Notes on Jacquet-Langlands' theory

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Preface

The purpose of these notes would have been better explained if we had chosen another title, namely, "Jacquet-Langlands' theory made easy"; it occurred to us at the last moment that a more pedestrian choice would be more prudent, since after all the author is in a rather bad position to judge ...

These notes cover a very large part of §§2, 3, 5, 6, 9, 10 and 11 of Jacquet-Langlands' work, Automorphic Forms on GL(2), VII + 548 pp., 1970, Springer (Lecture Notes in Mathematics, No. 114). Since the volume of our notes is about one fifth of 548 pp., it is not to be expected that we have been able here to explain everything. In fact, we have entirely omitted the explicit construction of discrete series from quadratic extensions or quaternion algebra (§4 of J. L.), the connection with zêta functions of matrix algebras (§13), and the most interesting, or at any rate newest, part of their work, namely, the relations between the "spectra" of a quaternion algebra and a 2×2 matrix algebra. The reader who is sufficiently interested by the present notes will of course have to go back to Jacquet and Langlands anyway.

We have given full proofs in §1 and nearly complete ones in §3, but not in §2. For the bibliography, we refer the reader to Jacquet and Langlands, where references will be found.

These notes have been written after lectures on the same subject at The Institute for Advanced Study, where we found from September, 1969, to April, 1970, a very welcome atmosphere of quiet intellectual work. It is for us a great pleasure to express here our deep gratitude not only for the conveniences we were provided with, but also for the fact that we were spared the duty to thank the U. S. Air Force for its main contribution to Culture and Civilization, namely, the highly palatable Napalm-and-Mathematics cocktail that is the mark of the times in the most advanced country of the world.

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§1. Representations of the GL_2 group of a $\mathfrak{p}$-adic field

In this section we denote by F a non-archimedean locally compact field, by $\mathcal{O}_F$ and $E_F = \mathcal{O}_F^*$ the ring of integers and group of units of F, and we choose once and for all a non-trivial character τ_F of the additive group, which will be used to define Fourier transforms. The prime ideal of F will be denoted by $\mathfrak{p}$ and a generator of it by ϖ. The largest ideal on which τ_F is trivial will be $\mathfrak{p}^{-d}$ for a certain integer d. There is an absolute value on F defined for instance by the relation $d(ax) = |a|dx$, where dx is an invariant measure on the additive group of F. We shall assume dx is chosen in such a way that the Fourier inversion formula can be written as

$$\hat{f}(y) = \int f(x)\overline{\tau}_F(xy)dx \Rightarrow f(x) = \int \hat{f}(y)\tau_F(xy)dy$$

for nice functions, e.g., for $f \in \mathcal{S}(F)$, the space of locally constant functions with compact support on F. The invariant measure d^*x of F^* will be chosen in such a way that

$$\int_{E_F} d^*x = 1,$$

so that $d^*x = c|x|^{-1}dx$ with a constant c, the value of which is unimportant for the time being.

We shall put

$$G_F = GL(2, F), \qquad M_F = GL(2, \mathcal{O}_F),$$

so that M_F is a maximal compact (and open) subgroup of G_F. The set of locally constant functions with compact support on G_F will be denoted by $\mathcal{H}_F$; it is an algebra (the <u>Hecke algebra</u> of G_F) under convolution product

$$f*g(x) = \int_{G_F} f(xy^{-1})g(y)d^*y ,$$

where d^*y denotes an invariant measure on G_F such that $\int_{M_F} d^*y = 1$.

1. Admissible representations

Let π be a linear representation of G_F on a complex vector space $\mathcal{V}$. For every (finite dimensional) irreducible continuous representation ϑ of the compact group M_F, let $\mathcal{V}(\vartheta)$ be the set of all $\xi \in \mathcal{V}$ which transform under $\pi(M_F)$ according to a finite multiple of ϑ. The representation π will be called admissible if

(1) $$\mathcal{V} = \oplus \mathcal{V}(\vartheta) \quad \text{and} \quad \dim \mathcal{V}(\vartheta) < +\infty.$$

Equivalent conditions: every $\xi \in \mathcal{V}$ is fixed under some open subgroup of G_F, and the set of all $\xi \in \mathcal{V}$ which are fixed under a given open subgroup of G_F is finite-dimensional. These conditions arise in a natural way from the study of automorphic functions as well as from general representation theory. For such a representation we can define, for every $f \in \mathcal{H}_F$, a linear operator $\pi(f)$ on $\mathcal{V}$ by

(2) $$\pi(f)\xi = \int_{G_F} f(x)\pi(x)\xi \, d^*x$$

(the "integral" of course reduces to a finite sum--look at the open stabilizer of ξ in G_F). Hence π extends to a representation of the group algebra $\mathcal{H}_F$, with two properties which characterize, as can easily be proved, the representations of $\mathcal{H}_F$ which can be obtained in that way: (i) for every $\xi \in \mathcal{V}$ there is an $f \in \mathcal{H}_F$ such that $\pi(f)\xi = \xi$; (ii) every $\pi(f)$ maps $\mathcal{V}$ on a finite-dimensional subspace of $\mathcal{V}$. Such representations of $\mathcal{H}_F$ will still be called admissible.

Let π be an admissible representation of G_F on $\mathcal{V}$. We may consider the representation $g \mapsto {}^t\pi(g^{-1})$ on the dual space $\mathcal{V}^* = \prod \mathcal{V}(\vartheta)^*$. The subspace of those $\xi^* \in \mathcal{V}^*$ which are invariant under some open subgroup of G_F is evidently

(3) $$\check{\mathcal{V}} = \oplus \mathcal{V}(\vartheta)^* ;$$

we denote by $\check{\pi}(g)$ the restriction of ${}^t\pi(g)^{-1}$ to $\check{\mathcal{V}}$. We thus get an admissible representation of G_F on $\check{\mathcal{V}}$, which we call the contragredient of π.

If a subspace $\mathcal{V}_1$ of $\mathcal{V}$ is invariant under $\pi(G_F)$, i.e., under $\pi(\mathcal{H}_F)$, then the subspace $\check{\mathcal{V}}_1^{\perp}$ of all $\xi^* \in \check{\mathcal{V}}$ which are orthogonal to $\mathcal{V}_1$ is invariant under $\check{\pi}$, and we have $(\mathcal{V}_1^{\perp})^{\perp} = \mathcal{V}_1$. Thus we get a one-to-one correspondence between invariant subspaces of $\mathcal{V}$ and of $\check{\mathcal{V}}$. A representation with no non-trivial invariant subspaces will be called <u>irreducible</u>. The purpose of this section is to classify these representations, and to associate to each irreducible admissible representation a "local zeta function" which will more or less characterize it.

Note finally that the Schur lemma is valid for irreducible admissible representations: if an operator $T \in \mathcal{L}(\mathcal{V})$ commutes with π, then it operates in every $\mathcal{V}(\vartheta)$, hence must have eigenvectors, so that T is a scalar.

2. <u>The Kirillov model: preliminary construction</u>

Our first goal (number 2 to 4) will be to show that every admissible irreducible representation of G_F can be realized in a very concrete way on a space of functions on F^*, the multiplicative group of non-zero elements of F. For finite dimensional representations the problem is not interesting--a finite dimensional irreducible admissible representation π of G_F is one-dimensional, and given by $\pi(g) = \chi(\det g)$ for some character* χ of F^*. In fact, the finiteness of $\dim \mathcal{V}$ implies that the kernel of π is an <u>open</u> hence non-trivial invariant subgroup of G_F; but any non-trivial invariant subgroup of $GL(2,F)$ contains $SL(2,F)$; hence π is trivial on $SL(2,F)$, the space of π is one-dimensional by Schur's lemma, and we get the result by taking into account the fact that every $g \in GL(2,F)$ is the product of something in $SL(2,F)$ and the matrix $\begin{pmatrix} \det g & 0 \\ 0 & 1 \end{pmatrix}$. We shall thus consider infinite-dimensional representations only. For such representations the following theorem will be proved:

<u>Theorem 1. Let π be an irreducible admissible representation of G_F on an infinite-dimensional vector space $\mathcal{V}$. Then there exists one and only one space $\mathcal{V}'$ of complex valued functions on F^*, and one and only one representation</u>

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* By a character we mean a continuous homomorphism in $\mathbb{C}^*$. Characters such that $|\chi(x)| = 1$ will be called <u>unitary</u>.

π' of G_F on $\mathcal{V}'$, satisfying the following two conditions: π' is equivalent to π, and we have

(4) $$\pi'\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi'(x) = \tau_F(bx)\xi'(ax) \qquad (a,\ x \in F^*,\ b \in F)$$

for every $\xi' \in \mathcal{V}'$.

Furthermore each function in $\mathcal{V}'$ is locally constant, and vanishes outside some compact subset of F; each locally constant function which vanishes outside some compact subset of F^* belongs to $\mathcal{V}'$, and the space $\mathcal{S}(F^*)$ of such functions has finite codimension in $\mathcal{V}'$.

Suppose for a moment we have constructed $\mathcal{V}'$ and π', and let $\xi \mapsto \xi'$ denote an isomorphism of $\mathcal{V}$ on $\mathcal{V}'$ compatible with π and π'; hence

(5) $$\eta = \pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi \Rightarrow \eta'(x) = \tau_F(bx)\xi'(ax).$$

Consider the linear form L on $\mathcal{V}$ given by $L(\xi) = \xi'(1)$; we evidently have

(6) $$L\left[\pi\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\xi\right] = \tau_F(b)L(\xi) \quad \text{for all } \xi \in \mathcal{V} \text{ and } b \in F,$$

and furthermore,

(7) $$\xi'(x) = L\left[\pi\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\xi\right] \quad \text{for all } \xi \in \mathcal{V} \text{ and } x \in F^*.$$

From (6) it follows that

(8) $$\int_{\mathfrak{g}^{-n}} \overline{\tau_F(x)} \cdot L\left[\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\xi\right]dx = L(\xi)\int_{\mathfrak{g}^{-n}} dx$$

for each n; if we consider in $\mathcal{V}$ the subspace $\mathcal{V}_o$ of all vectors ξ such that

(9) $$\int_{\mathfrak{g}^{-n}} \overline{\tau_F(x)}\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\xi \, dx = 0 \quad \text{for all large } n,$$

then it is clear that $\mathcal{V}_o \subset \mathrm{Ker}(L)$. The main step in the proof will be to show that

(10) $$\dim(\mathcal{V}/\mathcal{V}_o) = 1.$$

If we can prove (10), and then denote by L a non-zero linear form vanishing on $\mathcal{V}_o$, then we shall get the space $\mathcal{V}'$ by associating to every $\xi \in \mathcal{V}$ the function (7), and the existence and uniqueness of $\mathcal{V}'$ and π' will easily follow as we shall see later.

