Automorphic Forms on GL(2)

Andrew Snowden

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Foreword

1. I wrote these notes to learn the material covered in the book of Jacquet and Langlands [JL] titled "Automorphic forms on GL(2)." The book contains a great deal of information, but I found it to be somewhat disorganized. The point of these notes is simply to reorganize the material in that book to make it easier for me (and possibly others) to understand.

2. Most of the statements and proofs of propositions in these notes are taken directly from other sources. I have tried to be good at referencing exactly from where and from which source everything I took originally comes, but I may have missed a few things.

3. Broadly speaking, these notes cover the following material:

1. Most of chapter one of Jacquet-Langlands: all of the theory of GL(2) over a non-archimedean field and most of the theory over archimedean fields.
2. Some of the first three sections of chapter two of Jacquet-Langlands.
3. None of the last section of chapter two or any of chapter 3 of Jacquet-Langlands.
4. Most of the first three chapters of Shimura's book [Sh] (and none of the rest of the book).

The notes therefore cover a large part of Bump's book [Bu] since most of the material in that book is contained in the above list. I hope to expand on these notes over time to include more.

Version Information

1 Version 0.1. First release version; released September 20th, 2005; 111 pages. Rough account of contents:

1. A very small amount of background material.
2. Some GL(1) theory.
3. Statements related to the Weil representations (no proofs).
4. Extensive amounts of material on the representation theory of GL(2) over a non-archimedean local field and the real field.
5. Statements of the main results (no proofs) for the representation theory of GL(2) over the complex field.

2 Version 0.2. Released October 24th, 2005; 138 pages. Changes from previous version:

1. Includes much more background material: distributions on smooth manifolds, restricted direct and tensor products, module theory for idempotented algebras.
2. Added proofs for the "basic" case of GL(1) over a local field.
3. Added a short amount of material on global theory.

3 Version 0.3. Released Januray 19th, 2006; 187 pages. Changes from previous version:

1. Added sections "Finite functions on locally compact groups" and "Simple constituents of composite modules" to the "Complements" chapter (Chapter 1).
2. Added chapter "The classical theory of automorphic forms on GL(2)" (Chapter 3). Rough account of contents:
   (a) Some point-set topology preliminaries.
   (b) Definition of upper half plane and the action of $\mathrm{PSL}(2,\mathbb{R})$ on it.
   (c) Discussion of the space $\Gamma\backslash\mathfrak{H}^*$, e.g., it is a locally compact Hausdorff space with a complex structure. Fuchsian groups of the first kind.
   (d) The group $\mathrm{SL}(2,\mathbb{Z})$ and its congruence subgroups: some group theory and genus calculations.
   (e) Discussion on "abstract" Hecke algebras, closely following Shimura.
   (f) Definition of automorphic form on the upper half plane. Dimension of certain spaces of automorphic forms. The action of Hecke algebras on automorphic forms.
3. Added section "Representation theory of TDLC groups" to the beginning of the "Representations of $\mathrm{GL}(2,F)$ in the non-archimedean case" chapter (Chapter 5). There are now a lot of redundencies with the following section which should be fixed in the future. I imagine the section on TDLC groups will eventually be removed from this chapter and placed in an earlier chapter.
4. Added a lot to the "Representations of $\mathrm{GL}(2,A)$" chapter:
   (a) An automorphic cuspidal representation is a constituent of $\mathscr{A}_0(\eta)$ for some η.

(b) Statement of admissibility and complete reducibility of $\mathscr{A}_0(\eta)$, but not proof.

(c) The Fourier expansion of a cusp form.

(d) Automorphic cuspidal representations have Whittaker models.

(e) The multiplicity one theorem.

(f) Automorphic representations which are not cuspidal.

Chapter 1

Complements

1.1 The representation theory of compact groups

1. In this section we very briefly go over some facts and definitions about representations of compact topological groups.

2. Let K be a compact topological group. We let $\hat{K}$ denote the set of isomorphism classes of finite dimensional irreducible representations of K. An element of $\hat{K}$ is called a *K-type*. We say a finite dimensional representation V of K has type $\sigma \in \hat{K}$ is V has isomorphism type σ.

3. Let V be a representation of K and let σ be a K-type. We define the *σ-isotypic* component of V, denoted $V(\sigma)$, to be the sum of all finite dimensional stable subspaces of V which are irreducible and of isomorphism type σ. We say that σ is a type of V if $V(\sigma)$ is nonzero.

4 Proposition (K-V Prop 1.18). *Let V be a representation of K and let σ be a K-type. Then $V(\sigma)$ is isomorphic to a direct sum of representations of type σ; in any such decomposition, the cardinality of terms in the direct sum is equal. Thus the multiplicity of σ in V is a well defined (possibly infinite) cardinal.*

Note that in Knapp-Vogan the proposition is stated only for compact Lie groups, but their proof works for any compact group.

5. Let V be a representation of K and let v be an element of V. We say that v is *K-finite* if the following two conditions hold:

1. v lies in a finite dimensional stable subspace W of V.
2. The map $K \to \mathrm{GL}(W)$ is continuous (and thus, if K is a Lie group, smooth).

Following Knapp-Vogan, we say V is *locally K-finite* if all of its elements are K-finite.

6. Let $C(K)$ be the space of measurable complex valued functions on K. It is an algebra with respect to convolution (using the unique bi-invariant normalized Haar measure).

7. Let σ be a K-type. We define an element $\xi = \xi_\sigma$ of $C(K)$ as follows: if (π, V) is any representation of type σ then

$$\xi(g) = (\deg \sigma) \operatorname{tr} \pi^{-1}(g).$$

The function ξ is easily verified to be an idempotent of $C(K)$. If $\sigma' \neq \sigma$ then ξ_σ and $\xi_{\sigma'}$ are orthogonal idempotents. Thus if $\sigma_1, \ldots, \sigma_r$ are distinct K-types then $\xi = \sum_i \xi_{\sigma_i}$ is an idempotent; we call such idempotents *elementary*. More precsiely, we say that ξ is the elementary idempotent corresponding to $\sigma_1, \ldots, \sigma_r$.

8 Proposition (K-V Prop. 1.18, 1.20). *Let (π, V) be a locally finite representation of K.*

1. *Let f be an element of $C(K)$, v an element of V and W a finite dimensional stable subspace in which V lies. Define*
$$\pi(f)v = \int_K f(g)\pi(g)v dg$$
where dg is the unique normalized Haar measure on K (note that the integral is well-defined since the integrand takes values in the finite dimensional space W). Then $f \mapsto \pi(f)$ gives a representation of the algebra $C(K)$ on V.

2. *If $\xi = \xi_\sigma$ then $\pi(\xi)$ is a projection operator onto $V(\sigma)$.*

3. *V decomposes into a direct sum of the $V(\sigma)$.*

9. Let W be a finite dimensional representation of K with types $\sigma_1, \ldots, \sigma_r$. Let ξ be the elementary idempotent corresponding to $\sigma_1, \ldots, \sigma_r$; we also say ξ is the elementary idempotent of W. The proposition implies $\pi(\xi) = 1$ on W.

1.2 Distributions on smooth manifolds

10. This section is loosely based on the discussion of distributions in Knapp-Vogan. See appendix B in particular. We omit proofs.

11. Throughout this section the word "manifold" will mean a smooth real manifold with a countable base.

1.2.1 The topological vector spaces $C^\infty(X)$

12. If X is a manifold we let $C^\infty(X)$ denote the space of smooth complex valued functions on X; in this section we discuss the topology on this vector space.

13. We let $\mathrm{VF}^\infty(X)$ denote the Lie algebra of smooth vector fields on X (recall that a smooth vector field is a smooth section of the tangent bundle TX) and we let $\mathrm{VF}_c^\infty(X)$ denote the Lie subalgebra consisting of those vector fields with compact support. The action of a smooth vector field on a smooth function gives a Lie algebra representation of $\mathrm{VF}^\infty(X)$ on $C^\infty(X)$.

14. We let $\mathscr{U}^\infty(X)$ denote the universal enveloping algebra of $\mathrm{VF}^\infty(X)$ and we let $\mathscr{U}_c^\infty(X)$ denote the universal enveloping algebra of $\mathrm{VF}_c^\infty(X)$ (recall that the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is the quotient of the tensor algebra of $\mathfrak{g}$ by the relations $XY - YX = [X, Y]$). The Lie algebra representation of $\mathrm{VF}^\infty(X)$ on $C^\infty(X)$ mentioned in the previous article implies that $C^\infty(X)$ is naturally a module over the algebra $\mathscr{U}^\infty(X)$.

15. An element D of $\mathscr{U}_c^\infty(X)$ defines a semi-norm on $C^\infty(X)$ by
$$\|f\|_D = \max |Df|$$
(note that Df is a smooth function with compact support on X, so it makes sense to take its maximum).

16. We define the topology on $C^\infty(X)$ to be the weakest topology for which all of the semi-norms $\|\cdot\|_D$ (with D in $\mathscr{U}_c^\infty(X)$) are continuous.

17. There is a slight variant of the above definition which is sometimes useful. If D is an element of $\mathscr{U}^\infty(X)$ and K is a compact subset of X define a semi-norm on $C^\infty(X)$ by
$$\|f\|_{D,K} = \max_K |Df|.$$
It is easily seen that the topology given by this family of semi-norms is the same as the topology defined above.

18 Proposition (K-V Prop. B.7). *Let X be a manifold. Then a countable collection of the seminorms $\|\cdot\|_D$ suffices to define the topology on $C^\infty(X)$. Thus $C^\infty(X)$ is a Frechet space and therefore metrizable.*

19 Proposition. *Let X be a manifold. Then the map*

$$C^\infty(X) \times C^\infty(X) \to C^\infty(X)$$

given by pointwise multiplication of functions is continuous .

If D is a vector field on X and f and g are elements of $C^\infty(X)$ then $D(fg) = fDg + gDf$. From this one sees that there is a Leibnitz formula for more general elements D of $\mathscr{U}^\infty(X)$. It is then easy to see that if f and g are small in a seminorm $\|\cdot\|_D$ then so is fg.

1.2.2 Distributions on manifolds

20. We let $\mathscr{E}'(X)$ denote the dual of the topological vector space $C^\infty(X)$, that is, $\mathscr{E}'(X)$ is the vector space of all continuous linear functionals on $C^\infty(X)$. For reasons that will be explained momentarily, elements of $\mathscr{E}'(X)$ are called *distributions of compact support* on X.

21. If T is an element of $\mathscr{E}'(X)$ and f an element of $C^\infty(X)$ we write

$$\langle T, f\rangle$$

or, when variable names need emphasis,

$$\int f(x) dT(x)$$

to indicate the value of the functional T at f.

22. Let T be an element of $\mathscr{E}'(X)$. We define the *support* of T, denoted $\operatorname{supp} T$, to be the following subset of X: a point x is *not* in $\operatorname{supp} T$ if there exists an open neighborhood U of x and a function ϕ in $C^\infty(X)$ with support contained in U such that $\langle T, \phi\rangle \neq 0$. Clearly, the complement of $\operatorname{supp} T$ is an open set and so $\operatorname{supp} T$ is closed.

23 Proposition (K-V Prop. B.15). *Elements of $\mathscr{E}'(X)$ have compact support.*

24. Proposition 23 partially explains the name "distributions of compact support." To fully explain the name, we must introduce the space $C_c^\infty(X)$: it is, of course, the space of functions in $C^\infty(X)$ with compact support. This space has a topology which we do not define; suffice it to say that it is *not* the subspace topology inherited from $C^\infty(X)$. It is in fact stronger (or at least, no weaker). In other words, the inclusion map

$$i : C_c^\infty(X) \to C^\infty(X)$$

is continuous but not a homeomorphism onto its image.

Let $\mathscr{E}(X)$ be the dual of the topological vector space $C_c^\infty(X)$. Elements of $\mathscr{E}(X)$ are called *distributions* on X. One may define the support of a distribution in much the same way that we defined the support of an element of $\mathscr{E}'(X)$. Since i is continuous there is a pullback map

$$i^* : \mathscr{E}'(X) \to \mathscr{E}(X).$$

The point of all of this is: the image of i^* is precisely the space of distributions on X which have compact support. This explains why elements of $\mathscr{E}'(X)$ are called "distributions of compact support."

25. Note that distributions of compact support push forward: if $f : X \to Y$ is a map of smooth manifolds and T is a distribution of compact support on X then we obtain a distribution of compact support f_*T on Y via the formula

$$\langle f_*T, \phi\rangle = \langle T, f^*\phi\rangle$$

where ϕ is an element of $C^\infty(Y)$ and $f^*\phi$ is its pullback, *i.e.*, the composition ϕf. Thus, by definition, f_* and f^* are adjoint.

26. Let f be an element of $C^\infty(X)$. It follows from proposition 19 that the endomorphism of $C^\infty(X)$ given by multiplication by f is continuous. If T is a distribution on X of compact support we may pullback T by this endomorphism to get a distribution of compact support on X which we denoted by fT. If ϕ is an element of $C^\infty(X)$ then we have

$$\langle fT, \phi\rangle = \langle T, f\phi\rangle.$$

Thus $\mathscr{E}'(X)$ is a module over $C^\infty(X)$.

1.2.3 Examples of distributions

27. Let X be a smooth manifold and let dx be a Borel measure on X which assigns finite volume to compact sets. Let D be an element of $\mathscr{U}_c^\infty(X)$. For ϕ in $C^\infty(X)$ define

$$\langle T, \phi\rangle = \int_X (D\phi)(x)dx.$$

Then T is a distribution of compact support on X. Note that, in fact, $\operatorname{supp} T = \operatorname{supp} D \cap \operatorname{supp}(dx)$. Most distributions that we will encounter are of this form. We give two special cases.

1. If D is a smooth function f and dx is given by a volume form then $\langle T, \phi\rangle$ is given by integrating ϕ against f over X. This construction yields a map

   $$C_c^\infty(X) \to \mathscr{E}'(X).$$

   If T is an element of the image of this map, we will sometimes simply refer to T as a smooth function with compact support.

2. If dx is the point-mass measure supported at x then T is the evaluation of $D\phi$ at x. Thus, for instance, if $D = 1$ then $\langle T, \phi\rangle = \phi(x)$ and T is the famous *Dirac distribution* supported at x.

1.2.4 Distributions on product spaces

28 Proposition (K-V Thm B.20). *Let X and Y be smooth manifolds and let T be a distribution of compact support on X.*

1. *Given an element f of $C^\infty(X \times Y)$ the function f' on Y given by*

   $$f'(y) = \int f(x, y)dT(x)$$

   is an element of $C^\infty(Y)$.

2. *The map*

   $$C^\infty(X \times Y) \to C^\infty(Y)$$

   given by $f \mapsto f'$ is continuous.

29 Proposition (K-V Thm. B.20). *Let X and Y be smooth manifolds and let S and T be distributions of compact support on X and Y.*

1. *There exists a unique distribution of compact support $S \times T$ on $X \times Y$ such that if ϕ is an element of $C^\infty(X)$ and ψ is an element of $C^\infty(Y)$ and $\phi \times \psi$ is the function on $X \times Y$ whose value at (x, y) is $\phi(x)\psi(y)$ then*

   $$\langle S \times T, \phi \times \psi\rangle = \langle S, \phi\rangle\langle T, \psi\rangle.$$

2. *(Fubini's theorem.) For all f in $C^\infty(X \times Y)$ we have*

   $$\int_{X\times Y} f(x, y)d(S \times T)(x, y) = \int_X \int_Y f(x, y)dT(y)dS(x) = \int_Y \int_X f(x, y)dS(x)dT(y).$$

1.2.5 Distributions supported on submanifolds

30. Let X be a manifold and Y a closed submanifold. We let $\mathscr{E}'(X,Y)$ denote the space of distributions of compact support on X with support contained in Y.

31. Let $C^\infty(X,Y)$ denote the space of germs of functions smooth functions on X near Y, that is, $C^\infty(X,Y)$ is the quotient of $C^\infty(X)$ obtained by identifying f and g if there exists an open set U containing Y such that the restrictions of f and g to U are equal. There is a natural map

$$C^\infty(X) \to C^\infty(X,Y).$$

If T is an element of $\mathscr{E}'(X,Y)$ and f is an element of $C^\infty(X)$ then $\langle T, f\rangle$ only depends on the class of f in $C^\infty(X,Y)$.

32. Again let X be a manifold and Y a closed submanifold. Let $i : Y \to X$ be the inclusion. Let $\pi : NY \to Y$ be the normal bundle of i, that is the quotient of $i^*(TX)$ by TY. Let $t : NY \to X$ be a tubular neighborhood of Y in X. Note that this allows us to write $i^*(TX) = TY \oplus NY$, *i.e.*, the tubular neighborhood tells us how to split the exact sequence

$$0 \longrightarrow TY \longrightarrow i^*(TX) \longrightarrow NY \longrightarrow 0.$$

Let D be a smooth section of NY. We can pullback D via π to obtain a section $NY \to \pi^*(NY)$. Note that $\pi^*(NY)$ is canonically a sub-bundle of $t^*(TX)$. Thus, from D we get a section $\pi^*D : NY \to t^*(TX)$. Now let χ be a bump function which is 1 on Y and has support contained in NY. Then $D' = \chi\pi^*D$ may be regarded as a smooth section of TX. Note that if D_1 and D_2 are two sections of NY then D_1' and D_2' are commuting vector fields on X. Thus, letting $\mathscr{U}^\infty(X,Y)$ denote the smooth sections of $\mathrm{Sym}(NY)$, the above construction yields a map

$$\mathscr{U}^\infty(X,Y) \to \mathscr{U}^\infty(X).$$

One should think of elements of $\mathscr{U}^\infty(X,Y)$ as differential operators in a neighborhood of Y which act in the direction normal to Y.

33. If T is a distribution of compact support on X and D is an element of $\mathscr{U}^\infty(X,Y)$ we let $D \times T$ be the distribution of compact support on X given by

$$\langle D \times T, \phi\rangle = \langle T, D\phi\rangle.$$

34 Proposition (due to Schwartz, K-V Thm. B.28). *Let X be a manifold and $i : Y \to X$ a closed submanifold. Then the map*

$$\mathscr{U}^\infty(X,Y) \otimes \mathscr{E}'(Y) \to \mathscr{E}'(X;Y)$$

given by $(D,T) \mapsto D \times i_*T$ is an isomorphism (the tensor product is over $C^\infty(Y)$).

35. Intuitively, proposition 34 says that a distribution of compact support on X with support contained in Y is made up of distributions of compact support on Y together with derivatives in the normal direction to Y.

1.2.6 Distributions on Lie groups

36. We now assume that the manifold in question, which we now denoted by G, is a Lie group. This gives us two maps

$$\mu : G \times G \to G, \qquad \iota : G \to G$$

the multiplication and inversion in the group. Using these maps we get several operations on distributions, which we now discuss.

37. Let g be an element of G and let T be a distribution of compact support on G. Let $L(g)$ and $R(g)$ be the left and right multiplication by g maps on X (*i.e.*, $L(g)(h) = gh$ and $R(g)(h) = hg$). We define $\lambda(g)T$ to be the pushforward of T under $L(g)$; we define $\rho(g)T$ to be the pushforward of T under $R(g^{-1})$ (note the inverse). Since $L(g_1 g_2) = L(g_1)L(g_2)$ and $R(g_1 g_2) = R(g_2)R(g_1)$, it follows that λ and ρ are representations of X on the space $\mathscr{E}'(G)$; we call these the *left regular representation* and the *right regular representation*.

38. We also define left and right regular representations of G on $C^\infty(G)$. In this case, if f is an element of $C^\infty(G)$ then $\lambda(g)f$ is the pullback of f under $\lambda(g^{-1})$ while $\rho(g)f$ is the pullback of f under $R(g)$. Note that

$$(\lambda(g)f)(h) = f(g^{-1}h), \qquad (\rho(g)f)(h) = f(hg).$$

Clearly, we have

$$\langle \lambda(g)T, \phi\rangle = \langle T, \lambda(g^{-1})\phi\rangle, \qquad \langle \rho(g)T, \phi\rangle = \langle T, \rho(g^{-1})\phi\rangle.$$

39. Let T be a distribution of compact support on G. We define the *transpose* of T, denoted $T^\vee$, to be the pushforward of T under ι. If f is an element of $C^\infty(G)$ then we define the transpose of f, denoted $f^\vee$, to be the pullback of f under ι. Note that $f^\vee(x) = f(x^{-1})$. By the adjointness of ι_* and ι^* we see that

$$\langle T^\vee, f\rangle = \langle T, f^\vee\rangle.$$

40. Let S and T be distributions of compact support on G. We define the *convolution* of S and T, denoted $S * T$, to be the pushforward of $S \times T$ under the multiplication map, that is,

$$S * T = \mu_*(S \times T).$$

Some comments:

1. If ϕ belongs to $C^\infty(X)$ then

   $$\langle S * T, \phi\rangle = \langle S \times T, \mu^*\phi\rangle = \int_{G\times G} \phi(xy)dS(x)dT(y).$$

2. If S and T are given by integrating smooth functions f and g of compact support against the left Haar measure, then

   $$\langle S * T, \phi\rangle = \int_G \int_G \phi(xy)f(x)g(y)dxdy = \int_G \phi(y)\left[\int_G f(x)g(x^{-1}y)dx\right]dy$$

   so that $S * T$ is given by integrating $f * g$ against the left Haar measure. Thus convolution of distributions naturally extends convolution of functions.

