$$\int_{\mathcal{O}_F} \chi_1(y)|y|^{-s} d^\times y = 0.$$

If $\chi_1$ is an unramified character with $\chi_1(\varpi) = \alpha_1$ then we have

$$\begin{aligned}\int_{\mathcal{O}_F} \chi_1(y)|y|^{-s} d^\times y &= \sum_{n=0}^{\infty} \alpha_1^n p^{-sn} \\ &= (1 - \alpha_1 p^{-s})^{-1}\end{aligned}$$

Thus

$$Z(s,f_0) = c_1(1-\alpha_1 p^{-s})^{-1} + c_2(1-\alpha_2 p^{-s})^{-1}$$

so that

$$\frac{Z(s,f_0)}{L(s,\pi)} = c_1(1-\alpha_2 p^{-s}) + c_2(1-\alpha_1 p^{-s})$$

is a polynomial function on $p^{-s}$. Here $L(s,\pi) = (1-\alpha_1 p^{-s})^{-1}(1-\alpha_2 p^{-s})^{-1}$. Notice that since $\alpha_1 \neq \alpha_2$, the scalars $c_1, c_2$ can be choosen such that $c_1(1-\alpha_2 p^{-s}) + c_2(1-\alpha_1 p^{-s}) = 1$. In this case we have

$$p(s,f_0) = 1.$$

In the above argument, we only use the asymptotic expansion of $f$ around 0 based on the Jacquet module of $i_B^G(\chi)$. The same argument works with special representation and supercuspidal representation as well. In the supercuspidal case $L(s,\pi) = 1$ and we can choose $f_0 = 1_{\mathcal{O}_F^\times}$. □

More generally, for all characters $\xi : F^\times \to \mathbb{C}^\times$, we can define

$$Z(s, f, \xi) = \int_{F^\times} f(y)\chi(y)|y|^{s-1/2} d^\times y \tag{4.10.3}$$

as well as the factor $L(s, \pi, \chi)$ so that

$$Z(s, f, \chi) = p(s, f, \chi) L(s, \pi, \chi). \tag{4.10.4}$$

**Proposition 4.10.2.** *Let $(\pi, V)$ be an irreducible smooth representation of $G$ and let $\chi$ be a character of $F^\times$. Then for all but at most two values of $s$ modulo $2\pi i/\log(q)$, the dimension of the space of linear functionals $\Lambda : V \to \mathbb{C}$ satisfying*

$$L\left(\pi \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix} v\right) = \chi(y)|y|^s L(v) \tag{4.10.5}$$

*is equal to one.*

*Proof.* Let $L_1, L_2$ denote two linearly independent linear form satisfying the equation 4.10.5. Since $V(N) = C_c^\infty(F^\times)$ and the space of $\chi$-eigendistribution on $F^\times$ is one-dimensional, $L_1$ and $L_2$ are proportional on $V(N)$. There exist $c_1, c_2$ not both zero so that $c_1L_1 + c_2L_2$ factors through $V_N$. But $\dim V_N \leq 2$, and there are at most two characters of $F^\times$ that appears in $V_N$. If $y \mapsto \chi(y)|y|^s$ is not one of these characters, the space of functionals $\Lambda$ satisfying (4.10.5) is of dimension not greater than one. By the above proposition, we know that $f \mapsto p_\chi(f, s)$ is a non zero element of this space. Therefore, its dimension is at least one. □

**Theorem 4.10.3.** *Let $(\pi, V)$ be an infinite-dimensional irreducible smooth representation of $G$ with central character $\omega$. Let $\chi$ be a character of $F^\times$. We identify $V$ with its Kirillov model. There exists a meromorphic function $\gamma(s, \pi, \xi, \psi)$ such that for all $f \in V$ we have*

$$Z(1 - s, \pi(w_1)f, \omega^{-1}\chi^{-1}) = \gamma(s, \pi, \chi, \psi) Z(s, f, \chi) \tag{4.10.6}$$

*where $w_1$ is the matrix*

$$w_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{4.10.7}$$

*Proof.* Both $f\Lambda \mapsto L_1(f) = Z(s, f, \chi)$ and $f \mapsto L_2(f) = Z(1 - s, \pi(w_1)f, \omega^{-1}\chi^{-1})$ are linear functional on $V$ that satisfy

$$L\left(\pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} f\right) = \chi(a)^{-1}|a|^{-s+1/2} L(f). \tag{4.10.8}$$

First we check this property for $L_1$.

$$\begin{aligned} L_1(\pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} f) &= \int_{F^\times} f(ay)\chi(y)|y|^{s-1/2} d^\times y \\ &= \chi(a)^{-1}|a|^{-s+1/2} L(f). \end{aligned}$$

Now we will check this property for $L_2$.

$$\begin{aligned} L_2(\pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} f) &= \int_{F^\times} \left( \pi(w_1)\pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} f \right)(y)\, (\omega^{-1}\chi^{-1})(y)|y|^{-s+1/2} d^\times y \\ &= \int_{F^\times} \omega(a) \left( \pi \begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix} \pi(w_1) f \right)(y)\, \omega^{-1}(y)\chi^{-1}(y)|y|^{-s+1/2} d^\times y \\ &= \int_{F^\times} (\pi(w_1)f)(a^{-1}y)\, \omega^{-1}(a^{-1}y)\chi^{-1}(y)|y|^{-s+1/2} d^\times y \\ &= \chi(a)^{-1}|a|^{-s+1/2} L_2(f) \end{aligned}$$

For all $s$ except two at complex values, the space of such linear functionals is one dimensional. There exists a proportionality constant $\gamma(s,\pi,\xi,\psi)$ between the two linear form which depend holomorphically on $s$. Both $Z(1-s,\pi(w_1)f,\omega^{-1}\chi^{-1})$ and $Z(s,f,\chi)$ are meromorphic function on $s$, then so is $\gamma(s,\pi,\chi,\psi)$. □

**Theorem 4.10.4.** *There exists an invertible holomorphic function $\sigma(s,\pi,\psi)$ such that*

$$\gamma(s,\pi,\chi,\psi) = \frac{L(1-s,\tilde{\pi},\chi^{-1})}{\epsilon(s,\pi,\chi,\psi)L(s,\pi,\chi)}$$

*where $\tilde{\pi}$ is the contragredient representation of $\pi$.*

*Proof.* By definition, we have

$$\gamma(s,\pi,\chi,\psi) = \frac{Z(1-s,\pi(w_1)f,\omega^{-1}\chi^{-1})}{Z(s,f,\chi)}$$

for all functions $f : F^\times \to \mathbb{C}$ in the Kirillov model of $\pi$. Let consider the function $\tilde{f} : F^\times \to \mathbb{C}$ given by

$$\tilde{f}(y) = \pi(w_1)f(y)\omega^{-1}(y).$$

We observe that $\tilde{f}$ is an element of the Kirillov model of the contragrendient representation $\tilde{\pi}$ of $\pi$. If $\pi = i_B^G(\chi_1,\chi_2)$ is an unramified principal series, there exists $c_1, c_2 \in \mathbb{C}$ such that

$$\pi(w_1)f(y) = \delta^{1/2}(y)(c_1\chi_1(y) + c_2\chi_2(y))$$

for small $|y|$. We have $\omega(y) = \chi_1\chi_2(y)$ so that

$$\tilde{f}(y) = \delta^{1/2}(y)(c_1\chi_1^{-1}(y) + c_2\chi_2^{-1}(y))$$

belongs to the Kirillov model of the contragredient representation $\tilde{\pi} = i_B^G(\chi_1^{-1},\chi_2^{-1})$. The same argument works in other cases. In fact $f \mapsto \tilde{f}$ induces a bijection from the Kirillov model of $\pi$ on the Kirillov model of $\tilde{\pi}$.

We have

$$Z(1-s,\pi(w_1)f,\omega^{-1}\chi^{-1}) = p(s,\tilde{f},\chi^{-1})L(1-s,\tilde{\pi},\chi^{-1})$$

where $p(s,\tilde{f},\chi^{-1})$ is some polynomial in $q^{\pm s}$ which is equal one for some $f$.

We have

$$\gamma(s,\pi,\chi,\psi) = \frac{L(1-s,\tilde{\pi},\chi^{-1})}{\epsilon(s,\pi,\chi,\psi)L(s,\pi,\chi)}$$

with

$$\epsilon(s,\pi,\chi,\psi) = \frac{p(s,\tilde{f},\chi^{-1})}{p(s,f,\chi)}.$$

We exploit the fact the $\epsilon$ does not depend on $f$ to prove that it is an invertible holomorphic function. We can choose $f$ such that $p(s, f, \chi) = 1$ i.e. $\epsilon$ is a holomorphic function on $s$. We can also choose $f$ such that $p(s, \tilde{f}, \chi^{-1}) = 1$ which implies that $\epsilon$ is inverse to a holomorphic function. So $\epsilon$ is an invertible holomorphic function. □

4.11. **Formal Mellin transform.**

## 5. Automorphic representations on adelic groups

In this chapter $\mathbb{A}$ will denote the ring of adèles of $\mathbb{Q}$. Let $\mathbb{A}^{\text{fin}}$ denote the ring of finite adèles. We have $\mathbb{A} = \mathbb{A}^{\text{fin}} \times \mathbb{R}$. More generally, if $F$ is a global field, we will denote $\mathbb{A}_F$ its ring of adèles. We have $\mathbb{A}_F = \mathbb{A}_F^{\text{fin}} \times \mathbb{A}_F^{\infty}$.

In this chapter, we will use the letter $G$ for the group $\text{GL}_2$.

5.1. **Adèles and idèles.** The absolute values of the field of rational numbers $\mathbb{Q}$ is either archimedean or non archimedean, thus associated a prime number $p$ or archimedean. Recall that the $p$-adic absolute value of $m/n$ with $m, n \in \mathbb{Z}$ is $|m/n| = p^{-\text{ord}_p(m)+\text{ord}_p(n)}$ where $\text{ord}_p$ of an integer is the order of the largest power of $p$ dividing it. While the completion of $\mathbb{Q}$ with respect to the archimedean absolute value is the field $\mathbb{R}$ of real numbers, the completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with the $p$-adic absolute value is the field of $p$-adic numbers. It is the fraction field of the ring $\mathbb{Z}_p$ of $p$-adic integers that is the completion of $\mathbb{Z}$ with respect to the $p$-adic absolute value.

The ring $\mathbb{A}$ of adèles of $\mathbb{Q}$ is the restricted product

$$\mathbb{A} = \check{\prod}_{p \in \mathcal{P}} \mathbb{Q}_p \times \mathbb{R}$$

whose elements are $x_{\mathbb{A}} = (x_p, x)_{p \in \mathcal{P}}$ with $x \in \mathbb{R}$, $x_P \in \mathbb{Q}_p$ for all $p$ and $x_p \in \mathbb{Z}_p$ for all but finitely many $p \in \mathcal{P}$. We also write $\mathbb{A} = \mathbb{A}^{\text{fin}} \times \mathbb{R}$ where $\mathbb{A}^{\text{fin}} = \check{\prod}_{p \in \mathcal{P}} \mathbb{Q}_p$ is called the ring of finite adèles. We equip $\mathbb{A}$ and $\mathbb{A}^{\text{fin}}$ with the Tychonoff product topology which makes it locally compact. The compact subring $\prod_{p \in \mathcal{P}} \mathbb{Z}_p$ is the profinite completion $\hat{\mathbb{Z}}$ of the ring of integers $\mathbb{Z}$. The field of rational numbers $\mathbb{Q}$ is contained in $\mathbb{A}$ as a discrete subring.