For the time being we start with the subspace $\mathcal{V}_o$ defined by condition (9) and denote by X the factor space $\mathcal{V}/\mathcal{V}_o$ and by L the canonical map from $\mathcal{V}$ to X. For every $\xi \in \mathcal{V}$ we consider on F^* the X-valued function (7).

<u>Lemma 1. The relation</u> $\eta = \pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi$ <u>implies</u> $\eta'(t) = \tau_F(bt)\xi'(at)$.

We have to show that $\mathcal{V}_o$ contains $\pi\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\eta - \tau_F(bt)\pi\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix}\xi$ i.e., that

(11) $$\int_{\mathcal{G}^{-n}} \overline{\tau_F(x)}\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\left[\pi\begin{pmatrix} ta & tb \\ 0 & 1 \end{pmatrix} - \tau_F(bt)\pi\begin{pmatrix} ta & 0 \\ 0 & 1 \end{pmatrix}\right]\xi \cdot dx = 0$$

for large n, which is clear (take n large enough so that $tb \in \mathcal{G}^{-n}$, and replace the integration variable x by x-bt in the first term of the difference).

<u>Lemma 2. Each function ξ' is locally constant, and vanishes outside a compact subset of F.</u>

For every $a \in F^*$ sufficiently close to 1 we have $\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\xi = \xi$ and hence $\xi'(xa) = \xi'(x)$ for all x, from which the first assertion follows. Similarly there is in F a non-zero ideal $\mathfrak{A}$ such that $\pi\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\xi = \xi$ for all $b \in \mathfrak{A}$, hence $\xi'(x) = \tau_F(bx)\xi'(x)$, which of course implies the second property.

<u>Lemma 3. The map $\xi \mapsto \xi'$ is injective.</u>

Assume $\xi' = 0$, i.e., $\pi\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\xi \in \mathcal{V}_o$ for all $t \neq 0$. We see at once that for every $t \neq 0$ we have

(12) $$\int_{\mathcal{G}^{-n}} \overline{\tau_F(tx)}\,\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\xi \cdot dx = 0 \quad \text{for all large } n.$$

The first step is to prove that <u>the function</u> $\varphi(x) = \pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\xi$ <u>is constant</u>. In fact, there is a non-zero ideal $\mathfrak{A} \subset \mathcal{G}$ such that $\pi\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\xi = \xi$ for all $u \in 1 + \mathfrak{A}$. Defining

(13) $$\varphi_n(t) = \int_{\mathcal{G}^{-n}} \tau_F(tx)\varphi(x)dx$$

we then see at once that $\varphi_n(tu) = \pi\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}\varphi_n(t)$ for all n, all t, and all u in $1 + \mathfrak{A}$. Since any compact subset K of F^* is a finite union of cosets mod* $1+\mathfrak{M}$, it follows that $\varphi_n(t) = 0$ for all $t \in K$ provided n is large enough, because we assume (12). But let ψ be a Schwartz-Bruhat function on F, and suppose its Fourier transform

(14) $$\hat{\psi}(t) = \int_F \overline{\tau_F(tx)}\psi(x)dx$$

vanishes at $t = 0$, hence outside of a compact subset K of F^*. Since ψ vanishes outside of $\mathcal{G}^{-n}$ for large n, we get, by making use of Fourier inversion formula,

(15) $$\int_F \psi(x)\varphi(x)dx = \int_{\mathcal{G}^{-n}} \varphi(x)dx \int_K \hat{\psi}(t)\tau_F(tx)dt = \int_K \varphi_n(t)\hat{\psi}(t)dt = 0$$

for large n. Hence the function $\varphi(x)$, which is translation invariant under an open subgroup of F, is orthogonal to all $\psi \in \mathcal{S}(F)$ which are orthogonal to the function 1. It follows that $\varphi(x)$ is constant, i.e., that

(16) $$\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\xi = \xi \quad \text{for all } x \in F.$$

The second step in the proof of Lemma 3 is to show that (16) <u>implies</u> $\xi = 0$. Let H be the subgroup of all $g \in G_F$ such that $\pi(g)\xi = \xi$. It is open and contains the subgroup U_F of all matrices of the form $\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}$. Since H is open, it is not contained in the subgroup $\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}$ of G_F, hence H intersects the "big cell" $c \neq 0$, and since H contains U_F, it follows from the "Bruhat décomposition" that H contains a matrix $g = \begin{pmatrix}0 & *\\ * & 0\end{pmatrix}$. But then H must contain the subgroup generated by U_F and such a matrix, namely, $SL(2,F)$. But the set of all $\xi \in \mathcal{V}$ that are fixed under $SL(2,F)$ is an invariant subspace of $\mathcal{V}$ on which $GL(2,F)$ operates as a commutative group. There can thus be no such $\xi \neq 0$ if $\dim \mathcal{V} > 1$, q.e.d.

Lemma 3 makes it possible to identify each vector $\xi \in \mathcal{V}$ with the corresponding function ξ', and from now on we shall write ξ and $\xi(x)$ instead of ξ' and $\xi'(x)$, so that the elements of $\mathcal{V}$ will be certain X-valued functions on F^* on which G_F operates through π in such a way that $\pi\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\xi(x) = \tau_F(bx)\xi(ax)$. The canonical map $L : \mathcal{V} \to X$ can now be identified with $\xi \mapsto \xi(1)$.

<u>Lemma 4. The space $\mathcal{S}_X(F^*)$ of X-valued locally constant functions with compact support on F^* is contained in $\mathcal{V}$. Furthermore, $\pi\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\xi - \xi \in \mathcal{S}_X(F^*)$ for all $b \in F$ and $\xi \in \mathcal{V}$.</u>

The last assertion is obvious since $\tau_F(bx) - 1$ vanishes in a neighborhood of zero for every $b \in F$. To show that $\mathcal{S}_X(F^*) = \mathcal{S}(F^*) \otimes X$ is contained in $\mathcal{V}$, it will be enough to prove that, for vectors $c \in X$ which generate X, all functions $x \mapsto \varphi(x)c$, with $\varphi \in \mathcal{S}(F^*)$, belong to $\mathcal{V}$. But the subspace $\mathcal{S}_c(F^*)$ of those $\varphi \in \mathcal{S}(F^*)$ such that $\mathcal{V}$ contains the function $\varphi(x)c$ is of course stable under the operators(*)

$$(17) \qquad \{x \mapsto \varphi(x)\} \mapsto \{x \mapsto \tau_F(bx)\varphi(ax)\} \qquad (a \in F^*,\ b \in F).$$

Hence it will be enough to prove that (i) the space $\mathcal{S}(F^*)$ is irreducible under the above operators, and that (ii) one has $\mathcal{S}_c(F^*) \neq 0$ for enough vectors $c \in X$.

To prove (i) let $\mathcal{H}$ be a subspace of $\mathcal{S}(F^*)$ invariant under the operators (17). Since every $\xi \in \mathcal{S}(F^*)$ is invariant under a subgroup of finite index of the group E_F of units of F, it is clear that $\mathcal{H} = \Sigma\mathcal{H}(\chi)$ where we denote, for every character χ of E_F, by $\mathcal{H}(\chi)$ the set of all $\xi \in \mathcal{H}$ such that $\xi(xu) = \xi(x)\chi(u)$. If we define

$$(19) \qquad \chi_*(x) = \begin{cases}\chi(x) & \text{if } x \in E_F\\ 0 & \text{if } x \notin E_F,\end{cases}$$

then $\mathcal{S}(F^*)$ is generated by the functions $\chi_*(ax)$ for all χ and all $a \in F^*$. To prove that $\mathcal{H} = \mathcal{S}(F^*)$ is $\mathcal{H} \neq 0$, it will be enough to prove that $\chi_* \in \mathcal{H}$ for every χ.

But there is a $\chi' \neq \chi$ such that $\mathcal{H}(\chi') \neq 0$ [otherwise we would have $\mathcal{H}(\chi) = \mathcal{H}$, which is not compatible with the behavior of the additive characters of F]. Choose such a $\chi' \neq \chi$ and a non-zero $\xi' \in \mathcal{H}(\chi')$. Since $\mathcal{H}$ is invariant under (17), $\mathcal{H}(\chi)$ contains, for all $a \neq 0$ and b, the function

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(*) The use of this notation springs from an attempt by the author to bridge the generation gap. We hope it will have a good reception.

(20) $$\xi(x) = \int_{E_F} \tau_F(bxu)\xi'(axu)\overline{\chi(u)}d^*u = \gamma(bx, \chi'\overline{\chi})\xi'(ax)$$

with a Gaussian sum defined by

(21) $$\gamma(x, \lambda) = \int_{E_F} \tau_F(xu)\lambda(u)d^*u$$

for every character λ of E_F. But it is well known that if λ is non-trivial then

(22) $$\gamma(x, \lambda) \neq 0 \Longleftrightarrow v_{\mathcal{G}}(x) = -d-f(\lambda)$$

where $f(\lambda)$ is the exponent of $\mathcal{G}$ in the conductor of λ. Since $\chi \neq \chi'$ in (20) we can thus choose $b \neq 0$ in such a way that $\gamma(bx, \chi'\overline{\chi}) \neq 0 \Longleftrightarrow x \in E_F$. We can also choose a such that $\xi'(ax) \neq 0$ if $x \in E_F$. Then (20) is evidently proportional to χ_*; hence $\mathcal{H} = \mathcal{S}(F^*)$ and the irreducibility of $\mathcal{S}(F^*)$ under the operators (17). is proved.

The proof of property (ii) is similar. To prove that $\mathcal{S}_c(F^*) \neq 0$ for enough vectors $c \in X$ we may of course limit ourselves to vectors c for which there is in $\mathcal{V}$ a function ξ' such that $\xi'(1) = c$, and satisfying a relation $\xi'(xu) = \xi'(x)\chi'(u)$. Consider then such a ξ' and choose any character $\chi \neq \chi'$; clearly $\mathcal{V}$ still contains the function $\gamma(bx, \chi'\overline{\chi})\xi'(ax)$ given by (20). If we choose $a = 1$ and a suitable b, we thus get in $\mathcal{V}$ a non-zero function proportional to $\chi_*(x)c$, from which it follows that $\mathcal{S}_c(F^*) \neq 0$.