3. Since μ is associative it follows that convolution of distributions is as well. Thus $\mathscr{E}'(G)$ forms an associative algebra under convolution. The Dirac distribution supported at the identity element of X is an identity element under convolution. The algebra $\mathscr{E}'(G)$ is not commutative unless G is.

4. We have

   $$\operatorname{supp}(S * T) \subset (\operatorname{supp} S)(\operatorname{supp} T)$$

   where the product on the right is pointwise product of sets.

41. We can also define the convolution of a distribution of compact support T with an element f of $C_c^\infty(G)$; the result is an element of $C_c^\infty(G)$. This can be done directly by defining

$$(f * T)(g) = \langle T^\vee, \lambda(g^{-1})f\rangle, \qquad (T * f)(x) = \langle T^\vee, \rho(g)f\rangle.$$

Some comments:

1. $f * T$ and $T * f$ are smooth functions by proposition 28; it is clear that they have compact support.

2. It is easily verified that if S is the distribution of compact support given by integrating f against the left Haar measure then $S * T$ and $T * S$ are the distributions of compact support given by integrating $f * T$ and $T * f$ against the left Haar measure.

3. The following two identities are easily verified

$$\langle T * S, f\rangle = \langle T, f \star S^\vee\rangle = \langle S, T^\vee \star f\rangle.$$

42. The following easily verified formula relate the right and left regaulr representations to convolution with the Dirac distribution δ_g supported at g:

$$\begin{aligned} f * \delta_g &= \rho(g^{-1})f, &\quad \delta_g * f &= \lambda(g)f \\ T * \delta_g &= \rho(g^{-1})T, &\quad \delta_g * T &= \lambda(g)T \end{aligned}$$

43. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}_{\mathbb{C}}$ its complexification and $\mathscr{U}$ the universal enveloping algebra of $\mathfrak{g}_C$. Elements of $\mathscr{U}$ may be treated as distributions supported at the origin (in fact, by a theorem of Schwartz, all distributions with supported at the origin are of this form). For the sake of clarity, we will sometimes write $\delta(X)$ for the distribution corresponding to X.

We define the transpose map on $\mathscr{U}$ to be the unique anti-involution which is equal to multiplication by -1 on $\mathfrak{g}$. The identites

$$\delta(X)^\vee = \delta(X^\vee), \qquad \delta_g * \delta(X) * \delta_{g^{-1}} = \rho(g)\lambda(g)\delta(X) = \delta((\operatorname{Ad} g)X)$$

follow easily.

44. Note that, since everything is smooth, the regular representations of G on $C^\infty(G)$ and $\mathscr{E}'(G)$ can be differentiated to yield representations of the Lie algebra $\mathfrak{g}$ of G on these spaces. We thus get representations of $\mathfrak{g}_{\mathbb{C}}$ and $\mathscr{U}$ as well. Some comments:

1. We have
$$\langle \lambda(X)T, \phi\rangle = \langle T, \lambda(X^\vee)\phi\rangle, \qquad \langle \rho(X)T, \phi\rangle = \langle T, \rho(X^\vee)\phi\rangle.$$

2. If X is an element of $\mathscr{U}$ and f is an element of $C^\infty(X)$ then $\rho(X)f$ is the unique left invariant differential operator whose evaluation at the identity is $\delta(X)f$. Similarly, $\lambda(X)f$ is the unique right invariant operator equal to $\delta(X^\vee)f$ at the identity. The identities
$$(\rho(X)f)(g) = \langle \delta(X), \lambda(g^{-1})f\rangle, \qquad (\lambda(X)f)(g) = \langle \delta(X), \rho(g)f\rangle$$
follow easily from this description.

3. For f in $C^\infty(G)$, T in $\mathscr{E}'(G)$ and X in $\mathscr{U}$ we have
$$\begin{aligned} f * \delta(X) &= \rho(X^\vee)f, &\quad \delta(X) * f &= \lambda(X)f \\ T * \delta(X) &= \rho(X^\vee)T, &\quad \delta(X) * T &= \lambda(X)T \end{aligned}$$

4. It is now easy to see that the map $\delta : \mathscr{U} \to \mathscr{E}'(X)$ is in fact an algebra homomorphism, that is
$$\delta(X) * \delta(Y) = \delta(XY).$$

45 Proposition (K-V Prop. 1.68). *Let G be a Lie group and $i : H \to G$ a closed subgroup. Then the map*
$$\mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{h})} \mathscr{E}'(H) \to \mathscr{E}'(G, H)$$
sending $X \otimes T$ to $\delta(X) * (i_* T)$ is an isomorphism.

46 Proposition (K-V Prop. 1.26, 1.31). *Let K be a* compact *Lie group.*

1. *The smooth functions on K which are K-finite under both λ and ρ form a two sided ideal of $\mathscr{E}'(K)$.*

2. *Any element of $\mathscr{E}'(K)$ which is K-finite under either λ or ρ is in fact a smooth function and K-finite under both λ and ρ.*

The first assertion is clear; we prove the second. Let T be a distribution which is K-finite under λ contained in the finite dimensional λ-stable subspace W. Let ξ be the elementary idempotent of W. Note that ξ is a smooth on K which is K-finite under both λ and ρ. Now, if ϕ is any element of $C^\infty(K)$ then we have

$$\begin{aligned}\langle\lambda(\xi)T,\phi\rangle &= \int_K \xi(g)\langle\lambda(g)T,\phi\rangle dg = \int_K \xi(g)\langle T,\lambda(g^{-1})\phi\rangle dg \\ &= \int_K \xi(g)(\phi * T^\vee)(g)dg = \langle\xi,\phi * T^\vee\rangle = \langle\xi * T,\phi\rangle\end{aligned}$$

Thus $T = \lambda(\xi)T = \xi * T$ is smooth and K-finite under both λ and ρ by the first statement.

1.3 Basic algebraic and topological constructs

1.3.1 Limits of topological spaces

47 Lemma. *Let Y be a space and $(X_\alpha)_{\alpha\in I}$ a direct system of spaces. Then the natural map*

$$h : \operatorname{colim}\big(X_\alpha \times Y\big) \to \big(\operatorname{colim} X_\alpha\big) \times Y$$

is a homeomorphism.

Let A be the colimit of the $X_\alpha \times Y$ *as sets*. Let π_1 be the natural projection map

$$\pi_1 : \coprod \big(X_\alpha \times Y\big) \to A$$

to be the natural projection map. Recall that $\operatorname{colim}\big(X_\alpha \times Y\big)$ is the set A with the the strongest topology such that π_1 is continuous.

Let B be the colimit of the X_α *as sets*. Let π_2 be the natural projection map

$$\pi_2 : \coprod X_\alpha \to B$$

Once again, $\operatorname{colim} X_\alpha$ is the set B with the the strongest topology such that π_2 is continuous. One easily checks that $\big(\operatorname{colim} X_\alpha\big) \times Y$ is the set $B \times Y$ with the strongest topology such that $\pi_2 \times \mathrm{id}$ is continuous.

Now, we have the following commutative diagram:

$$\begin{tikzcd} \coprod\big(X_\alpha \times Y\big) \arrow[r] \arrow[d, "\pi_1"'] & \big(\coprod X_\alpha\big) \times Y \arrow[d, "\pi_2\times\mathrm{id}"] \\ A \arrow[r, "h"] & B \times Y \end{tikzcd}$$

The upper horizontal map is the canonical isomorphism. Note that h is a bijection of sets, so to prove the lemma it suffcies to show h is an open map. However, this follows immediately from the definition of the topologies and the commutativity of the diagram.

48 Lemma. *Let $(X_\alpha)_{\alpha\in I}$ and $(Y_\beta)_{\beta\in J}$ be two direct systems of spaces. Then the natural maps*

$$\operatorname*{colim}_{(\alpha,\beta)\in I\times J}\big(X_\alpha \times Y_\beta\big) \to \operatorname*{colim}_{\alpha\in I}\operatorname*{colim}_{\beta\in J}\big(X_\alpha \times Y_\beta\big) \to \big(\operatorname{colim} X_\alpha\big) \times \big(\operatorname{colim} Y_\beta\big)$$

are homeomorphisms.

First map: this is true in any category; apply $\operatorname{Hom}(-,Z)$ and check that the two inverse limits of sets agree. Alternatively, it is easy to see that the map is a bijection; to check it is a homeomorphism it suffices to show that it is an open map. A set U is open on the left hand side if and only if $U \cap (X_\alpha \times Y_\beta)$ is open for all $(\alpha,\beta) \in I \times J$; U is open on the right hand side if and only if $U \cap \operatorname{colim}_{\beta\in J}(X_\alpha \times Y_\beta)$ is open for all α, which amounts to saying $U \cap (X_\alpha \times Y_\beta)$ is open for all α and β. Thus the map is open.

Second map: apply lemma 47 twice.

49 Lemma. *Let $(X_\alpha)_{\alpha\in I}$ and $(Y_\alpha)_{\alpha\in I}$ be two direct systems of spaces over a cofiltered index set I. Then the natural map*

$$\operatorname{colim}\big(X_\alpha \times Y_\alpha\big) \to \big(\operatorname{colim} X_\alpha\big) \times \big(\operatorname{colim} Y_\alpha\big)$$

is a homeomorphism.

We have

$$\operatorname*{colim}_{\alpha\in I}\big(X_\alpha \times Y_\alpha\big) \to \operatorname*{colim}_{(\alpha,\alpha')\in I\times I}\big(X_\alpha \times Y_{\alpha'}\big) \to \big(\operatorname{colim} X_\alpha\big) \times \big(\operatorname{colim} Y_\alpha\big).$$

Since I is cofiltered, the diagonal embedding of I in $I \times I$ is cofinal; hence the first map is a homeomorphism. The second map is a homeomorphism by lemma 48.

1.3.2 Restricted direct products: a categorical approach

50. Let $\mathcal{C}$ be a fixed category. We shall make several assumptions about $\mathcal{C}$:

1. $\mathcal{C}$ has arbitrary products;
2. $\mathcal{C}$ has finite fibre products;
3. $\mathcal{C}$ has arbitrary filtered colimits;
4. filtered colimits commute with finite products, that is, the natural map

   $$\operatorname{colim}(X_\alpha \times Y) \to (\operatorname{colim} X_\alpha) \times Y$$

   is an isomorphism.

Note that the category of topological spaces satisfies these axioms, and in fact, this is the archetype of the construction.

51. Let Σ be a set of indices and for each $v \in \Sigma$ let $i_v : Y_v \to X_v$ be a morphism of $\mathcal{C}$. We write P for the product of the X_v and π_v for the natural projection map $P \to X_v$. For a subset S of Σ we write

$$P_S = \prod_{v\in S} X_v \times \prod_{v\notin S} Y_v.$$

If S is a subset of S' then we get a natural morphism $i_{SS'} : P_S \to P_{S'}$. In particular, (since $P = P_\Sigma$) for all S there is a morphism $i_S : P_S \to P$.

52. Let $\mathcal{C}'$ be the following category: the objects are tuples (Z, f_v) where Z is an object in $\mathcal{C}$ and for each v in Σ, f_v is a morphism $Z \to X_v$; a morphism $\phi : (Z, f_v) \to (Z', f'_v)$ in $\mathcal{C}'$ is given by a morphism $\phi : Z \to Z'$ in $\mathcal{A}$ such that the diagram

$$\begin{tikzcd} Z \arrow[r, "\phi"] \arrow[dr, "f_v"'] & Z' \arrow[d, "f'_v"] \\ & X_v \end{tikzcd}$$

commutes for each v.

53. Note that P is a final object in $\mathcal{C}'$, and that this fact is exactly equivalent to the universal property for the product.

54. We now define a functor $\hat{\ } : \mathcal{C}' \to \mathcal{C}'$ together with a natural transformation $\Phi : \hat{\ } \to \mathrm{id}$.

Let (Z, f_v) be an object in $\mathcal{C}'$. Taking the product of the f_v gives a map $f : Z \to P$. Let Z_S be the fibre product of Z with P_S (think of this as the inverse image of P_S under f). Clearly if S is a subset of S' then there is a natural map $Z_S \to Z_{S'}$. Let $\hat{Z} = \operatorname{colim}_S Z_S$, where the direct limit is over *finite* subsets S of Σ. The maps f_v induce maps $\hat{f}_v : \hat{Z} \to X_v$. The functor $\hat{\ }$ thus assigns to (Z, f_v) the object $(\hat{Z}, \hat{f}_v)$.

Clearly, if $\phi : (Z, f_v) \to (Z', f'_v)$ then ϕ induces morphisms $Z_S \to Z'_S$ for each S and thus induces a map on the direct limits, $\hat{\phi} : \hat{Z} \to \hat{Z}'$. This is what $\hat{\ }$ does to morphisms.

For each S we have the projection map $Z_S \to Z$, and so we get a natural map $\hat{Z} \to Z$. It is clear that the diagram

$$\begin{tikzcd} \hat{Z} \arrow[r] \arrow[dr, "\hat{f}_v"'] & Z \arrow[d, "f_v"] \\ & X_v \end{tikzcd}$$

is commutative, and so we have a morphism $\Phi(X, f_v) : (\hat{Z}, \hat{f}_v) \to (Z, f_v)$. This is the natural transformation $\Phi : \hat{\cdot} \to \mathrm{id}$.

55. Note that there are now two definitions of P_S, the one in article 51 and the one obtained from the object (P, π_v) of $\mathcal{C}'$ as described in article 54, but that they agree.

56 Proposition. *The functor $\hat{\cdot}$ is idempotent in the sense that for any object (Z, f_v) the natural map $\Phi(\hat{Z}, \hat{f}_v) : (\hat{\hat{Z}}, \hat{\hat{f}}_v) \to (\hat{Z}, \hat{f}_v)$ is an isomorphism.*

We have

$$(\hat{Z})_S = \hat{Z} \times_P P_S = \operatorname*{colim}_{S'} Z_{S'} \times P_S = \operatorname*{colim}_{S'} (Z \times_P P_{S'}) \times_P P_S = \operatorname*{colim}_{S'} Z \times_P (P_{S'} \times_P P_S).$$

Now, if S' contains S then $P_{S'} \times_P P_S$ is equal to P_S. Thus, in this case, the S' term of the colimit is just Z_S. Therefore the natural map $(\hat{Z})_S \to Z_S$ is an isomorphism. This proves the proposition.

57. An object (Z, f_v) of $\mathcal{C}'$ is *admissible* if the map $\Phi(Z, f_v) : (\hat{Z}, \hat{f}_v) \to (Z, f_v)$ is an isomorphism.

58. Let $\hat{\mathcal{C}}'$ be the full subcategory of $\mathcal{C}'$ consisting of the admissible objects. We may think of the functor $\hat{\cdot}$ as a projection operator $\mathcal{C}' \to \hat{\mathcal{C}}'$.

59 Proposition. *The object $(\hat{P}, \hat{\pi}_v)$ is a final object in $\hat{\mathcal{C}}'$.*

Let (Z, f_v) be admissible. As (P, π_v) is final in $\mathcal{C}'$ we get a map $(Z, f_v) \to (P, \pi_v)$. Applying $\hat{\cdot}$, and the fact that (Z, f_v) is admissible, gives a map $(Z, f_v) \to (\hat{P}, \hat{\pi}_v)$. The uniqueness of such a map is clear.

60. The *restricted direct product* of the X_v with respect to the Y_v (and the i_v), denoted

$$\prod_{v \in \Sigma} (X_v : Y_v)$$

is defined to be the object $\hat{P}$ together with the maps $\hat{\pi}_v : \hat{P} \to X_v$. According to proposition 59 it has the following universal property: given any space Z and an admissible family of maps $f_v : Z \to X_v$ there exists a unique map $Z \to \hat{P}$ making the obvious diagram commute.

1.3.3 Restricted direct products of topological spaces

61. We now consider the case when the category $\mathcal{C}$ is the category of topological spaces, Y_v is a subspace of X_v and i_v is the inclusion map. Note that, in spite of all the abstract language of the previous section, the restricted direct product of the X_v with respect to the Y_v is simply the direct limit of the spaces P_S where S ranges over the finite subsets of Σ.

62 Proposition. *Let X_v be a family of spaces and Y_v a family of subspaces, and let a point x_v be given in each Y_v. Let $v_0 \in \Sigma$. The embedding of X_{v_0} into $\hat{P}$ (which sets the vth corrdinate to x_v if $v \neq v_0$) is continuous.*

Define $f_v : X_{v_0} \to X_v$ to be the identity map if $v = v_0$ and otherwise to be the map sending everything to x_v. If S is a finite set of indices containing v_0 then we clearly have $(X_{v_0})_S = X_{v_0}$; therefore $\hat{X}_{v_0} = X_{v_0}$ and (X_{v_0}, f_v) is admissible. The resulting map $X_{v_0} \to \hat{P}$ is that of the statement of the proposition.

63 Proposition. *Let X_v be a family of spaces and Y_v a family subspaces. For each v let a map $\mu_v : X_v \times X_v \to X_v$ be given such that $\mu_v(Y_v \times Y_v) \subset Y_v$. Then there exists a map $\mu : \hat{P} \times \hat{P} \to \hat{P}$ making the diagram*

$$\begin{tikzcd} \hat{P} \times \hat{P} \arrow[r, "\mu"] \arrow[d, "\pi_v \times \pi_v"'] & \hat{P} \arrow[d, "\pi_v"] \\ X_v \times X_v \arrow[r, "\mu_v"] & X_v \end{tikzcd}$$

commute for each v.

Let $S, S' \subset \Sigma$ be two finite subsets, and let S'' be a finite subset containing both S and S'. We have

$$P_S \times P_{S'} \longrightarrow P_{S''} \times P_{S''} \longrightarrow P_{S''} \longrightarrow \hat{P}.$$

The first map is just the product of inclusions. The second map is the product of the product maps; the fact that it maps into $P_{S''}$ uses the property that μ_v takes $Y_v \times Y_v$ into Y_v. If we now take the direct limit in S' and then in S, and use lemma 47 we get a map $\mu : \hat{P} \times \hat{P} \to \hat{P}$ with the requisite properties.

64 Corollary. *Let X_v be a family of topological groups (resp. rings) and Y_v a family of topological subgroups (resp. subrings). Then the restricted direct product of the X_v with respect to the Y_v is again a topological group (resp. ring).*

65 Proposition. *Let X_v be a family of spaces and Y_v a family of* open *subspaces. Then P_S is an open subspace of $\hat{P}$.*

A set $U \subset \hat{P}$ is open if and only if $U \cap P_{S'}$ is open in $P_{S'}$ for all finite subsets $S' \subset \Sigma$. By the definition of the product topology and the fact that Y_v is open in X_v it follows that $P_S \cap P_{S'}$ is open in $P_{S'}$, and so P_S is open in $\hat{P}$.

66 Proposition. *Let X_v be a family of locally compact spaces and Y_v a family of compact open subspaces. Then the restricted direct product of the X_v with respect to the Y_v is locally compact.*

This follows at once since the P_S are compact and open.

1.3.4 Restricted tensor products

67. Let Σ be an index set and for each $v \in \Sigma$ let V_v be a given vector space. Furthermore, let an element x_v of V_v be given for all v such that x_v is nonzero if v is outside a finite set Σ_0.

68. For a finite subset $S \subset \Sigma$ let V_S denote the tensor product of the V_v over $v \in S$. If $S \subset S'$ then there is a natural map $V_S \to V_{S'}$ given by tensoring with x_v for $v \in S' \setminus S$. (Note that if $\Sigma_0 \subset S$ then this map is injective.) We thus have a direct system of vector spaces. The colimt of this system is the *restricted tensor product* of the V_v with respect to the x_v. We write

$$\bigotimes_{v \in \Sigma} (V_v : x_v)$$

for the restricted tensor product.

69 Proposition. *Let V_v be a family of vector spaces and let x_v and y_v be familes of elements which differ on a finite set. Then we have a natural isomorphism*

$$\bigotimes_v (V_v : x_v) \cong \bigotimes_v (V_v : y_v).$$

Let S_0 be the finite set of indices where $x_v \neq y_v$. Then if $S_0 \subset S \subset S'$ the two maps $V_S \to V_{S'}$ given by tensoring with the x_v and tensoring with the y_v are the same. Since the the direct system consisting of the V_S with $S \supset S_0$ is cofinal in the direct system V_S (with no restriction on S), the result follows.

70. Let Σ be an index set. For each $v \in \Sigma$ let be given a vector space V_v, an element x_v of V_v, a topological group G_v, a subgroup K_v and a representation $\pi_v : G_v \to \mathrm{GL}(V_v)$ such that x_v is nonzero outside a finite set and for all but finitely many v the vector x_v is fixed by K_v. Then, in the obvious way, we get a representation

$$\otimes_v \pi_v : \prod_v (G_v : K_v) \to \mathrm{GL}\Big(\bigotimes_v (V_v : x_v) \Big)$$

which we call the *restricted tensor product* of the representations π_v.

1.3.5 Strong topologies on the points of schemes

71. Let $X = \operatorname{spec} A$ be an affine scheme over a commutative ring k and let R be a commutative topological k-algebra. We wish to give the set $X(R)$ of R-valued points of X a topology. If we pick a presentation for A, *i.e.*, write $A = k[(x_i)_{i \in I}]/(f_j)_{j \in J}$, then we may identify $X(R)$ with the subspace of $R^{\prod I}$ satifying the equations $f_j = 0$ for $j \in J$. We may thus give $X(R)$ the topology of this subspace. It is easily verified that this is independent of the presentation.