**Lemma 5.1.1.** *We have $\mathbb{Q} \cap \prod_{p \in \mathcal{P}} \mathbb{Z}_p = \mathbb{Z}$ and $\mathbb{Q} \backslash \mathbb{A} / \prod_{p \in \mathcal{P}} \mathbb{Z}_p = \mathbb{R}/\mathbb{Z}$. In particular $\mathbb{Q} \backslash \mathbb{A}$ is a compact group.*

*Proof.* A rational number $m/n$ with $(m, n) = 1$ that is $p$-integral for all prime $p$ as $n$ has no prime factors. Thus $\mathbb{Q} \cap \hat{\mathbb{Z}} = \mathbb{Z}$. A coset in $\mathbb{A}^{\text{fin}} / \prod_{p \in \mathcal{P}} \mathbb{Z}_p$ can be represented by a finite collection of fractions $(m_p/p^{r_p})_{p \in S}$ indexed by a finite set $S$ of prime numbers. The sum $m/n = \sum_{p \in \Sigma} m_p/p^{r_p}$ satisfies the property that $m/n \equiv m_p/p^{r_p}$ for all $p$. This implies that $\mathbb{A}^{\text{fin}} = \mathbb{Q} + \prod_{p \in \mathcal{P}} \mathbb{Z}_p$. It also follows that $\mathbb{Q} \backslash \mathbb{A} / \prod_{p \in \mathcal{P}} \mathbb{Z}_p = \mathbb{R}/\mathbb{Z}$. □

The group $\mathbb{A}^{\times}$ of idèles of $\mathbb{Q}$ is the restricted product

$$\mathbb{A}^{\times} = \check{\prod}_{p \in \mathcal{P}} \mathbb{Q}_p^{\times} \times \mathbb{R}^{\times}$$

whose elements are $x_{\mathbb{A}} = (x_p, x)_{p \in \mathcal{P}}$ with $x \in \mathbb{R}^{\times}$, $x_P \in \mathbb{Q}_p^{\times}$ for all $p$ and $x_p \in \mathbb{Z}_p^{\times}$ for all but finitely many $p \in \mathcal{P}$. We also write $\mathbb{A}^{\times} = \mathbb{A}^{\text{fin}\times} \times \mathbb{R}^{\times}$ where $\mathbb{A}^{\text{fin}\times}$ is the group of finite idèles.

For $x_A \in \mathbb{A}^\times$, let us consider the norm $|x_A| = \prod_{p\in\mathcal{P}} |x_p|_p |x_\infty|_\infty$ the infinite product is well defined because all but finitely many of its terms is equal to one. Let $\mathbb{A}^1$ be the subgroup of elements of norm one in $\mathbb{A}^\times$.

**Lemma 5.1.2.** *We have $\mathbb{Q}^\times \cap \prod_{p\in\mathcal{P}} \mathbb{Z}_p^\times = \{\pm 1\}$ and $\mathbb{Q}^\times\backslash\mathbb{A}^\times/\prod_{p\in\mathcal{P}} \mathbb{Z}_p^\times = \mathbb{R}_+^\times$. The group of norm one idèles $\mathbb{A}^1$ contains $\mathbb{Q}^\times$ as a cocompact subgroup i.e. the quotient $\mathbb{Q}^\times\backslash\mathbb{A}^1$ is a compact group.*

*Proof.* A rational number that is $p$-integral and whose inverse is also $p$-integral must be $\pm 1$. For every prime $p$, there is an isomorphism $\mathbb{Q}_p^\times/\mathbb{Z}_p^\times = \mathbb{Z}$ given by the $p$-adic valuation. The quotient $\mathbb{A}^{\mathrm{fin}\times}/\prod_{p\in\mathcal{P}} \mathbb{Z}_p$ is given by a collection of integers $(r_p)_{p\in\mathcal{P}}$ that vanish except for finitely many of them. Since this class can be represented by the rational number $\prod_{p\in\mathcal{P}} p^{r_p}$, we have $\mathbb{Q}^\times\backslash\mathbb{A}^{\mathrm{fin}\times}/\prod_{p\in\mathcal{P}} \mathbb{Z}_p = 1$. It follows that $\mathbb{Q}^\times\backslash\mathbb{A}^\times/\prod_{p\in\mathcal{P}} \mathbb{Z}_p^\times = \mathbb{R}^\times/\{\pm 1\}$ that is isomorphic to $\mathbb{R}_+^\times$ by the application $x \mapsto |x|$.

In order to prove $\mathbb{Q}^\times \subset \mathbb{A}^1$, it is enough to prove the product formula

$$|x|_\infty \prod_{p\in\mathcal{P}} |x|_p = 1$$

for all $x \in \mathbb{Z} - \{0\}$. But this follows immediately from the decomposition in prime factors $x = \pm \prod p^{r_p}$. We have $|x|_p = p^{-r_p}$ for all prime $p$ and $|x|_\infty = \prod p^{r_p}$. According to the isomorphism $\mathbb{Q}^\times\backslash\mathbb{A}^\times/\prod_{p\in\mathcal{P}} \mathbb{Z}_p^\times = \mathbb{R}_+^\times$ above, we have $\mathbb{Q}^\times\backslash\mathbb{A}^1/\prod_{p\in\mathcal{P}} \mathbb{Z}_p^\times = 1$. This implies the compacity of $\mathbb{Q}^\times\backslash\mathbb{A}^1$. □

For every number field $F$, the ring of adèles $\mathbb{A}_F$ is defined in a similar manner. An absolute value of $F$ is either archimedean or non archimedean. For an archimedean absolute value $v$, an infinite place, the completion $F_v$ is either the field of real numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$. A non archimedean absolute value $v$, a finite place, is up to normalization is an extension of the $p$-adic valuation on $\mathbb{Q}$; the $v$-adic completion $F_v$ is a finite extension of $\mathbb{Q}_p$. We will denote by $\mathcal{O}_v$ its ring of integers. The ring of adèles $\mathbb{A}_F$ is the restricted product of all those local fields $F_v$ whose elements are $(x_v)$ with $x_v \in F_v$ for all $v$ and $x_v \in \mathcal{O}_v$ for all but finitely many $v$. Again, $\mathbb{A}_F = F_\infty \times \mathbb{A}_{F,\mathrm{fin}}$ where $F_\infty$ is the product of the completion at infinite places and the ring of finite adèles $\mathbb{A}_{F,\mathrm{fin}}$ is the restricted product of the completions at all finite places. We denote $\mathcal{O}_{\mathrm{fin}} = \prod_v \mathcal{O}_v$ the product over all the finite places $v$ of $F$ of the valuation ring $\mathcal{O}_v$ in $F_v$. Let $\mathcal{O}_F$ denote the ring of integers of $F$.

**Proposition 5.1.3.** *The quotient $F\backslash\mathbb{A}_F$ is a compact group.*

*Proof.* We have $F \cap \mathcal{O}_{\mathrm{fin}} = \mathcal{O}_F$, the ring of integers of $F$. It follows that $F\backslash\mathbb{A}_F/\mathcal{O}_{\mathrm{fin}} = \mathcal{O}_F\backslash F_\infty$. Here $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$ is a $r$-dimensional real vector space that contains $\mathcal{O}_F$ as a complete lattice. The quotient $\mathcal{O}_F\backslash F_\infty$ is thus compact. □

**Proposition 5.1.4.** *$\mathbb{A}_{F,1}^\times$ contains $F^\times$ as a subgroup and the quotient $F^\times\backslash\mathbb{A}_{F,1}^\times$ is a compact group.*

*Proof.* We will see that $F^\times\backslash\mathbb{A}_{F,1}^\times/\mathcal{O}_{\mathrm{fin}}^\times$ is a compact group. Consider the projection on the finite idèles

$$\mathrm{pr}_{\mathrm{fin}} : F^\times\backslash\mathbb{A}_{F,1}^\times/\mathcal{O}_{\mathrm{fin}}^\times \to F^\times\backslash\mathbb{A}_{F,\mathrm{fin}}^\times/\mathcal{O}_{\mathrm{fin}}^\times.$$

The latter group is the finite by the theorem of finiteness of ideal classes. The kernel of $\mathrm{pr}_{\mathrm{fin}}$ is $(F^\times \cap \mathcal{O}_{\mathrm{fin}}^\times)\backslash F_\infty^\times$. We have $F^\times \cap \mathcal{O}_{\mathrm{fin}}^\times = \mathcal{O}_F^\times$, the group of units of $\mathcal{O}_F$. The compacity of the quotient $\mathcal{O}_F^\times \backslash F_\infty^1$ is a formulation of the Dirichlet unit theorem. □

The group $F^\times\backslash\mathbb{A}_F^\times$ is called the group of idèles classes. This is a locally compact group equipped with an absolute value

$$F^\times\backslash\mathbb{A}_F^\times \to \mathbb{R}^+$$

whose kernel $F^\times\backslash\mathbb{A}_F^1$ is a compact group.

5.2. **The case of function fields.** Let $F$ be the field of rational functions on a smooth projective curve connected $C$ defined over a finite field $k$. The places of $F$ are the closed points of $C$. Of course, there is no archimedean place. The ring of adèles $\mathbb{A}_F$ contains a discrete subring $F$ and a compact subring $\mathcal{O}_{\mathbb{A}_F} = \prod_{v\in|C|}\mathcal{O}_v$ where $\mathcal{O}_v$ is the completion of the local ring $\mathcal{O}_{C,v}$ of $C$ at $v$.

**Proposition 5.2.1.** *We have* $F \cap \mathcal{O}_{\mathbb{A}_F} = \mathrm{H}^0(C, \mathcal{O}_C)$ *and* $F\backslash\mathbb{A}_F/\mathcal{O}_{\mathbb{A}_F} = \mathrm{H}^1(C, \mathcal{O}_C)$. *In particular, they are both finite dimensional $k$-vector spaces.*

**Proposition 5.2.2.** *The quotient* $\mathbb{A}_F^\times/\mathcal{O}_{A_F}^\times$ *can be identified with the group of divisors of $C$, the double quotient* $F^\times\backslash\mathbb{A}_F^\times/\mathcal{O}_{A_F}^\times$ *with the group* $\mathrm{Pic}_C$ *of isomorphism classes of line bundle on $C$. The degree of a line bundle is a homomorphism* $\deg : F^\times\backslash\mathbb{A}_F^\times/\mathcal{O}_{A_F}^\times \to \mathbb{Z}$ *whose kernel, the group* $\mathrm{Pic}_C^0$ *of line bundles of degree* 0*, is finite.*

The absolute value on $F^\times\backslash\mathbb{A}_F^\times$ is given by $|x| = q^{-\deg(x)}$ where $q$ is the cardinal of the base field $k$. Its kernel is the compact group $F^\times\backslash\mathbb{A}_F^1$ that is an extension of the finite group $\mathrm{Pic}_C^0$ by the compact group $\mathcal{O}_{A_F}^\times$. The choice of a line bundle of degree one provides a splitting $\mathrm{Pic}_C = \mathrm{Pic}_C^0 \times \mathbb{Z}$.

5.3. **Strong approximation theorem.**

**Theorem 5.3.1.** $\mathrm{SL}_2(\mathbb{Q})$ *is dense in* $\mathrm{SL}_2(\mathbb{A}^{\mathrm{fin}})$.

*Proof.* Let $M$ denote the closure of $\mathrm{SL}_2(\mathbb{Q})$ in $\mathrm{SL}_2(\mathbb{A}^{\mathrm{fin}})$. We first prove that $M$ contains $\mathrm{SL}(\mathbb{Q}_p)$ embedded as the $p$-component of $\mathrm{SL}_2(\mathbb{A}^{\mathrm{fin}})$. It is enough to prove that $M$ contains the subgroups $N(\mathbb{Q}_p)$ and $N^-(\mathbb{Q}_p)$ since these subgroups generate $\mathrm{SL}_2(\mathbb{Q}_p)$. But this statement derives from the density of $\mathbb{Q}$ in $\mathbb{A}^{\mathrm{fin}}$.