3. The commutativity lemma $\simeq \mathcal{O}_P^{\alpha}$

For every character χ of E_F, every $t \in F$ and every $a \in X$ define

(23) $$\chi_{t,a}(x) = \begin{cases} \chi(t^{-1}x)a & \text{if } x \in tE_F \\ 0 & \text{if } x \notin tE_F \end{cases} .$$

These functions generate the vector space $\mathcal{S}_X(F^*)$, and every $\xi \in \mathcal{S}_X(F^*)$ can be written as a series (in fact, a finite sum)

(24) $$\xi = \sum_{tE_F} \sum_{\chi} \chi_{t,a} \quad \text{where} \quad a = a(t, \chi) = \int_{E_F} \xi(tu)\overline{\chi(u)}du$$

if we assume the total mass of the Haar measure of E_F is 1. Consider now the action on $\mathcal{S}_X(F^*) \subset \mathcal{V}$ of the operator $\pi(w)$, where

(25) $$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

for every $t \in F^*$ and every character χ of E_F, we define in X a linear operator

(26) $$J_\pi(t,\chi) : a \mapsto \pi(w)\chi_{t,a}(1) = L[\pi(w)\chi_{t,a}].$$

If we put $\chi_a = \chi_{1,a}$ then we evidently have $\chi_{t,a}(x) = \chi_a(t^{-1}x)$, hence

(27) $$J_\pi(t,\chi)a = L[\pi(w)\pi\begin{pmatrix} 1/t & 0 \\ 0 & 1 \end{pmatrix}\chi_a] = L[\pi\begin{pmatrix} 1 & 0 \\ 0 & 1/t \end{pmatrix}\pi(w)\chi_a]$$
$$= \omega_\pi(t^{-1})L[\pi\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\pi(w)\chi_a] = \omega_\pi(t^{-1})\pi(w)\chi_a(t),$$

where ω_π is the character of F^* defined by

(28) $$\omega_\pi(t)1 = \pi\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.$$

Formula (27) and Lemma 2 show that each function $J_\pi(t,\chi)a$ is locally constant and vanishes outside a compact subset of F; it is furthermore clear that

(29) $$J_\pi(xu,\chi) = J_\pi(x,\chi)\overline{\chi(u)}$$

for every character χ of E_F. By making use of (24) we get

$$\pi(w)\xi(1) = \sum_{tE_F}\sum_{\chi} J_\pi(t,\chi)a = \sum J_\pi(t,\chi)\int_{E_F}\xi(tu)\overline{\chi(u)}du ,$$

hence

(30) $$\pi(w)\xi(1) = \sum_\chi\int_{F^*} J_\pi(y,\chi)\xi(y)d^*y$$

for every $\xi \in \mathcal{S}_X(F^*)$ - a substitute for the more pleasant formula

(31) $$\pi(w)\xi(1) = \int J_\pi(y)\xi(y)d^*y$$

which we cannot write at this point. If we now apply (30) to the function $\pi\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix}\xi$ instead of ξ, we get at once

(32) $$\pi(w)\xi(x) = \omega_\pi(x)\sum_\chi\int_{F^*} J_\pi(xy,\chi)\xi(y)d^*y ;$$

each integral converges in a trivial way, and the series is actually a finite sum.

Lemma 5. The family of operators $J_\pi(x, \chi)$ is commutative.

To prove this lemma we define $u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for every $t \in F$ and $h(t) = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}$ for every $t \in F^*$ and start from the relation

(33) $$wu(t)w^{-1} = u(-1/t)wh(t)u(-1/t), \qquad t \in F^* .$$

We shall compute the function

(34) $$\eta_t = \pi[wu(t)w^{-1}]\xi = \pi[u(-1/t)wh(t)u(-1/t)]\xi$$

for a given $\xi \in \mathscr{S}_X(F^*)$ by making use of Lemma 1 and relation (32). Using the right hand side of (33) we find at once

(35) $$\eta_t(x) = \tau_F(-x/t)\omega_\pi(x)\sum_\chi \int J_\pi(xy, \chi)\omega_\pi(1/t)\tau_F(-ty)\xi(t^2y)d^*y$$
$$= \omega_\pi(x/t)\sum_\chi \int J_\pi(xy/t^2, \chi)\tau_F[-t^{-1}(x+y)]\xi(y)d^*y.$$

To compute the same function from the left hand side of (33) we write

(36) $$\eta_t = \pi[wu(t)w^{-1}]\xi = \pi(w)[\pi(u(t))\pi(w^{-1})\xi - \pi(w^{-1})\xi] + \xi$$
$$= \omega_\pi(-1)\pi(w)[\pi(u(t))\pi(w)\xi - \pi(w)\xi] + \xi$$

and observe that $\pi[u(t)]\pi(w)\xi - \pi(w)\xi$ belongs to $\mathscr{S}_X(F^*)$ although $\pi(w)\xi$ may not. Using (32) twice we thus get

(37) $$\eta_t(x) = \xi(x) + \omega_\pi(-x)\sum_{\chi'} \int J_\pi(xz, \chi')[\tau_F(tz)-1]\omega_\pi(z)d^*z \times \sum_{\chi''}\int J_\pi(zy, \chi'')\xi(y)d^*y$$
$$= \xi(x) + \omega_\pi(-x)\sum_{\chi', \chi''}\iint J_\pi(xz, \chi')J_\pi(zy, \chi'')\xi(y)[\tau_F(tz)-1]\omega_\pi(z)d^*yd^*z.$$

If we choose any two $t_1, t_2 \in F^*$ and compute $\eta_{t_1} - \eta_{t_2}$ we thus get

(38)
$$\omega_\pi(-t_1)^{-1}\sum_\chi\int J_\pi(xy/t_1^2,\chi)\tau_F[-t_1^{-1}(x+y)]\xi(y)d^*y -$$
$$- \omega_\pi(-t_2)^{-1}\sum_\chi\int J_\pi(xy/t_2^2,\chi)\tau_F[-t_2^{-1}(x+y)]\xi(y)d^*y =$$
$$= \sum_{\chi',\chi''}\iint J_\pi(xz,\chi')J_\pi(zy,\chi'')\xi(y)[\tau_F(t_1z) - \tau_F(t_2z)]\omega_\pi(z)d^*yd^*z.$$

Since the kernels $J_\pi(xy/t^2,\chi)\tau_F[t^{-1}(x+y)]$ in the left hand side are symmetric functions of x and y, the same must be true in the right hand side, i.e., we must have

(39)
$$\sum_{\chi',\chi''}\int J_\pi(xz,\chi')J_\pi(zy,\chi'')[\tau_F(t_1z) - \tau_F(t_2z)\omega_\pi(z)d^*z =$$
$$= \sum_{\chi',\chi''}\int J_\pi(zy,\chi'')J_\pi(xz,\chi')[\tau_F(t_1z) - \tau_F(t_2z)]\omega_\pi(z)d^*z.$$

Looking at the way both sides transform under $x\mapsto xu$ or $y\mapsto yu$ for $u\in E_F$, we see that for any two characters χ' and χ'' of E_F the function

(40) $$\varphi(z) = \omega_\pi(z)[J_\pi(xz,\chi')J_\pi(yz,\chi'') - J_\pi(yz,\chi'')J_\pi(xz,\chi')]$$

(where x and y are arbitrary elements of F^*) must satisfy

(41) $$\int\varphi(z)[\tau_F(t_1z) - \tau_F(t_2z)]d^*z = 0 \quad \text{for all } t_1,t_2\in F^*;$$

since $\varphi(z)a$, for every $a\in X$, is a locally constant function on F^* which vanishes outside a compact subset of F, it follows at once from (41) that φ is orthogonal to all functions in $\mathscr{S}(F^*)$, hence $\varphi = 0$, which concludes the proof of the lemma.

Lemma 6. The space X is one-dimensional.

We first prove that

(42) $$\mathcal{V} = \mathcal{V}_* + \pi(w)\mathcal{V}_* \quad \text{where } \mathcal{V}_* = \mathscr{S}_X(F^*).$$

In fact $\mathcal{V}$ is spanned by the subspaces $\pi(g)\mathcal{V}_*$ since it is irreducible; but $\mathcal{V}_*$ is stable under the operators $\pi\begin{pmatrix}* & * \\ 0 & *\end{pmatrix}$; Bruhat's decomposition thus shows that $\mathcal{V}$ is sum of $\mathcal{V}_*$ and the subspaces $\pi[u(t)w]\mathcal{V}_*$. Since we know that $\pi[u(t)w]\xi - \pi(w)\xi$ belongs to $\mathcal{V}_*$ for every $t\in F$ and every $\xi\in\mathcal{V}$, our assertion follows.

We now prove that <u>if a linear operator</u> A <u>on</u> X <u>commutes with every</u> $J_\pi(x,\chi)$ <u>then</u> A <u>is a scalar</u>. To see that we consider the operator T_A (defined on functions $F^* \to X$) given by $T_A\xi(x) = A(\xi(x))$; we shall prove that $\mathcal{V}$ is invariant under T_A and that T_A induces in $\mathcal{V}$ an operator which commutes with π (hence a scalar, from which it will evidently follow that A itself is a scalar). In fact, any $\xi \in \mathcal{V}$ can be written, as we have just seen, as $\xi = \xi' + \pi(w)\xi''$ with two functions ξ' and ξ'' in $\mathcal{S}_X(F^*)$. We then have

$$T_A\xi(x) = A[\xi'(x) + \pi(w)\xi''(x)]$$

(43)

$$= A(\xi'(x)) + A[\omega_\pi(x)\sum_\chi \int_{F^*} J_\pi(xy,\chi)\xi''(y)d^*y];$$

and since A commutes with the $J_\pi(x,\chi)$ we see (use the fact that the series and integrals above are actually finite sums) that

(44) $$T_A\xi(x) = A(\xi'(x)) + \omega_\pi(x)\sum_\chi \int_{F^*} J_\pi(xy;\chi)A(\xi''(y))d^*y;$$

since the function $x \mapsto A(\xi(x)) = T_A\xi(x)$ is still in $\mathcal{S}_X(F^*)$ for every $\xi \in \mathcal{S}_X(F^*)$, this can be written as $T_A(\xi' + \pi(w)\xi'') = T_A\xi' + \pi(w)T_A\xi''$, which of course concludes the second part of the proof since $\pi(w)^2 = \pm 1$.

Finally, the above argument and Lemma 5 show that in particular all operators $J_\pi(x,\chi)$ are scalars, hence that every linear operator A on X commutes with the $J_\pi(x,\chi)$, hence is a scalar. This implies $\dim(X) = 1$, q.e.d.