72. If X is a general scheme over k one may still give a natural topology to the set $X(R)$; however, the construction is a bit more involved and we will not need it, so we refrain from describing it here.

73 Proposition. *Let X be an affine scheme of finite type over k and let R be a compact Hausdorff topological k-algebra. Then $X(R)$ is a compact Hausdorff space.*

Let $X = \operatorname{spec} A$ and write $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. The space R^n is compact Hausdorff. Since R is Hausdorff the subspace define by $f_j = 0$ is closed, and therefore their intersection is closed. Thus $X(R)$, being a closed subset of a compact Hausdorff space, is compact Hausdorff.

74 Proposition. *Let X be an affine scheme of finite type over k, let R be a topological k-algebra and let T be an open (resp. closed) subalgebra of R. Then $X(T)$ is an open (resp. closed) subset of $X(R)$.*

We have $X(R)$ as a closed subset of R^n; similarly $X(T)$ is a closed subset of T^n. The result follows from two facts: 1) T^n is open (resp. closed) in R^n, and 2) $X(T) = X(R) \cap T^n$.

75 Proposition. *Let X be an affine scheme of finite type over k and let R_α be a* filtered *direct system of topological rings. Then*

$$X(\operatorname{colim} R_\alpha) = \operatorname{colim} X(R_\alpha).$$

In other words, for schemes of finite type, "formation of R-valued points commutes with filtered colimts in R."

76 Proposition. *Let X be an affine scheme over k and let R_α be an inverse system of rings. Then*

$$X(\lim R_\alpha) = \lim X(R_\alpha).$$

In other words, "formation of R-valued points commutes with limits in R." Note in particular the case when the index category is discrete so that the result takes the form

$$X\Big(\prod R_\alpha \Big) = \prod X(R_\alpha).$$

77 Proposition. *Let X be an affine scheme of finite type over k. Let R_v be a family of topological rings and T_v a family of subrings for $v \in \Sigma$. Then*

$$X\Big(\prod (R_v : T_v) \Big) = \prod (X(R_v) : X(T_v)).$$

In other words, for schemes of finite type, "formation of R-valued points commutes with restricted direct products."

This follows immediately from propositions 75 and 76 since the restricted direct product is a direct limit (*i.e.*, filtered colimit) of products.

1.3.6 The adeles and related groups

78. Let F be a global field (*i.e.*, a number field or a function field) and let Σ be its set of places. Let F_v be the completion of F at v, and let $\mathcal{O}_v$ be the ring of integers in F_v. The restricted direct product

$$A = \prod_{v \in \Sigma} (F_v : \mathcal{O}_v)$$

is the *adele ring* of F. As F_v is locally compact and $\mathcal{O}_v$ is compact open, the adele ring is locally compact (*cf.* proposition 66).

We write Σ_f for the set of finite places and A_f for the corresponding restricted direct product. This is the ring of *finite adeles* of F. We also write F_∞ for the product of the F_v over the infinite places. (These constructs only comes into play when F is a number field).

If S is a finite set of places, we write

$$A_S = \prod_{v \in S} F_v \times \prod_{v \notin S} \mathcal{O}_v;$$

this is called the ring of S-*adeles*. It is locally compact and open in A.

79. Again let F be a global field and Σ its set of places. If F is a number field then let $k = \mathcal{O}$ be the ring of integers; if F is a function field then let k be the constant subfield. If G is an (affine) algebraic group over k then $G(A)$ is naturally a topological group (where A is the adele ring of F).

80 Proposition. *Let G be an affine group scheme of finite type over k. Then we have*

$$G(A) = \prod (G(F_v) : G(\mathcal{O}_v)).$$

The group $G(\mathcal{O}_v)$ is a compact open subgroup of $G(F_v)$. The group $G(A)$ is locally compact.

The expression of $G(A)$ as a direct product follows immediately from proposition 77. That $G(\mathcal{O}_v)$ is compact follows from proposition 73 and the fact that $\mathcal{O}_v$ is compact; that $G(\mathcal{O}_v)$ is open in $G(F_v)$ follows from proposition 74. The local compactness of $G(A)$ now follows from proposition 66.

81. The group $A^\times = \mathrm{GL}(1, A)$ is called the *idele group* of F. Algebraically it is the group of units of A, but it does not have the subspace topology (it is homeomorphic to the closed subspace of A^2 defined by $xy = 1$).

1.4 Idempotented algebras and their modules

1.4.1 Idempotented algebras

82. Note that much of the discussion of this section is taken from Bump chapter 3, section 4.

83. In the context of idempotented algebras, the word "ring" (resp. "algebra") will always mean by default a possibly noncommutative ring (resp. algebra) with or without unit.

84. Recall that an element e of a ring R is an *idempotent* if $e^2 = e$. The idempotents form a partially ordered set by defining $f \leq e$ if $ef = fe = f$. Two idempotents e and e' are *orthogonal* if $ee' = e'e = 0$. Not that if $f \geq e$ then $f = e + e'$ where $e' = f - e$ is orthogonal to e.

85. An *idempotented algebra* over a field k is a k-algebra H together with a collection E of idempotents of H satisfying the conditions:

1. For all $e_1, e_2 \in E$ there exists $f \in E$ with $e_1, e_2 \leq f$ (in other words, E is cofiltered as a partially ordered set).
2. For all $x \in H$ there exists $e \in E$ with $ex = xe = x$.

When we speak of an idempotent of an idempotented algebra (H, E), we will by default mean an element of E (as opposed to a generic idempotent of the ring H).

86. If H is a ring and e an idempotent, we write $H[e]$ for the ring eHe. In this ring e is a unit. Note that if $e \leq f$ then $H[e]$ is a subring of $H[f]$.

87. If (H_1, E_1) and (H_2, E_2) are idempotented algebras over k then their tensor product (over k) H is again an idempotented algebra over k: the set E is taken to be $E_1 \otimes E_2$, that is, all elements of the form $e_1 \otimes e_2$ with e_1 in E_1 and e_2 in E_2.

88. More generally, let Σ be some index set and for each v in Σ let be given an idempotented algebra (H_v, E_v) and a distinguished idempotent $e_v^\circ \in E_V$. Then the restricted tensor product H of the H_v with respect to the e_v° is again an idempotented algebra: for the set E we take all tensors $\otimes e_v$ where $e_v = e_v^\circ$ for almost all v. We write

$$H = \bigotimes (H_v : e_v^\circ)$$

to indicate that H is the restricted tensor product of the H_v with respect to the e_v°.

1.4.2 Modules over idempotented algebras

89. If M is a (left) module over the idempotented algebra H, we write $M[e]$ for the $H[e]$-module eM. Note that if $e \leq f$ then $M[e]$ is an $H[e]$-submodule of $M[f]$.

90. Let H be an idempotented algebra and M an H-module. We say M is *smooth* if the canonical map

$$\operatorname*{colim}_{e \in E} M[e] \to M$$

is an isomorphism (this can be taken in the category of H-modules). This is equivalent to the condition that for all $x \in M$ there exists $e \in E$ with $ex = x$.

91. We say that an H-module M is *admissible* if it is smooth and $M[e]$ is finite dimensionsal as a vector space over k for all $e \in E$.

92. Note that H is always smooth as a module over itself (by the definition of an idempotented algebra) but may or may not be admissible.

93. Note that the properties "smooth" and "admissible" are preserved by taking submodules and quotients.

94. If M_1 and M_2 are modules over H_1 and H_2 then $M = M_1 \otimes M_2$ is naturally a module over $H = H_1 \otimes H_2$. It is clear that H is smooth (resp. admissible) if and only if both M_1 and M_2 are smooth (resp. admissible).

95. More generally, let Σ be some index set and let

$$H = \bigotimes (H_v : e_v^\circ)$$

be the tensor product of idempotented algebras index by Σ (*cf.* article 88). For each v in Σ let be given data (M_v, m_v°) where M_v is a module over H_v and m_v° is a nonzero element of M_v which (for almost all v) is fixed by e_v°. Then the restricted tensor product M of the M_v with respect to the m_v° is naturally a module over H. It is clear that M is smooth if and only if each M_v is smooth. If $e = \otimes e_v$ is an element of E then

$$M[e] = \otimes (M[e_v] : e_v m_v^\circ).$$

From this expression it is easy to see that M is admissible if and only if each M_v is admissible and for almost all v the module $M[e_v^\circ]$ is one dimensional.

96. Again, let Σ be an index set and let $H = \bigotimes(H_v : e_v^\circ)$. We say that a family of modules (M_v) is *admissible* if each M_v is an admissible H_v-module and for almost all v the module $M_v[e_v^\circ]$ is one dimensional. Given an admissible family of modules (M_v), we define its tensor product $M = \bigotimes M_v$ to be the restricted tensor product

$$M = \bigotimes(M_v : m_v^\circ)$$

where (for almost all v) m_v° is any nonzero vector in $M_v[e_v^\circ]$. It is easy to see that the isomorphism class of M does not depend on the choice of the m_v°. By the previous article the module M is an admissible H-module.

97. Before ending this section, we would like to point out explicitly the following important fact, the proof of which is trivial.

98 Lemma (Bump Prop. 3.4.7). *Let H be an idempotented algebra, let M be a module over H, let e and e' are orthogonal idempotents and let $f = e + e'$. Then*

$$M[f] = M[e] \oplus M[e'].$$

1.4.3 The contragrediant module

99. Throughout this section H will denote a fixed idempotented algebra and ι will denote a fixed anti-involution of H. We assume that E is closed under ι.

100. Let M be a module over H. We define the *dual* of M, denoted M^*, to be the module consisting of all linear maps $M \to k$ and where the action of H is given by

$$\langle x, rx^* \rangle = \langle r^\iota x, x^* \rangle.$$

101. Let M be an H-module. We say an element x^* of the dual module M^* is *smooth* if there exists an idempotent in E stabilizing x^*. Note that for such an idempotent e we have

$$\langle x, x^* \rangle = \langle e^\iota x, x^* \rangle$$

for all x in M. Thus if M is admissible then x^* may safely be regarded as an element of $M[e^\iota]^*$.

102. Let M be an H-module. We define the *contragrediant* of M, denoted $\widetilde{M}$, to be the submodule of M^* consisting of all the smooth linear functionals on M. Note that $\widetilde{M} = HM^*$.

If $f : M \to N$ is a map of modules there is an induced map on duals $f^* : N^* \to M^*$. Since this is a map of H-modules, the image of $\widetilde{N}$ under f^* is contained in $\widetilde{M}$. We thus get a map $\widetilde{f} : \widetilde{N} \to \widetilde{M}$, called the *contragrediant* of f. In this way, the contagrediant is a functor.

103 Proposition. *We have the following:*

1. *The contragrediant of any module is smooth.*
2. *If M is any module then $\widetilde{M}[e]$ is naturally isomorphic to $M[e^\iota]^*$.*
3. *The contragrediant of an admissible module is again admissible.*

1) By definition, a smooth element of M^* is stabilized by an element of E; thus every element of $\widetilde{M}$ is stabilized by an element of E, and so $\widetilde{M}$ is smooth.

2) First note that we have a decomposition of M as $M[e^\iota] \oplus M'$ where M' is the submodule consisting of elements of M which are stabilized by idempotents orthogonal to e^ι. We now define maps between $\widetilde{M}[e]$ and $M[e^\iota]^*$ in both directions.

A. An element of $\widetilde{M}[e]$ is a linear form on M; it can be restricted to a linear form on $M[e^\iota]$. This is the map $\widetilde{M}[e] \to M[e^\iota]^*$.

B. Thanks to the decomposition $M = M[e^\iota] \oplus M'$ a linear form on $M[e^\iota]$ may be extended to a linear form on M by defining it to be zero on M'. Note that a form obtained in this way will be fixed by e and will thus be an element of $\widetilde{M}[e]$. We have thus given a map $M[e^\iota]^* \to \widetilde{M}[e]$.

It is clear that these two maps are mutual inverses of each other, whence the proposition.

3) This follows immediately from parts 1 and 2.

104 Proposition. *If M is an admissible module then the natural map $M \to \widetilde{M}^*$ induces an isomorphism of M with its double contragrediant.*

Let e be an element of E. Two applications of proposition 103 yield

$$\widetilde{\widetilde{M}}[e] = \widetilde{M}[e^\iota]^*, \qquad \widetilde{M}[e^\iota] = M[e]^*.$$

By admissibility, $M[e]$ is finite dimensional and thus naturally isomorphic to its double dual. Thus we have a natrual isomorphism

$$\widetilde{\widetilde{M}}[e] = M[e].$$

As a consequence of natruality, this isomorphism commutes with the canonical inclusion maps $M[e] \to M[f]$ for $e \leq f$. Taking the direct limit over E gives a canonicall isomorphism of M with its double contragrediant.

105 Proposition. *The contragrediant functor is an exact functor on the category of admissible H-modules.*

It is clear that the contragrediant is left exact. We must show that if N is a submodule of M then the map $\widetilde{M} \to \widetilde{N}$ is surjective. Thus let x^* be an element of $\widetilde{N}$. Let e be an idempotent stabilizing x^* and write $N = N[e] \oplus N'$, $M = M[e] \oplus M'$, as before. Note that $N[e] \subset M[e]$ and $N' \subset M'$. Since x^* is fixed by e it annihiliates N'. Thus we can extend x^* to a smooth form on M which annihilates M'. This proves the proposition.

106 Proposition. *Let M be an admissible H-module. Then M is simple if and only if $\widetilde{M}$ is.*

Let M_1 be a submodule of M and let $M_2 = M/M_1$. Since the contragrediant is exact (proposition 105) it follows that $\widetilde{M_1} = \widetilde{M}/\widetilde{M_2}$. Thus if M_1 is a nonzero proper submodule of M then $\widetilde{M_2}$ is a nonzero proper subomdule of $\widetilde{M}$. Therefore if $\widetilde{M}$ is simple then M is simple. Applying the same reasoning to the contragrediant of M and identifying the double contragrediant of M with M (proposition 104), we deduce the result.

107 Proposition. *Let M and N be two admissible H-modules and let β be a nonzero bilinear form on $M \times N$ which satisfies $\beta(m, rn) = \beta(r^\iota n, m)$ for all m in M, n in N and r in H.*

1. *The natural map $M \to N^*$ induced by β has its image contained in $\widetilde{N}$.*
2. *If M or N is simpe then β is nondegenerate.*
3. *If β is nondegenerate then M is isomorphic to the contragrediant of N.*

The first two assertions are clear. As to the third, if β is nondegenerate then the maps $f_1 : M \to \widetilde{N}$ and $f_2 : N \to \widetilde{M}$ are injective. It follows from proposition 105 that $\widetilde{f_1}$ and $\widetilde{f_2}$ are surjective. However, if we identify the double contragrediant of M with M and do the same for N then $\widetilde{f_1} = f_2$ and $\widetilde{f_2} = f_1$. Thus f_1 and f_2 are both isomorphisms.

1.4.4 Review of standard module theory

108 Lemma. *Let R be a k-algebra with unit, let M be a semisimple R-module, let $K = \mathrm{End}_R(M)$ and let f belong to $\mathrm{End}_K(M)$. For every x in M there exists α in R such that $f(x) = \alpha x$.*

Note that since M is semisimple the submodule Rx is a direct summand; thus there is a R-linear projection map $\pi : M \to Rx$. It is clear that π belongs to K. Therefore f commutes with π and so we see that $f(x) = f(\pi x) = \pi f(x)$ belongs to Rx. This completes the proof.

109 Proposition (due to Burnside; Bump Thm. 3.4.1). *Let k be an algebraically closed field, let R be a k-algebra with unit and let M be a finite dimensional simple R-module. Then the homomorphism $R \to \mathrm{End}_k(M)$ is surjective.*

Let n be the dimension of M over k. Since k is algebraically closed and M is simple we have $\mathrm{End}_R(M) = k$. Furthermore $M^{\oplus n}$ is semisimple as an R-module and $K = \mathrm{End}_R(M^{\oplus n})$ is isomorphic to $M_n(k)$. Let f be a given element of $\mathrm{End}_k(M)$. The endomorphism $f^{\oplus n}$ of $M^{\oplus n}$ is clearly K-linear and so by lemma 108 for any x in $M^{\oplus n}$ there exists α in R such that $f^{\oplus n}(x) = \alpha x$. Let $e_1, \ldots, e_n$ be a basis of M over k, let $x = (e_1, \ldots, e_n)$ and let α be such that $f^{\oplus n}(x) = \alpha x$. Then $f(e_i) = \alpha e_i$ and so the image of α under $R \to \mathrm{End}_k(M)$ is f. This completes the proof.

110 Proposition (Bump Prop. 3.4.1). *Let A and B be k-algebras with units, let R be their tensor product and let P be a finite dimensional simple R-module. Then there exists a simple A-module M and a simple B-module N, both unique up to isomorphism, such that P is isomorphic to a quotient of $M \otimes N$.*

Identify A and B with subalgebras of R in the standard way. Assume $P \neq 0$; since P is finite dimensional it contains a nonzero simple A-submodule M' (take an A-stable subspace of minimal nonzero dimension). Note that for any b in B the subspace bM' of P is stable under the action of A (since $ab = ba$) and furthermore bM' is isomorphic to M' as an A-module. Since bM' and $b'M'$ are simple A-modules they must either be equal or disjoint (since their intersection is stable under A). We thus find that as an A-module P is equal to the direct sum of bM' as b ranges over some finite subset of B; thus, abstractly, P is isomorphic to $A^{\oplus d}$ for some d. It follows from Schur's lemma that the module M in the statement of the proposition must be isomorphic to M' and thus its isomorphism class is uniquely determined. We take $M = M'$. The same reasoning shows that the isomorphism class of N is uniquely determined, but we will select N in a different fashion.

Let $N_1 = \mathrm{Hom}_A(M, P)$ and regard N_1 as a B-module via the B-module structure on P. Define a map $\lambda : M \otimes N_1 \to P$ by $\lambda(n \otimes m) = n(m)$. This is obviously a k-linear map. We now verify that it is in fact an R-module map. We have

$$\lambda((a \otimes b)(m \otimes n)) = \lambda(am \otimes bn) = (bn)(am) = ba \cdot n(m) = (a \otimes b)\lambda(m \otimes n).$$

Now, N_1 has a simple submodule N. It follows from the previous paragraph (which gives the structure of N_1) that $M \otimes N$ is not annihilated by λ. Thus its image in P is nonzero and therefore is all of P since P is simple. Thus P is a quotient of $M \otimes N$.

111 Proposition (Bump Prop. 3.4.2). *Let k be an algebraically field, let A and B be k-algebras with unit and let $R = A \otimes B$.*

1. *If M and N are finite dimensional simple modules over A and B then $M \otimes N$ is a simple R-module.*
2. *Every finite dimensional simple R-module is of the form $M \otimes N$ where M and N are uniquely determined (up to isomorphism) finite dimensional simple module over A and B.*

1) The maps $A \to \mathrm{End}_k(M)$ and $B \to \mathrm{End}_k(M)$ are surjective by proposition 109. It thus suffices to show that $M \otimes N$ is simple as an $\mathrm{End}_k(M) \otimes \mathrm{End}_k(N)$ module. However, the natural map $\mathrm{End}_k(M) \otimes \mathrm{End}_k(N) \to \mathrm{End}_k(M \otimes N)$ is an isomorphism; since any module is simple over its (vector space) endomorphism ring the first statement follows.

2) This follows at once from the first statement and proposition 110.

112 Proposition. *Let R be an algebra with unit over the algebraically closed field k and let M and N be finite dimensional simple R-modules. Then M is* not *isomorphic to N if and only if there exists an element of R which acts neutrally on M and annihilates N.*

Clearly there can exist no such element if M and N are isomorphic. Thus assume M is not isomorphic to N. First note that $R/\operatorname{ann} M$ canonically isomorphic to $\mathrm{End}_k M$ by proposition 109; similarly for N. This observation has two consequences: 1) both $\operatorname{ann} M$ and $\operatorname{ann} N$ are maximal two sided ideal of R (since $\mathrm{End}_k(M)$ is a ring of matrices, which is simple); and 2) $\operatorname{ann} M$ is not equal to $\operatorname{ann} N$ (if they were equal then $\mathrm{End}_k(M)$ would equal $\mathrm{End}_k(N)$; but this ring has only one finite dimensional simple module up to isomorphism, and so M and N would be isomorphic). It thus follows that $\operatorname{ann} M + \operatorname{ann} N = R$ and so we can find x and y in $\operatorname{ann} M$ and $\operatorname{ann} N$ such that $x + y = 1$. Therefore $1 - x = y$ acts neutrally on M while it annihlates N.

1.4.5 First properties of modules over idempotented algebras

113 Proposition. *Let H be an idempotented algebra over the closed field k. Let T be a k-linear endomorphism of the simple admissible H-module M which commutes with T. Then T is a scalar.*

Since T commutes with H it maps $M[e]$ into itself. Since this is a finite dimensional vector space over an algebraically closed field, T has an eigenvector, say with eigenvalue λ. Thus $T-\lambda$ is an endomorphism of M commuting with H which has nonzero kernel; since the kernel is a nonzero H-submodule of M it is all of M. Therefore $T = \lambda$.