Let $S$ be a finite set of primes. It is enough to prove that $M$ contains $\prod_{p\in S}\mathrm{SL}_2(\mathbb{Q}_p) \times \prod_{p\notin S}\mathrm{SL}_2(\mathbb{Z}_p)$ because by enlarging $S$, these groups cover $\mathrm{SL}_2(\mathbb{A}^{\mathrm{fin}})$. Since $M$ contains already $\mathrm{SL}_2(\mathbb{Q}_p)$ for $p \in S$, it is enough to prove that $M$ contains the profinite group $\prod_{p\notin S}\mathrm{SL}_2(\mathbb{Z}_p)$. This is equivalent to prove that $M \cap \prod_{p\notin S}\mathrm{SL}_2(\mathbb{Z}_p)$ maps onto the finite quotient $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ for every integer $n$ prime to $S$. But this follows from the fact $\mathrm{SL}_2(\mathbb{Z}_p) \subset M$ for all $p|n$ that we already know. □

**Corollary 5.3.2.** *For every compact open subgroup $K_0$ of* $\mathrm{SL}(\mathbb{A}^{\mathrm{fin}})$*, let us denote* $\Gamma_0 = \mathrm{SL}_2(\mathbb{Q}) \cap K_0$*.The embedding of* $\mathrm{SL}_2(\mathbb{R})$ *as the infinite component of* $\mathrm{SL}_2(\mathbb{A})$ *induces a homeomorphism*

$$\Gamma\backslash\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})/K_0.$$

*Proof.* We first prove that the map

$$\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})/K_0$$

is surjective. An element of

$$\mathrm{SL}_2(\mathbb{A})/K_0 = (\mathrm{SL}_2(\mathbb{A}^{\mathrm{fin}})/K_0) \times \mathrm{SL}_2(\mathbb{R}).$$

can be written as $x = (x^{\mathrm{fin}}, g_\infty)$ with $x^{\mathrm{fin}} \in \mathrm{SL}_2(\mathbb{A}^{\mathrm{fin}})/K_0$ and $g_\infty \in \mathrm{SL}_2(\mathbb{R})$. The density of $\mathrm{SL}_2(\mathbb{Q})$ in $\mathrm{SL}_2(\mathbb{A}^{\mathrm{fin}})$ implies that there exists $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ such that whose image in $\mathrm{SL}_2(\mathbb{A}^{\mathrm{fin}})/K_0$ is $x^{\mathrm{fin}}$. The transformation formula $\gamma^{-1}(x^{\mathrm{fin}}, g_\infty) = (1, \gamma^{-1}g_\infty)$ implies the desired surjectivity. Now let $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Q})$ having the same image $x^{\mathrm{fin}} \in \mathrm{SL}_2(\mathbb{A}^{\mathrm{fin}})/K_0$ then $\gamma_2 = \gamma_1\gamma$ where

$$\gamma \in \mathrm{SL}_2(\mathbb{Q}) \cap K_0 = \Gamma_0.$$

It follows that the map

$$\Gamma\backslash\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})/K_0$$

is a homeomorphism. □

For every positive integer $N$ let $K_0(N) = \prod_p K_0(N)_p$ where $K_0(N)_p$ be the subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ of matrix with congruent to an upper triangular matrices modulo $N$. We have

$$\Gamma_0(N) = \mathrm{SL}_2(\mathbb{Q}) \cap K_0(N).$$

**Proposition 5.3.3.** *We have a homeomorphism*

$$\Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{R}) \simeq Z(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K_0(N).$$

*Proof.* This is the combination of the above corollary and the fact that $\mathbb{Q}$ has class number one. □

## 5.4. **Automorphic representations and automorphic forms.**

**Proposition 5.4.1.** *The quotient space $G(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})$ has finite measure.*

For every unitary character $\omega : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$, we consider the space $L^2(G(F)\backslash G(\mathbb{A}), \omega)$ of $G(F)$-functions $\phi$ on $G(\mathbb{A})$, that transform by the character $\omega$ with respect to the action of the center $Z$, and whose module $|\phi|$ is a square integrable function on $G(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})$.

A function $\phi$ is said to be cuspidal if

$$\int_{N(F)\backslash N(\mathbb{A})} \phi(ng)dn = 0$$

for all $g \in G(\mathbb{A})$. The subspace of cuspidal functions $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}), \omega)$ is a closed subspace of $L^2(G(F)\backslash G(\mathbb{A}), \omega)$. Both $L^2(G(F)\backslash G(\mathbb{A}), \omega)$ and $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}), \omega)$ are Hilbert representations of $G(\mathbf{A})$.

**Theorem 5.4.2.** *The space $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}), \omega)$ decomposes as a Hilbert direct sum of irreducible invariant subspaces.*

Let $\mathcal{A}(G, \omega)$ denote the space of smooth functions $\phi : G(\mathbb{A}) \to \mathbb{C}$ such that

1. $\phi$ transforms under the action of $Z(\mathbb{A})$ according to $\omega$,
2. $\phi$ is $K$-finite with respect to any compact subgroup $K = K_{\mathrm{fin}}K_\infty$ where $K_{\mathrm{fin}}$ is a compact open subgroup of $G(\mathbb{A}^{\mathrm{fin}})$ and $K^{\mathrm{fin}}$ is the maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$,

(3) $\phi$ is $Z(U(\mathfrak{g}))$-finite
(4) $\phi$ has moderate growth i.e. $|\phi(g)| \leq C||g||^N$ for some constant $C \in \mathbb{R}_+$ and $N \in \mathbb{N}$.

Such a function $\phi$ is called an automorphic form. The $K$-finiteness implies that there exists a compact open subgroup $K_0$ of $G(\mathbb{A}^{\mathrm{fin}})$. If $\omega = 1$ then $\phi$ can be seen as an automorphic function

$$\phi : \Gamma_0 \backslash \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}.$$

As we have already seen in the framework on automorphic representations on real groups, a $L^2$-automorphic functions that are $K$-finite and $Z(U(\mathfrak{g}))$-finite are analytic and will automatically have moderate growth. We will also consider the space

$$\mathcal{A}_{\mathrm{cusp}}(G, \omega) = \mathcal{A}(G) \cap L^2_{\mathrm{cusp}}(G(F) \backslash G(\mathbb{A}), \omega)$$

of cuspidal automorphic functions. The spaces $\mathcal{A}(G, \omega)$ and $\mathcal{A}_{\mathrm{cusp}}(G, \omega)$ are smooth $G(\mathbf{A}^{\mathrm{fin}})$-representations and $(\mathfrak{g}, K_\infty)$-modules.

**Definition 5.4.3.** *An irreducible admissible $G(\mathbb{A})$-module is a restricted tensor product*

$$\pi = \bigoplus_p \pi_p$$

*with $p$ runs over all places of $\mathbb{Q}$ where*

(1) *$(\pi_\infty, V_\infty)$ is an irreducible admissible $(\mathfrak{g}, K)$-module*
(2) *for all finite place $p$, $(\pi_p, V_p)$ is an irreducible admissible representation of $G(\mathbb{Q}_p)$,*
(3) *for all but finitely many $p$, $\pi_p$ is unramified i.e $\dim(V_p^{K_p}) = 1$.*

*By restricted tensor products we mean the space of vector of the form*

$$v = \bigotimes_p v_p$$

*where $v_p \in V_p^{K_p}$ for almost all prime $p$.*

**Theorem 5.4.4.** *Let $\pi$ be an irreducible $G(\mathbb{A})$-invariant subspace of $L^2(G(F) \backslash G(\mathbb{A}), \omega)$. The $\pi \cap \mathcal{A}(G, \omega)$ is an admissible $G(\mathbb{A})$-module in the above sense.*

## 5.5. **Fourier expansion and Whittaker models.**

**Theorem 5.5.1.** *Let $(\pi, V)$ be an automorphic cuspidal representation of $G$ with*

$$V \subset \mathcal{A}_0(G(F) \backslash G(\mathbb{A}), \omega)$$

*where $\omega : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ is a unitary character of the center. If $\phi \in V$ and $g \in G(\mathbb{A})$, let*

$$W_\phi(g) = \int_{F \backslash \mathbb{A}} \phi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) \psi(-x) dx.$$

*The space $W_V$ of functions $W_\phi$ is a Whittaker model of $\pi$. We have the Fourier expansion*

$$\phi(g) = \sum_{a \in F^\times} W_\phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g\right). \tag{5.5.1}$$

*Proof.* The function $f : \mathbb{A} \to \mathbb{C}$

$$f(x) = \phi\left(\begin{bmatrix}1 & x\\0 & 1\end{bmatrix} g\right)$$

is smooth and periodic with respect to the discrete cocompact subgroup $F$. It admits the Fourier expansion

$$f(x) = \sum_{a\in F} c(a)\psi(ax)$$

since $F$ can be identified with the Pontryagin dual of $F\backslash\mathbb{A}$. The coefficient $c(a)$ is given by the integral

$$c(a) = \int_{F\backslash\mathbb{A}} \phi\left(\begin{bmatrix}1 & x\\0 & 1\end{bmatrix}\right)\psi(-ax)dx.$$

Since $\phi$ is cuspidal $c(0) = 0$. For $a \in F^\times$, we have

$$\begin{aligned}
W_\phi\left(\begin{bmatrix}a & 0\\0 & 1\end{bmatrix} g\right) &= \int_{F\backslash\mathbb{A}} \phi\left(\begin{bmatrix}1 & x\\0 & 1\end{bmatrix}\begin{bmatrix}a & 0\\0 & 1\end{bmatrix} g\right)\psi(-x)dx\\
&= \int_{F\backslash\mathbb{A}} \phi\left(\begin{bmatrix}a & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & a^{-1}x\\0 & 1\end{bmatrix} g\right)\psi(-x)dx\\
&= \int_{F\backslash\mathbb{A}} \phi\left(\begin{bmatrix}1 & x\\0 & 1\end{bmatrix}\right)\psi(-ax)dx.\\
&= c(a)
\end{aligned}$$

where we have used the invariance property of $\phi$ with respect to the translation of $\begin{bmatrix}a & 0\\0 & 1\end{bmatrix}$. □

5.6. **Multiplicity one.** We will prove the strong multiplicity one theorem of Piatetski-Shapiro.

**Theorem 5.6.1.** *Let $(\pi, V)$ and $(\pi', V')$ be two automorphic cuspidal representations of $G$ i.e.*

$$V, V' \subset \mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}), \omega)$$

*for some unitary character $\omega$ of the center. Assume that $\pi_v \simeq \pi'_v$ for all infinite places and for all but finitely many finite places. Then $V = V'$.*

*Proof.* If $\pi_v \simeq \pi'_v$ for all $v$ then they have the same model of Whittaker. The equality $V = V'$ follows from (5.5.1).

Now we suppose that $\pi_v \simeq \pi'_v$ for all infinite places and for all but finitely many finite places. Consider their Whittaker model $\bigoplus_v \mathrm{Wh}(V_v)$ and $\bigoplus_v \mathrm{Wh}(V'_v)$ with $\mathrm{Wh}(V_v) = \mathrm{Wh}(V'_v)$ for almost all places including the infinite place. We choose Whittaker functions $W = \bigotimes_v W_v$ and $W' = \bigotimes_v W'_v$ as follows

(1) $W_\infty = W'_\infty$ is an eigenvector of $K_\infty$
(2) for all $p$ such that $\mathrm{Wh}(V_p) = \mathrm{Wh}(V'_p)$ is unramified we choose $W_v = W'_v$ being a normalized $K_v$-invariant vector,
(3) for finitely remaining $v$ we choose $W_v$ and $W'_v$ such that they have the same Kirillov function $F_v^\times \to \mathbb{C}$ which is a smooth compactly supported function.