4. <u>The finiteness property</u>

Since X is one-dimensional we may identify it (in a non-canonical way) with $\underline{C}$, and replace $\mathcal{S}_X(F^*)$ by $\mathcal{S}(F^*)$. We thus get an identification of $\mathcal{V}$ with a space of complex-valued functions on F^* (in fact, locally constant and zero outside compact subsets of F) on which π operates in such a way that $\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi(x) = \tau_F(bx)\xi(ax)$; and we know that $\mathcal{S}(F^*) \subset \mathcal{V}$. We are now going to prove that $\dim(\mathcal{V}/\mathcal{S}(F^*))$ is finite. Because of (42) it would of course be enough to prove that

(44) $$\dim[\mathcal{V}_* / \pi(w)\mathcal{V}_* \cap \mathcal{V}_*] < +\infty ;$$

if we denote by $\mathcal{V}_*(\chi) = \mathcal{S}(F^*, \chi)$, for every character χ of E_F, the subspace of all $\xi \in \mathcal{V}_*$ such that $\xi(xu) = \xi(x)\chi(u)$, then in order to prove (44) it will clearly be enough to prove the following two lemmas:

Lemma 7. *The space $\pi(w)\mathcal{V}_* \cap \mathcal{V}_*(\chi)$ has finite codimension in $\mathcal{V}_*(\chi)$ for every character χ of E_F.*

Lemma 8. *The space $\pi(w)\mathcal{V}_*$ contains $\mathcal{V}_*(\chi)$ for almost all characters χ of E_F.*

To prove Lemma 7 we observe that it is actually enough to show that the subspace $\mathcal{H} = \pi(w)\mathcal{V}_* \cap \mathcal{V}_*(\chi)$ of $\mathcal{V}_*(\chi) = \mathcal{S}(F^*, \chi)$ is *non-zero*. In fact, every linear form λ on $\mathcal{S}(F^*, \chi)$ is given by a formula

$$\lambda(\xi) = \sum_{n \in \mathbb{Z}} \lambda_n \xi(\varpi^n) \tag{45}$$

where ϖ is a generator of $\mathcal{G}$, and where the λ_n are arbitrary complex coefficients. Since $\mathcal{H}$ is invariant under (multiplicative) translations, it is clear that if $\mathcal{H} \neq 0$ then all λ orthogonal to $\mathcal{H}$ satisfy a non-trivial recursive relation

$$\sum_1^p \alpha_i \lambda_{n-i} = 0 . \tag{46}$$

But the space of solutions λ of (46) is finite-dimensional, hence the lemma.

Before we start the proof of the fact that

$$\pi(w)\mathcal{V}_* \cap \mathcal{V}_*(\chi) \neq 0$$

we observe that

$$\pi[u(t)w^{-1}]\xi - \pi(w^{-1})\xi - \pi[h(t)u(-1/t)]\xi \in \mathcal{V}_* \cap \pi(w)\mathcal{V}_* \tag{47}$$

for all $t \in F^*$ and $\xi \in \mathcal{V}_*$. First of all it is clear that $\mathcal{V}_*$ contains $\pi[u(t)w^{-1}]\xi - \pi(w^{-1})\xi$ and $\pi[h(t)u(-1/t)]\xi$, hence it remains to prove that the left hand side belongs to $\pi(w)\mathcal{V}_* = \pi(w^{-1})\mathcal{V}_*$, i.e., that

$$\pi[wu(t)w^{-1}] - \xi - \pi[wh(t)u(-1/t)]\xi \in \mathcal{V}_* ; \tag{48}$$

but this follows from (33) since we can then write the above expression as $\pi[u(-1/t)]\eta - \eta - \xi$, with $\eta = \pi[wh(t)u(-1/t)]\xi$. Hence (47).

The value of (47) at $x \in F^*$ is

$$(49) \qquad \omega_\pi(-1)[\tau_F(tx)-1]\pi(w)\xi(x) - \omega_\pi(t^{-1})\tau_F(-tx)\xi(t^2x) \ ;$$

since $\pi(w)\mathcal{V}_* \cap \mathcal{V}_*$ is invariant under the operators $h(t)$, we conclude that for all $t \in F^*$, $\xi \in \mathcal{V}_*$ and characters χ of E_F the space $\pi(w)\mathcal{V}_* \cap \mathcal{V}_*(\chi)$ contains the function

$$(50) \qquad x \mapsto \omega_\pi(-1)\int [\tau_F(txu)-1]\pi(w)\xi(xu)\cdot \overline{\chi}(u)d^*u - \omega_\pi(t^{-1})\int \tau_F(-txu)\xi(t^2xu)\overline{\chi}(u)d^*u.$$

Choosing

$$(51) \qquad \xi(x) = \chi'_*(x) = \begin{cases} \chi'(x) & \text{if } x \in E_F \\ 0 & \text{if } x \notin E_F \end{cases}$$

where χ' is another character of E_F, we have

$$(52) \qquad \pi(w)\xi(x) = \omega_\pi(x)J_\pi(x, \chi')$$

and thus see that $\pi(w)\mathcal{V}_* \cap \mathcal{V}_*(\chi)$ contains the function

$$(53) \quad x \longmapsto \omega_\pi(-x)J_\pi(x,\chi')\int [\tau_F(txu)-1]\omega_\pi(u)\overline{\chi\chi'}(u)d^*u - \omega_\pi(t^{-1})\chi'_*(t^2x)\int \tau_F(-txu)\chi'\overline{\chi}(u)\cdot d^*u$$

i.e., the function

$$(54) \quad \omega_\pi(-x)J_\pi(x,\chi')[\gamma(tx, \omega_\pi\overline{\chi\chi'}) - \delta(\omega_\pi\overline{\chi\chi'})] - \omega_\pi(t^{-1})\chi'_*(t^2x)\gamma(-tx, \overline{\chi}\chi')$$

with Gaussian sums γ given by (20), and the obvious meaning for the Dirac symbol δ. We shall show that, (if $\mathcal{V}_* \neq \mathcal{V}$), it is always possible to choose a χ' such that (54) does not identically vanish; this will prove Lemma 7, as we have seen.

The lemma will thus be proved if we can choose x, χ' and t in such a way that

$$(55) \qquad J_\pi(x,\chi') \neq 0, \ \ \gamma(tx, \omega\overline{\chi\chi'}) - \delta(\omega\overline{\chi\chi'}) \neq 0, \ \ 2v(t) + v(x) \neq 0$$

because for such a choice of x, χ' and t the first term in (54) will be nonzero, while the second one will vanish since $\chi'_*(t^2x) = 0$ as soon as $2v(t) + v(x) \neq 0$.

Now for every character χ' of E_F there is at least one integer $n(\chi')$ such that

(56) $$v(x) = n(\chi') \Rightarrow \gamma(x; \overline{\omega\chi\chi'}) - \delta(\overline{\omega\chi\chi'}) \neq 0$$

[and in fact exactly one if $\chi' \neq \omega\overline{\chi}$]. The problem is thus to choose x, χ' and t such that

(57) $$2v(t) + v(x) \neq 0, \quad v(t) + v(x) = n(\chi'), \; J_\pi(x, \chi') \neq 0.$$

But if we have $v(t) + v(x) = n(\chi')$ and $2v(t) + v(x) = 0$ then $v(x) = 2n(\chi')$. Hence the problem is to choose χ' and x such that

(58) $$J_\pi(x, \chi') \neq 0 \quad \text{and} \quad v(x) \neq 2n(\chi').$$

If this is not possible then all functions $J_\pi(x, \chi')$ belong to $\mathcal{S}(F^*)$, and we evidently have then $\pi(w)\mathcal{V}_* \subset \mathcal{V}_*$, i.e., $\mathcal{V}_* = \mathcal{V}$, which contradicts our assumption [or proves the lemma!].

We still have to prove Lemma 8. This is clear if $\pi(w)\mathcal{V}_* \cap \mathcal{V}_*(\chi)$ contains a function whose support reduces to one single class mod E_F. To prove that such is the case for almost every χ, we consider the function (54) with $\chi' = \text{id}$, and assume the conductors of $\omega_\pi\overline{\chi}$ and $\overline{\chi}$ are the same (which is true as soon as the conductor of χ is large enough, hence for almost all χ). Let $\mathfrak{g}^f$ be this conductor. In the expression (54), which now reduces to

(59) $$\omega_\pi(-x)J_\pi(x, \text{id})\gamma(tx, \omega_\pi\overline{\chi}) - \omega_\pi(t^{-1})\text{id}_*(t^2x)\gamma(-tx, \overline{\chi}),$$

the second term is 0 except if

(60) $$2v(t) + v(x) = 0, \; v(t) + v(x) = -d-f,$$

i.e., except if $v(t) = d+f$ and $v(x) = -2(d+f)$. But $J(x, \text{id}) = 0$ if $v(x)$ is large negative. If f is large enough and if t is such that $v(t) = d+f$, we thus see that (59) is non-zero if and only if $v(x) = -2(d+f)$; this concludes the proof of Lemma 8.

To conclude the proof of theorem 1 we still have to prove the uniqueness of the space $\mathcal{V}'$ and of the representation π' on $\mathcal{V}'$. If we use again temporarily the notation of theorem 1 then it is clear that all we need to prove is that there is, up to a constant factor, at most one mapping $\xi \mapsto \xi'$ from $\mathcal{V}$ to a space of complex valued functions on F^* such that

(61) $$\eta = \pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi \Longrightarrow \eta'(x) = \tau_F(bx)\xi'(ax).$$

But consider, for such a mapping, the linear form $L(\xi) = \xi'(1)$ on $\mathcal{V}$; we clearly have

(62) $$\xi'(x) = L[\pi\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\xi],$$

as well as

(63) $$L[\pi\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\xi] = \tau_F(b)L(\xi).$$

By (62) the map $\xi \mapsto \xi'$ is uniquely determined by L. Hence it is enough to prove there is on $\mathcal{V}$ (up to a constant factor) at most one linear form L satisfying (63). But as we have seen after the statement of theorem 1 such a linear form vanishes on the subspace $\mathcal{V}_o$. Since $\dim(\mathcal{V}/\mathcal{V}_o) = 1$, the result follows.

5. Whittaker functions

Let π be an irreducible admissible representation of G_F. If the space $\mathcal{V}$ of π is made up of complex valued functions on F^* on which π operates in such a way that $\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi(x) = \tau_F(bx)\xi(ax)$, then π will be called a Kirillov representation of G_F (or the Kirillov model of the corresponding class of irreducible representations), and the space $\mathcal{V}$ of π will be denoted by $\mathcal{K}(\pi)$. Each class of irreducible admissible representations of G_F contains exactly one Kirillov representation.

Let π be a Kirillov representation of G_F. For every $\xi \in \mathcal{K}(\pi)$ consider the function

(64) $$W_\xi(g) = \pi(g)\xi(1) = L[\pi(g)\xi]$$

on G_F; we get a bijection $\xi \mapsto W_\xi$ of $\mathcal{K}(\pi)$ on a space $\mathcal{W}(\pi)$ of functions on G_F satisfying

(65) $$W[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g] = \tau_F(x)W(g)$$

and locally constant; clearly π acts on $\mathcal{W}(\pi)$ through right translations. The elements of $\mathcal{W}(\pi)$ will be called the <u>Whittaker functions</u> of π, and $\mathcal{W}(\pi)$ will be the <u>Whittaker space</u> of π.