114 Proposition (Bump Prop. 3.4.5, 4.2.3). *Let M be a smooth module over the idempotented algebra H. Then M is simple if and only if $M[e]$ is a simple $H[e]$-module for all e in E (note that 0 counts as a simple module).*

Let N be a proper nonzero submodule of M. Since both N and M are admissible we have

$$M = \bigcup M[e], \qquad N = \bigcup N[e]$$

and thus for some e we have that $N[e]$ is a proper nonzero $H[e]$-submodule of $M[e]$. Therefore if $M[e]$ is simple for all e then M is simple.

Now suppose that N is a proper nonzero $H[e]$-submodule of $M[e]$. Let n be an element of $HN \cap M[e]$. By the definition of HN, there exist elements n_i in N and elements r_i in H such that

$$n = \sum_i r_i n_i.$$

Since e acts as the identity on elements of $M[e]$ we have

$$n = en = \sum_i (er_i e) n_i.$$

But er_ie belongs to $H[e]$; since N is stable under $H[e]$ it thus follows that n belongs to N. Thus $HN \cap M[e] \subset N$ and it follows that HN is a proper nonzero submodule of M. Therefore, if M is simple then $M[e]$ is simple for all e.

115 Proposition (Bump Prop. 3.4.6, 4.2.7). *Let H be an idempotented algebra over the closed field k. Let M and N be simple admissible modules over H. Then M is isomorphic to N if and only $M[e]$ is isomorphic to $N[e]$ for all e in E.*

If M or N is zero the proposition is trivial. Thus let M and N be nonzero and fix an element e_0 of E so that $M[e_0]$ and $N[e_0]$ are nonzero. Let f_0 be the given isomorphism $M[e_0] \to N[e_0]$. Note that since $M[e_0]$ and $N[e_0]$ are simple (*cf.* proposition 114) the isomorphism f_0 is unique up to scalar (*cf.* proposition 113). In particular, if $e \geq e_0$ and f is the given isomorphism $M[e] \to N[e]$ then the restriction of f to $M[e_0]$ is a scalar multiple of f_0. Thus, by rescaling if necessary, we can assume that for each $e \geq e_0$ we are given an isomorphism $M[e] \to N[e]$ and that these morphisms are compatible with the inclusion maps $M[e] \to M[e']$ and $N[e] \to N[e']$ for $e' \geq e$. In other words, we are given isomorphisms of the directed systems $(M[e])_{e\in E}$ and $(N[e])_{e\in E}$. Since the limits of these systems are M and N (by the definition of smooth), it follows that M and N are isomorphic.

116 Proposition. *Let H be an idempotented algebra over the closed field k and let M and N be nonzero simple admissible modules over H. Then M is isomorphic to N if and only if there exists an idempotent e such that $M[e]$ and $N[e]$ are nonzero and isomorphic.*

Let the stated condition be satisfied. Let $f \geq e$ be a second idempotent. If $M[f]$ and $N[f]$ are not isomorphic then by proposition 112 we can find r in $H[f]$ acting neutrally on $M[f]$ and annihilating $N[f]$. However, then ere is an element of $H[e]$ which acts neutrally on $M[e]$ and annilihilates $N[e]$, contradicting the hypothesis that they are isomorphic. Thus $M[f]$ is isomorphic to $N[f]$ for all $f \geq e$; proposition 115 therefore implies that M is isomorphic to N. (Note that proposition 115 actually states that we must establish that $M[e]$ and $N[e]$ are isomorphic for all e; it is clear that a cofinite set of e suffices.)

117 Proposition (Bump Thm. 3.4.2). *Let H_1 and H_2 be idempotented algebras over the closed field k and let H be their tensor product.*

1. *If M_1 and M_2 are simple admissible modules over H_1 and H_2 then $M = M_1 \otimes M_2$ is a simple admissible module over H.*

2. *If M is a simple admissible module over H then there exist unique (up to isomorphism) simple modules M_1 and M_2 over H_1 and H_2 such that M is isomorphic to $M_1 \otimes M_2$.*

We will use proposition 114 without note.

1) We have

$$M[e_1 \otimes e_2] = M_1[e_1] \otimes M_2[e_2].$$

The modules $M_1[e_1]$ and $M_2[e_2]$ are simple. Thus $M[e]$ is simple for all e and so M is simple. Admissibility is clear (and has already been discussed).

2) Assume that M is nonzero. Thus there exists an idempotent $e_1^\circ \otimes e_2^\circ$ of H such that $M[e_1^\circ \otimes e_2^\circ]$ is nonzero. We say that an idempotent e_i of E_i is *large enough* if $e_i \geq e_i^\circ$; similarly an idempotent $e = e_1 \otimes e_2$ of E is *large enough* if both of its parts are. If e is large enough then $M[e]$ is nonzero.

Sublemma A. Let $e = e_1 \otimes e_2$ be an idempotent of H. Note that $H[e] = H_1[e_1] \otimes H_2[e_2]$. The $H[e]$-module $M[e]$ is simple and therefore (by proposition 111) there exists a unique simple $H_1[e_1]$-module $M_1(e_1, e_2)$ and a unique simple $H_2[e_2]$-module $M_2(e_1, e_2)$ such that

$$M[e] = M_1(e_1, e_2) \otimes M_2(e_1, e_2).$$

If e is large enough then $M_i(e_1, e_2)$ is nonzero.

Sublemma B. We now show that if $e = e_1 \otimes e_2$ is large enough then $M_1(e_1, e_2)$ only depends on e_1 (up to isomorphism), and similarly for M_2.

Let $f_2 \geq e_2$, let $e_2' = f_2 - e_2$, let $f = e_1 \otimes f_2$ and let $e' = e_1 \otimes e_2'$. Note that f is large enough, though e' may not be. We have (using lemma 98)

$$M_1(e_1, f_2) \otimes M_2(e_1, f_2) = M[f] = M[e] \oplus M[e'] = \big(M_1(e_1, e_2) \otimes M_2(e_1, e_2)\big) \oplus \big(M_1(e_1, e_2') \otimes M_2(e_1, e_2')\big).$$

Thus $M[f]$, regarded as an $H_1[e_1]$-module is semisimple; using the leftmost side of the above identity we see that its only simple constituent is $M_1(e_1, f_2)$, while using the rightmost side we see that its simple constituents are $M_1(e_1, e_2)$ and (if nonzero) $M_1(e_1, e_2')$. We have therefore shown that whenever $f_2 \geq e_2$ we have $M_1(e_1, e_2) \cong M_1(e_1, f_2)$. Since the set E_2 is cofiltered (*i.e.*, given e_2 and e_2' there exists f_2 with $f_2 \geq e_2, e_2'$) it follows that if e_1 and e_2 are large enough then $M_1(e_1, e_2)$ does not depend on e_2 (up to isomorphism). We select some module $M_1(e_1)$ in this isomorphism class. Similarly for $M_2(e_2)$. We thus have

$$M[e] = M_1(e_1) \otimes M_2(e_2).$$

Sublemma C. Let $e = e_1 \otimes e_2$ be large enough and let $f = f_1 \otimes e_2$ with $f_1 \geq e_1$. There is a canonical inclusion of $H[e]$-modules $M[e] \to M[f]$. Note that as $H_1[e_1]$-modules $M[e]$ is isomorphic to a direct sum of p copies of $M_1(e_1)$ while $M[f]$ is isomorphic to a direct sum of q copies of $M_1(f_1)$ (for some positive integers p and q). Thus

$$1 \leq \dim \mathrm{Hom}_{H_1[e_1]}(M[e], M[f]) = pq \dim \mathrm{Hom}_{H_1[e_1]}(M_1(e_1), M_1(f_1))$$

and therefore

$$\dim \mathrm{Hom}_{H_1[e_1]}(M_1(e_1), M_1(f_1)) \geq 1.$$

Sublemma D. Let e and e' be orthogonal idempotents and let $f = e + e'$. Since $H[e]$ acts as zero on $M[e']$ the only $H[e]$-module map $M[e] \to M[e']$ is the zero map. Therefore (using lemma 98 and the fact that $M[e]$ is simple) we see that

$$\dim \mathrm{Hom}_{H[e]}(M[e], M[f]) = \dim \mathrm{Hom}_{H[e]}(M[e], M[e]) = 1.$$

Sublemma E. Let $e = e_1 \otimes e_2$ be large enough and let $f = f_1 \otimes e_2$ with $f_1 \geq e_1$. We have a canonical injection

$$\begin{aligned} &\mathrm{Hom}_{H_1[e_1]}(M_1(e_1), M_1(f_1)) \otimes \mathrm{Hom}_{H_2[e_2]}(M_2(e_2), M_2(e_2)) \\ &\qquad \to \mathrm{Hom}_{H[e]}(M_1(e_1) \otimes M_2(e_2), M_1(f_1) \otimes M_2(e_2)) \\ &\qquad = \mathrm{Hom}_{H[e]}(M[e], M[f]). \end{aligned}$$

Now since $M_2(e_2)$ is a simple $H_2[e_2]$-module the dimension of $\mathrm{Hom}_{H_2[e_2]}(M_2(e_2), M_2(e_2))$ is one. Sublemma D states that the dimension of the right side is equal to one. Therefore, we deduce

$$\dim \mathrm{Hom}_{H_1[e_1]}(M_1(e_1), M_1(f_1)) \leq 1.$$

Combining this result with sublemma C, we see that in fact

$$\dim \mathrm{Hom}_{H_1[e_1]}(M_1(e_1), M_1(f_1)) = 1.$$

Sublemma F. For all e_1 large enough select an arbitrary nonzero element $\lambda(e_1^\circ, e_1)$ of the one dimensional space $\mathrm{Hom}_{H_1[e_1^\circ]}(M_1(e_1^\circ), M_1(e_1))$. If $f_1 \geq e_1$ defined $\lambda(e_1, f_1)$ to be the unique element of $\mathrm{Hom}_{H_1[e_1]}(M_1(e_1), M_1(f_1))$ such that

$$\lambda(e_1^\circ, f_1) = \lambda(e_1, f_1)\lambda(e_1^\circ, e_1).$$

With such choices made, the modules $M_1(e_1)$ together with the maps λ form a direct system. Their direct limit is a simple module M_1 which clearly satisfies $M_1[e_1] \cong M_1(e_1)$ for e_1 large enough. We apply the same construction to obtain a module M_2.

Completion of proof. If $e = e_1 \otimes e_2$ is large enough then

$$M[e] \cong M_1(e_1) \otimes M_2(e_2) \cong (M_1 \otimes M_2)[e]$$

and so by proposition 115 (together with the first part of the present proposition, which implies that $M_1 \otimes M_2$ is simple), we have

$$M \cong M_1 \otimes M_2.$$

This proves the existence aspect of the second statement.

We must still show that M_1 and M_2 are unique up to isomorphism. Assume M_1' and M_2' are two simple modules satisfying $M \cong M_1' \otimes M_2'$. Then we have

$$M[e] \cong M_1'[e_1] \otimes M_2'[e_2]$$

and so, by the uniquessness of $M_1(e_1)$ and $M_2(e_2)$, we have

$$M_1(e_1) \cong M_1'[e_1], \qquad M_2(e_2) \cong M_2'[e_2].$$

Thus $M_1[e_1] \cong M_1'[e_1]$ and $M_2[e_2] \cong M_2'[e_2]$ and therefore, by proposition 115, we have $M_1 \cong M_1'$ and $M_2 \cong M_2'$.

1.4.6 The tensor product theorem

118 Lemma. *Let R_v be a family of rings with unit and let R be their restricted tensor product (with respect to their identity elements). Let $\gamma : R \to k$ be a ring homomorphism. Then there exist ring homomorphisms $\gamma_v : R_v \to k$ such that $\gamma(\otimes r_v) = \prod \gamma_v(r_v)$. (Note that ring homomorphisms must carry the identity element to 1.)*

The ring R_v may be identified in an obvious way with a subring of R. Under this identification, $\otimes r_v$ is identified with $\prod r_v$. Now, γ may be restricted to obtain a homomorphism $\gamma_v : R_v \to k$ and since γ is a ring homomorphism, we have

$$\gamma(\otimes r_v) = \gamma(\textstyle\prod r_v) = \prod \gamma(r_v) = \prod \gamma_v(r_v)$$

and the proposition is proved.

119 Theorem (Bump Thm. 3.4.4). *Let Σ be an index set and for each v in Σ let be given an idempotented algebra (H_v, E_v) over a fixed algebraically closed field k and an idempotent e_v° of E_v. Let H be the restricted tensor product of the H_v with respect to the e_v°.*

1. *The tensor product of an admissible family (M_v) of simple H_v-modules is a simple H-module.*
2. *Every simple H-module is the tensor product of a unique admissible family of simple modules.*

1) Let $M = \bigotimes M_v$ be the tensor product of the admissible family of simple modules (M_v). Let $e = \otimes e_v$ be an idempotent of H and let S be the finite set of v for which $e_v \neq e_v^\circ$. Then

$$M[e] \cong \bigotimes_{v \in S} M[e_v],$$

the isomorphism being given by tensoring an element of the right side with $\otimes_{v \notin S} m_v^\circ$. By proposition 114 each of the factors $M[e_v]$ is simple and so, by proposition 117, $M[e]$ is simple. Thus, by another application of proposition 114 we deduce that M is simple.

2) First some notation. If S is a subset of Σ we denote by H_S the restricted tensor product of the H_v with v in S. If S' is the complement of S we write H'_S in place of $H_{S'}$. Note that if S is any subset of Σ then $H = H_S \otimes H'_S$.

Now let M be an admissible simple H-module. Let S be a subset of Σ and let S' its complement. By proposition 117 there exists a unique simple H_S-module $M(S)$ and a unique simple H'_S-module $M'(S)$ such that M is isomorphic to $M(S) \otimes M'(S)$. Note that if S is the disjoint union of S_1 and S_2 then $M(S)$ is isomorphic to $M(S_1) \otimes M(S_2)$. If $S = \{v\}$ we write $M(v)$ and $M'(v)$ in place of $M(S)$ and $M'(S)$. We will show that $M(v)$ is an admissible family of simple modules, the tensor product of which is M.

Let $e = \otimes e_v$ be an idempotent for which $M[e]$ is nonzero. Note that $M(S)[e_S] = M[e](S)$, where the notation has the obvious meaning. By the definition of the restricted tensor product, $M[e]$ is the colimit over the cofiltered system of finite subsets S of Σ of the spaces $M[e](S)$. Since the result of such a limit can have dimension greater than 1 if and only if it does at some finite stage, it follows that if $M[e]$ has dimension greater than 1 then so does $M(v)[e_v]$ for some v. Thus if the dimension of $M[e]$ is greater than 1 we can write $M[e] = M(v)[e_v] \otimes M'(v)[e'_v]$ where the dimension of $M(v)[e_v]$ is greater than one and the dimension of $M'(v)[e'_v]$ is strictly smaller than that of $M[e]$. By continuing in this manner, we see that (regardless of the dimension of $M[e]$) there exists a finite subset S_1 of Σ such that $M[e] = M(S_1)[e_{S_1}] \otimes M'(S_1)[e'_{S_1}]$ where the dimension of $M'(S_1)[e'_{S_1}]$ is one.

Let S_2 be the finite set of v for which $e_v \neq e_v^\circ$ and let S be the union of S_1 and S_2; it is a finite set. For v not in S the module $M(v)[e_v^\circ]$ is one dimensional; thus the family $(M(v))$ is admissible. It suffices to verify that its tensor product is isomorphic to M. Proposition 117 shows

$$M \cong \left(\bigotimes_{v \in S} M(v) \right) \otimes M'(S)$$

Thus it suffices to show that the tensor product of the admissible family $(M(v))_{v \notin S}$ is isomorphic to $M'(S)$. To ease notation we now assume that S is empty. Thus $e = \otimes e_v^\circ$ and $M[e]$ is one dimensional.

Let m denote a nonzero element of $M[e]$. Since $M[e]$ is ne dimensional, there is a ring homomorphism $\gamma : H[e] \to k$ such that $hm = \gamma(h)m$. By lemma 118 there exist homomorphisms $\gamma_v : H_v[e_v^\circ] \to k$ which factor γ. It is clear from the decomposition $M[e] = M(v)[e_v^\circ] \otimes M'(v)[(e_v^\circ)']$ that $H_v[e_v]$ acts on $M(v)[e_v^\circ]$ via γ_v.

Now let N be the tensor product of the admissible family $(M(v))$. If $r = \otimes r_v$ is an element of $H[e]$ and $n = \otimes n_v$ is an element of $N[e]$ then

$$rn = \otimes r_v n_v = \otimes \gamma_v(r_v) n_v = \prod \gamma_v(r_v) \otimes n_v = \gamma(r)n.$$

Thus $N[e]$ and $M[e]$ are isomorphic and nonzero. Therefore, proposition 116 implies that M is isomorphic to N. This completes the proof of the theorem.

1.5 Two results on Gaussian sums

1.5.1 The first result

120. The following proposition is a fairly standard result on Gaussian sums. We use the following notations: F is a non-archimedean local field, $\mathscr{O}_F$ is its ring of integers, U_F is the unit group of $\mathscr{O}_F$, $\mathfrak{p}$ is the maximal ideal of $\mathscr{O}_F$ and ϖ is a generator for $\mathfrak{p}$.

121 Proposition. *Let ψ be a nontrivial additive character of F with conductor $\mathfrak{p}^{-m}$. Let μ be a character of U_F. Let*

$$\eta(\mu, x) = \int_{U_F} \mu(\epsilon)\psi(\epsilon x) d\epsilon$$

where $d\epsilon$ is the normalized Haar measure on U_F.

1. *If μ is nontrivial with conductor $1 + \mathfrak{p}^n$ then*

$$\eta(\mu, \varpi^p a) = \begin{cases} c\mu^{-1}(a) & p = -n - m \\ 0 & p \neq -n - m \end{cases}$$

where a is in U_F and c is a nonzero constant.

2. *If μ is the trivial character then*

$$\eta(\mu, \varpi^p a) = \begin{cases} 1 & p > -m - 1 \\ -|\varpi|(1 - |\varpi|)^{-1} & p = -m - 1 \\ 0 & p < -m - 1 \end{cases}$$

where, again, a is an element of U_F.

1) Let ϕ be the function on F whose value at x is zero if x is not in U_F and is $\mu(x)$ if x is in U_F. Let $\hat{\phi}$ be its Fourier transform:

$$\hat{\phi}(x) = \int_F \phi(y)\psi(xy) dy = \int_{U_F} \mu(y)\psi(xy) dy = \eta(\mu, x).$$

It is clear that

$$\hat{\phi}(\epsilon x) = \mu^{-1}(\epsilon)\hat{\phi}(x) \tag{1}$$

for all $x \in F$ and $\epsilon \in U_F$.

We now examine $\hat{\phi}(\varpi^p)$. We have

$$\hat{\phi}(\varpi^p) = \int_{U_F} \mu(y)\psi(\varpi^p y) dy = \sum_{y \in U_F/(1+\mathfrak{p}^n)} \int_{1+\mathfrak{p}^n} \mu(y\epsilon)\psi(\varpi^p y\epsilon) d\epsilon.$$

We now use the fact that $\mu(y\epsilon) = \mu(y)$ since μ has conductor $1 + \mathfrak{p}^n$. Also we change the integral over $1 + \mathfrak{p}^n$ to an integral over $\mathfrak{p}^n$; the Haar measure does not change. We thus have

$$\hat{\phi}(\varpi^p) = \sum_{y \in U_F/(1+\mathfrak{p}^n)} \mu(y) \int_{\mathfrak{p}^n} \psi(\varpi^p y(1 + \epsilon)) d\epsilon = \sum_{y \in U_F/(1+\mathfrak{p}^n)} \mu(y)\psi(\varpi^p y) \int_{\mathfrak{p}^n} \psi(\varpi^p y\epsilon) d\epsilon.$$

Now, $\epsilon \mapsto \psi(\varpi^p y\epsilon)$ is a nontrivial character of $\mathfrak{p}^n$ if and only if $p < -n - m$. We thus have

$$\int_{\mathfrak{p}^n} \psi(\varpi^p y\epsilon) d\epsilon = \begin{cases} 0 & p < -n - m \\ c_0 & p \geq -n - m \end{cases}$$

where c_0 is the volume of $\mathfrak{p}^n$. Thus $\hat{\phi}(\varpi^p)$ vanishes for $p < -n - m$ while it equals

$$c_0 \sum_{y \in U_F/(1+\mathfrak{p}^n)} \mu(y)\psi(\varpi^p y)$$

for $p \geq -n - m$. Now say that $p > -n - m$. Then $\psi(y\varpi^p)$ is well defined modulo $1 + \mathfrak{p}^{n-1}$ and

$$\hat{\phi}(\varpi^p) = c_0 \sum_{y \in U_F/(1+\mathfrak{p}^{n-1})} \left[\psi(y\varpi^p) \sum_{y' \in (1+\mathfrak{p}^{n-1})/(1+\mathfrak{p}^n)} \mu(yy')\right] = 0$$

because $y' \mapsto \mu(yy')$ is a nontrivial character of $(1 + \mathfrak{p}^{n-1})/(1 + \mathfrak{p}^n)$.