The last condition can be made possible because Kirillov model of any infinite dimensional representation contains $C_c^\infty(F_v^\times)$.

By 5.5.1 the series

$$\phi(g) = \sum_{a\in F^\times} W\left(\begin{bmatrix} a & 0 \\ 0 & 1\end{bmatrix} g\right)$$

and

$$\phi'(g) = \sum_{a\in F^\times} W'\left(\begin{bmatrix} a & 0 \\ 0 & 1\end{bmatrix} g\right)$$

are convergent to nonzero function $\phi, \phi' : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$. By our choice of $W$ and $W'$, they agree on $B(\mathbb{A})$. They are both invariant on the right by some $K_0$ where $K_0$ is some compact open subgroup of $G(\mathbb{A}^{\mathrm{fin}})$ and is a eigenvector of $K_\infty$ with the same eigenvalue. It follows that $\phi$ and $\phi'$ agree over

$$B(\mathbb{A})K_0K_\infty.$$

The strong approximation theorem implies that

$$G(F)B(\mathbb{A})K_0K_\infty = G(\mathbb{A}).$$

Since both $\phi$ and $\phi'$ are $G(F)$-invariant on the left, it follows $\phi = \phi'$.

The irreducible representations $V$ and $V'$ sharing a nonzero vector, are the equal. □

## 5.7. **Hecke theory from the Jacquet-Langlands point of view.**

**Theorem 5.7.1.** *Let $\pi = \bigotimes_v \pi_v$ be an irreducible admissible $G(\mathbb{A})$-module that occurs in $\mathcal{A}_{\mathrm{cusp}}(G,\omega)$. Define $L(s,\pi) = \prod_v L(s,\pi_v)$. Then*

1. *$L(s,\pi)$ and $L(s,\tilde{\pi})$ converge in a right half plane and can be holomorphically continued to $\mathbb{C}$.*
2. *They are bounded in any finite vertical strip.*
3. *$L(s,\pi) = \epsilon(s,\pi)L(1-s,\tilde{\pi})$ with $\epsilon(s,\pi) = \prod \epsilon(s,\pi_v,\psi_v)$ for any nontrivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}$.*

*Proof.* Let $V$ denote the space of the automorphic cuspidal representation $\pi$. For each $\phi \in V$, we consider the integral

$$Z(s,\phi) = \int_{F^\times\backslash\mathbb{A}^\times} \phi\left(\begin{bmatrix} a & 0 \\ 0 & 1\end{bmatrix}\right) |a|^{s-1/2} d^\times a. \tag{5.7.1}$$

Since $\phi$ is rapidly decreasing when $|a| \to \infty$, this integral is absolutely convergent for all $s \in \mathbb{C}$ and defines a holomorphic function in $s$ which is bounded on every vertical strip.

The Fourier expansion of $\phi$

$$\phi(g) = \sum_{a\in F^\times} W_\phi\left(\begin{bmatrix} a & 0 \\ 0 & 1\end{bmatrix} g\right)$$

allows us to unfold this integral as

$$I(s,\phi) = \int_{\mathbb{A}^\times} W_\phi\left(\begin{bmatrix} a & 0 \\ 0 & 1\end{bmatrix}\right) |a|^{s-1/2} d^\times a. \tag{5.7.2}$$

If $\phi$ is a decomposable vector, its Whittaker function is an infinite product

$$W_\phi(g) = \prod_v W_{\phi,v}(g_v)$$

for all $g \in G(\mathbf{A})$. Here for all most all $v$, $W_{\phi,v}$ is the normalised Whittaker function and $g_v \in G(\mathcal{O}_v)$ so that $W_{\phi,v}(g_v) = 1$. For $\Re(s) \geq 0$, (5.7.2) is an Eulerian product

$$\prod_v \int_{F_v^\times} W_{\phi,v}\left(\begin{bmatrix} a_v & 0 \\ 0 & 1 \end{bmatrix}\right) |a_v|^{s-1/2} d^\times a_v.$$

Thus for $\Re(s) \geq 0$, we have

$$Z(s,\phi) = \prod_v Z(s, W_{\phi,v}).$$

We have $Z(s, W_{\phi,v}) = L(s,\pi_v)p(s, W_{W,\phi,v})$ where $p(s, W_{W,\phi,v})$ is a polynomial which is equal to one if $W_{\phi,v}$ is the normalized unramified Whittaker function. It follows that

$$Z(s,\phi) = L(s,\pi) \prod_{v \text{ ramified}} p(s, W_{\phi,v})$$

and thus $L(s,\pi_v)$ admit a meromorphic continuation. At the ramified place, it is possible to choose $W_{\phi,v}$ so that $p(s, W_{\phi,v}) = 1$ which implies that $L(s,\pi)$ is holomorphic if $\pi$ is a automorphic cuspidal representation.

Using the automorphy of $\phi$, we have another development of $Z(\phi, s)$ as Euler product

$$\begin{aligned} \phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right) &= \phi\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \omega(a)\phi\left(\begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) \end{aligned}$$

Thus

$$\begin{aligned} Z(\phi,s) &= \int_{F^\times\backslash\mathbb{A}_F^\times} \phi\left(\begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)\omega(a)|a|^{s-1/2} d^\times a \\ &= \int_{F^\times\backslash\mathbb{A}_F^\times} (\pi(w_1)\phi)\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right)\omega^{-1}(a)|a|^{1/2-s} d^\times a \end{aligned}$$

after the change of variable $a \mapsto a^{-1}$. For $\Re(s) << 0$, we have

$$\begin{aligned} Z(\phi,s) &= \prod_v Z(1-s, \omega^{-1}(\pi(w_1)W_{\phi,v})) \\ &= \prod_v L(1-s,\tilde{\pi}_v)\epsilon(s,\pi_v,\psi_v)p(s,W_{\phi,v}) \end{aligned}$$

according to the local functional equation. Here $\epsilon(s,\pi_v,\psi_v)$ and $p(s, W_{\phi,v})$ are equal to 1 for almost all $v$. This implies that $L(s,\tilde{\pi})$ has meromorphic continuation to all the complex plane and that we have the functional equation

$$L(1-s,\tilde{\pi})\epsilon(s,\pi) = L(s,\pi)$$

where $\epsilon$ is an exponential function. □

For all character $\chi : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$, the same argument prove that for all automorphic cuspidal representation $\pi$ the $L$-function $L(s, \pi, \chi)$ has holomorphic continuation as well as $L(1 - s, \tilde{\pi}, \chi^{-1})$ and we have the functional equation

$$L(s, \pi, \chi) = \epsilon(s, \pi, \chi)L(1 - s, \tilde{\pi}, \chi^{-1}).$$

Moreover, the argument can be reversed to prove the converse theorem.

**Theorem 5.7.2.** *Let $\pi$ be an irreducible admissible $G(\mathbb{A})$-module so that $L(s, \pi, \chi)$ is holomorphic and satisfies the above functional equation for all character $\chi$. Then $\pi$ is automorphic cuspidal.*

## Appendix A. Review on compact Riemann surfaces

Let $X$ be a compact Riemann surface. By GAGA theorem, $X$ is a smooth projective curve. Concretely, this means that the field of meromorphic functions $F$ on $X$ is finite extension of the field of fractions $\mathbb{C}(t)$ of the polynomial ring $\mathbb{C}[t]$. In fact any non constant meromorphic function $f \in F$ defines a finite morphism $f : X \to \mathbf{P}^1_{\mathbb{C}}$ and by assigning $t \mapsto f$, we make $F$ a finite extension of $\mathbb{C}(t)$.

A.1. **Divisors.** The group of divisors $\mathrm{Div}(X)$ is the free abelian group with basis the set of points $x \in X$. Its elements are finite linear combinations $\sum_i d_i x_i$ with $d_i \in \mathbb{Z}$ and $x \in X$. The application $\sum_i d_{ix_i} \mapsto \sum_i d_i$ defines a homomorphism $\deg : \mathrm{Div}(X) \to \mathbb{Z}$. We denote $\mathrm{Div}^0(X)$ the group of divisors of degree 0. A divisor $D \in \mathrm{Div}(X)$ is said to be effective if $D = \sum_i d_i x_i$ with $d_i \geq 0$. We will write simply $D \geq 0$ if $D$ is effective.

For every $f \in F$, we have a divisor $\mathrm{div}(f) = \sum_x \nu_x(f) x$ where $\nu_x(f)$ is the vanishing or pole order of $f$ at the point $x$. If $u_x$ is a parameter of $X$ at $x$ then $f \sim u_x^{\nu_x}$ up to the multiplication by invertible function at $x$. We have $\mathrm{div}(f) \in \mathrm{Div}^0(X)$.

For every Zariski open subset $U \subset X$, we also the group $\mathrm{Div}(U)$ and the notion of efective divisors of $U$. For every meromorphic function $f \in F$, we also have a divisor $\mathrm{div}_U(f) = \sum_{x \in U} \nu_x(f) x$.

We will denote $\mathcal{O}_X$ the structural sheaf of the the algebraic curve $X$ whose generic fiber is $F$. For every Zariski open subset $U \subset X$, the sections of $\mathcal{O}_X(U)$ are regular algebraic functions on $U$. In other words

$$\Gamma(U, \mathcal{O}_X) = \{f \in F \mid \mathrm{div}_U(f) \geq 0\}.$$

For every $x$, the local ring of germs of regular functions at $x$ is

$$\mathcal{O}_{X,x} = \{f \in F \mid \nu_x(f) \geq 0\}.$$

This is a regular local ring of dimension one i.e. its maximal ideal is generated by one element. A generator of the maximal ideal of $\mathcal{O}_{X,x}$ is called a parameter of $X$ at $x$.

More generally, for every $D \in \mathrm{Div}(X)$, the sheaf $\mathcal{O}_X(D)$ is defined by

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in F \mid D|_U + \mathrm{div}_U(f) \geq 0\}.$$

For every $x$, the group of germs of its sections regular at $x$ is

$$\mathcal{O}_X(D)_x = \{f \in F \mid d_x + \nu_x(f) \geq 0\}.$$

Its is clear that for every $U \subset X$, $\mathrm{H}^0(U, \mathcal{O}_X(D))$ is a $\mathrm{H}^0(U, \mathcal{O}_X)$-module and for every $x \in X$, $\mathcal{O}_X(D)_x$ is a $\mathcal{O}_{X,x}$-free module of rank one. This means that for every $D \in \mathrm{Div}(X)$, $\mathcal{O}_X(D)$ is a locally free $\mathcal{O}_X$-module of rank one, in other words a line bundle over $X$.

The following statement is a consequence of the Riemann-Roch theorem and the duality of Serre that we will recall later.