If π is an irreducible admissible representation on an "abstract" vector space $\mathcal{V}$, then as we have seen there is on $\mathcal{V}$ essentially <u>one</u> non-zero linear form L such that

(66) $$L[\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\xi] = \tau_F(x)L(\xi)$$

for all $x \in F$ and $\xi \in \mathcal{V}$; and the choice of such an L defines an isomorphism $\xi \mapsto \xi'$ of $\mathcal{V}$ on the Kirillov space $\mathcal{K}(\pi)$ of π, given by

(67) $$\xi'(x) = L[\pi\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}\xi].$$

The Whittaker function W_ξ is then given by

(68) $$W_\xi(g) = (\pi(g)\xi)'(1) = L[\pi(g)\xi].$$

In particular, suppose $\mathcal{V}$ is contained in the space of solutions of (65) and that G_F operates on $\mathcal{V}$ through right translations. We may then choose for L the linear form $L(\varphi) = \varphi(e)$; it satisfies (66) because each $\varphi \in \mathcal{V}$ satisfies (65), and it is not zero everywhere on $\mathcal{V}$ because $\varphi(g) = L[\pi(g)\varphi]$ since $\pi(g)$ is the right translation defined by g. We then have $W_\varphi = \varphi$, and thus $\mathcal{V} = \mathcal{W}(\pi)$. In other words we get the following

<u>Corollary of Theorem 1. Let π be an irreducible admissible infinite dimensional representation of G_F. Then there is in the vector space of solutions of</u>

(69) $$W[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g] = \tau_F(x)W(g)$$

<u>one and only one right invariant subspace on which the right translations define a representation isomorphic to π, namely, the Whittaker space $\mathcal{W}(\pi)$ of π.</u>

This result will play a fundamental role in the applications to automorphic forms.

6. <u>A theorem on the contragredient of a representation</u>

Let π be an admissible representation of G_F on a vector space $\mathcal{V}$, and suppose we have another admissible representation π' on another vector space $\mathcal{V}'$, as well as a non-degenerate bilinear form $<\xi, \xi'>$ on $\mathcal{V} \times \mathcal{V}'$ such that

$$<\pi(g)\xi, \ \pi'(g)\xi'> = <\xi, \xi'>. \tag{70}$$

Then π' is isomorphic to the contragredient $\check{\pi}$ of π defined in No. 1. In fact, we get from (70) a homomorphism from $\mathcal{V}'$ into $\check{\mathcal{V}}$ by associating to every $\xi' \in \mathcal{V}'$ the linear form $\xi \mapsto <\xi, \xi'>$ on $\mathcal{V}$; and this homomorphism transforms π' into $\check{\pi}$. Hence it remains to prove that it is bijective. But we have $\mathcal{V}' = \oplus \mathcal{V}'(\vartheta)$ and $\check{\mathcal{V}} = \oplus \check{\mathcal{V}}(\vartheta)$ as in No. 1, and the homomorphism of $\mathcal{V}'$ into $\check{\mathcal{V}}$ evidently maps $\mathcal{V}'(\vartheta)$ into $\check{\mathcal{V}}(\vartheta)$ for every ϑ. On the other hand, since the canonical bilinear form on $\mathcal{V} \times \check{\mathcal{V}}$ and the given form on $\mathcal{V} \times \mathcal{V}'$ are invariant and non-degenerate, it is clear that we can identify $\check{\mathcal{V}}(\vartheta)$ and $\mathcal{V}'(\vartheta)$ with the dual of the finite dimensional vector space $\mathcal{V}(\check{\vartheta})$, where $\check{\vartheta}$ is the contragredient of ϑ in the usual sense. Hence the homomorphism $\mathcal{V}' \to \check{\mathcal{V}}$ under consideration induces a bijection $\mathcal{V}'(\vartheta) \to \check{\mathcal{V}}(\vartheta)$ for every ϑ, which shows that it is an isomorphism as was to be proved.

We shall now use these trivial remarks and theorem 1 to prove

<u>Theorem 2. Let π be an irreducible admissible representation of G_F. Then $\check{\pi}$ is equivalent to the representation</u>(*) $g \mapsto \omega_\pi(g)^{-1}\pi(g)$, <u>and the Kirillov space $\mathcal{K}(\check{\pi})$ is the set of functions $\omega_\pi(x)^{-1}\xi(x)$ with $\xi \in \mathcal{K}(\pi)$. Furthermore the invariant duality between $\mathcal{K}(\pi)$ and $\mathcal{K}(\check{\pi})$ is given by the bilinear form $<\xi, \eta>$ such that</u>

$$<\xi, \eta> = \int \xi_1(x) \cdot \eta(-x) d^* x + \int \xi_2(x) \cdot \check{\pi}(w)\eta(-x) d^* x \tag{71}$$

<u>if</u> $\xi = \xi_1 + \pi(w)\xi_2$ <u>with</u> $\xi_1, \xi_2 \in \mathcal{S}(F^*)$ <u>and</u> $\eta \in \mathcal{K}(\check{\pi})$.

(*) We put $\omega(g) = \omega(\det g)$ for every $g \in G_F$ and every character ω of F^*.

To prove that $\check{\pi}$ is equivalent to the representation

$$\pi'(g) = \omega_\pi(g)^{-1}\pi(g) \tag{72}$$

it is enough (since π is admissible and irreducible) to construct on $\mathcal{K}(\pi)$ a non-degenerate bilinear form $<\xi,\eta>_\pi$ such that $<\pi(g)\xi, \pi'(g)\eta>_\pi = <\xi,\eta>_\pi$. The construction and study of this form will be cut into several steps. In what follows we put $\mathcal{V} = \mathcal{K}(\pi)$ and $\mathcal{V}_* = \mathcal{S}(F^*) \subset \mathcal{V}$ as in the proof of Lemma 6.

Step 1. We define

$$<\xi,\eta>_\pi = \int \xi(x)\eta(-x)\omega_\pi(x)^{-1}d^*x \quad \text{if} \quad \xi \in \mathcal{V}_*, \ \eta \in \mathcal{V}; \tag{73}$$

the integral converges in a trivial way. We first show that

$$<\pi(w)\xi,\eta>_\pi = <\xi,\pi(w)^{-1}\eta>_\pi \quad \text{if} \quad \xi \in \mathcal{V}_* \cap \pi(w)\mathcal{V}_* \ \text{and} \ \eta \in \mathcal{V}_*. \tag{74}$$

In fact, we have by No. 3

$$\begin{aligned} \pi(w)^{-1}\eta(x) &= \omega_\pi(-1)\pi(w)\eta(x) = \\ &= \omega_\pi(-x)\sum_\chi \int J_\pi(xy,\chi)\eta(y)d^*y \end{aligned} \tag{75}$$

since $\eta \in \mathcal{S}(F^*)$, and thus

$$\begin{aligned} <\xi,\pi(w)^{-1}\eta>_\pi &= \int \xi(x)\omega_\pi^{-1}(x)d^*x\cdot\omega_\pi(x)\Sigma\int J_\pi(-xy,\chi)\eta(y)d^*y \\ &= \Sigma\int\int J_\pi(-xy,\chi)\xi(x)\eta(y)d^*xd^*y \\ &= \int\eta(-y)\omega_\pi(y)^{-1}d^*y\cdot\omega_\pi(y)\Sigma\int J_\pi(xy,\chi)\xi(x)d^*x \\ &= \int\pi(w)\xi(y)\cdot\eta(-y)\omega_\pi(y)^{-1}d^*y = <\pi(w)\xi,\eta>_\pi \end{aligned} \tag{76}$$

hence the result. Note that this kind of formal computation is justified as soon as $\xi,\eta \in \mathcal{S}(F^*)$, because the summation over the characters χ of E_F is actually a finite sum.

Step 2. We now observe that $\mathcal{V} = \mathcal{V}_* + \pi(w)\mathcal{V}_*$ and define $<\xi,\eta>_\pi$ on the whole of $\mathcal{V}\times\mathcal{V}$ by

$$<\xi,\eta>_\pi = <\xi_1,\eta>_\pi + <\xi_2,\pi(w)^{-1}\eta>_\pi \tag{77}$$

if $\xi = \xi_1 + \pi(w)\xi_2$ with $\xi_1,\xi_2 \in \mathcal{V}_*$, and $\eta \in \mathcal{V}$. This definition makes sense because if $\xi_1 + \pi(w)\xi_2 = 0$ then $\xi_1 \in \mathcal{V}_* \cap \pi(w)\mathcal{V}_*$, so that if we write $\eta = \eta_1 + \pi(w)\eta_2$ with $\eta_1, \eta_2 \in \mathcal{V}_*$ we get

$$<\xi_1, \eta>_\pi + <\xi_2, \pi(w)^{-1}\eta>_\pi =$$

(78)
$$\begin{aligned} &= <\xi_1, \eta_1>_\pi + <\xi_1, \pi(w)\eta_2>_\pi + <\xi_2, \pi(w)^{-1}\eta_1>_\pi + <\xi_2, \eta_2>_\pi \\ &= <\xi_1, \eta_1>_\pi + <\pi(w)^{-1}\xi_1, \eta_2>_\pi + <\pi(w)\xi_2, \eta_1>_\pi + <\xi_2, \eta_2>_\pi \\ &= <\xi_1, \eta_1>_\pi - <\xi_2, \eta_2>_\pi - <\xi_1, \eta_1>_\pi + <\xi_2, \eta_2>_\pi = 0 \,; \end{aligned}$$

we have of course made use of Step 1.

Step 3. We prove that

(79)
$$<\pi(w)\xi, \eta>_\pi = <\xi, \pi(w)^{-1}\eta>_\pi$$

for all $\xi, \eta \in \mathcal{V}$. In fact, if we write $\xi = \xi_1 + \pi(w)\xi_2$ and apply definition (77), we get

(80)
$$\begin{aligned} <\pi(w)\xi, \eta>_\pi &= <\xi_1, \pi(w)^{-1}\eta>_\pi + \omega_\pi(-1)<\xi_2, \eta>_\pi \\ <\xi, \pi(w)^{-1}\eta>_\pi &= <\xi_1, \pi(w)^{-1}\eta>_\pi + <\xi_2, \pi(w)^{-2}\eta>_\pi, \end{aligned}$$

hence the result.