We have thus shown that $\hat{\phi}(\varpi^p)$ vanishes except for $p = -n - m$. It cannot vanish for $p = n + m$ for then $\hat{\phi}$ would be zero by (1). Thus we have

$$\eta(\mu, \varpi^p a) = \hat{\phi}(\varpi^p a) = \begin{cases} c\mu^{-1}(a) & p = -n - m \\ 0 & p \neq -n - m \end{cases}$$

where a is an element of U_F and $c = \hat{\phi}(\varpi^p)$ is a nonzero constant. This proves the first part.

2) Let

$$A_k = \int_{\mathfrak{p}^k} \psi(x)dx \qquad a_k = \int_{U_F} \psi(\varpi^k \epsilon)d\epsilon.$$

On the one hand we have

$$A_k = \begin{cases} \frac{|\varpi|^k}{1-|\varpi|} & k \geq -m \\ 0 & k < -m \end{cases}$$

since ψ is trivial if and only if $k \geq -m$. The quantity $|\varpi|^k/(1 - |\varpi|)$ is the volume assigned to $\mathfrak{p}^k$ by the Haar measure on F which assigns volume 1 to U_F. On the other hand we have

$$A_k = \sum_{r=k}^{\infty} \int_{\varpi^r U_F} \psi(x)dx = \sum_{r=k}^{\infty} |\varpi|^r a_r.$$

We thus have

$$a_k = |\varpi|^{-k}(A_k - A_{k+1}) = \begin{cases} 1 & k > -m - 1 \\ -|\varpi|(1 - |\varpi|)^{-1} & k = -m - 1 \\ 0 & k < -m - 1 \end{cases}$$

It is clear that $\eta(1, \epsilon\varpi^k) = a_k$ if ϵ is in U_F; this proves the proposition.

1.5.2 The second result

122. We keep the same definitions as the previous section, and add some more: K will denote a quaternion division algebra over F, $\mathscr{O}_K$ its ring of integers, $\mathfrak{q}$ its prime ideal, β a generator of $\mathfrak{q}$ and $|\cdot|_K$ the norm on K given by $|a|_K = d(ax)/dx$. Thus $|\beta|_K = |\omega|_F^2$. We let U_K denote the group of elements x in $K^\times$ with $|x|_K = 1$.

123 Proposition. *Let ψ be a nontrivial additive character of K with conductor $\mathfrak{q}^{-m}$. Let (Ω, U) be a finite dimensional representation of U_K. Let*

$$\eta(\Omega, x) = \int_{U_K} \psi(\epsilon x)\Omega(\epsilon)d\epsilon$$

where $d\epsilon$ is the normalized Haar measure on U_K.

1. *If Ω is nontrivial with conductor $1 + \mathfrak{q}^n$ then*

$$\eta(\Omega, a\beta^p) = \begin{cases} A\Omega^{-1}(a) & p = -n - m \\ 0 & p \neq -n - m \end{cases}$$

where a belongs to U_K and A is a nonzero endomorphism of U.

2. *If Ω is the trivial representation then*

$$\eta(\Omega, a\beta^p) = \begin{cases} 1 & p > -m - 1 \\ -|\beta|_K(1 - |\beta|_K)^{-1} & p = -m - 1 \\ 0 & p < -m - 1 \end{cases}$$

where, again, a belongs to U_K.

3. *If Ω is nontrivial and is the restriction of an irreducible representation of $K^\times$ and the character ψ is symmetric (i.e., $\psi(xy) = \psi(yx)$) then the matrix A of part 1 is a scalar multiple of $\Omega(\beta^{n+m})$. In other words, there exists a nonzero scalar $\eta_0(\Omega;\psi)$ such that for any x in $\beta^{-n-m}U_K$ we have*

$$\eta(\Omega, x) = \eta_0(\Omega;\psi)\Omega^{-1}(x).$$

The first two assertions can be proved using exactly the same methods used to prove proposition 121. We now prove the third statement. If x and y are arbitrary elements of $K^\times$ then

$$\Omega(y)\eta(\Omega, x)\Omega^{-1}(y) = \int_{U_K} \psi(\epsilon x)\Omega(y\epsilon y^{-1})d\epsilon = \int_{U_K} \psi(y^{-1}(\epsilon yxy^{-1})y)\Omega(\epsilon)d\epsilon = \eta(\Omega, yxy^{-1}).$$

If we now write $\eta(\Omega, a\beta^{-n-m}) = A\Omega^{-1}(a)$ for a in U_K (per part 1 of the proposition) and take $x = \beta^{-n-m}$ in the above calculation, we find

$$\Omega(y)A\Omega^{-1}(y) = \eta(\Omega, yxy^{-1}) = \eta(\Omega, yxy^{-1}x^{-1}x) = A\Omega^{-1}(yxy^{-1}x^{-1})$$

since the commutator $yxy^{-1}x^{-1}$ belongs to U_K. A little manipulation now gives

$$\Omega(y)A\Omega(x) = A\Omega(x)\Omega(y).$$

Thus $A\Omega(\beta^{-n-m})$ commutes with $\Omega(y)$ for all y in $K^\times$ and therefore, by Schur's lemma, is a constant (which is necessarily nonzero because A is nonzero and $\Omega(\beta^{-n-m})$ is invertible). This proves the proposition.

1.6 Finite functions on certain locally compact groups

124. A complex valued continuous function on a locally compact abelian group will be called *finite* if the space spanned by its translates are finite dimensional. The space of finite functions forms a complex vector space (an algebra in fact). We write $\mathscr{F}(H)$ for the space of finite functions on the group H.

125. We are interested in the space of finite functions on groups of the form

$$H = H_0 \times \mathbb{Z}^m \times \mathbb{R}^n \tag{2}$$

where H_0 is a compact abelian group. We think of $\mathbb{Z}$ as a subgroup of $\mathbb{R}$. There are a number of obvious finite functions: 1) quasi-characters of H; 2) the projection functions ξ_i given by

$$\xi_i(h_0, x_1, \ldots, x_{n+m}) = x_i;$$

and 3) any polynomial expression in these functions. The point of this section is to prove the following proposition:

126 Proposition (J-L Lemma 8.1). *Let H be a group of the form (2). Then the functions of the form $\chi \prod_{i=1}^{n+m} \xi_i^{p_i}$ where the p_i are nonnegative integers form a basis for the space of finite functions.*

127 Proposition (J-L Lemma 8.1.1, 8.1.2). *Let H_1 and H_2 be two locally compact abelian groups. Then the natural map*

$$\mathscr{F}(H_1) \otimes_{\mathbb{C}} \mathscr{F}(H_2) \to \mathscr{F}(H_1 \times H_2)$$

which takes $f_1 \otimes f_2$ to the function whose value at (h_1, h_2) is $f_1(h_1)f_2(h_2)$ is an isomorphism.

Injective: Let f_i be a basis for $\mathscr{H}$ and g_i a basis for $\mathscr{H}$. We must show that the images of the $f_i \otimes g_j$ are linearly independent. Thus let

$$\sum a_{ij} f_i(h_1) g_j(h_2) = 0$$

be a linear relationship. This may be rewritten as

$$\sum_j \left(\sum_i a_{ij} f_i(h_1)\right) g_j(h_2) = 0.$$

For each h_1 this is a linear dependence amongst the g_j, which are linearly independent; it follows that the coefficients must vanish, *i.e.*, for each j we have

$$\sum_i a_{ij} f_i = 0.$$

But the f_i are linearly independent, and so $a_{ij} = 0$. This proves the the map is injective.

Surjective: Let f be a given finite function on $H_1 \times H_2$ and let V be a finite dimensional space of finite functions on $H_1 \times H_2$ containing f and stable under translation. Note that for any function g in V the function $h_1 \mapsto g(h_1, 0)$ is a finite function on H_1 (and similarly $h_2 \mapsto g(0, h_2)$ is a finite function on H_2).

For any element ξ of $H_1 \times H_2$ we obtain a linear functional on V by $g \mapsto g(\xi)$. Since the zero function is the only function annihilated by all these functionals, we may pick $\xi_1, \ldots, \xi_p$ so that the resulting functionals are a basis for the dual space of V. Let $f_1, \ldots, f_p$ be the corresponding dual basis of V.

The function $h \mapsto f(h + h')$, for h' fixed, is an element of V. We may thus write

$$f(h + h') = \sum \lambda_i(h') f_i(h)$$

where (by the definition of the basis f_i) we have $\lambda_i(h') = f(h' + \xi_i)$. Note therefore that λ_i is an element of V. If we now let ϕ_i be the finite function on H_1 given by $\phi_i(h_1) = \lambda_i(h_1, 0)$ and ψ_i the finite function on H_2 given by $\psi_i(h_2) = f_i(0, h_2)$ then we have

$$f(h_1, h_2) = \sum \phi_i(h_1) \psi_i(h_2)$$

which proves the surjectivity.

128. According to proposition 127, to prove proposition 126 we need only establish the special cases where H is compact, $H = \mathbb{Z}$ and $H = \mathbb{R}$.

129 Proof of proposition 126 for H compact (J-L pg. 280). We must show that the characters of H form a basis for the space of finite functions on H.

We first show that the characters of H are linearly independent (this is a standard result, but we include a proof). Assume we had a nontrivial linear dependence. Let

$$\sum_{i=1}^{r} a_i \chi_i(h) = 0.$$

be a linear dependence such that all the a_i are nonzero and r is minimal. Changing h to $h + g$, we obtain the relation

$$\sum_{i=1}^{r} a_i \chi_i(g) \chi_i(h) = 0.$$

Take g so that $\chi_1(g) \neq \chi_2(g)$. Multiplying the first relation by $\chi_1(g)$ and subtracting the second gives

$$\sum_{i=2}^{r} b_i \chi_i(h) = 0$$

where $b_i = (\chi_1(g) - \chi_i(g)) a_i$. Since $b_2 \neq 0$ this is a nontrivial dependence with fewer terms than the original dependence. This is a contradiction since the original dependence was assumed minimal; therefore the characters are linearly independent.

We now prove that the characters span $\mathscr{F}(H)$. Let V be an arbitrary finite dimensional subspace of $\mathscr{F}(H)$. It suffices to show that V is spanned by the characters it contains. As in the proof of proposition 127, let ξ_i be elements of H which yield a basis for the dual of V and let f_i be a dual basis. We have

$$\rho(g) f_i = \sum_{ij} \lambda_{ij}(g) f_j$$

where $\rho(g) f$ is the translate of f by g. From the expression $\lambda_{ij}(g) = f_i(g + \xi_j)$ we see that λ_{ij} is continuous. This shows that the representation of H on V by ρ is continuous (since the matrix of $\rho(g)$

with respect to the basis f_i is $\lambda_{ij}(g)$). We therefore have a continuous representation of the compact abelian group H on V. Since representations of compact groups are completely reducible, and the only irreducible representations of an abelian group are one dimensional, it follows that V breaks up into a direct sum of one dimensional subspaces which are stable under translation. However, it is clear that any such space is spanned by a character. It follows that V has a basis consisting of characters. This completes the proof.

130 Proof of proposition 126 for $H = \mathbb{Z}$. The quasi-characters of $\mathbb{Z}$ have the form $n \mapsto e^{an}$ where a is a complex number. It is clear that the functions $n \mapsto e^{an}n^p$ are linearly independent; we must prove they span the space of finite functions on $\mathbb{Z}$.

Thus let f be a finite function on $\mathbb{Z}$ and let V be a finite dimensional space containing f and stable under translation. Let p be the dimension of V. Since the $p+1$ functions $f(n), f(n+1), \ldots, f(n+p)$ all belong to V there must be a dependence relation, *i.e.*, f must satisfy a difference equation

$$\sum_{i=0}^{p} a_i f(n+i) = 0.$$

It is easy to see that a function satisfying such an equation has the requisite form (for instance, the generating function $\sum_{n=0}^{\infty} f(n)x^n$ is obviously a rational function).

131 Proof of proposition 126 for $H = \mathbb{R}$. The quasi-characters of $\mathbb{R}$ have the form $x \mapsto e^{ax}$ where a is a complex number. Again, it is clear that the functions $x \mapsto e^{ax}x^p$ are linearly independent; we must prove they span the space of finite functions on $\mathbb{R}$.

Thus let f be a finite function and let V be a finite dimensional space containing f and stable under translation. As we have done twice before, pick ξ_i in $\mathbb{R}$ which give a basis of V^* and let f_i in V be the dual basis. We thus have

$$\rho(x)f_i = \sum_i \lambda_{ij}(x) f_j$$

where $\lambda_{ij}(x) = f_i(x + \xi_j)$. It therefore follows that the λ_{ij} are continuous and so ρ is continuous as well (as a map $\mathbb{R} \to \mathrm{GL}(V)$). Since any continuous map $\mathbb{R} \to \mathrm{GL}(V)$ is automatically smooth, it follows that ρ is smooth; therefore the λ_{ij} are smooth and so the f_i are smooth and so all elements of V are smooth. Furthermore, V is closed under differentiation because

$$f_i' = \lim_{x \to 0} \frac{\rho(x)f_i - f_i}{x} = \sum_i \lambda_{ij}'(0) f_j.$$

Let p be the dimension of V. Since the $p+1$ functions $f, f^{(1)}, \ldots, f^{(p)}$ all belong to V (where $f^{(i)}$ is the ith derivative of f), it follows that there must be a linear dependence, *i.e.*, f must satisfy a differential equation

$$\sum_{i=0}^{p} a_i f^{(i)} = 0.$$

It is easy to see that any solution of this equation has the requisite form, which completes the proof.

1.7 Simple constituents of composite modules

132. Let H be an algebra. Recall that a *constituent* of an H-module A is a subquotient of A, that is, a constituent of A is of the form V/U where $U \subset V \subset A$ are submodules. In this section we prove several results of the following general form: if A is an H-module which is "composed" of other H-modules A_α (*e.g.*, A could be the direct sum of the A_α) then any simple module which is a constituent of A is a constituent of A_α for some α.

133 Proposition. *Let H be an algebra.*

1. *Let $A \to B$ be a surjection of H-modules. Then any constituent of B is a constituent of A.*
2. *Let $A \to B$ be an injection of H-modules. Then any constituent of A is a constituent of B.*
3. *Let*
$$A \xrightarrow{f} B \xrightarrow{g} C$$
be an exact sequence of H-modules. Then any simple constituent of B is a constituent of either A or C.

The first two statements are trivial. We prove the third. By the first two statements, we may assume that f is injective and g is surjective, so that A may identified with a submodule of B and C may be identified with the quotient B/C. Now, let $U \subset V \subset B$ be submodules such that V/U is simple. We separate two cases:

Case 1: $V \cap A \subset U$. The kernels of the projections $U, V \to C$ are both equal to $V \cap A$. We thus have

$$V/U \cong (V/(V \cap A))/(U/(V \cap A)).$$

Since $V/(V \cap A)$ and $U/(V \cap A)$ are isomorphic to the images of V and U in C, we find that V/U is a constituent of C.

Case 2: $V \cap A \not\subset U$. Since V/U is simple there are no intermediate submodules to U and V. Thus $V = U + (V \cap A)$ and we have

$$V/U = (V \cap A)/(U \cap A)$$

which shows that V/U is a constituent of A.

134 Proposition. *Let H be an algebra and let A be an H-module which is the union of submodules A_α (as α ranges in some index set). Then any simple consistuent of A is a constituent of A_α for some α.*

Let $U \subset V \subset A$ be submodules so that V/U is a simple module. We may find α so that $V \cap A_\alpha$ is not contained in U; since V/U is simple, it follows that $V = U + (V \cap A_\alpha)$ and so

$$V/U \cong (V \cap A_\alpha)/(U \cap A_\alpha).$$

Thus V/U is a constituent of A_α.

135 Proposition. *Let H be an algebra and let A be an H-module which is the direct sum of submodules A_α as α ranges over an index set I. Then any simple constituent of A is a constituent of A_α for some α.*

Since A is the union of spaces which are direct sums of finitely many of the A_α, we may assume, by proposition 134, that I is finite. Furthermore, by induction, we may assume that I has two elements. Thus $A = A_1 \oplus A_2$ and we must show that a simple constituent of A is either a constituent of A_1 or of A_2. This follows from proposition 133.

136 Proposition. *Let H_v be an idempotented algebra, for each v in some index set Σ. Let (A_v) be an admissible family of simple modules and let (B_v) be an admissible family of modules. If $\otimes A_v$ is a constituent of $\otimes B_v$ then A_v is a constituent of B_v for each v.*

Chapter 2

Review of $\mathrm{GL}(1)$ theory

2.1 The basic local theory

2.1.1 Notation and definitions

1. The following notations will be in use in this section:

1. F will denote a local field.
2. We write $|\cdot|_F$ for the canonical "analytic" norm on the locally compact topological ring F. To be explicit:
   (a) If F is non-archimedean and ϖ is a generator for the maximal ideal of F then $|\varpi|_F = q^{-1}$ where q is the cardinality of the residue class field of F.
   (b) If F is $\mathbb{R}$ then $|\cdot|$ is the standard absolute value.
   (c) If F is $\mathbb{C}$ then $|x|$ equals $x\bar{x}$.
3. We let ψ_F be a nontrivial additive character of F.
4. We let G be the group $\mathrm{GL}(1)$ and write G_F for its F-valued points (*i.e.*, the group $F^\times$).

2. If F is non-archimedean we say that ψ is *unramified* if the largest ideal of F on which it is trivial (*i.e.*, its conductor) is the ring of integers of F. Likewise, if F is non-archimedean and ω is a quasi-character of $F^\times$ then ω is *unramified* if the largest subgroup on which it is trivial is U_F (the norm 1 group).

3. By definition, an *irreducible admissible representation* of G_F is a quasi-character of $F^\times$ (*i.e.*, a continuous homomorphism $F^\times \to \mathbb{C}^\times$).

2.1.2 The functions $L(s,\omega)$ and $Z(s,\phi,\omega)$

4. We now define the *local L-function* of an irreducible admissible representation ω of G_F.

1. If F is non-archimedean then
$$L(s,\omega) = \begin{cases} \dfrac{1}{1-\omega(\varpi)q^{-s}} & \omega \text{ unramified} \\ 1 & \omega \text{ ramified} \end{cases}$$
where $q = |\varpi|_F^{-1}$ is the cardinality of the residue field. Note that if ω is unramified then $\omega(\varpi)$ is independent of the generator ϖ.
2. If F is the real field and $\omega(x) = (\operatorname{sgn} x)^m |x|_F^r$ where m is 0 or 1 then
$$L(s,\omega) = \pi^{-\frac{1}{2}(s+r+m)}\Gamma(\tfrac{1}{2}(s+r+m)).$$

3. If F is the complex field and $\omega(z) = |z|_{\mathbb{C}}^r z^n \bar{z}^m$ then

$$L(s,\omega) = 2(2\pi)^{-s+r+m+n}\Gamma(s+r+m+n).$$

5. If ϕ is an element of $\mathscr{S}(F)$ we define the zeta function

$$Z(s,\phi,\omega) = \int_{F^\times} \omega(a)\phi(a)|a|_F^s d^\times a.$$

6 Proposition. *Let ω be an admissible representation of G_F.*

1. *For all ϕ in $\mathscr{S}(F)$ the integral defining $Z(s,\phi,\omega)$ is absolutely convergent in some half plane $\Re s > s_0$.*
2. *For all ϕ in $\mathscr{S}(F)$ the ratio*
$$\frac{Z(s,\phi,\omega)}{L(s,\omega)} \tag{1}$$
can be analytically continued to an entire function.
3. *There exists ϕ such that the quotient (1) is equal to 1.*
4. *$L(s,\omega)$ is the unique Euler factor satisfying these properties. (Recall that an Euler factor is a function of the form $1/P(q^{-s})$ where P is a polynomial with constant term 1.)*

We have

$$Z(s,\phi,\omega) = \sum_{k\in\mathbb{Z}} |\varpi|_F^k \omega(\varpi)^k \int_{U_F} \omega(\epsilon)\phi(\epsilon\varpi^k)d\epsilon.$$

Since ϕ belongs to $\mathscr{S}(F)$ the quantity $\phi(\epsilon\varpi^k)$ is equal to 0 for $k \ll 0$ and is equal to $\phi(0)$ for $k \gg 0$. Thus we have

$$\begin{aligned} Z(s,\phi,\omega) &= \sum_{k=N_1}^{\infty} |\varpi|_F^k \omega(\varpi)^k \phi(0) \int_{U_F} \omega(\epsilon)d\epsilon + \sum_{k=N_2}^{N_1-1} |\varpi|_F^k \omega(\varpi)^k \int_{U_F} \omega(\epsilon)\phi(\epsilon\varpi^k)d\epsilon \\ &= \frac{|\varpi|_F^{N_1}\omega(\varpi)^{N_1}}{1-\omega(\varpi)q^{-s}}\phi(0)\int_{U_F}\omega(\epsilon)d\epsilon + P(q^{-s}) \end{aligned}$$

where P is a Laurent polynomial. Thus $Z(s,\phi,\omega)$ is a rational function in q^s and so the integral defining it converges for $\Re s$ sufficiently large. Furthermore, we see that if ω is ramified the integral above vanishes and so $Z(s,\phi,\omega)$ is just a Laurent polynomial in q^{-s}; if ω is unramified then

$$Z(s,\phi,\omega) = \phi(0)|\varpi|_F^{N_1}\omega(\varpi)L(s,\omega) + P(q^{-s}).$$

In either case, it is clear that $Z(s,\phi,\omega)/L(s,\omega)$ is an entire funciton. Thus statements 1 and 2 are proved.