**Proposition A.1.1.** *For every $D \in \mathrm{Div}(X)$ the vector space*

$$\Gamma(X, \mathcal{O}_X(D)) = \{f \in F \mid D + \mathrm{div}(f) \geq 0\}$$

*is a finite dimensional. If $\deg(D) < 0$, we have $\Gamma(X, \mathcal{O}_X(D)) = 0$. If $\deg(D) > 2g-2$ where $g$ is the genus of $X$, we have*

$$\dim \Gamma(X, \mathcal{O}_X(D)) = 1 - g + \deg(D).$$

A.2. **Line bundles.** This theorem generalizes to any line bundle because every line bundle $\mathcal{L}$ is of the form $\mathcal{O}_X(D)$. More precisely, let $a \in \mathcal{L} \otimes_{\mathcal{O}_X} F$ be a meromorphic section of $\mathcal{L}$. For every $x \in X$, let $\nu_x(a)$ be the integer so that we have the equivalence relation $a \sim l_x u_x^{\nu_x(a)}$ up to multiplication by an invertible unction at $x$. Here $l_x$ is a generator of $\mathcal{L}_x$ as $\mathcal{O}_{X,x}$-module and $u_x$ is a generator of the maximal ideal of $\mathcal{O}_x$. Let define $D = \mathrm{div}(a) = \sum_{x \in X} \nu_x(a)x$. Then we have a canonical isomorphism $\mathcal{L} = \mathcal{O}_X(D)$. Over the generic fiber, this is the isomorphism between one-dimensional $F$-vector spaces $F \to \mathcal{L} \otimes_{\mathcal{O}_X} F$ assigning $1 \mapsto a$. Even if any line bundle $\mathcal{L}$ on $X$ is of the form $\mathcal{L} = \mathcal{O}_X(D)$ for some divisor $D$, line bundle is not naturally equipped with a meromorphic section so the line bundles and divisors are not equivalent notions. Nevertheless the integer $\deg(\mathrm{div}(a))$ does not depend on the choice of the meromorphic section $a$ of $\mathcal{L}$ since two different meromorphic section differ by a meromorphic function. Therefore $\deg(\mathcal{L}) = \deg(\mathrm{div}(a))$ is well defined.

**Theorem A.2.1** (Riemann-Roch)**.** *The cohomology groups* $\mathrm{H}^i(X, \mathcal{L})$ *are finite dimensional and vanish if* $i \notin \{0, 1\}$. *We have*

$$\dim \mathrm{H}^0(X, \mathcal{L}) - \dim \mathrm{H}^1(X, \mathcal{L}) = 1 - g + \deg(\mathcal{L}).$$

The sheaf of 1-forms $\Omega_{X/k}$ of $X$ over $k$ is a line bundle over $X$ of degree $2g - 2$. For every affine open subset $U \subset X$ with ring of regular functions $A = \Gamma(U, \mathcal{O}_X)$, we have $\Gamma(U, \Omega_{X/k}) = \Omega_{A/k}$ where $\Omega_{A/k}$ is the $A$-module of Kahler differentials.

**Theorem A.2.2** (Serre)**.** *There is a canonical non degenerate pairing between* $\mathrm{H}^0(X, \mathcal{L})$ *and* $\mathrm{H}^1(X, \mathcal{L}^{-1} \otimes \Omega_{X/k})$. *In particular, we have a canonical isomorphism* $\mathrm{H}^1(X, \Omega_{X/k}) \to \mathbb{C}$.

A.3. **Covering.** Let us now review the Hurwitz theorem. Let $f : Y \to X$ be a finite morphism between smooth projective curves over $\mathbb{C}$. Pick a point $y \in Y$ with image $x \in X$, and let $u_x$ be a local parameter of $X$ at $x$ and $u_y$ a local parameter of $Y$ at $y$. We say that $f$ is étale at $y$ if $u_x$ is a local parameter of $Y$ at $y$, in other words $u_x$ and $u_y$ differ by an invertible function at $y$. Since our base fields in $\mathbb{C}$, this happens for all but finitely many points $y \in Y$. A non étale point $y \in Y$ is also called a ramified point. There exists an integer $e_y$, the ramification index, such that $u_x \sim u_y^{e_y}$ up to an invertible function.

**Theorem A.3.1** (Hurwitz)**.** *Let* $f : X \to Y$ *be a finite morphism of degree* $d$ *between smooth projective curves over* $\mathbb{C}$. *We have the relation*

$$2g_Y - 2 = d(2g_X - 2) + \sum_y (e_y - 1)$$

*where we sum over all ramification points of* $Y$.

One can pull back a 1-form from $X$ to $Y$. This defines homomorphism $f^*\Omega_X \to \Omega_Y$ which which is an injective map whose cokernel is a torsion sheaf $\Omega_{Y/X}$ supported by the ramified points. We have $\deg(f^*\Omega_X) = d(2g_X - 2)$ and $\deg(\Omega_Y) = 2g_Y - 2$. It is not difficult to evaluate the length of the cokernel. Let $y \in Y$ be a ramified point. We derives from the relation $u_x \sim u_y^{e_y}$ that $du_x \sim u_y^{e_y - 1} du_y$ which means that the length of the direct factor of $\Omega_{Y/X}$ supported by $y$ is $e_y - 1$. The Hurwitz theorem follows.

A.4. **Adelic desctiption.** Following Weil, any line bundle admit adelic description as follows. Recall that $F_x$ is the completion of $F$ with respect to the topology defined by the maximal ideal of the local ring $\mathcal{O}_{X,x}$ and $\mathcal{O}_x$ its ring of integers. In constrast of the local ring $\mathcal{O}_{X,x}$ remembers about the curve $X$ because its field of fractions is $F$, the completed local ring $\mathcal{O}_x$ is always isomorphic to the ring of formal series of one variable. Let $\mathbb{A}_F$ denote the ring of tuples $(f_x)_{x\in X}$ with $f_x \in F_x$ for all $x \in X$ and $f_x \in \mathcal{O}_x$ for all but finitely many $x \in X$. In particular $\mathcal{L}_{\mathbb{A}}$ contains both $F$ and $\prod_x \mathcal{O}_x$.

We can attach to line bundle $\mathcal{L}$ on $X$ the following adelic data. First we have $L = \mathcal{L}\otimes_{\mathcal{O}_X} F$ is a one-dimensional $F$-vector space. At each point $x \in X$, the one-dimensional vector space $L_x = L \otimes_F F_x$ is equipped with a $\mathcal{O}_x$-submodule $\mathcal{L}_x = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_x$ which is a free $\mathcal{O}_x$-module of rank one. The adelic data $(L, (\mathcal{L}_x)_{x\in X})$ is required to satisfy the following condition : for every nonzero element $a \in L$, $a$ is generator of $\mathcal{L}_x$ for all but finitely many $x \in X$. We denote $L_{\mathbb{A}} = L \otimes_F \mathbb{A}_F$ that contains both $L$ and $\prod_x \mathcal{L}_x$. We observe that the adelization of line bundle has an obvious generalization to vector bundles.

**Proposition A.4.1.** *The cohomology groups are given by the following formula*

$$\mathrm{H}^0(X,\mathcal{L}) = L \cap \prod_x \mathcal{L}_x \tag{A.4.1}$$

$$\mathrm{H}^1(X,\mathcal{L}) = L_{\mathbb{A}}/(L + \prod_x \mathcal{L}_x). \tag{A.4.2}$$

For each $x \in X$, we have a canonical map $\Omega_X \otimes_{\mathcal{O}_X} F_x \to \mathbb{C}$ given by the residue at $x$. By taking the sum of residue, we have a map $L_{\mathbb{A}} \to \mathbb{C}$ whose restriction to $\prod_x \Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_x$ vanish by construction, and whose restriction to $\Omega_{X,x} \otimes_{\mathcal{O}_X} F$ vanishes by the residue theorem. This defines a canonical map $\mathrm{H}^1(X,\mathcal{L}) \to \mathbb{C}$ that appears in Serre's theorem.

## Appendix B. Fourier transform and the Poisson summation formula

See [9, chapter 5] for more details and proofs.

### B.1. Schwartz functions and the Fourier transform.

A Schwartz function on $\mathbb{R}$ is a smooth (indefinitely differentiable) functions $f : \mathbb{R} \to \mathbb{C}$ so that $f$ along with all its derivatives $f', f^{(2)}, \dots$ are *rapidly decreasing*, in the sense that

$$\sup_{x\in\mathbb{R}} |x|^k |f^{(\ell)}(x)| < \infty \text{ for every } k, \ell \geq 0 \tag{B.1.1}$$

We denote $\mathcal{S}(\mathbb{R})$ the space of all Schwartz functions. Smooth functions with compact support are also Schwartz functions.

Simple example of Schwartz functions are $P(x)e^{-x^2}$ where $P(x)$ is a polynomial function. The main property of the Schwartz class of function is its stability under the Fourier transform $f \mapsto \hat{f}$ with

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-2\pi i x y} f(x) dx.$$

**Theorem B.1.1.** *If $f \in \mathcal{S}(\mathbb{R})$ then $\hat{f} \in \mathcal{S}(\mathbb{R})$.*

*Proof.* It can be easily checked that the Fourier transform of a Schwartz function is bounded. The theorem derives from the fact that the Fourier transform exchanges diffentation and multiplication

$$y^k \left(\frac{d}{dy}\right)^{\ell} \hat{f}(y)$$

is the Fourier transform of

$$\frac{1}{(2\pi i)^k}\left(\frac{d}{dx}\right)^k(-2\pi ix)^\ell f(x)$$

which is also a Schwartz function. □

B.2. **The Poisson summation formula.**

**Theorem B.2.1.** *Let $f\in\mathcal{S}(\mathbb{R})$ be a Schwartz function. The series*

$$\sum_{n=-\infty}^{\infty} f(n) \text{ and } \sum_{n=-\infty}^{\infty}\hat{f}(n).$$

*are absolutely convergent. Moreover, we have the equality*

$$\sum_{n=-\infty}^{\infty} f(n)=\sum_{n=-\infty}^{\infty} f(n).$$

*Proof.* [9, p.154]. □

It is sometimes useful to relax the rapidly decreasing condition. A function $f$ is said to be a *sufficiently decreasing condition* if there exist constant $\epsilon>0$ and $A>0$ such that

$$|f(x)|<\frac{A}{1+|x|^{1+\epsilon}}.$$

The series $\sum_{n=-\infty}^{n=\infty} f(n)$ is absolutely convergent as long as the function $f$ is sufficiently decreasing. In fact the Poisson summation formula and the proof [9, p.154] holds if both $f$ and $\hat{f}$ are sufficiently decreasing.

**Proposition B.2.2.** *For any function $f$ that is twice continously derivable and if $f,f',f^{(2)}$ have moderate decrease, the Poisson summation formula holds.*

## Appendix C. Review on Haar measures

C.1. **Locally compact groups.** Let $G$ be a locally compact topological group. We denote the action of $G$ on itself by translation on the left by the formula $l(g_1)g=g_1g$. The action by translation on the right is given by $r(g_1)g=gg_1^{-1}$. These actions induces two actions of $G$ on the space $C_c(G)$ of continuous functions with compact support, given by $l(g_1)f(g)=f(g_1^{-1}g)$ and $r(g_1)f(g)=f(gg_1)$. By duality, we have action on the left and on the right of $G$ on the space of linear functionals of $C_c(G)$.

**Proposition C.1.1** (Haar)**.** *There exists one and up to a factor only one left-invariant measure $d_{l(G)}g$ and we call it a left-invariant Haar measure.*

Obviously, there is also a right invariant measure $d_{r(G)}$ well defined up to a positive factor. Left and right invariant measures are related by the Haar modulus. The right translation $r(g_1)d_{l(G)}$ is still a left-invariant measure, so that there is a unique positive constant $\delta_G(g_1)$ such that $r(g_1)d_{l(G)}g=\delta_G(g_1)d_{l(G)}g$, and the map $g_1\mapsto\delta_G(g_1)$ defines a character $G\to\mathbb{R}^+$.