Step 4. Computation (76) shows that

(81)
$$\int \pi(w)\xi(x)\cdot\eta(-x)\omega_\pi(x)^{-1}d^*x = \int \xi(x)\cdot\pi(w^{-1})\eta(-x)\cdot\omega_\pi(x)^{-1}d^*x$$

for any two $\xi, \eta \in \mathcal{V}_*$. The right hand side is $<\xi, \pi(w)^{-1}\eta>_\pi$ by (73), hence $<\pi(w)\xi, \eta>_\pi$ by (77); hence formula (73) is still valid if $\xi \in \pi(w)\mathcal{V}_*$ provided $\eta \in \mathcal{V}_*$, from which we conclude that we still have

(82)
$$<\xi, \eta>_\pi = \int \xi(x)\eta(-x)\omega_\pi(x)^{-1}d^*x \quad \text{if} \quad \xi \in \mathcal{V}, \ \eta \in \mathcal{V}_*.$$

Step 5. We prove that

(83)
$$<\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\xi, \eta>_\pi = <\xi, \pi'\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\eta>_\pi$$

for all $\xi, \eta \in \mathcal{V}$, i.e., that

(84)
$$<\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\xi, \eta>_\pi = \omega_\pi(a)<\xi, \pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\eta>_\pi.$$

If $\xi \in \mathcal{V}_*$ we can use (73) to compute both sides and then the result is obtained at once by replacing $\xi(x)$ by $\xi(ax)$ in (73) and then x by $a^{-1}x$ in the integral. If $\xi \in \mathcal{V}$ is not in $\mathcal{V}_*$ we are (with obvious notation) reduced to proving that

(85) $$\langle\pi\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\pi(w)\xi_2, \eta\rangle_\pi = \omega_\pi(a)\langle\pi(w)\xi_2, \pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix}\eta\rangle_\pi ;$$

but

(86) $$\begin{aligned}\langle\pi\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\pi(w)\xi_2, \eta\rangle_\pi &= \langle\pi(w)\pi\begin{pmatrix} 1 & 0 \\ 0 & a\end{pmatrix}\xi_2, \eta\rangle_\pi = \\ &= \langle\pi\begin{pmatrix} 1 & 0 \\ 0 & a\end{pmatrix}\xi_2, \pi(w)^{-1}\eta\rangle_\pi \quad \text{by Step 3} \\ &= \omega_\pi(a)\langle\pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix}\xi_2, \pi(w)^{-1}\eta\rangle_\pi \\ &= \omega_\pi(a)\cdot\omega_\pi(a^{-1})\langle\xi_2, \pi\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\pi(w)^{-1}\eta\rangle_\pi \quad \text{because } \xi_2 \in \mathcal{V}_* \\ &= \langle\xi_2, \pi(w)^{-1}\pi\begin{pmatrix} 1 & 0 \\ 0 & a\end{pmatrix}\eta\rangle_\pi \\ &= \langle\pi(w)\xi_2, \pi\begin{pmatrix} 1 & 0 \\ 0 & a\end{pmatrix}\eta\rangle_\pi \quad \text{by Step 3} \\ &= \omega_\pi(a)\langle\pi(w)\xi_2, \pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix}\eta\rangle_\pi ,\end{aligned}$$ q. e. d.

<u>Step</u> 6. We prove that

(87) $$\langle\pi(u)\xi, \eta\rangle_\pi = \langle\xi, \pi(u)^{-1}\eta\rangle_\pi \quad \text{for all } \xi,\eta \in \mathcal{V}$$

if $u = \begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}$. Since $\pi(u)\xi(x) = \tau_F(bx)\xi(x)$, this formula is clear if we can compute the scalar products by means of (73), i. e., if ξ <u>or</u> η belongs to $\mathcal{V}_*$ (Step 4). It is thus clear that we are reduced to prove

(88) $$\langle\pi(u)\pi(w)\xi_2, \pi(w)\eta_2\rangle_\pi = \langle\pi(w)\xi_2, \pi(u)^{-1}\pi(w)\eta_2\rangle_\pi$$

in case $\xi_2, \eta_2 \in \mathcal{V}_*$, which by Step 3 reduces to

(89) $$\langle\pi(w^{-1}uw)\xi_2, \eta_2\rangle_\pi = \langle\xi_2, \pi(w^{-1}uw)^{-1}\eta_2\rangle_\pi .$$

Now we have

(90) $$\begin{aligned} w^{-1}uw = \begin{pmatrix} 1 & 0 \\ -b & 1\end{pmatrix} &= \begin{pmatrix} 1 & -b^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} b^{-1} & 0 \\ 0 & b\end{pmatrix}w\begin{pmatrix} 1 & -b^{-1} \\ 0 & 1\end{pmatrix} \\ &= \begin{pmatrix} b & 0 \\ 0 & b\end{pmatrix}\begin{pmatrix} 1 & -b^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} b^{-2} & 0 \\ 0 & 1\end{pmatrix}w\begin{pmatrix} 1 & -b^{-1} \\ 0 & 1\end{pmatrix} \\ &= \begin{pmatrix} b & 0 \\ 0 & b\end{pmatrix}u'hwu' \end{aligned}$$

with obvious notations. Hence

$$\begin{aligned}
<\pi(w^{-1}uw)\xi_2, \eta_2>_\pi &= \omega_\pi(b)<\pi(u'hwu')\xi_2, \eta_2>_\pi \\
&= \omega_\pi(b)<\pi(hwu')\xi_2, \pi(u')^{-1}\eta_2>_\pi \quad \text{because } \eta_2 \in \mathcal{V}_* \\
&= <\pi(wu')\xi_2, \pi(h)^{-1}\pi(u')^{-1}\eta_2>_\pi \quad \text{by Step 5} \qquad (91)\\
&= <\pi(u')\xi_2, \pi(w)^{-1}\pi(h)^{-1}\pi(u')^{-1}\eta_2>_\pi \quad \text{by Step 3} \\
&= <\xi_2, \pi(u')^{-1}\pi(w)^{-1}\pi(h)^{-1}\pi(u')^{-1}\eta_2>_\pi \quad \text{because } \xi_2 \in \mathcal{V}_*,
\end{aligned}$$

hence the result.

To conclude the proof of the theorem, we observe that we have proved identity

$$<\pi(g)\xi, \eta>_\pi = <\xi, \pi'(g)^{-1}\eta>_\pi \qquad (\xi, \eta \in \mathcal{V}) \tag{92}$$

for matrices g which generate G_F, so that it is valid for all $g \in G_F$. On the other hand the bilinear form $<\xi,\eta>_\pi$ on $\mathcal{K}(\pi) \times \mathcal{K}(\pi)$ is non-degenerate, because a function orthogonal to $\mathcal{V}_*$ is zero by formula (73).

We have thus shown thus far that $\check{\pi}$ is equivalent to π'. To conclude the proof, we observe that we have

$$\pi'\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi(x) = \omega_\pi(a)^{-1}\tau_F(bx)\xi(ax) \tag{93}$$

for all $\xi \in \mathcal{K}(\pi)$. To get the Kirillov model of π' or $\check{\pi}$ we must get rid of the factor $\omega_\pi(a)$ in the above formula, which can be done at once by transforming $\mathcal{K}(\pi)$ and π' under the mapping T_π given by $T_\pi\xi(x) = \omega_\pi(x)^{-1}\xi(x)$.

Since a given representation has only one Kirillov realization, it follows that $\mathcal{K}(\check{\pi}) = T_\pi(\mathcal{K}(\pi))$ and that the Kirillov realization of π is given by

$$\check{\pi}(g) = T_\pi \circ \pi'(g) \circ T_\pi^{-1} = \omega_\pi(g)^{-1} T_\pi \circ \pi(g) \circ T_\pi^{-1}. \tag{94}$$

Since the duality $<\xi,\eta>$ between $\mathcal{K}(\pi)$ and $\mathcal{K}(\pi')$ is given by $<\xi,\eta> = <\xi, T_\pi^{-1}\eta>_\pi$, the proof of the theorem is now complete.

7. Supercuspidal representations*

Let π be a given irreducible admissible representation of G_F. We

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*Much more general results will be found in Harish-Chandra, Harmonic Analysis on Reductive $\mathfrak{p}$-adic Groups (Lecture Notes by G. van Dijk).

shall say that π is <u>supercuspidal</u> if $\mathcal{K}(\pi) = \mathcal{S}(F^*)$, i.e., if all functions $\xi(x)$ in the Kirillov model $\mathcal{K}(\pi)$ of π vanish around 0.

It is easy to see that an equivalent property is the fact that for every $\xi \in \mathcal{K}(\pi)$ we have

(95) $$\int_{\mathfrak{q}^{-n}} \pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\xi \cdot dx = 0 \quad \text{for } n \text{ large.}$$

In fact, it is clear in all cases that the left hand side of (95) is the function

(96) $$y \mapsto \int_{\mathfrak{q}^{-n}} \tau_F(xy)\xi(y)dx = \xi(y)\int_{\mathfrak{q}^{-n}} \tau_F(xy)dx \; ;$$

if $\mathfrak{q}^{-d}$ is the largest ideal on which τ_F is trivial, then

(97) $$\int_{\mathfrak{q}^{-n}} \tau_F(xy)dx \neq 0 \iff y \in \mathfrak{q}^{n-d};$$

for the expression (96) to be identically zero for n large, it is thus necessary and sufficient that $\xi(y) = 0$ in some neighborhood of zero, hence the result. If π is a supercuspidal representation then $\mathcal{K}(\check{\pi}) = \mathcal{S}(F^*)$ since $\mathcal{K}(\check{\pi})$ is obtained by multiplying all functions $\xi \in \mathcal{K}(\pi) = \mathcal{S}(F^*)$ by the locally constant function $\omega_\pi(x)$. Hence $\check{\pi}$ is also supercuspidal, and the invariant duality between $\mathcal{K}(\pi)$ and $\mathcal{K}(\check{\pi})$ reduces here to the bilinear form

(98) $$\langle \xi, \eta \rangle = \int_{F^*} \xi(x)\eta(-x)d^*x$$

on $\mathcal{S}(F^*)$, which thus satisfies

(99) $$\langle \pi(g)\xi, \check{\pi}(g)\eta \rangle = \langle \xi, \eta \rangle .$$

Let Z_F be the center of G_F (of course Z_F is isomorphic to F^*). Then for any two $\xi, \eta \in \mathcal{K}(\pi)$ the "coefficient" $\langle \pi(g)\xi, \eta \rangle$ of π is a locally constant function on G_F, whose support is <u>compact</u> <u>mod</u> Z_F. In fact, we have $G_F = M_F H_F M_F$ where M_F is compact and H_F is the diagonal subgroup of G_F, which is the product of Z_F and the subgroup $\begin{pmatrix}* & 0\\ 0 & 1\end{pmatrix}$; but

(100) $$\langle \pi\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\xi, \eta \rangle = \int_{F^*} \xi(tx)\eta(-x)d^*x$$

belongs to $\mathcal{S}(F^*)$ as a function on t; hence the result.

This property of supercuspidal representations is a characteristic one. In fact, let π be an irreducible admissible representation on a vector space $\mathcal{V}$, and assume the function $\langle\pi(g)\xi, \eta\rangle$ has compact support mod Z_F for all $\xi \in \mathcal{V}$ and $\eta \in \check{\mathcal{V}}$. We may assume $\mathcal{V} = \mathcal{K}(\pi)$ and $\check{\mathcal{V}} = \mathcal{K}(\check{\pi})$ and take ξ in $\mathcal{S}(F^*)$; the function

(101) $$\langle\pi\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\xi, \eta\rangle = \int \xi(tx)\eta(-x)d^*x$$

must then vanish outside a compact subset of F^* for all $\xi \in \mathcal{S}(F^*)$ and all $\eta \in \mathcal{K}(\check{\pi})$. Evidently it follows from this condition that $\mathcal{K}(\check{\pi}) = \mathcal{S}(F^*)$, q.e.d. In short:

Theorem 3. Let π be an irreducible admissible representation of G_F on a vector space $\mathcal{V}$. The following conditions are equivalent:

(i) π is supercuspidal, i.e., $\mathcal{K}(\pi) = \mathcal{S}(F^*)$

(ii) $\int_{\mathfrak{p}^{-n}} \pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\xi \cdot dx = 0$ for n large for every $\xi \in \mathcal{V}$

(iii) the function $\langle\pi(g)\xi, \eta\rangle$ has compact support mod Z_F for all $\xi \in \mathcal{V}$, $\eta \in \check{\mathcal{V}}$.