As to statement 3, if ω is unramified then taking ϕ equal to the characteristic function of U_F works; otherwise take ϕ equal to the characteristic function of U_F multiplied by ω^{-1}.

We now prove statement 4. Assume $L'(s)$ were another Euler factor satisfying statements 2 and 3. Take ϕ such that $Z(s,\phi,\omega) = L(s,\omega)$; since $Z(s,\phi,\omega)/L'(s)$ is entire we see that $L(s,\omega)/L'(s)$ is entire. Similarly, $L'(s)/L(s,\omega)$ is entire. Thus the rational function (of q^{-s}) $L(s,\omega)/L'(s)$ and its reciprocal are entire; therefore it is a constant. The condition on the constant term of an Euler factor implies that $L' = L$, proving uniqueness.

2.1.3 The local functional equation

7 Theorem. *Let ω be an admissible representation of G_F.*

1. *There exist factors $\epsilon(s,\omega,\psi_F)$ such that for all ϕ in $\mathscr{S}(F)$ we have*
$$\frac{Z(1-s,\hat{\phi},\omega^{-1})}{L(1-s,\omega^{-1})} = \epsilon(s,\omega,\psi_F)\frac{Z(s,\phi,\omega)}{L(s,\omega)} \tag{2}$$
where $\hat{\phi}$ is the Fourier transform of ϕ with respect to ψ_F.

2. *The factor $\epsilon(s,\omega,\psi_F)$ is of the form ab^s.*

3. *If F is non-archimedean and both ω and ψ_F are unramified then*
$$\epsilon(s,\omega,\psi_F) = 1.$$

4. *If F is the real field and $\omega(x) = (\operatorname{sgn} x)^m |x|_{\mathbb{R}}^r$ where m is 0 or 1 and $\psi_{\mathbb{R}}(x) = e^{2\pi i \alpha x}$ then*
$$\epsilon(s,\omega,\psi_{\mathbb{R}}) = (i \operatorname{sgn}\alpha)^m |\alpha|_{\mathbb{R}}^{s+r-1/2}.$$

5. *If F is the complex field and $\omega(z) = |z|_{\mathbb{C}}^r z^n \bar{z}^m$ and $\psi_{\mathbb{C}}(z) = e^{4\pi i \Re(zw)}$ then*
$$\epsilon(s,\omega,\psi_{\mathbb{C}}) = i^{m+n}\omega(w)|w|_{\mathbb{C}}^{s-1/2}.$$

8. The identity (2) is called the *local functional equation* for GL(1).

9. The factors $\epsilon(s,\omega,\psi_F)$ are called *ϵ-factors*. Sometimes it is more convenient to use *γ-factors*, defined by
$$\gamma(s,\omega,\psi_F) = \frac{L(1-s,\omega^{-1})}{L(s,\omega)}\epsilon(s,\omega,\psi_F).$$
With the γ-factors, the local functional equation takes the form
$$Z(1-s,\hat{\phi},\omega^{-1}) = \gamma(s,\omega,\psi_F)Z(s,\phi,\omega).$$

2.2 The local theory associated to a quaternion algebra

2.2.1 Representations of quaternion algebras

10. We retain the notation from the previous section and add some more. In this section K will denote the unique quaternion division algebra over F. We let ν be the "algebraic" norm on K given by $\nu(x) = xx^\iota$ (where ι is conjugation on K). We denote by K_1 and by U_K the inverse images under ν of 1 and U_F respectively. There is a short exact sequence
$$1 \longrightarrow K_1 \longrightarrow U_K \xrightarrow{\nu} U_F \longrightarrow 1.$$
The groups K_1 and U_K are compact.

We let $|\cdot|_K$ be the canonical "analytic" norm on the locally compact ring K given by $|a|_K = d(ax)/dx$ where dx is any additive Haar measure. If β is a local uniformizer for K (*i.e.*, a generator of the prime ideal of K) and ϖ is the corresponding local uniformizer for F (*i.e.*, is equal to $\nu(\beta)$) then
$$|\beta|_K = |\varpi|_F^2, \qquad |\varpi|_K = |\varpi|_F^4.$$
Note that the powers of β form a set of coset representatives for $K^\times$ modulo U_K.

11. If χ is a character of $F^\times$ and ν is the norm on K we denote the one dimensional representation $g \mapsto \chi(\nu(g))$ by χ also. We say this representation is *ramified* or *unramified* according to whether χ is ramified or unramified. If Ω is a representation of $K^\times$ we denote by $\chi \otimes \Omega$ the representation $g \mapsto \chi(g)\Omega(g)$.

12. Schur's lemma holds for finite dimensional irreducible representations Ω of $K^\times$, that is, for such representation any operator commuting with $K^\times$ is in fact a scalar. In particular, since $F^\times$ lies in the center of $K^\times$, there is a quasi-character ω of $F^\times$ such that

$$\Omega(a) = \omega(a)I$$

for a in $F^\times$. We call ω the *central quasi-character* of Ω.

Note that if ω is the central quasi-character of Ω then $\chi^2\omega$ is the central quasi-character of $\chi \otimes \Omega$

13. We say that Ω is *unitary* if there is an invariant hermitian form on the space of the representation. Since $K^\times/F^\times$ is compact, an irreducible finite dimensional representation is unitary if and only if its central quasi-character is a character.

14. Note that if Ω is a finite dimensional irreducible representation of $K^\times$ of degree greater than 1 then there is no nonzero vector fixed by all of K_1 (this follows since $K^\times/K_1$ is abelian).

15. Let Ω be a finite dimensional representation of $K^\times$ on the space U.

1. We let $\mathscr{S}(K, U)$ be the space of all locally constant compactly supported functions on K with values in U.
2. We let $\mathscr{S}(K, \Omega)$ be the subspace consisting of functions Φ which satisfy $\Phi(xh) = \Omega^{-1}(h)\Phi(x)$ for h in K_1.
3. We let $\mathscr{S}^U(K, \Omega)$ be the subspace consisting of functions Φ which satisfy $\Phi(xh) = \Omega^{-1}(h)\Phi(x)$ for h in U_K.

Note that $\mathscr{S}^U(K, \Omega) \subset \mathscr{S}(K, \Omega)$. Note also that if Ω is irreducible of degree greater than 1 or is degree 1 and ramified then any function Φ in $\mathscr{S}(K, \Omega)$ vanishes at 0 (since $\Phi(0)$ is fixed by all of K_1).

16 Lemma (J-L Lemma 4.1). *The commutator subgroup of $K^\times$ is the norm one subgroup K_1.*

It is clear that K_1 contains the commutator subgroup. Suppose that x belongs to K_1. If $x = x^\iota$ then $x^2 = xx^\iota = 1$ so that $x = \pm 1$. If $x \neq x^\iota$ then $F(x)$ is a quadratic separable extension of F. Thus in all cases there is a separable quadratic extension L of F contained in K and containing x. By Hilbert's theorem 90 there is a y in L such that $x = y/y^\iota$. Also, there is an element σ of K such that $\sigma z \sigma^{-1} = z^\iota$ for all z in L. Thus $x = y\sigma y^{-1}\sigma^{-1}$ is in the commutator subgroup.

17. Lemma 16 implies that any one dimensional representation of $K^\times$ is the representation associated to a quasi-character of $F^\times$.

2.2.2 The functions $L(s, \Omega)$ and $Z(s, \Phi, \Omega)$

18. In this section we basically carry out the GL(1) theory with $F^\times$ replaced by $K^\times$. Thus we consider irreducible finite dimensional representations of $K^\times$ in place of irreducible finite dimensional representations of $F^\times$ (*i.e.*, quasi-characters).

19. Let (Ω, U) be a finite dimensional irreducible representation of $K^\times$. For Φ in $\mathscr{S}(K)$, u in U and $\widetilde{u}$ in $\widetilde{U}$ we define the zeta function

$$Z(s, \Phi, \Omega; u, \widetilde{u}) = \int_{K^\times} |a|_K^{s/2+1/4} \Phi(a) \langle \Omega(a)u, \widetilde{u} \rangle d^\times a.$$

For Φ in $\mathscr{S}(K, U)$ we also define the zeta functions

$$Z(s, \Phi, \Omega) = \int_{K^\times} |a|_K^{s/2+1/4} \Omega(a)\Phi(a) d^\times a.$$

and

$$Z(s, \Phi, \Omega^{-1}) = \int_{K^\times} |a|_K^{s/2+1/4} \Omega^{-1}(a)\Phi(a) d^\times a.$$

Of couse these two types of zeta functions are almost the same: we have

$$Z(s,\Phi,\Omega;u,\widetilde{u}) = \langle Z(s,\Phi u,\Omega),\widetilde{u}\rangle$$

and

$$Z(s,\Phi,\widetilde{\Omega};\widetilde{u},u) = \langle Z(s,\Phi u,\Omega^{-1}),\widetilde{u}\rangle.$$

We shall sometimes find one type of zeta function more convenient than the other.

20. We now define an L-funciton $L(s,\Omega)$:

1. If Ω is the one dimensional representation corresponding to the quasi-character χ of $F^\times$ we define $L(s,\Omega) = L(s,\alpha_F^{1/2}\chi)$, where the L-function on the right is the GL(1) L-function of §2.1.2.
2. If Ω has degree greater than one then we define $L(s,\pi) = 1$.

21 Proposition (J-L Lemma 4.2.5, Thm. 4.3). *Let (Ω,U) be a finite dimensional irreducible representation of $K^\times$.*

1. *The integrals defining $Z(s,\Phi,\Omega;u,\widetilde{u})$ converge in some half plane $\Re s > s_0$.*
2. *The ratio*
$$\frac{Z(s,\Phi,\Omega;u,\widetilde{u})}{L(s,\Omega)} \tag{3}$$
can be analytically continued to an entire function.
3. *The variables Φ, u and $\widetilde{u}$ can be selected so that the quotient 3 is equal to 1.*
4. *$L(s,\Omega)$ is the unique Euler factor satifying 2 and 3.*

22. We first give a lemma. We let U_K denote the group of elements x in $K^\times$

23 Lemma. *Given Φ in $\mathscr{S}(K,U)$ put*

$$\Phi_1(x) = \int_{U_K} \Omega(h)\Phi(xh)dh$$

where dh is the normalized Haar measure on U_K. The Fourier transform of Φ_1 is given by

$$\Phi_1'(x) = \int_{U_K} \Omega^{-1}(h)\Phi'(hx)dh.$$

The functions Φ_1 and $x \mapsto \Phi_1'(x^\iota)$ belong to $\mathscr{S}^U(K,\Omega)$. We have

$$Z(s,\Phi,\Omega) = Z(s,\Phi_1,\Omega), \qquad Z(s,\Phi',\Omega^{-1}) = Z(s,\Phi_1',\Omega^{-1}).$$

This is a simple computation and is left to the reader.

24 Proof of proposition 21. We prove the statements for $Z(s,\Phi,\Omega)$; these imply the statements as given in the proposition.

1, 2) By lemma 23 it suffices to these statements when Φ belongs to $\mathscr{S}^U(K,\Omega)$. If $\deg\Omega > 1$ or $\deg\Omega = 1$ and Ω is ramified they are obvious when one notes that functions in $\mathscr{S}(K,\Omega)$ have compact support contained in $K^\times$.

Now consider the case when Ω is the one dimensional representation corresponding to the unramified quasi-character χ of $F^\times$. Let β be a generator for the prime ideal of K. Then

$$Z(s,\Phi,\Omega) = \int_{K^\times} |a|_K^{s/2+1/4}\chi(a)\Phi(a)d^\times a = \sum_n |\beta|_K^{ns/2+n/4}\chi(\beta)^n\Phi(\beta^n).$$

Since Φ has compact support in K the terms of the sum are 0 for n sufficiently large and negative. Since Φ is locally constant we have $\Phi(\beta^n) = \Phi(0)$ for n sufficiently large. Thus

$$\begin{aligned} Z(s, \Phi, \Omega) &= \Phi(0) \sum_{n \geq n_2} |\varpi|_F^{ns+n/2} \chi(\varpi)^n + \sum_{n_1 < n < n_2} |\varpi|_F^{ns+n/2} \chi(\varpi)^n \Phi(\beta^n) \\ &= \Phi(0) |\varpi|_F^{n_2 s + n_2/2} \chi(\varpi)^{n_2} L(s, \Omega) + \sum_{n_1 < n < n_2} |\varpi|_F^{ns+n/2} \chi(\varpi)^n \Phi(\beta^n) \end{aligned}$$

and the first two statements follow.

3) Let Φ_u be the function which is 0 outside of U_K (the unit group of $\mathscr{O}_K$) and on U_K is given by $\Phi_u(x) = \Omega^{-1}(x)u$. Then

$$Z(s, \Phi_u, \Omega) = cu$$

where c is the volume of U_K with respect to $d^\times a$.

4) This is clear.

2.2.3 The local functional equation

25 Theorem (J-L Lemma 4.2.5, Thm 4.3). *Let (Ω, U) be a finite dimensional irreducible representation of $K^\times$.*

1. *There exist ϵ-factors such that for any Φ in $\mathscr{S}(K, U)$, any u in U and any $\widetilde{u}$ in $\widetilde{U}$ we have*

$$\frac{Z(1-s, \Phi', \widetilde{\Omega}; \widetilde{u}, u)}{L(1-s, \widetilde{\Omega})} = \epsilon(s, \Omega, \psi) \frac{Z(s, \Phi, \Omega; u, \widetilde{u})}{L(s, \Omega)}$$

   where Φ' is the Fourier transform of Φ.

2. *The factors $\epsilon(s, \Omega, \psi)$ are of the form ab^s.*

3. *If the degree of Ω is one and Ω is unramified then*

$$\epsilon(s, \Omega, \psi) = \frac{|\varpi|_F^{-m(s+1/2)} \chi(\varpi)^m}{1 - |\varpi|_F^2}$$

   where $\mathfrak{q}^{-m}$ is the conductor of ψ_K.

4. *If the degree of Ω is greater than 1 then*

$$\epsilon(s, \Omega, \psi) = \eta_0(\Omega^{-1}; \psi) |\varpi|_F^{(n+m)(s-3/2)}$$

   where $\mathfrak{q}^{-m}$ is the conductor of ψ_K, $1 + \mathfrak{q}^n$ is the conductor of the restriction of Ω to K_1 (which is necessarily nontrivial) and $\eta_0(\Omega^{-1}; \psi)$ is as in §1.5.2, proposition 123.

26. Note that the functional equation of theorem 25 is equivalent to the functional equation

$$\frac{Z(1-s, \Phi', \Omega^{-1})}{L(1-s, \widetilde{\Omega})} = \epsilon(s, \Omega, \psi) \frac{Z(s, \Phi, \Omega)}{L(s, \Omega)}$$

for all Φ in $\mathscr{S}(K, U)$. This is the form in which we will prove it.

27. It is sometimes more convenient to use γ-factors in place of ϵ-factors. They are defined by

$$\gamma(s, \Omega, \psi) = \frac{L(1-s, \widetilde{\Omega})}{L(s, \Omega)} \epsilon(s, \Omega, \psi).$$

28 Proof of theorem 25. We let β be a generator for the maximal ideal of $\mathscr{O}_K$.

Sublemma. Let Φ be an element of $\mathscr{S}^U(K,\Omega)$. Then

$$\Phi'(\beta^k) = \int_K \psi(\beta^k x)\Phi(x)dx = \sum_\ell |\beta|_K^\ell \int_{K_1} \psi(\beta^{k+\ell}\epsilon)\Phi(\beta^\ell\epsilon)d\epsilon = \sum_\ell |\beta|_K^\ell \int_{K_1} \psi(\beta^{k+\ell}\epsilon)\Omega^{-1}(\epsilon)\Phi(\beta^\ell)d\epsilon$$

and so

$$\Phi'(\beta^k) = \sum_\ell |\beta|_K^\ell \eta(\Omega^{-1}, \beta^{k+\ell})\Phi(\beta^\ell). \tag{4}$$

We now prove the proposition, in two cases. Note that by lemma 23 it suffices to take Φ in $\mathscr{S}^U(K,\Omega)$.

Case 1: $\deg\Omega > 1$ *or* $\deg\Omega = 1$ *and* Ω *ramified.* By §1.5.2, proposition 123, the terms of the sum (4) are zero except the term with $k+\ell = -n-m$ which is equal to $\eta_0(\Omega^{-1};\psi)\Omega(\beta^{-n-m})$. We thus have

$$\Phi'(\beta^k) = |\beta|_K^{-n-m-k}\eta_0(\Omega^{-1};\psi)\Omega(\beta^{-n-m})\Phi(\beta^{-n-m-k}).$$

Therefore

$$\begin{aligned} Z(1-s,\Phi',\Omega^{-1}) &= \sum_k |\beta|_K^{(1-s)k/2+k/4}\Omega(\beta^{-k})\Phi'(\beta^k) \\ &= \eta_0(\Omega^{-1};\psi)\sum_k |\beta|_K^{-sk/2-k/4-n-m}\Omega(\beta^{-k-n-m})\Phi(\beta^{-n-m-k}) \\ &= \eta_0(\Omega^{-1};\psi)|\varpi|_F^{(n+m)(s-3/2)}\sum_k |\beta|_K^{sk/2}\Omega(\beta^k)\Phi(\beta^k) \\ &= \epsilon(s,\Omega,\psi)Z(s,\Phi,\Omega) \end{aligned}$$

and this case is proved.

Case 2: $\deg\Omega = 1$ *and* Ω *unramified.* Let Ω be the one dimensional representation corresponding to the unramified quasi-character χ of $F^\times$ and identify U with $\mathbb{C}$. Without loss of generality we may assume χ to be unitary and thus the trivial character. Let Φ and Ψ be arbitrary elements of $\mathscr{S}(K)$. We first prove

$$Z(1-s,\Phi',\Omega^{-1})Z(s,\Psi,\Omega) = Z(s,\Phi,\Omega)Z(1-s,\Psi',\Omega^{-1}). \tag{5}$$

If $-\frac{1}{2} < \Re s < \frac{3}{2}$ then both terms on the left side are given by definite integrals, and we have

$$Z(1-s,\Phi',\Omega^{-1})Z(s,\Psi,\Omega) = \iint |x|_K^{3/4-s/2}|y|_K^{s/2+1/4}\Phi'(x)\Psi(y)d^\times x d^{\times y}$$

where the each integral is over $K^\times$. Expressing Φ' as an integral, the right hand side becomes

$$\iiint |x|_K^{3/4-s/2}|y|_K^{s/2+1/4}\psi(xz)\Phi(z)\Psi(y)dzd^\times xd^\times y$$

where all the integrals are over $K^\times$ (the Fourier transform is actually integrated over K, but throwing out zero does not change anything). If we now change x to xz^{-1} and write $dz = |z|_K d^\times z$, we obtain

$$Z(1-s,\Phi',\Omega^{-1})Z(s,\Psi,\Omega) = \iiint |x|_K^{3/4-s/2}|z|_K^{s/2+1/4}|y|_K^{s/2+1/4}\psi(x)\Phi(x)\Psi(y)d^\times zd^\times xd^\times y.$$

Since this is symmetric in Φ and Ψ we can conclude the identity (5).

Now let Ψ be the the characteristic function of U_K. Let $d^\times a$ be the Haar measure which gives U_K volume 1. Since Ω is trivial on U_K we find

$$Z(s,\Psi,\Omega) = 1.$$

Applying §1.5.2, proposition 123 we find

$$\Psi'(a\beta^k) = \begin{cases} 1 & k > -m-1 \\ c & k = -m-1 \\ 0 & k < -m-1 \end{cases}$$

where $c = -|\beta|_K(1-|\beta|_K)^{-1}$. An easy computation now gives

$$Z(s,\Psi',\Omega^{-1}) = \frac{\chi(\beta)^m|\varpi|_F^{-m(s+1/2)}}{1-|\varpi|_f^2}\frac{L(s,\widetilde{\Omega})}{L(1-s,\Omega)}.$$

Inserting this into (5) we deduce the stated functional equation.

29 Note. The proof given here when $\deg \Omega > 1$ is different than the proof given in J-L Lemma 4.2.4; I could not understand why the distribution β in the proof given in J-L is a smooth function.

Chapter 3

The classical theory of automorphic forms on $\mathrm{GL}(2)$

3.1 Preliminaries on topological groups

3.1.1 Generalities

1 Lemma (Shimura Lemma 1.2). *Let S be a non-empty locally compact Hausdorff space and let $\{V_i\}_{i\geq 1}$ be a countable collection of closed subspaces, the union of which is S. Then at least one of the V_i contains an interior point.*

Assume no V_i has an interior point; we will derive a contradiction. Let W_1 be a non-empty open subset of S with compact closure. Having defined $W_1, \ldots, W_i$, define W_{i+1} to be an open set whose closure is contained in $W_i - V_i$. The W_i then form a decreasing chain of compact sets and so it follows that their intersection is nonempty. However, their intersection is disjoint from any V_i, which is a contradiction.