**Lemma C.1.2.** *If $d_{l(G)}g$ is a left invariant measure then $\delta_G(g)d_{l(G)}g$ is a right invariant measure.*

The group $G$ is said to be unimodular if the Haar modulus is trivial. This is the case for instant if $G$ is its owned derived group $[G,G]$ so that all characters $G \to \mathbb{R}^+$ ought to be trivial. It is also the case if $G$ is compact because any there is no non trivial compact subgroup in $\mathbb{R}^+$. In this case, we will write $dg$ for both $d_{l(G)}g$ and $d_{r(G)}g$.

In the case of Lie group i.e. a topological group equipped with a manifold structure. Here manifold can be defined over $F = \mathbb{R}, \mathbb{C}$ or a $p$-adic field. Let $\mathfrak{g}$ be the tangent space of $G$ at the origin. Given by a non zero linear form $\bigwedge^{\dim(G)} \mathfrak{g} \to F$ where $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{Q}_p$ respectively, we have a left (resp. right) invariant volume form $\omega_{l(G)}$ (resp. $\omega_{r(G)}$) by left (resp. right) translation. These forms are related by

$$\omega_{r(G)} = \det(\mathrm{ad}(g))\omega_{l(G)}$$

where $\mathrm{ad}(g)$ denote the adjoint action. Recall that $\mathrm{ad}(g)$ is the derivation of the conjugation action $x \mapsto gxg^{-1}$ of $G$ on itself.

We also have densities $|\omega|_{l(G)}$ and $|\omega|_{r(G)}$ associated to these volume forms. They are of course related by

$$\omega_{r(G)} = |\det(\mathrm{ad}(g))|\omega_{l(G)}$$

and also the left (right.) invariant measure $d_{l(G)}g$ (resp. $d_{r(G)}g$).

C.2. **Quotient space.** Let $H$ be a closed subgroup of a locally compact group $G$. For all characters $\chi : H \to \mathbb{C}^\times$, we consider the space $C_c(H\backslash G, \chi)$ consisting of locally constant function $f : G \to \mathbb{C}$ that satisfy the equation $f(hg) = \chi(h)f(g)$ for all $h \in H$ and $g \in G$ and such that there exists a compact $K_f$ so that $f$ is supported on $HK_f$.

**Proposition C.2.1.** (1) *We have a linear surjective map*

$$\Lambda : C_c(G) \to C_c(H\backslash G, \delta_H) \tag{C.2.1}$$

*that assigns to every $f \in C_c(G)$ the function $(\Lambda f)(x) = \int_H f(hg) d_{l(H)}h$.*

(2) *This application satisfies the property*

$$\Lambda(l(h)f) = \Lambda(f)$$

*and for every continuous linear map $C_c(G) \to E$ invariant under $l(H)$ must factorize through $\Lambda$. In particular the left invariant measure as a linear functional $C_c(G) \to \mathbb{C}$ factorizes through $C_c(H\backslash G, \delta_H)$.*

(3) *If $G$ is unimodular, the induced linear map $C_c(H\backslash G, \delta_H) \to \mathbb{C}$ is invariant under the right action of $G$ on $H\backslash G$. All such $G$-invariant linear forms are proportional.*

*Proof.* The main observation here is that $\Lambda$ does not commute with the left translation by $h_1 \in H$ if $\delta_H$ is non trivial. Let us check first the invariant property $\Lambda(l(h_1^{-1})f) = \Lambda(f)$ for all $h_1 \in H$. For all $g \in G$, we have

$$\begin{aligned} \Lambda(l(h_1^{-1})f)(g) &= \int_H f(h_1 hg) d_{l(H)}h \\ &= \int_H f(h'g) d_{l(H)}h' \\ &= \Lambda(f)(g). \end{aligned}$$

For every $h_1 \in H$, we have

$$\begin{aligned}(l(h_1^{-1})(\Lambda f))(g) &= (\Lambda f)(h_1 g) \\ &= \int_H f(hh_1 g) d_{l(H)} h \\ &= \delta_H(h_1) \int_H f(h' g) d_{l(H)} h' \\ &= \delta_H(h_1) \Lambda f(g)\end{aligned}$$

where we used the change of variables $h' = hh_1$ and the transformation rule of measures $\delta_H(h_1) d_{l(H)} h' = d_{l(H)} h$.

Let $f \in C_c(G)$ be a function supported by a compact set $K_f$. The function $\Lambda f$ is then supported by $HK_f$. Thus $\Lambda$ defines a linear map $C_c(G) \to C_c(H\backslash G, \delta_H)$.

For every $x \in H\backslash G$, there exists an open subset $U_x$ so that the quotient map $G \to H\backslash G$ admits a section over $u_x : U_x \to G$. We can identify the preimage of $U_x$ in $G$ with $H \times u_x(U_x)$. We can also assume $U_x$ is contained in a compact subset of $H\backslash G$. Let $\phi \in C_c(H\backslash G, \delta_H^{-1})$ so that $f$ is supported by $HK_f$ where $K_f$ is a compact subet of $G$. Since the image of $K_f$ in $H\backslash G$ is compact, there exists a finite number of $x \in H\backslash G$ so that the union of $U_x$ covers the image of $K_f$. Using the partition of unity, we an assume that $\phi$ is supported on $H \times \epsilon_x(U_x)$. We can now choose a function $f$ in the form $f = f_H \otimes \phi$ so that $f_H$ is continuous and compactly supported and such that $\int_H f_H(h) d_{l(H)} h = 1$. Since $U_x$ is contained in a compact, $f$ is compactly supported in $G$ and we have $\Lambda f = \phi$.

We can prove that for every continuous linear map $C_c(G) \to E$ that is invariant under $l(H)$ has to factorise through $\Lambda : C_c(G) \to C_c(H\backslash G, \delta_H)$. In particular the left invariant measure $C_c(G) \to \mathbb{C}$ factorises through a canonical linear map $I : C_c(H\backslash G, \delta_H) \to \mathbb{C}$.

Assume that $G$ is unimodular i.e. $d_{l(G)} g = d_{r(G)} g$. The straightforward calculation

$$\begin{aligned}I(r(g_1)\Lambda(f)) &= \int_G f(gg_1) dg \\ &= \int_G f(g') dg' \\ &= L(\Lambda(f)).\end{aligned}$$

shows that the map $I : C_c(H\backslash G, \delta_H) \to \mathbb{C}$ is invariant under the right action of $G$ □

We are mainly interested in the case where there exists a compact subgroup $K$ of $G$ so that $G = HK$. In that case the support condition is automatically satisfied so that we have $C_c(H\backslash G, \delta_H) = C(H\backslash G, \delta_H)$. Moreover, the linear form $L : C(H\backslash G, \delta_H) \to \mathbb{C}$ can also defined more explicitly as integration over $K$.

**Proposition C.2.2.** *Suppose that there exists a compact subgroup $K$ so that $G = HK$. Then the restriction to $K$ defines an isomorphism*

$$C(H\backslash G, \delta_H) \to C(H \cap K\backslash K).$$

*The linear form $I : C(H\backslash G, \delta_H) \to \mathbb{C}$ and the integration on $K$ with respect to its invariant measure are proportional with respect to this identification.*

*Proof.* Let $f \in C(H\backslash G, \delta_H)$. It derives from $G = HK$ and $f(hk) = \delta_H(h) f(k)$ that $f$ is uniquely determined by its restriction to $K$. Since $K$ is compact, its closed subgroup $H \cap K$

is also compact. It follows that the restriction of the character $\delta_H$ to $H \cap K$ is trivial. Thus the restriction to $K$ defines a linear bijection $C(H\backslash G, \delta_H) \to C(H \cap K\backslash K)$.

The linear form $I : C(H\backslash G, \delta_H) \to \mathbb{C}$ defines a linear form $C(H \cap K\backslash K)$ that is $K$-invariant on the right. It must be proportional with the linear form $C(H \cap K\backslash K)$ defined by integration over $K$. □

## Appendix D. Operators

For more details, see [8, chapter VI].

### D.1. Compact operators.

**Definition D.1.1.** *Let $X, Y$ be Banach spaces. A bounded operator $T : X \to Y$ is called compact if it takes bounded subset in $X$ into a precompact set in $Y$ i.e. a subset of $Y$ whose closure is compact. In other words, for every bounded sequences $\{x_n\} \subset X$, $\{Tx_n\}$ has a convergent subsequence in $Y$.*

Important examples of compact operator are integral operator on a compact Haussdorff space, in particular on the unit interval $[0, 1]$. Let $K(x, y)$ be a continuous function of two variables $x, y \in [0, 1]$. The operator

$$(K\phi)(x) = \int_0^1 K(x, y)\phi(y)dy$$

is a bounded operator on the Banach space $C[0, 1]$ with the uniform norm

$$||\phi||_\infty = \max_{x\in[0,1]} \phi(x).$$

We have indeed

$$||K\phi||_\infty \leq \max_{x,y\in[0,1]} K(x, y)||\phi||_\infty.$$

Let $B_1$ be the set of $\phi \in C^\infty[x, y]$ with norm $||\phi||_\infty \leq 1$. The operator $K$ takes $B_1$ into an equicontinuous family. By compacity, for every $\epsilon > 0$, there exists $\delta > 0$ so that $|K(x, y) - K(x', y)| < \epsilon$ for every $y \in [0, 1]$ as long as $|x - x'| < \delta$. This implies $|(K\phi)(x) - (K\phi)(x')| \leq \epsilon$ for every $\phi \in B_1$. By the Ascoli theorem, a bounded equicontinuous family of functions on $[0, 1]$ is precompact. In other words, $K$ is a compact operator on $[0, 1]$.

**Lemma D.1.2.** *Let $S, T \in \mathcal{L}(X)$. IF $S$ or $T$ is compact then $ST$ is compact. In other words, compact operators form a left and right ideal of $\mathcal{L}(X)$.*

**Proposition D.1.3.** *Operators of finite rank are compact and a norm limit of operators of finite rank is also compact. Inversely, every compact operator on a separable Hilbert space is the norm limit of a sequence of operators of finite rank.*

*Proof.* Balls in finite dimensional vector spaces are compact, and therefore operators of finite rank are compact.

Let $T : E \to F$ be norm limit of compact operators $T_n : E \to F$. Let $x_m$ be a bounded sequence in $E$. After the extracion of a subsequence, we can assume that for every integers $m_1, m_2 \geq n$, we have $|Tx_{m_1} - Tx_{m_2}| \leq 1/n$. It follows that $Tx_m$ is a Cauchy sequence which ought be convergent as $F$ is complete.

Let $\{x_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathsf{H}$. It suffices to prove that the finite rank operators

$$v \mapsto \sum_{i=1}^{n} \langle x_i, v\rangle T x_i$$

converge to $T$ with respect to the norm topology. It suffices to prove that the norm of the restriction of $T$ to the Hilbert subspace generated by $x_{n+1}, x_{n+2}, \ldots$ tends to zero as $n$ goes to infinity. This derives from the compacity of $T$. $\square$

**Proposition D.1.4.** *Let $M$ be a locally compact topological space equipped with a measure, and $\mathsf{H} = L^2(M)$. For every $K \in L^2(M \times M)$, the operator $\phi \mapsto A_K(\phi)$ with*

$$A_K(\phi)(x) = \int_M K(x, y)\phi(y)dy$$

*is a compact operator.*

*Proof.* It follows from the Cauchy-Swartz inequality that $||A_K|| \leq ||K||_2$. Thus the operators $A_K$ is continuous.