8. Introduction to the principal series

As we shall see, all irreducible non-supercuspidal representations of G_F can be explicitly described in a simple way. The first step is to define, for any two characters μ_1, μ_2 of F^*, a representation ρ_{μ_1, μ_2} of G_F as follows: the space $\mathcal{B}_{\mu_1, \mu_2}$ of ρ_{μ_1, μ_2} is the set of all locally constant functions φ on G_F such that

(102) $$\varphi\left[\begin{pmatrix} t' & * \\ 0 & t'' \end{pmatrix}g\right] = \mu_1(t')\mu_2(t'')|t'/t''|^{1/2}\varphi(g),$$

and the group operates on $\mathcal{B}_{\mu_1, \mu_2}$ through right translations. We thus get a series of admissible representations, which we shall refrain from calling the "principal series" because not all of them are irreducible (see theorem 6 below). The first basic fact is the following:

Theorem 4. If an irreducible admissible representation π of G_F is not

<u>supercuspidal, then it is a subrepresentation of</u> ρ_{μ_1, μ_2} <u>for some choice of</u> μ_1, μ_2.

In fact, consider the Kirillov space $\mathcal{K}(\pi) \supset \mathcal{S}(F^*)$ of π. Then $\mathcal{S}(F^*)$ is invariant under the operators $\pi\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, so that they operate on the <u>finite-dimensional</u> space $\mathcal{K}(\pi)/\mathcal{S}(F^*)$; furthermore the matrices $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ operate trivially there because $[\tau_F(bx)-1]\xi(x)$ is always in $\mathcal{S}(F^*)$. We conclude at once that if $\mathcal{S}(F^*) \neq \mathcal{K}(\pi)$ there is on $\mathcal{K}(\pi)$ a non-zero linear form B and characters μ_1, μ_2 of F^* such that

(103) $$B[\pi\begin{pmatrix} t' & * \\ 0 & t'' \end{pmatrix}\xi] = \mu_1(t')\mu_2(t'')|t'/t''|^{1/2}B(\xi).$$

We then get an isomorphism of π into ρ_{μ_1, μ_2} by associating to every $\xi \in \mathcal{K}(\pi)$ the function

(104) $$\varphi_\xi(g) = B[\pi(g)\xi],$$

which evidently belongs to $\mathcal{B}_{\mu_1, \mu_2}$, q. e. d.

<u>Theorem</u> 5. <u>The contragredient of</u> ρ_{μ_1, μ_2} <u>is</u> $\rho_{-\mu_1, -\mu_2}$ (where $-\mu = \mu^{-1}$).

We need first of all some remarks on invariant measures on groups. Let P be a closed subgroup of a locally compact unimodular group G. If P is unimodular, there is an invariant measure on $P\backslash G$. In the general case, consider the character β_P of P given by

(105) $$d_\ell(pp_o^{-1}) = d_\ell(p_o p p_o^{-1}) = \beta_P(p_o)d_\ell p,$$

where $d_\ell p$ is a left invariant measure on P. Let L(G, P) be the space of continuous functions on G such that

(106) $$\varphi(pg) = \beta_P(p)\varphi(g)$$

and whose support is compact mod P. Then there exists on L(G, P) essentially one positive linear form which is invariant under right translations. If we denote it by

(107) $$\varphi \to \oint_{P\backslash G} \varphi(g)d\dot{g},$$

we have decomposition formula

(108) $$\int_G \varphi(g)dg = \oint_{P\backslash G} d\dot{g} \int_P \varphi(pg)d_\ell p$$

for every continuous function φ with compact support on G. Finally if M is a closed subgroup of G such that $P \cap M$ is compact and $G = MP$ up to a set of measure zero, then

(109) $$\oint_{P\backslash G} \varphi(g)d\dot{g} = \int_M \varphi(mx)d_r m \quad \text{for all } x \in G \text{ and } \varphi \in L(G, P).$$

The "twisted" invariant measure (107) is useful in particular in the following context. Let F and F' be two topological vector spaces in duality, and suppose we are given two continuous representations μ and μ' of P on F and F'; suppose they are contragredient to each other, i. e., that $<\mu(p)a, \mu'(p)a'> = <a, a'>$ for all $p \in P$, $a \in F$ and $a' \in F'$. Denote by $L(G, P, \mu)$ the vector space of all continuous mappings $\varphi : G \rightarrow F$ which satisfy

(110) $$\varphi(pg) = \mu(p)\beta_P^{1/2}(p)\varphi(g)$$

and have compact support mod P; define in a similar way the space $L(G, P, \mu')$; these spaces are stable under right translations by elements of G (and right translations on $L(G, P, \mu)$ more or less define the representation on G "induced" by μ; see below). It is now clear that

(111) $$<\varphi(pg), \varphi'(pg)> = \beta_P(p)<\varphi(g), \varphi'(g)>$$

for all $\varphi \in L(G, P, \mu)$ and $\varphi' \in L(G, P, \mu')$; hence we can define a duality between these two vector spaces by

(112) $$<\varphi, \varphi'> = \oint_{P\backslash G} <\varphi(g), \varphi'(g)>d\dot{g},$$

and this bilinear form is invariant under right translations; this means that the representations of G induced by μ and μ' are more or less (i. e., depending on your definition of "contragredience") contragredient to each other.

We now prove Theorem 5. If we consider the subgroup $P = P_F$ of all triangular matrices $p = \begin{pmatrix} t' & * \\ 0 & t'' \end{pmatrix}$ in G, then we evidently have $\beta_P(p) = |t'/t''|$.

We can thus define a pairing

(113) $$<\varphi, \psi> \doteq \oint_{P_F \backslash G_F} \varphi(g)\psi(g)d\dot{g} = \int_{M_F} \varphi(m)\psi(m)dm$$

between the spaces $\mathcal{B}_{\mu_1, \mu_2}$ and $\mathcal{B}_{-\mu_1, -\mu_2}$, which satisfies

(114) $$<\rho_{\mu_1, \mu_2}(g)\varphi, \rho_{-\mu_1, -\mu_2}(g)\psi> = <\varphi, \psi>.$$

To conclude the proof it remains to prove that the bilinear form $<\varphi, \psi>$ is non-degenerate. But the restrictions to M_F of the functions $\varphi \in \mathcal{B}_{\mu_1, \mu_2}$ are the locally constant functions on M_F such that

(115) $$\varphi[\begin{pmatrix} u' & v \\ 0 & u'' \end{pmatrix} m] = \mu_1(u')\mu_2(u'')\varphi(m)$$

for all u', $u'' \in E_F$ and $v \in \mathcal{O}_F$, the ring of integers of F; and the restrictions of the $\psi \in \mathcal{B}_{-\mu_1, -\mu_2}$ are similarly characterized, with $\mu_1(u')^{-1} = \overline{\mu_1(u')}$ and $\mu_2(u'')^{-1} = \overline{\mu_2(u'')}$ instead. Hence the restrictions to M_F of the functions ψ are the conjugate functions of the restrictions of the φ. Thus (113) reduces to the $L^2(M_F)$ scalar product, and the proof is now complete.

9. A lemma on Fourier transforms

We are now going to show that there exists a Kirillov model for the representation ρ_{μ_1, μ_2} on the vector space $\mathcal{B}_{\mu_1, \mu_2}$, even though ρ_{μ_1, μ_2} may not be irreducible; this construction will furthermore lead to complete results as to the decomposition of ρ_{μ_1, μ_2}.

Since the "big cell" is everywhere dense in G_F, it is clear from (102) that every $\varphi \in \mathcal{B}_{\mu_1, \mu_2}$ is uniquely determined by the function $x \longmapsto \varphi[w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}]$ on the additive group F; in fact, the decomposition

(116) $$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c^{-1} \cdot \det g & * \\ 0 & c \end{pmatrix} w^{-1} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix} \text{ if } c \neq 0$$

shows that

(117) $$\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu_1(\det g)|\det g|^{1/2}\mu^{-1}(c)|c|^{-1}\Phi(d/c) \text{ if } c \neq 0,$$

where we put

(118) $$\varphi[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}] = \Phi(x).$$

The function $\Phi(x)$ is clearly locally constant (in fact, it is translation invariant under an open subgroup of F), and its behavior at infinity is given by

(119) $$\Phi(x) = \varphi(e)\mu^{-1}(x)|x|^{-1} \quad \text{for } |x| \text{ large enough.}$$

In fact, we have $w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -x^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}x^{-1} & 0\\ 0 & x\end{pmatrix}\begin{pmatrix}1 & 0\\ -x^{-1} & 1\end{pmatrix}$ for $x \neq 0$, hence $\varphi[w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}] = \mu(x)^{-1}|x|^{-1}\varphi\begin{pmatrix}1 & 0\\ -x^{-1} & 1\end{pmatrix}$ by (102); but since φ is locally constant we have $\varphi\begin{pmatrix}1 & 0\\ -x^{-1} & 1\end{pmatrix} = \varphi(e)$ for $|x|$ large, whence (119). Conversely, it is easy to see that every locally constant function Φ on F such that $\mu(x)|x|\Phi(x)$ is constant for $|x|$ large, is given by (118) with a function $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$.

To get a Kirillov model for the representation ρ_{μ_1,μ_2} we shall associate to every $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ the function

(120) $$\xi_\varphi(x) = \mu_2(x)|x|^{1/2}\int\varphi[w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}]\overline{\tau_F(xy)}dy = \mu_2(x)|x|^{1/2}\int\Phi(y)\overline{\tau_F(xy)}dy,$$

which is nearly the Fourier transform of (118); it will be seen in a moment that this Fourier transform does make sense if we consider Φ as a distribution(*) and that this Fourier transform is actually a function on F^* (not always on F). Taking that for granted for the time being it is "of course not difficult" to see that the mapping $\varphi \mapsto \xi_\varphi$ is injective, and that if we look upon ρ_{μ_1,μ_2} as a representation of G_F on the space of functions (120), then the fundamental condition for Kirillov's models, namely

(121) $$\rho_{\mu_1,\mu_2}\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\xi(x) = \tau_F(bx)\xi(ax),$$

is satisfied, if only because formula (120) has been selected with (121) in sight.

(*) It even makes sense in the traditional way if $\int_{|x|\geq 1}|\mu(y)^{-1}|d^*y < +\infty$, i.e., if $|\mu(x)| = |x|^\sigma$ with $\sigma > 0$. The case $\sigma < 0$ could be reduced to the previous one by using theorems 2 and 3. Unfortunately, the case $\sigma = 0$ cannot be handled in that simple way. Similar convergence problems arise, not surprisingly, in the construction of the well-known "intertwining operators".