2 Proposition (Shimura Thm. 1.1). *Let G be a locally compact Hausdorff group with a countable base of open sets acting transitively on the locally compact Huasdorff space S. Let s be any point in S and let K be its isotropy group. Then the natural map $G/K \to S$ is a homeomorphism.*

It is clear that $G/K \to S$ is a continuous bijection; it thus suffices to show that it is an open map. It therefore suffices to show that if U is any open set in G and g an element of U then gs is an interior point of Us. Let V be a compact neighborhood of the identity such that $V = V^{-1}$ and $gV^2 \subset U$. Since Vs is a continuous image of the compact set V it is compact, and therefore closed in S. By our assumption, S is the union of sets g_iVs for some countable collection $\{g_i\}$ of elements of G. By lemma 1 it follows that some g_iVs contains an interior point; hence Vs contains an interior point. Thus we can find an element v of V and an open set $\overline{W}$ of S such that $vs \in \overline{W} \subset Vs$. Let W be the inverse image of $\overline{W}$ in G; thus W is an open subset of G contained in V and such that $\overline{W} = Ws$. Now, we have

$$gs = gv^{-1}vs \in gW^{-1}Ws \subset gV^2s \subset Us$$

Since $gW^{-1}Ws = gW^{-1}\overline{W}$ is open in S, it follows that gs is an interior point of Us.

3 Proposition (Shimura Prop. 1.3). *Let G be a Hausdorff group acting on a locally compact Hausdorff space S. Then $G\backslash S$ is compact if and only if there exists a compact subset C of S such that $S = GC$.*

Let π denote the projection map $S \to G\backslash S$. If $S = GC$ then $G\backslash S = \pi(C)$ and so is compact. To prove the converse, cover S by open sets U_i with compact closures. The sets $\pi(U_i)$ are open and cover the compact space $G\backslash S$; thus there exists a finite index set I such that the $\pi(U_i)$ with i in I cover $G\backslash S$. We may then take C to be the union of the closures of the U_i with i in I.

4 Proposition (Shimura Prop. 1.4). *Let G be a Hausdorff group and Γ a subgroup which is locally compact in the subspace topology. Then Γ is closed in G. In particular, if Γ is discrete then it has no limit point in G.*

Let C be a compact neighborhood of the identity in Γ. Let U be an open neighborhood of the identity in G such that $U \cap \Gamma \subset C$. Let x be an element of the closure of Γ in G. We must show that x belongs to Γ.

Let V be a neighborhood of x in G such that $V^{-1}V \subset U$. Then $(V \cap \Gamma)^{-1}(V \cap \Gamma) \subset C$. Let y be an element of $V \cap \Gamma$ (such a y necessarily exists). Then

$$y^{-1}(V \cap \Gamma) \subset (V \cap \Gamma)^{-1}(V \cap \Gamma) \subset C$$

and so $V \cap \Gamma \subset yC$. It is clear that x belongs to the closure of $V \cap \Gamma$ and thus to the closure of yC. However, yC is already closed (since it is compact); thus x belongs to yC and therefore to Γ.

5 Proposition (Shimura Prop. 1.5). *Let G be a locally compact Hausdorff group and let K be a compact subgroup of G. Let S be the quotient space G/K and let $h : G \to S$ be the natural map. If $\overline{A}$ is a compact subset of S then its inverse image A under h is compact in G.*

The group K acts on the locally compact space A; the quotient, $\overline{A}$, is compact. Therefore, by proposition 49, there exists a compact set C in A such that $A = CK$. Thus A is the continuous image (under the multiplication map) of the compact set $C \times K$ and is therefore compact.

6 Proposition (Shimura Prop. 1.6). *Let G, K, S and h be as in proposition 5 and let Γ be a subgroup of G. Then the following two statements are equivalent:*

1. *Γ is a discrete subgroup of G;*
2. *For any two compact subsets $\overline{A}$ and $\overline{B}$ of S there are only finitely many elements γ of Γ such that $\gamma\overline{A}$ meets $\overline{B}$.*

($1 \implies 2$) Let $\overline{A}$ and $\overline{B}$ be compact subsets of S and let A and B be their inverse images under h (they are compact by proposition 5). If γ is an element of Γ and γA meets B then γ belongs to $\Gamma \cap (BA^{-1})$ (and conversely). Since $\Gamma \cap (BA^{-1})$ is a discrete subset of the compact space BA^{-1} it is finite.

($2 \implies 1$) Let V be a compact neighborhood of the identity in G and let s be the image of the identity in S. Let $\overline{A} = \{s\}$ and let $\overline{B} = h(V)$; they are compact subsets of S. If γ belongs to $\Gamma \cap V$ then clearly $\gamma t \in h(V)$, that is to say $\gamma\overline{A}$ and $\overline{B}$ meet. Thus $\Gamma \cap V$ is a finite set and so Γ is discrete.

3.1.2 Discrete Subgroups

7. For this section we make the following assumptions:

1. G is a locally compact Hausdorff group;
2. K is a compact subgroup of G;
3. $S = G/K$ is the quotient space;
4. $h : G \to S$ is the projection map;
5. Γ is a discrete subgroup of G.

8 Proposition (Shimura Prop. 1.7). *Every z in S has an open neighborhood U such that for any γ in Γ the set γU meets U if and only if γ stabilizes z.*

Let V be a compact neighborhood of z. By proposition 6 there are only finitely many elements γ of Γ such that γV meets V. Enumerate these elements as $\gamma_1, \ldots, \gamma_r$ such that for $1 \leq i \leq s$ we have $\gamma_i z = z$ and for $s < i \leq r$ we have $\gamma_i z \neq z$. For $i > s$ let V_i and W_i be disjoint neighborhoods of z and $g_i z$. We may now take U to be any neighborhood of z contained in $V \cap (\cap_{i>s}(V_i \cap g_i^{-1}W_i))$.

9 Proposition (Shimura Prop. 1.8). *If two points z and z' of S do not lie in the same orbit under Γ then there exist neighborhoods U of z and U' of z' such that for all γ in Γ the sets γU and U' are disjoint.*

Let V and V' be compact neighborhoods of z and z'. The set of γ in Γ for which γV and V' meet is a finite set (*cf.* proposition 6). Enumerate the elements of this set as $\gamma_1, \ldots, \gamma_r$. Since z and z' are not equivalent we have $\gamma_i z \neq z'$ and therefore we can find neighborhoods U_i of $\gamma_i z_1$ and U_i' of z' which are disjoint. We may now take U to be any neighborhood of z contained in $V \cap (\cap_{i=1}^r g_i^{-1} U_i)$ and we may take U' to be any neighborhood of z' contained in $V' \cap (\cap_{i=1}^r U_i')$.

10 Corollary. *The quotient space $\Gamma \backslash S$ is Hausdorff.*

11 Proposition (Shimura Prop. 1.9). *The space $\Gamma \backslash S$ is compact if and only if $\Gamma \backslash G$ is compact.*

If $\Gamma \backslash S$ is compact then there exists a compact set C of S such that $S = \Gamma C$ (*cf.* proposition 3). Thus $G = \Gamma h^{-1}(C)$. Since $h^{-1}(C)$ is compact (*cf.* proposition 5) it follows that $\Gamma \backslash G$ is compact as well. To prove the converse simple note that $\Gamma \backslash S$ is a continuous image of $\Gamma \backslash G$.

3.1.3 Miscellany

12 Proposition (Shimura Prop 1.10). *Let G_1 be a locally compact Hausdorff group, G_2 a compact Hausdorff group, Γ a closed subgroup of $G_1 \times G_2$ and Γ_1 the projection of Γ to G_1.*

1. *Γ_1 is closed in G_1.*
2. *$\Gamma \backslash (G_1 \times G_2)$ is compact if and only if $\Gamma_1 \backslash G_1$ is compact.*
3. *If Γ is discrete in $G_1 \times G_2$ then Γ_1 is discrete in G_1.*

Let V be a compact neighborhood of the identity in G_1. Then $(V \times G_2) \cap \Gamma$ is compact; its image under the projection map $G_1 \times G_2 \to G_1$ is $V \cap \Gamma_1$. Thus $V \cap \Gamma_1$ is compact and so Γ_1 is locally compact. By proposition 4 it follows that Γ_1 is closed. If Γ is discrete then $(V \times G_2) \cap \Gamma$ is finite and so $V \cap \Gamma_1$ is finite as well; thus Γ_1 is discrete.

We now prove the second assertion. If $\Gamma_1 \backslash G_1$ is compact then there exists a compact subset C of G_1 such that $G_1 = \Gamma C$ (*cf.* proposition 3). It then follows that the quotient map $C \times G_2 \to \Gamma \backslash (G_1 \times G_2)$ is surjective; since $C \times G_2$ is compact it follows that $\Gamma \backslash (G_1 \times G_2)$ is compact. To prove the converse simply note that $\Gamma_1 \backslash G_1$ is a continuous image of $\Gamma \backslash (G_1 \times G_2)$.

13. Recall that two subgroups Γ and Γ' of a group G are said to be *commensurable* if $\Gamma \cap \Gamma'$ is of finite index in both Γ and Γ'.

14 Proposition. *We have the following:*

1. *Commensurability is an equivalence relation.*
2. *If Γ and Γ' are commensurable subgroups of a topological group G then Γ is discrete if and only if Γ' is discrete.*
3. *If Γ and Γ' are commensurable closed subgroups of a locally compact Huasdorff group G then $\Gamma \backslash G$ is compact if and only if $\Gamma' \backslash G$ is compact.*

These statements are all easily proved and therefore left to the reader.

3.2 The action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathfrak{H}$

3.2.1 Definitions

15. Let $\mathfrak{H}$ denote the upper half plane in $\mathbb{C}$, *i.e.*,

$$\mathfrak{H} = \{z \in \mathbb{C} | \Im z > 0\}.$$

It is a complex manifold. It also carries the structure of a Riemannian metric. We do not give a precise definition of the metric; suffice it to say that the geodesics in $\mathfrak{H}$ are semi-circles centered somewhere on the real axis.

16. The group $\mathrm{GL}^+(2, \mathbb{R})$ of 2×2 matrices with real coefficients and positive determinant acts on $\mathbb{CP}^1$ via linear fractional transformations, that is, if

$$\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a typical element of $\mathrm{GL}^+(2, \mathbb{R})$ and z is a typical element of $\mathbb{CP}^1$ then

$$\sigma z = \frac{az + b}{cz + d}.$$

It is clear that $\mathbb{RP}^1$ is taken into itself under this action; it thus follows (since $\mathfrak{H}$ and $\mathrm{GL}^+(2, \mathbb{R})$ are both connected) that $\mathfrak{H}$ is also taken into itself under the action.

17. The action of $\mathrm{GL}^+(2, \mathbb{R})$ on $\mathfrak{H}$ is both both holomorphic and isometrical. The action is not faithful, for it is clear that scalar matrices act trivially. The quotient group

$$\mathrm{PSL}(2, \mathbb{R}) = \mathrm{GL}^+(2, \mathbb{R})/\mathbb{R}^\times = \mathrm{SL}(2, \mathbb{R})/\{\pm 1\}$$

however, does act faithfully. In fact, $\mathrm{PSL}(2, \mathbb{R})$ is both the group of holomorphic automorphisms of $\mathfrak{H}$ and the group of orientation preserving isometries of $\mathfrak{H}$.

18. The action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathfrak{H}$ is easily seen to be transitive. We thus obtain a surjection

$$\mathrm{SL}(2, \mathbb{R}) \to \mathfrak{H}, \qquad \sigma \mapsto \sigma i$$

where $i = \sqrt{-1}$ (of course, we could have selected any point in $\mathfrak{H}^*$; i is just particularly convenient). It is easily verified that the stabilzer of i is $\mathrm{SO}(2, \mathbb{R})$; we thus obtain a diffeomorphism

$$\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R}) \to \mathfrak{H}.$$

19. For an element σ of $\mathrm{GL}(2, \mathbb{R})$ we write

$$\sigma = \begin{bmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{bmatrix}$$

and put

$$j(\sigma, z) = c_\sigma z + d_\sigma.$$

A useful identity is

$$\Im(\sigma z) = (\det \sigma)|j(\sigma, z)|^{-2} \Im z. \tag{1}$$

20. Finally, we mention that the measure $y^{-2}dxdy$ on the upper half plane is invariant under the action of $\mathrm{GL}^+(2, \mathbb{R})$. This can be directly verified.

3.2.2 Classification of linear fractional transformations

21 Proposition (Shimura Prop. 1.12, 1.13). *Let σ be an element of* $\mathrm{SL}(2,\mathbb{R})$ *which is not equal to* ± 1*. There are three possibilities:*

1. *σ is* parabolic*: σ stabilizes exactly one point in $\mathbb{RP}^1$ and no points in $\mathfrak{H}$, the trace of σ is ± 2 and σ is conjugate (in* $\mathrm{SL}(2,\mathbb{C})$*) to a matrix of the form* $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$*;*

2. *σ is* ellitpic*: σ stabilizes exactly one point in $\mathfrak{H}$ and no points in $\mathbb{RP}^1$, the tace of σ is less than 2 in absolute value and σ is conjugate to a matrix of the form* $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ *with λ and μ distinct values of unit modulus;*

3. *σ is* hyperbolic*: σ stabilizes exactly two points in $\mathbb{RP}^1$ and no points in $\mathfrak{H}$, the trace of σ is greater than 2 in absolute value and σ is conjugate to a matrix of the form* $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ *with $|\lambda| \neq |\mu|$.*

Write σ as

$$\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The statements concerning conjugates of σ follows from the other statements and the theory of Jordan canonical form. We break the proof into two cases.

Case 1: $c = 0$. We have $d = a^{-1}$. The trace of σ is $a + 1/a$; this is equal to ± 2 if and only if $a = \pm 1$ and is otherwise greater than 2 in absolute value. For an element z of $\mathbb{C}$ we have $gz = a^2 z + ba$. It is thus clear that if $|\operatorname{tr}\sigma| = 2$ (*i.e.*, $a = \pm 1$) then σ stabilizes only ∞, while if $|\operatorname{tr}\sigma| > 2$ (*i.e.*, $a \neq \pm 1$) then σ stabilizes exactly two points in $\mathbb{RP}^1$, namely ∞ and $ba/(1 - a^2)$.

Case 2: $c \neq 0$. First note that ∞ will never be stabilized. An element z of $\mathbb{C}$ is stabilized by σ if and only if it satisfies the equation

$$cz^2 + (a - d)z + b = 0.$$

The solutions to this equation are

$$\frac{a - d \pm \sqrt{\Delta}}{2c}$$

where

$$\Delta = (\operatorname{tr}\sigma)^2 - 4.$$

Thus according to $|\operatorname{tr}\sigma| = 2$, $|\operatorname{tr} g| < 2$ and $|\operatorname{tr}\sigma| > 2$ there is exactly one solution (and it is real), exactly two solutions (one in the upper half plane and one in the lower half plane) and exactly two solutions (both of which are real). This proves the proposition.

22 Proposition (Shimura Prop. 1.14). *Let σ be an element of* $\mathrm{SL}(2,\mathbb{R})$ *such that σ^n is not ± 1. Then σ and σ^n are of the same type (i.e., both are parabolic, both are elliptic or both are hyperbolic).*

This follows immediately from the characterization of the types by their Jordan form.

3.3 The space $\Gamma\backslash\mathfrak{H}^*$ for discrete subgroups Γ

23. Throughout this section Γ denotes a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$. If σ is an element of $\mathrm{SL}(2,\mathbb{R})$ then we write $\overline{\sigma}$ for its image in $\mathrm{PSL}(2,\mathbb{R})$; similarly $\overline{\Gamma}$ denotes the image of Γ in $\mathrm{SL}(2,\mathbb{R})$.

3.3.1 Elliptic points, cuspidal points and the space $\mathfrak{H}^*$

24. The group Γ acts on both $\mathfrak{H}$ and $\mathbb{RP}^1$. We define special points in these spaces with respect to the action of Γ:

1. We say an element of $\mathfrak{H}$ is *elliptic* if it is stabilized by an elliptic element of Γ;

2. We say an element of $\mathbb{RP}^1$ is a *cusp* if it is stabilized by a parabolic element of Γ.

It is clear that the action of Γ preserves the property of being an elliptic point or cusp (*i.e.*, if z is a cusp so is γz for all γ in Γ).

25 Proposition (Shimura Prop. 1.16). *The stabilizer in* Γ *of an elliptic point is a finite cyclic group.*

Let z be an elliptic point and let τ be an element of $\mathrm{SL}(2,\mathbb{R})$ such that $\tau(i) = z$. Since the stabilizer of i in $\mathrm{SL}(2,\mathbb{R})$ is $\mathrm{SO}(2,\mathbb{R})$ it follows that the stabilizer of z in Γ is $\tau\mathrm{SO}(2,\mathbb{R})\tau^{-1}\cap\Gamma$. As this is a discrete subgroup of $\mathrm{SO}(2,\mathbb{R})$, the circle group, it is finite and cyclic.

26. We define the *order* of an elliptic point to be the order of its stabilizer *in* $\mathrm{PSL}(2,\mathbb{R})$. This may or may not be equal to the order of the stabilizer in $\mathrm{SL}(2,\mathbb{R})$ (the two numbers may differ by a factor of 2); precisely, we have the following:

27 Proposition (Shimura Prop. 1.20). *If* σ *is an elliptic element of* Γ *of order* $2h$ *then* $\sigma^h = -1$ *and the elliptic point stabilized by* σ *has order* h.

This follows immediately from the Jordan form of σ.

28 Proposition (Shimura Prop. 1.17). *Let* s *be a cusp and let* Γ_s *be its stabilizer in* Γ. *Then* $\overline{\Gamma}_s$ *is infinite cyclic. Moreover, other than* ± 1, Γ_s *consists entirely of parabolic elements.*

By conjugating we may assume $s = \infty$. The stabilizer of s in $\mathrm{SL}(2,\mathbb{R})$ is the Borel subgroup

$$\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}.$$

Thus the parabolic elements of $\mathrm{SL}(2,\mathbb{R})$ which stabilize s form the group of unipotent matrices

$$\begin{bmatrix} \pm 1 & b \\ 0 & \pm 1 \end{bmatrix}.$$

The image of this group in $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to $\mathbb{R}$; its nontrivial discrete subgroups are infinite cyclic.

It therefore follows that the parabolic elements of Γ_s, together with $\Gamma\cap\{\pm 1\}$, form a group, whose image in $\mathrm{PSL}(2,\mathbb{R})$ is infinite cyclic. Let

$$\sigma = \begin{bmatrix} \pm 1 & h \\ 0 & \pm 1 \end{bmatrix}$$

be a generator of this group. It is clear that Γ_s does not contain any elliptic elements (since elliptic elements cannot stabilize points in $\mathbb{RP}^1$); we must show that it does not contain any hyperbolic elements. Thus assume that the hyperbolic element

$$\tau = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}$$

belongs to Γ_s. Replacing τ by τ^{-1}, we may assume that $|a| < 1$. But then

$$\tau\sigma\tau^{-1} = \begin{bmatrix} \pm 1 & a^2 h \\ 0 & \pm 1 \end{bmatrix}$$

is a parabolic element with $|a^2 h| < |h|$. It follows that $\overline{\tau\sigma\tau^{-1}}$ cannot be of the form $\overline{\sigma}^n$. Since this contradicts the fact that $\overline{\sigma}$ generates the group of parabolic elements in $\overline{\Gamma}_s$ it follows that no such τ can exist.

29 Proposition (Shimura Prop. 1.18). *The elements of finite order in* Γ *consist exactly of* $\Gamma\cap\{\pm 1\}$ *and the elliptic elements of* Γ.

It follows from the characterization of types by their Jordan form that an element of finite order is necessarily elliptic or ± 1; conversely, such elements have finite order by proposition 25.

30 Proposition (Shimura Prop 1.19). *The set of elliptic points of* Γ *has no limit point in* $\mathfrak{H}$.

Assume that z_n is a sequence of elliptic points converging to z. Let U be a neighborhood of z such that γU meets U only if γ stabilizes z (*cf.* proposition 8). For $n > n_0$ the point z_n will belong to U. Thus, if γ is an elliptic element which stabilizes z_n then γU and U intersect, whence γ stabilizes z. But elliptic elements have only one fixed point and so we must have $z_n = z$ for $n > n_0$. Thus z is not a limit point of the set of elliptic points.

31 Proposition (Shimura Prop. 1.30). *If Γ' is a subgroup of* $\mathrm{SL}(2,\mathbb{R})$ *which is commensurable with* Γ *(and therefore discrete, cf. proposition 14) then* Γ *and* Γ' *have the same cusps.*

It suffices to consider the case where Γ' is a subgroup of finite index in Γ. If s is a cusp of Γ' then it is clear a cusp of Γ. If s is a cusp of Γ then there exists a parabolic element σ in Γ stabilizing s. Since Γ' has finite index, there is a nonzero integer k such that σ^k belongs to Γ'. Since σ^k is also parabolic (*cf.* proposition 28) it follows that s is a cusp of Γ'.

32. We let $\mathfrak{H}^*$ denote the union of $\mathfrak{H}$ with the cusps of Γ. Since the action of Γ on $\mathbb{RP}^1$ takes cusps to cusps, we may regard Γ as acting on $\mathfrak{H}^*$.

33. We now define a topology on $\mathfrak{H}^*$. The subspace $\mathfrak{H}$ gets its usual topology. We must give a fundamental system of neighborhoods at a cusp s. If $s \neq \infty$ then we take a fundamental system to be all sets of the form

$$\{s\} \cup \{\text{the interior of a circle in } \mathfrak{H} \text{ tangent to the real axis at s}\}.$$

If $s = \infty$ then we take a fundamental system to be all sets of the form

$$\{s\} \cup \{z \in \mathfrak{H} | \Im z > c\}$$

where c is any positive real number. It is clear that this topology is Hausdorff and that Γ acts by homeomorphisms. The topology is not, however, locally compact, unless Γ has no cusps (*i.e.*, unless $\mathfrak{H}^* = \mathfrak{H}$).