Let $\{\phi_i\}_{i=1}^{\infty}$ be an orthonormal basis for $L^2(M)$. Then $\{\phi_i(x)\overline{\phi_j(y)}\}$ is an orthonormal basis for $L^2(M \times M)$ so

$$K = \sum_{i,j=1}^{\infty} \alpha_{ij}\phi_i(x)\overline{\phi_j(y)}.$$

Consider the kernels

$$K_n = \sum_{i,j=1}^{n} \alpha_{ij}\phi_i(x)\overline{\phi_j(y)}$$

whose associated operators $A_{K_n}$ are of finite rank. Since $K_n \to K$ as $n \to \infty$ in $L^2$-norm, $A_{K_n} \to A_K$ in norm topology as $n \to \infty$. It follows that $A_K$ is a compact operator. $\square$

**Theorem D.1.5** (The Fredholm alternative)**.** *If $A$ is a compact operator on $\mathsf{H}$, then either $(1 - A)^{-1}$ exists as bounded operator or $Av = v$ has a nonzero solution.*

*Proof.* Let $F$ be a finite rank operator such that $||A-F|| < 1$. Then the operator $1-(A-F)$ is invertible and its inverse is the infinite series $1 + \sum_{n=1}^{\infty}(A - F)^n$. We have

$$\begin{aligned}(1 - A)(F(1 + \sum_{n=1}^{\infty}(A - F)^n)) &= (1 - (A - F) - F)(1 + \sum_{n=1}^{\infty}(A - F)^n) \\ &= 1 - F(1 + \sum_{n=1}^{\infty}(A - F)^n)\end{aligned}$$

where $F(1 + \sum_{n=1}^{\infty}(A - F)^n)$ is a finite rank operator. It is thus enough to consider the case of finite rank operator instead of compact operators. In this case $(1 - A)$ is invertible if and only if the determinant of the restriction of $1 - A$ to the range of $A$, which is finite dimensional, is non zero. This is equivalent to the existence of a nonzero solution of the equation $Av = v$. $\square$

**Theorem D.1.6** (Analytic Fredholm theorem)**.** *Let $D$ be an open connected subset of $\mathbb{C}$. Let $A : D \to \mathcal{L}(\mathsf{H})$ be an analytic family of compact operators. Assume that there exists $z \in D$ so that $(1 - A(z))^{-1}$ exists as bounded operator. Then $(1 - A(z))^{-1}$ is meromorphic* *in $D$, analytic in the complement $D - S$ of a discrete set $S \subset D$. Its residue at every point $z \in S$ is an operator of finite rank. Moreover, at every point $z \in S$, there exists a nonzero solution of the equation $A(z)v = v$.*

*Proof.* Similar to the aproof of the Fredholm alternative, see [8, p.202]. □

**Theorem D.1.7** (Riesz-Schauder)**.** *Let $A$ be a compact operator on a Hilbert space $\mathsf{H}$. There exists a bounded subset $\sigma(A) \subset \mathbb{C}$ with no accumulation points but $0$ so that $(\lambda - A)^{-1}$ exists on $\mathbb{C} - \sigma(A)$. Moreover, every nonzero $\lambda \in \sigma(A)$ is an eigenvalue of finite multiplicity.*

## Appendix E. Abelian $L$-functions

E.1. **Characters and Hecke characters.** Let $F$ be a global field. We denote by $F_v$ its local fields and $\mathbb{A}_F$ its ring of adèles. We seek a description of characters $\chi : F_v^\times \to \mathbb{C}$ as well as characters of the idèles class group $F^\times \backslash \mathbb{A}^\times \to \mathbb{C}$. Our convention is that characters are continuous homomorphisms with values in $\mathbb{C}^\times$, and unitary characters are continuous homomorphisms with values in the unit circle $\mathbb{C}^1$.

Recall that the local field $F_v$ is equipped with an absolute value $x \mapsto |x| \in \mathbb{R}^+$. Its kernel $U_v$ is a compact subgroup of $F_v$. If $F_v = \mathbb{R}$, then $U_v = \{\pm 1\}$. If $F_v = \mathbb{C}$ then $U_v = \mathbb{C}^1$. If $F_v$ in non archimedean, then $U_v = \mathcal{O}_v^\times$. There exists a homomorphism $F_v^\times \to U_v$ noted by $x \mapsto u_v(x)$ constructed as follows. If $v$ is archimedean, $u_v(x)$ is the unique element of $U_v$ such that $x = u_{v(x)} y$ with $y \in \mathbb{R}^+$. In the non archimedean case, we need to make a further choice of a prime element $\epsilon_v \in F_v^\times$. Then $u_v(x)$ is the unique element of $U_v$ such that

$$x = u_v(x) \epsilon_v^{\mathrm{ord}_v(x)}.$$

Thus, $F_v = U_v \times \mathbb{R}^+$ if $v$ is archimedean and $F_v = U_v \times \mathbb{Z}$ if $v$ is archimedean.

A character $\chi : F_v \to \mathbb{C}^\times$ trivial on $U_v$ is called unramified. All unramified character is of the form

$$x \mapsto |x|^s = e^{s \log |x|}$$

for a certain complex number $s$. This complex number is well defined of $F_v$ is archimedean. In the non-archimedean case, it is only well defined modulo $2\pi i \log |\epsilon_v|$. The character $x \mapsto |x|^s$ is unitary if and only if $s$ is purely imaginary. In the archimedean case, the space of unramified characters is the complex plane $\mathbb{C}$ with the imaginary line $i\mathbb{R}$ as the space of unitary unramified characters. In the non archimedean, the space of unramified characters is the cylinder $\mathbb{C}/2\pi i \log |\epsilon_v|$ with the imaginary circle $i\mathbb{R}/2\pi i \log |\epsilon_v|$ as the space of unitary unramified characters.

The characters of the compact group $U_v$ is automatically unitary. They form the unitary dual $\hat{U}_v$ of $U_v$; $\hat{U}_v$ is a discrete group as $U_v$ is a compact group. If $F_v = \mathbb{R}$, $U_v = \{\pm 1\}$, its dual is also $\hat{U}_v = \{\pm 1\}$. If $F_v = \mathbb{C}$, $U_v = \mathbb{C}^1$, its dual is $\hat{U}_v = \mathbb{Z}$. If $F_v$ is non archimedean, $U_v$ is an inverse limit of finite group, its dual is a direct limit of finite groups.

**Proposition E.1.1.** *The characters $\chi : F_v \to \mathbb{C}^\times$ are all of the form $\chi(x) = \chi_u(u_v(x))|x|^s$ where $\chi_u$ is a character of the compact group $U_v$ and where $|x|^s$ is an unramified character, the complex number $s$ is completely determined by $\chi$ in the archimedean case but only modulo $2\pi i \log |\epsilon_v|$ in the non archimedean case. It is unitary if and only if $s$ is purely imaginary.*

*Proof.* The statement follows from the decomposition $F_v = U_v \times \mathbb{R}^+$ if $v$ is archimedean and $F_v = U_v \times \mathbb{Z}$ if $v$ is archimedean. □

A Hecke character is a character $\chi : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$. It induces a character $\chi_v : F_v^\times \to \mathbb{C}^\times$ for each place $v$ so that we have $\chi = \otimes_v \chi_v$. Since $\chi$ is continuous, $\chi_v$ is unramified for all but finitely many $v$. A Hecke character $\chi$ is called unramified if all its local components $\chi_v$ are unramified.

If $F$ is the field of rational functions of a curve $C$ as in 5.2, an unramified Hecke character is a character $\chi : \mathrm{Pic}_C \to \mathbb{C}^\times$. With a choice of a line bundle of degree one, we have a splitting $\mathrm{Pic}_C = \mathrm{Pic}_C^0 \times \mathbb{Z}$. In this case $\chi$ will also split as $\chi = \chi_0 \otimes |.|^s$ where $\chi_0$ is a character of the finite group $\mathrm{Pic}_C^0$ and where $|.|^s$ is the character $x \mapsto |x|^s = q^{-\deg(x)s}$.

E.2. **Eigendistributions for local fields.** In [3], Kudla gave an exposition of Tate's thesis following the treatment to Weil in his Bourbaki talk.

Let $S(F_v)$ be the space of Schwartz-Bruhat functions on the local field $F_v$; its dual $S(F_v)'$ is the space of tempered distributions. If $v$ is non archimedean, $S(F_v)$ is the space $C_c^\infty(F_v)$ of complexed valued locally constant functions with compact support. The tempered distributions are linear functionals on it. If $v$ is archimedean, $S(F_v)$ is the space of complexed valued smooth functions which, together with all its derivatives, are rapidly decreasing. This is a Fréchet space. The tempered distributions are the continuous linear functionals on it.

The group $F_v^\times$ acts on $F_v$ and induces an action on the space of Schwartz-Bruhat functions $S(F_v)$ as well as the space of tempered distributions $S(F_v)'$. For every character $\chi : F_v^\times \to \mathbb{C}^\times$, let us denote $S(F_v)'(\chi)$ the space of eigenvectors corresponding to the eigenvalue $\chi$.

**Proposition E.2.1.** *Assume $v$ non archimedean. For any character $\chi : F_v^\times \to \mathbb{C}^\times$, we have* $\dim S(F_v)'(\chi) = 1$.

*Proof.* We have an inclusion $C_c^\infty(F_v^\times) \hookrightarrow S(F_v)$ from the space of function on $F^\times$ with compact support into the space of Schwartz-Bruhat functions on $F_v$. By duality, there is an exact sequences

$$0 \to S(F_v)'_0 \to S(F_v)' \to C_c^\infty(F_v)' \to 0$$

where $S(F_v)'_0$ is the space of distributions on $F_v$ supported by 0. Since $v$ is non-archimedean, $S(F_v)'_0$ is one-dimensional and generated by the $\delta$-distribution $\delta_0$.

For evert character $\chi : F_v \to \mathbb{C}^\times$, this induces a left exact sequences of $\chi$-eigenspaces

$$0 \to S(F_v)'_0(\chi) \to S(F_v)'(\chi) \to C_c^\infty(F_v)'(\chi).$$

We have $\dim S(F_v)'_0(\chi_0) = 1$ for the trivial character $\chi_0$, and $\dim S(F_v)'_0(\chi) = 0$ for nontrivial characters $\chi \neq \chi_0$. The space $C_c^\infty(F_v)'(\chi)$ is one-dimensional. If $d^\times x$ is a Haar measure on $F_v^\times$, this space is generated by the distribution $\chi(x)d^\times x$. We have the inequality $1 \leq \dim S(F_v)'(\chi_0) \leq 2$ and in the second case we have $\dim S(F_v)'(\chi) \leq 1$. For nontrivial character, it remains to prove that $\dim S(F_v)'(\chi) \geq 1$ by constructing explicitly a nonzero element of its. For this, we will consider two different cases : $\chi$ is unramified and $\chi$ is ramified. Later on, we will consider separately the case of the trivial character.

The linear application $S(F_v) \to C_c^\infty(F_v^\times)$ mapping $f$ on the function $f(x) - f(\epsilon_v^{-1}x)$ is $F_v^\times$-equivariant. By composing with the application $C_c^\infty(F_v^\times) \to \mathbb{C}$ given by the integration against the measure $\chi(x)d^{\times x}$, we get an element $S(F_v)'(\chi)$ for any character $\chi : F^\times \to \mathbb{C}^\times$. This element may or may not be zero. Let's evaluate it on the function $f = 1_{\mathcal{O}_v}$. The function $x \mapsto f(x) - f(\epsilon_v^{-1}x)$ is the characteristic function $1_{\mathcal{O}_v^\times}$ of the maximal compact subgroup $\mathcal{O}_v^\times$ of $F_v^\times$. If $\chi$ is *unramified*, the integration of $1_{\mathcal{O}_v^\times}$ against $\chi(x)d^\times x$ is one. We defined in this case a nonzero element of $\dim S(F_v)'(\chi)$ that we will denote by $z_0(\chi) \in S(F_v)'(\chi)$. It follows that if $\chi$ in unramified and nontrivial $\dim S(F_v)'(\chi) = 1$.