To replace (120) by something more meaningful, we shall need the following lemma:

Lemma 9. *Let μ be a character of F^* and let $\mathcal{F}_\mu$ be the space of locally constant functions Φ on F such that $\Phi(x)\mu(x)|x|$ is constant for $|x|$ large. For every $\Phi \in \mathcal{F}_\mu$ define*

$$\hat{\Phi}(x) = \sum_{n \in \mathbb{Z}} \int_{v(y)=n} \Phi(y)\bar{\tau}_F(xy)dy \quad \text{for } x \in F^*. \tag{122}$$

Then the above series converges uniformly on every compact subset of F^*, and the mapping $\Phi \mapsto \hat{\Phi}$ is injective except if $\mu(x) = |x|^{-1}$, in which case its kernel is the set of constant functions in $\mathcal{F}_\mu$. The image $\hat{\mathcal{F}}_\mu$ of $\mathcal{F}_\mu$ under $\Phi \mapsto \hat{\Phi}$ is the set of locally constant functions ψ on F^* which vanish outside some compact subset of F, and whose behaviour in some neighborhood of 0 is given by the following formulas:

$$\psi(x) = \begin{cases} a\mu(x) + b & \text{if } \mu(x) \not\equiv 1, \ |x|^{-1} \\ av(x) + b & \text{if } \mu(x) \equiv 1 \\ b & \text{if } \mu(x) \equiv |x|^{-1}, \end{cases} \tag{123}$$

with arbitrary constants a and b.

It is clear that $\mathcal{F}_\mu$ is the direct sum of $\mathcal{S}(F)$ and the one-dimensional subspace spanned by the function

$$\Phi_\mu(x) = \begin{cases} \mu^{-1}(x)|x|^{-1} & \text{if } |x| \geq 1 \\ 0 & \text{if } |x| < 1. \end{cases} \tag{124}$$

The convergence of (122) and the behaviour of $\hat{\Phi}$ near 0 and ∞ are clear if $\Phi \in \mathcal{S}(F)$, so that the main part of the proof will be for Φ_μ.

The corresponding series (122) is clearly (up to an immaterial constant factor due to the choice of Haar measures)

$$\sum_{n \leq 0} \int_{v(y)=n} \bar{\tau}_F(xy)\mu^{-1}(y)d^*y; \tag{125}$$

assume first μ is ramified, and let $\mathfrak{g}^f$ be its conductor. Then

(126) $$\int_{v(y)=n} \bar{\tau}_F(xy)\mu^{-1}(y)d^*y \neq 0 \Longleftrightarrow v(x) = -d-f-n;$$

this makes obvious the fact that (125) converges uniformly on every compact subset of F^*, is locally constant, and vanishes for $|x|$ large. Furthermore we have

(127) $$\hat{\Phi}_\mu(x) = \int_{v(xy)=-d-f} \bar{\tau}_F(xy)\mu^{-1}(y)d^*y = a\mu(x) \quad \text{for} \quad v(x) \geq -d-f$$

where

(128) $$a = \int_{v(z)=-d-f} \tau_F(z)\mu^{-1}(z)d^*z \neq 0,$$

hence (123) in this case for all $\Phi \in \mathcal{F}_\mu$.

If now μ is unramified, so that $\mu(x) = |x|^s$ for some s, then we have

(129) $$\int_{v(y)=n} \bar{\tau}_F(xy)|y|^{-s}d^*y = q^{ns}\int_{\mathfrak{o}^*} \bar{\tau}_F(\varpi^n xu)du = q^{ns}\left[\int_{\mathfrak{o}} - \int_{\mathfrak{p}}\right] =$$
$$= q^{ns}[h(\varpi^n x) - |\varpi| h(\varpi^{n+1}x)]$$

where ϖ is a uniformizing variable, $q = N(\mathfrak{p})$, and

(130) $$h(x) = \int_{\mathfrak{o}} \bar{\tau}_F(xu)du = \begin{cases} 1 & \text{if } v(x) \geqslant -d \\ 0 & \text{if } v(x) < -d . \end{cases}$$

The series (122) thus reduces to

(131) $$\hat{\Phi}(x) = F_s(x) - |\varpi| F_s(\varpi x)$$

where

(132) $$F_s(x) = \sum_{n \leq 0} q^{ns}h(\varpi^n x) = \sum_{-d-v(x) \leq n \leq 0} q^{ns};$$

it is thus clear that (122) converges uniformly on compact subsets of F^*, is locally constant on F^*, and vanishes if $v(x) \leqslant -d-1$. If $q^s \neq 1$, i.e., if μ is non-trivial, then we have for $v(x) \geq -d$ a relation $F_s(x) = a'|x|^s + b'$ with $a' \neq 0$; hence $\hat{\Phi}(x) = a|x|^s + b$ with $a = a'(1 - |\varpi|^{s+1}) \neq 0$ if μ is not the character $x \mapsto |x|^{-1}$, and $a = 0$ if $\mu(x) = |x|^{-1}$. If $q^s = 1$ then $F_s(x) = v(x) + d + 1$ and $\hat{\Phi}_\mu(x) = av(x) + b$ with $a = 1 - |\varpi| \neq 0$, for $v(x) \geq -d$.

We have now proved everything except for the determination of the kernel of $\Phi \mapsto \hat{\Phi}$. For every $f \in \mathcal{S}(F^*)$ we have

(133) $$\int_{F^*} f(x)\hat{\Phi}(x)dx = \sum_n \int_F f(x)dx \int_{v(y)=n} \bar{\tau}_F(xy)\Phi(y)dy$$
$$= \sum \int_{v(y)=n} \hat{f}(y)\Phi(y)dy = \int_F \hat{f}(y)\Phi(y)dy,$$

which means that the Fourier transform of the distribution $\Phi(x)dx$ induces on F^* the measure $\hat{\Phi}(x)dx$. If $\hat{\Phi} = 0$ we thus see that the Fourier transform of $\Phi(x)dx$ must be proportional to the Dirac measure, which means that Φ must be constant--and this can happen if and only if $\mu(x) = |x|^{-1}$. This concludes the proof.

It is still useful to observe that if $|\mu(x)| = |x|^{\sigma}$ with $\sigma > 0$, then $\hat{\Phi}(x) = \int_F \Phi(y)\bar{\tau}_F(xy)dy$ with an absolutely convergent integral. If $\sigma > -1/2$ then Φ is square integrable on F, and $\hat{\Phi}$ is its Fourier transform in the L^2 sense. Finally, it is clear by (123) that the functions $\hat{\Phi}$ are integrable on F provided $\sigma > -1$, and that in this case we have $\Phi(x) = \int \hat{\Phi}(y)\tau_F(xy)dy$ for every $\Phi \in \mathcal{F}_{\mu}$.

10. The principal series and the special representations

We can now go back to the representation ρ_{μ_1,μ_2}. It follows from (119) that the representation space $\mathcal{B}_{\mu_1,\mu_2}$ is the same as $\mathcal{F}_{\mu}$ under the map $\varphi \to \Phi$ given by (118). With the same meaning for $\hat{\Phi}$ as in Lemma 9, let us associate to every $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ the function ξ_{φ} given by

(134) $$\xi_{\varphi}(x) = \mu_2(x)|x|^{1/2}\hat{\Phi}(x) = \mu_2(x)|x|^{1/2}\sum \int_{v(y)=n} \varphi[w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}]\bar{\tau}_F(xy)dy.$$

A trivial computation then shows that

(135) $$\varphi' = \rho_{\mu_1,\mu_2}\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\varphi \Longrightarrow \xi_{\varphi'}(x) = \tau_F(bx)\xi_{\varphi}(ax),$$

and since the functions ξ_{φ} are locally constant and vanish outside compact subsets of F, we see that the mapping $\varphi \mapsto \xi_{\varphi}$ yields a Kirillov model for the representation ρ_{μ_1,μ_2}, except for the fact that this mapping may not be one-to-one. This occurs if and only if $\mu(x) = |x|^{-1}$, in which case the kernel of this mapping is generated by the function φ for which $\Phi(x) = 1$, i.e., [use (117)] by the function

(136) $$\varphi_o(g) = \mu_1(\det g)|\det g|^{1/2}.$$

Of course the one-dimensional subspace generated by (136) is invariant under ρ_{μ_1,μ_2}, and $\varphi \mapsto \xi_\varphi$ then induces a bijection of the corresponding factor space on the space of functions ξ_φ.

Note that the image $\mathcal{K}_{\mu_1,\mu_2}$ of $\mathcal{B}_{\mu_1,\mu_2}$ under $\varphi \mapsto \xi_\varphi$ can be described from Lemma 9; it is the space of locally constant functions on F^* which vanish for $|x|$ large and whose behaviour near 0 is given by the following formulas, which follow at once from (123) and (124):

(137) $$\xi(x) = \begin{cases} |x|^{1/2}[a\mu_1(x) + b\mu_2(x)] & \text{if } \mu(x) \neq 1, |x|^{-1}, \\ |x|^{1/2}[a\mu_2(x)v(x) + b\mu_2(x)] & \text{if } \mu(x) = 1, \\ b|x|^{1/2}\mu_2(x) & \text{if } \mu(x) = |x|^{-1}. \end{cases}$$

This space contains always $\mathcal{S}(F^*)$ as a subspace of codimension 2, except if μ is the character $x \mapsto |x|^{-1}$ in which case $\mathcal{S}(F^*)$ has codimension 1. We shall now be able to decide whether ρ_{μ_1,μ_2} is irreducible or not:

<u>Theorem</u> 6. <u>The representation</u> ρ_{μ_1,μ_2} <u>is irreducible except if</u> $\mu(x) = |x|$ <u>or</u> $|x|^{-1}$. <u>If</u> $\mu(x) = |x|^{-1}$ <u>then</u> $\mathcal{B}_{\mu_1,\mu_2}$ <u>contains a one-dimensional invariant subspace, generated by the function</u> $g \mapsto \mu_1(\det g)|\det g|^{1/2}$, <u>and the representation on the factor space is irreducible.</u> <u>If</u> $\mu(x) = |x|$ <u>then</u> $\mathcal{B}_{\mu_1,\mu_2}$ <u>contains an irreducible subspace of codimension one, namely, the set of</u> φ <u>such that</u>

(138) $$\oint_{P_F \backslash G_F} \varphi(g)\mu_1^{-1}(\det g)|\det g|^{1/2}d\dot{g} = \int \varphi[w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}]dx = 0.$$

Since the kernel of $\varphi \mapsto \xi_\varphi$ is invariant in all cases, this mapping transforms ρ_{μ_1,μ_2} into a representation of G_F on the image space $\mathcal{K}_{\mu_1,\mu_2}$, and property (135) shows that if an invariant subspace of $\mathcal{K}_{\mu_1,\mu_2}$ contains a