Let us consider now the case of a ramified character. In this case, for every $f \in S(F_v)$, the integral

$$\int_{F_v^\times - \epsilon_v^n \mathcal{O}_v} f(x)\chi(x) d^\times x$$

is convergent and independent of $n$ for large $n$ since $f$ is constant on $\epsilon_v^n \mathcal{O}_v$ for $n$ large enough. This defines a nonzero element $z_0(\chi) \in S(F_v)'(\chi)$ for every ramified character $\chi$.

Finally, we consider the case of the trivial character $\chi_0$ ... □

By will write $z_0(s,\chi) = z_0(\chi|.|^s)$ to emphasize on the dependance on the twisting by $|.|^s$ when the complex number $s$ varies. For an unitary character $\chi$ of $F_v^\times$, the local zeta integral

$$z(s,\chi;f) := \int_{F_v^\times} f(x)\chi(x)|x|^s d^\times x \tag{E.2.1}$$

is absolutely convergent for all $f \in S(F_v)$ provided $\Re(s) > 0$. In this range, the distribution $f \mapsto z(s,\chi;f)$ defines a nonzero element $z(s,\chi) \in S(F_v)'(\chi|.|^s)$. A direct calculation shows that

$$z_0(s,\chi) = L_v(s,\chi)^{-1} z(s,\chi) \tag{E.2.2}$$

when the right hand side is defined i.e. for all $\Re(s) > 0$. This expression implies an analytic continuation of $z(s,\chi)$ as $L(s.\chi)z_0(s,\chi)$.

**Proposition E.2.2.** *Assume $v$ archimedean. For any character $\chi : F_v^\times \to \mathbb{C}^\times$, we have $\dim S(F_v)'(\chi) = 1$.*

*Proof.* As in the non archimedean case, we have an inclusion $C_c^\infty(F_v^\times) \hookrightarrow S(F_v)$ from the space of function on $F^\times$ with compact support into the space of Schwartz-Bruhat functions on $F_v$. By duality, there is an exact sequences

$$0 \to S(F_v)'_0 \to S(F_v)' \to C_c^\infty(F_v)' \to 0$$

where $S(F_v)'_0$ is the space of distributions on $F_v$ supported by 0. In contrast with the non archimedean case, the space of distributions supported by zero $S(F_v)'_0$ is no longer one-dimensional. If $v$ is real, $S(F_v)'_0$ is generated by the derivatives $D^k\delta_0$ of the $\delta$-function $\delta_0$. If $v$ is complex, $S(F_v)'_0$ is generated by $D^k\bar{D}^l\delta_0$.

The above sequence induces a left exact sequences of $\chi$-eigenspaces

$$0 \to S(F_v)'_0(\chi) \to S(F_v)'(\chi) \to C_c^\infty(F_v)'(\chi).$$

It is easy to see that the space $C_c^\infty(F_v)'(\chi)$ is one-dimensional. If $d^\times x$ is a Haar measure on $F_v^\times$, this space is generated by the distribution $\chi(x)d^\times x$. If $v$ is real, $S(F_v)'_0(\chi) \neq 0$ only for $\chi$ is one of the character $\chi_k(x) = x^{-k}$ for positive integers $k$ for which $S(F_v)'_0(\chi_k)$ is generated by $D^k\delta_0$. We have similar conclusion if $v$ is complex, with the characters $\chi_{k,l}(x) = x^{-k}\bar{x}^{-l}$.

As in the non archimedean case, we consider the zeta integral

$$z(s,\chi;f) := \int_{F_v^\times} f(x)\chi(x)|x|^s d^\times x \tag{E.2.3}$$

that is convergent assuming $\chi$ unitary and $\Re(s) > 0$. In contrast with the non archimedean case, it is the whole Taylor series of $f$ that accounts for the poles of $z(s,\chi;f)$ rather than just the value $f(0)$. The proposition follows from the following two lemmas. □

In the real case, a unitary character $\chi : \mathbb{R}^\times \to \mathbb{C}^1$ must be either the trivial character or the sign character. In both case, we put $L(s, \chi) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$. In the complex case, a unitary character $\chi : \mathbb{C} \to \mathbb{C}^1$ must be of the form $\chi(z) = z^n \bar{z}^{-n}$.

**Lemma E.2.3.** *The distribution $z_0(s, \chi) = L(s, \chi)^{-1} z(s, \chi)$ has an entire analytic continuation to the whole s-plane, and for all s, defines a basis vector for the space $S(F_v)'$.*

*Proof.* The proof is based from an elementary calculation contained in the following lemma. □

**Lemma E.2.4.** *(See [2, Lemma 3.1.7]) Suppose that $f$ is a continuous function on $[0, 1]$ which has a uniform convergent Taylor expansion*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

*Then*

$$\int_0^1 f(x) x^{s-1} dx$$

*has meromorphic continuation to all s, with poles only at the values $s = -k$. The residue at the pole $s = -k$ is the coefficient $a_k$.*

Fix a nontrivial additive unitary character $\psi : F_v \to \mathbb{C}^1$. This choice induces an isomorphism of $F_v$ with its dual $\hat{F}_v = \mathrm{Hom}(F_v, \mathbb{C}^1)$; we associate to an element $y \in F_v$ the unitary character $x \mapsto \psi(xy)$. The Fourier transform

$$\hat{f}(y) = \int_{F_v} f(x) \psi(xy) dx$$

of a function $f \in S(F_v)$ is well defined and again lies in $S(F_v)$. There is a unique choice of Haar measure $dx$ that is self-dual with respect to this Fourier transform so that $\hat{\hat{f}}(x) = f(-x)$. The Fourier transform on $S(F_v)$ gives rise to the Fourier transform on the dual space $S(F_v)'$ of tempered distributions.

**Proposition E.2.5.** *If $\lambda$ is an $\chi$-eigendistribution, its Fourier transform $\hat{\lambda}$ is an $\chi^{-1}\chi_1$-eigendistribution where $\chi_1$ is the absolute value character : $\chi_1(x) = |x|$.*

**Corollary E.2.6.** *There exists a nonzero constant $\epsilon(s, \chi, \psi)$ such that*

$$\hat{z}_0(1 - s, \chi) = \epsilon(s, \chi, \psi) z_0(s, \chi).$$

*This constant, called the local epsilon factor, depends on $\chi, \psi$ and holomorphically on s.*

For a given function $f \in S(F_v)$, the local functional equation can be written as the relation

$$z(1 - s, \chi^{-1}, \hat{f}) = \gamma(s, \chi, \psi) z(s, \chi, f)$$

where the gamma factor is defined as

$$\gamma(s, \chi, \psi) = \epsilon(s, \chi, \psi) \frac{L(1 - s, \chi^{-1})}{L(s, \chi)}.$$

In contrast with the $\epsilon$-factor, the $\gamma$-factor may have pole or zero.

E.3. **The global functional equation and the Poisson summation formula.** Let $F$ be a global field, $\mathbb{A}_F$ its ring of adèles. The space $S(\mathbb{A})$ of Schwartz-Bruhat functions on $\mathbb{A}$ contains all functions of the form $f = \otimes_v f_v$ where $f_v \in S(F_v)$ for all places $v$ and $f_v = 1_{\mathcal{O}_v}$ for almost all $v$. The finite linear such factorizable functions are dense on $S(\mathbb{A})$. Again the topological dual $S(\mathbb{A})'$ consisting on continuous linear functionals on $S(\mathbb{A})$ is the space of tempered distributions. Let $\lambda_v$ be a distribution on $S(F_v)$ such that $\langle \lambda_v, 1_{\mathcal{O}_v}\rangle = 1$ for almost all $v$, then there exists a tempered distribution $\lambda = \otimes \lambda_v$ such that $\langle \lambda, f\rangle = \prod_v \langle \lambda_v, f_v\rangle$.

**Proposition E.3.1.** *For all character $\chi$ of $\mathbb{A}_F^\times$, the space of global $\chi$-eigendistributions $S(\mathbb{A}_F)'(\chi)$ has dimensional one. For any $s \in \mathbb{C}$, $S(\mathbb{A}_F)'(\chi|.|^s)$ is generated by the standard distribution*

$$z_0(s,\chi) = \otimes_v z_0(s,\chi_v).$$

For a character $\chi : F^\times\backslash\mathbb{A}_F^\times$ and a function $f \in S(\mathbb{A})$, we can define the global zeta integral by

$$z(s,\chi,f) = \int_{\mathbb{A}^\times} f(x)\chi(x)|x|^s d^\times x. \tag{E.3.1}$$

If $f$ is factorizable $f = \otimes_v f_v$, this is a product of the local zeta integrals $z(s,\chi_v,f_v)$. Since $z_0(s,\chi_v,f_v) = 1$ as long as $\chi_v$ is unramified and $f_v = 1_{\mathcal{O}_v^\times}$ which happens for $v \notin S$ where $S$ is a finite set of places, the convergence of the infinite product $z(s,\chi,f)$ boils down to the convergence of the local zeta integrals and the convergence of the partial $L$-function $L^S(s,\chi) = \prod_{v\notin S} L_v(s,\chi_v)$. We know that this product is absolutely convergent for $\Re(s) > 1$. On this half-plane of convergence, we have

$$z(s,\chi) = \Lambda(s,\chi) z_0(s,\chi)$$

where $\Lambda(s,\chi) = \prod_v L_v(s,\chi_v)$ is the complete $L$-function.

We fix a global additive character $\psi : k\backslash\mathbb{A} \to \mathbb{C}^1$. By restricting to local components, we can write $\psi = \otimes_v \psi_v$ where $\psi_v$ is unitary character of $F_v$. This choice permits us to identify $\mathbb{A}$ with its dual $\mathbb{A} = \mathrm{Hom}(\mathbb{A},\mathbb{C}^1)$ compatibly with the local identifications $F_v = \mathrm{Hom}(F_v,\mathbb{C}^1)$. The global Fourier transform $S(\mathbb{A}) \to S(\mathbb{A})$ is compatible with the local ones i.e., if $f = \otimes_v f_v$ then $\hat{f} = \otimes_v \hat{f}_v$. It induces a Fourier transform of tempered distributions $S(\mathbb{A})' \to S(\mathbb{A})'$.

**Theorem E.3.2.** *The distribution $z(s,\chi)$ defined by the global zeta integral* (E.3.1) *for $\Re(s) > 1$ has a meromorphic continuation to the whole $s$-plane and satisfies the functional equation*

$$\hat{z}(1-s,\chi^{-1}) = z(s,\chi).$$

*Proof.* Since Schwartz functions have rapid decay at $\infty$, the integral

$$z_{>1}(s,\chi,f) = \int_{\mathbb{A}^\times, |x|>1} f(x)\chi(x)|x|^s d^\times x$$

is convergent for all $s$. □

Recall that E.2.6 has the form $\hat{z}_0(1-s,\chi_v^{-1}) = \epsilon_v(s,\chi_v,\psi_v) = z_0(s,\chi_v)$ where the local epsilon factors $\epsilon_v(s,\chi_v,\psi_v)$ are equal to one for almost all $v$. We define the global epsilon factor by $\epsilon(s,\chi) = \prod_v \epsilon_v(s,\chi_v,\psi_v)$ and obtain the functional equation

$$\hat{z}_0(1-s,\chi^{-1}) = \epsilon(s,\chi) z_0(s,\chi).$$

By comparing this expression with E.3.2, we get the functional equation of the complete $L$-function.

**Corollary E.3.3.**

$$\Lambda(s,\chi) = \epsilon(s,\chi)\Lambda(1-s,\chi^{-1}).$$

## References


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