3.3.2 The space $\Gamma\backslash\mathfrak{H}^*$ is Hausdorff

34 Proposition (Shimura Thm. 1.28). *The space* $\Gamma\backslash\mathfrak{H}^*$ *is Hausdorff.*

35. The proof of proposition 34 will take the rest of this section. We may assume that Γ has a cusp, for otherwise the proposition is trivial. By conjugating, we may assume that ∞ is a cusp. We let Γ_∞ be the stabilizer of ∞ in Γ; by proposition 28 we can find a generator $\pm\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$ of $\overline{\Gamma}_\infty$. Note that Γ_∞ is precisely the set of σ for which $c_\sigma = 0$.

36 Lemma (Shimura Lemma 1.23). *The quantity $|c_\sigma|$ depends only on the double coset $\Gamma_\infty\sigma\Gamma_\infty$.*

This is a simple calculation.

37 Lemma (Shimura Lemma 1.24). *Given $M > 0$ there are only finitely many double cosets $\Gamma_\infty\sigma\Gamma_\infty$ with $c_\sigma < M$.*

We are going to show that there exists a compact set $K = K(M)$ in $\mathfrak{H}$ such that for every σ in $\mathrm{SL}(2,\mathbb{R})$ there exists a σ'' in $\Gamma_\infty\sigma\Gamma_\infty$ such that $\sigma''(i)$ belongs to K.

Write

$$\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Let

$$\tau = \pm\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$$

be the generator of $\overline{\Gamma}_\infty$. We can choose an integer n such that $1 \leq d + nhc \leq 1 + |hc|$ (note that $1 + |hc| \leq 1 + |h|M$). Put $\sigma' = \sigma\tau^n$. We have

$$\sigma' = \pm\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & nh \\ 0 & 1 \end{bmatrix} = \pm\begin{bmatrix} a & b+nha \\ c & d+nhc \end{bmatrix}$$

and so $d_{\sigma'} = \pm(d + nhc)$. By (1) we have

$$\Im(\sigma'(i)) = \frac{1}{c_{\sigma'}^2 + d_{\sigma'}^2}$$

so that

$$\frac{1}{M^2+(1+|h|M)^2} \le \Im(\sigma'(i)) \le 1.$$

Pick an integer m such that $0 \le \Re(\sigma'(i)) + mh \le |h|$ and put $\sigma'' = \tau^m\sigma'$. Then the real part of $\sigma''(i)$ lies in $[0,|h|]$ and the imaginary part of $\sigma''(i)$ is the same as the imaginary part of $\sigma'(i)$. Thus we may take

$$K(M) = \{z \in \mathfrak{H} | \Re z \in [0,|h|] \text{ and } (M^2+(1+|h|M)^2)^{-1} \le \Im z \le 1\}.$$

To recap, we have shown that any double coset $\Gamma_\infty\sigma\Gamma_\infty$ with $c_\sigma < M$ has a representative σ'' with $\sigma''(i)$ in the compact set $K(M)$. However, by proposition 6 there are only finitely many σ'' in Γ such that $\sigma''(i)$ lies in $K(M)$. This proves the proposition.

38 Lemma (Shimura Lemma 1.25). *There exists a positive number r such that for all σ in $\Gamma - \Gamma_\infty$ we have $|c_\sigma| \ge r$. Furthermore, for all z in $\mathfrak{H}$ and σ in $\Gamma - \Gamma_\infty$ we have $(\Im z)(\Im(\sigma z)) \le r^{-2}$.*

The existence of r follows immediately from lemma 37. If σ belongs to $\Gamma - \Gamma_\infty$ then

$$\Im(\sigma z) = |c_\sigma z + d_\sigma|^{-2}\Im z \le |c_\sigma \Im z|^{-2}\Im z \le r^{-2}(\Im z)^{-1}.$$

39 Lemma (Shimura Lemma 1.26). *Every cusp s of Γ has a neighborhood U in $\mathfrak{H}^*$ such that for all σ in Γ the sets σU and U meet if and only if σ stabilizes s.*

It suffices (by conjugation) to consider $s = \infty$. Let r be as in lemma 38 and put $U = \{z \in \mathfrak{H} | \Im z > r^{-1}\}$. Then U is a neighborhood of z. If σ is any element of $\Gamma - \Gamma_\infty$ and z is an element of U then $\Im(\sigma z) < 1/r$, *i.e.*, U and σU do not meet.

40. Let s be a cusp of Γ and let U be a neighborhood as in lemma 39. Then two points in U are in the same Γ orbit if and only if they are in the same Γ_s orbit. Thus $\Gamma_s\backslash U$ may be regarded as a subset of $\Gamma\backslash\mathfrak{H}^*$. Also, note that U cannot contain any elliptic points.

41 Lemma (Shimura Lemma 1.27). *Let s be a cusp of Γ. For every compact subset K of $\mathfrak{H}$ there exists a neighborhood U of s in $\mathfrak{H}^*$ such that U and γK are disjoint, for every γ in Γ.*

Agian, it suffice to consider $s = \infty$. We can find two positive real numbers A and B such that all elements of K have imaginary part in the interval (A, B). Let r be as in lemma 38. Put

$$U = \{\infty\} \cup \{z \in \mathfrak{H} | \Im z > \mathrm{Max}(B, 1/(Ar^2))\}.$$

Let z belong to K. If σ is in $\Gamma - \Gamma_\infty$ then $\Im(\sigma z) < 1/(Ar^2)$ while if σ is in Γ_∞ then $\Im(\sigma z) = \Im z < B$. Thus U has the requisite property.

42 Lemma. *Let s and t be two Γ inequivalent cusps. Then there exist neighborhoods U and V of s in t in $\mathfrak{H}^*$ such that γU and V are disjoint for all γ in Γ.*

We may assume that $t = \infty$. Let $\pm\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$ be a generator for $\overline{\Gamma}_\infty$. Let u be a positive real number and define sets

$$\begin{aligned} L &= \{z \in \mathfrak{H} | \Im z = u\} \\ K &= \{z \in L | 0 \le \Re z \le |h|\} \\ V &= \{\infty\} \cup \{z \in \mathfrak{H} | \Im z > u\}. \end{aligned}$$

Since K is compact, we can find, by lemma 41, a neighborhood U of s so that K and γU are disjoint, for all γ in Γ. In fact, we may assume that the boundary of U is a circle which is tangent to the real axis at s. We now show that these choices of U and V satisfy the statement of the lemma.

Assume to the contrary that there exists an element γ of Γ such that γU meets V. Since $\gamma s \ne \infty$ it follows that the boundary of γU is a circle tangent to the real axis. Thus if γU meets V then it also meets L, and therefore some δK for δ in Γ_∞. But then $\delta^{-1}\gamma U$ meets K, which contradicts the definition of K. Thus no such γ exists and the proposition is proved.

43 Proof of proposition 34. We call points in $\Gamma\backslash\mathfrak{H}^*$ cusps or non-cusps in the obvious manner. We must prove three things:

1. Two non-cusps can be separated.
2. Two cusps can be separated.
3. A cusp and a non-cusp can be separated.

Since we know that $\Gamma\backslash\mathfrak{H}$ is Hausdorff (*cf.* proposition 9) it follows that we can separate two non-cusps. Lemma 42 is precisely the statement that two cusps can be separated. Lemma 41 implies that a cusp and a non-cusp can be separated.

3.3.3 The space $\Gamma\backslash\mathfrak{H}^*$ is locally compact

44 Proposition (Shimura Prop. 1.29). *The space* $\Gamma\backslash\mathfrak{H}^*$ *is locally compact.*

Since we know that $\Gamma\backslash\mathfrak{H}$ is locally compact it follows that all non-cusps in $\Gamma\backslash\mathfrak{H}^*$ have compact neighborhoods. Thus we must show that each cusp in $\Gamma\backslash\mathfrak{H}^*$ has a compact neighborhood. Let $\pi : \mathfrak{H}^* \to \Gamma\backslash\mathfrak{H}^*$ be the quotient map. It suffices to assume that ∞ is a cusp of Γ and show that $\pi(\infty)$ has a compact neighborhood.

By lemma 39, and the remark which follows, there exists a neighborhood $V = \{\infty\} \cup \{z \in \mathfrak{H} | \Im z > c\}$ of ∞ (where c is a positive number) such that $\Gamma_\infty\backslash V$ is identified with $\pi(V)$. If $\pm \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$ is a generator of $\overline{\Gamma}_\infty$ then we see that $\pi(V)$ coincides with the image of the compact set $\{z \in V | z = \infty \text{ or } 0 \leq \Re z \leq |h|\}$. Thus $\pi(V)$ is a compact neighborhood of $\pi(\infty)$.

3.3.4 The complex structure on $\Gamma\backslash\mathfrak{H}^*$

45. We will now put a complex structure on the quotient space $\Gamma\backslash\mathfrak{H}^*$. We do this by specifying for each point v in $\Gamma\backslash\mathfrak{H}^*$ an open neighborhood U_v in $\Gamma\backslash\mathfrak{H}^*$ together with a homeomorphism ϕ_v from U_v to an open subset of $\mathbb{C}$ in such a way that the transition functions $\phi_v\phi_w^{-1}$ are holomorphic.

We let $\pi : \mathfrak{H}^* \to \Gamma\backslash\mathfrak{H}^*$ be the quotient map.

46 (Shimura pg. 18). Let v be a given point in $\mathfrak{H}^*$. Let U be an open neighborhood of v in $\mathfrak{H}^*$ such that for any γ in Γ the sets γU and U meet if and only if γ belongs to Γ_v, the stabilizer of v (*cf.* lemma 39). Then we have a natural injection $\Gamma_v\backslash U \to \Gamma\backslash\mathfrak{H}^*$ the image U_v of which is an open neighborhood of $\pi(v)$. We must give a homeomorphism ϕ_v of U_v to open set of $\mathbb{C}$. We now separate three cases:

Case 1: v is neither an elliptic point nor a cusp. In this case Γ_v consists only of 1 and possibly -1; thus $\pi : U \to U_v$ is a homeomorphism. We let ϕ_v be the inverse of π restricted to U.

Case 2: v is an elliptic point. Let λ be a holomorphic isomorphism of $\mathfrak{H}$ with the unit disc taking v to 0, *e.g.*, $\lambda(z) = (z - v)/(z + v)$. If $\overline{\Gamma}_v$ is of order n then $\lambda\overline{\Gamma}_v\lambda^{-1}$ consists of the transformations $w \mapsto \zeta^k w$ for $k = 0, \ldots, n-1$, where $\zeta = e^{2\pi i/n}$. We thus define $\phi_v : \Gamma_v\backslash U \to \mathbb{C}$ by $\phi_v(\pi(z)) = \lambda(v)^n$. It is clear that ϕ_v is a homeomorphism to an open subset of $\mathbb{C}$.

Case 3: v is a cusp. Let ρ be an element of $\mathrm{SL}(2, \mathbb{R})$ which takes v to ∞. Then

$$\rho\Gamma_v\rho^{-1} \cdot \{\pm 1\} = \left\{ \pm \begin{bmatrix} 1 & nh \\ 0 & 1 \end{bmatrix} \middle| n \in \mathbb{Z} \right\}$$

where we take h to be positive. We define a homeomorphism ϕ_v of $\Gamma_s\backslash U$ to an open subset of $\mathbb{C}$ by $\phi_v(\pi(z)) = \exp(2\pi i\rho(z)/h)$.

47. Having defined (U_v, ϕ_v) for each point v in $\Gamma\backslash\mathfrak{H}^*$ one must now check that $\phi_v\phi_w^{-1}$ is holomorphic. We leave this routine verification to the reader.

48 Proposition. *Let* Γ' *be a subgroup of finite index in* Γ*. Then the natural map* $\Gamma'\backslash\mathfrak{H}^*$ *to* $\Gamma\backslash\mathfrak{H}^*$ *is a holomorphic ramified cover of degree* $[\overline{\Gamma} : \overline{\Gamma}']$.

Left to the reader.

3.3.5 Fuchsian groups of the first kind

49. A discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ is a *Fuchsian group of the first kind* if the space $\Gamma\backslash\mathfrak{H}^*$ is compact. Note that if Γ is such a group then $\Gamma\backslash\mathfrak{H}^*$ is a compact Riemann surface and thus an algebraic curve.

50 Proposition (Shimura Prop. 1.31). *Let Γ and Γ' be commensurable subgroups of $\mathrm{SL}(2,\mathbb{R})$. Then Γ is a Fuchsian group of the first kind if and only if Γ' is.*

It suffices to consider the case when Γ' is a finite index subgroup of Γ. We thus have a map $f : \Gamma'\backslash\mathfrak{H}^* \to \Gamma\backslash\mathfrak{H}^*$. The map f is a holomorphic ramified cover (*cf.* proposition 48), and therefore proper. There is nothing more that needs to be said.

51 Proposition (Shimura Prop. 1.32). *If Γ is a Fuchsian group of the first kind then the number of cusps and elliptic points in $\Gamma\backslash\mathfrak{H}^*$ is finite.*

Let C (resp. E) denote the set of cusps (resp. elliptic points) of Γ in $\mathfrak{H}^*$. For each z in $\mathfrak{H}$ we may take a neighborhood U_z such that U_z contains no elliptic points, with the possible excpetion of z itself (*cf.* lemma 30). For each s in C we can find a neighborhood U_s of s containing no elliptic points (*cf.* lemma 39 and following comment). Now, let $\pi : \mathfrak{H}^* \to \Gamma\backslash\mathfrak{H}^*$ be the quotient map. Then the number of points in $\pi(C)$ (resp. $\pi(E)$) is at most the number of $\pi(U_s)$ (resp. $\pi(U_z)$) which are needed to cover the compact space $\Gamma\backslash\mathfrak{H}^*$.

52. Let Γ be a Fuchsian group of the first kind and let Γ' be a subgroup of finite index. We have a commutative diagram

$$\begin{array}{ccc} \mathfrak{H}^* & \longrightarrow & \mathfrak{H}^* \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Gamma'\backslash\mathfrak{H}^* & \xrightarrow{f} & \Gamma\backslash\mathfrak{H}^* \end{array}$$

where the top map is the identity map and the other maps are the natural porjection maps. We have already said (*cf.* proposition 48) that f is a (ramified) covering map of degree $[\overline{\Gamma} : \overline{\Gamma}']$. We now briefly give some more properties relating f, Γ and Γ'.

53 Proposition (Hurwitz Formula). *Let $q_1, \ldots, q_r$ be the points in $\Gamma'\backslash\mathfrak{H}^*$ at which f ramifies and let e_i be the ramification index of f at q_i. If g is the genus of $\Gamma\backslash\mathfrak{H}^*$ and g' is the genus of $\Gamma'\backslash\mathfrak{H}^*$ then*

$$2g' - 2 = n(2g-2) + \sum_{i=1}^{r} e_i - 1$$

where $n = [\overline{\Gamma} : \overline{\Gamma}']$ is the degree of f.

This is a standard fact which we do not prove.

54 Proposition (Shimura Prop. 1.37). *Let $z \in \mathfrak{H}^*$, let $p = \pi(z)$ and let $f^{-1}(p) = \{q_1, \ldots, q_h\}$. Choose points w_i in $\mathfrak{H}^*$ such that $\pi'(w_i) = q_i$.*

1. *The ramification index e_i of f at q_i is $[\overline{\Gamma}_{w_i} : \overline{\Gamma}'_{w_i}]$. In particular, f only ramifies at elliptic points and cusps.*
2. *If $w_i = \sigma_i(z)$ with σ_i in Γ then $e_i = [\overline{\Gamma}_z : \sigma_i^{-1}\overline{\Gamma}'\sigma_i \cap \overline{\Gamma}_z]$ and $\overline{\Gamma} = \coprod_{i=1}^{h} \overline{\Gamma}'\sigma_i\overline{\Gamma}_z$.*
3. *If Γ' is a normal subgroup of Γ then $e_1 = \cdots = e_h = e$ and $[\overline{\Gamma} : \overline{\Gamma}'] = eh$.*

Left to the reader.

3.4 The group $\mathrm{SL}(2,\mathbb{Z})$ and its congruence subgroups

3.4.1 Group theory of $\mathrm{SL}(2,\mathbb{Z})$

55 Proposition (Shimura pg. 16). *The group* $\mathrm{SL}(2,\mathbb{Z})$ *is generated by the two matrices*

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Let G be the subgroup generated by S and T inside $\mathrm{SL}(2,\mathbb{Z})$. Observe that $-1 = S^2$ belongs to G, that all upper triangular matrices in $\mathrm{SL}(2,\mathbb{Z})$ belongs to G and that if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ belongs to G then so does $\begin{bmatrix} -c & -d \\ a & b \end{bmatrix} = S\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Assume that G is not all of $\mathrm{SL}(2,\mathbb{Z})$ and take an element $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $\mathrm{SL}(2,\mathbb{Z}) - G$ with $\mathrm{Min}(|a|,|c|)$ as small as possible. We may assume $0 < |c| \leq |a|$. We can then find integers q and r such that $a = cq + r$ with $0 \leq r < |c|$. We then have that $T^{-q}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} r & * \\ c & * \end{bmatrix}$ does not belong to G but $r = \mathrm{Min}(r,|c|) < |c| = \mathrm{Min}(|a|,|c|)$. This is a contradiction, so G must be all of $\mathrm{SL}(2,\mathbb{Z})$.

56 Proposition. *Let* G_t *denote the set of matrices in* $\mathrm{SL}(2,\mathbb{Z})$ *of trace* t. *Then for* $t \neq \pm 2$ *every matrix in* G_t *is conjugate to a matrix* σ *which satisfies*

$$|c_\sigma| \leq \tfrac{11}{6}|t| + 1, \qquad |a_\sigma - d_\sigma| \leq |c_\sigma|.$$

First note that if $c = 0$ then $t = \pm 2$, so that if $t \neq \pm 2$ then $c \neq 0$. Now, we have the formula

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a-c & a+b-c-d \\ c & c+d \end{bmatrix}.$$

Iterating this identity shows that any matrix σ is conjugate to a matirx σ' such that $c_\sigma = c_{\sigma'}$ and $|a_{\sigma'} - d_{\sigma'}| \leq |c_{\sigma'}|$. We also have the formula

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

which shows that any σ is conjugate to a σ' with $c_{\sigma'} = -b_\sigma$. Thus, in effect, we can "switch" the b and c entries of a matrix.

The idea now is the following: given a matrix, use conjugation by T to bring a and d within c of each other. Then, if b is smaller than c switch b and c. We need to determine under which conditions b will be smaller than c.

So, we have a matrix $\begin{bmatrix} a & b \\ d & d \end{bmatrix}$ satisfying

$$|a - d| \leq |c|, \qquad a + d = t \tag{2}$$

and we want to know when

$$|b| < |c| \tag{3}$$

holds. To begin with, since the determinant is equal to 1, we have

$$|b| \leq \frac{|ad| + 1}{|c|}.$$

Thus if

$$|c|^2 > |ad| + 1$$

then (3) holds. Using equation (2), we find that $|a|$ and $|d|$ are bounded above by $(|c| + t)/2$. Thus we see that if

$$|c|^2 > (|c| + t)^2/4 + 1$$

then (3) holds. Using a bit of algebra, it follows that if

$$|c| > \tfrac{11}{6}t + 1$$

then (3) holds. Thus so long as $|c|$ is strictly greater than $\frac{11}{6}t + 1$ then we can conjugate by S and make $|c|$ smaller. It follows that we can make $|c|$ less than or equal to $\frac{11}{6}t + 1$.

57 Proposition. *An elliptic element of* $\mathrm{SL}(2,\mathbb{Z})$ *is conjugate to precisely one of the following matrices:*

$$\pm\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \pm\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \pm\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

This is reduced to a short computation after applying proposition 56 with $t = 0, \pm 1$.

58 Proposition. *A parabolic element of* $\mathrm{SL}(2,\mathbb{Z})$ *is conjugate to a matrix of the form*

$$\pm\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$$

where h is an integer.

This follows from our computation of the cusps of $\mathrm{SL}(2,\mathbb{Z})$ given in the next section in proposition 61.

3.4.2 The action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathfrak{H}$

59. Let Γ be a subgroup of $\mathrm{SL}(2,\mathbb{R})$. We say a set F in $\mathfrak{H}$ is a *fundamental domain* for Γ if:

1. F is a connect open subset of $\mathfrak{H}$;
2. no two points of F are equivalent under Γ;
3. every point of $\mathfrak{H}$ is equivalent to some point in the closure of F.

It is known that every discrete subgroup Γ has a fundamental domain, but we will not need this result.

60 Proposition (Shimura pg. 16). *Let F be the subset of $\mathfrak{H}$ consisting of those elements z such that $|z| > 1$ and $|\Re z| < 1/2$. Then F is a fundamental domain for* $\mathrm{SL}(2,\mathbb{Z})$.

It is clear that F is a connected open subset of $\mathfrak{H}$. Let us now verify that no two points in F are equivalent. Assume that z and z' are two distinct points in F which are equivalent. We may assume $\Im z \le \Im z'$. Say $z' = \sigma z$ with $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. Then we have $\Im z \le \Im z' = (\Im z)/|cz+d|^2$. Thus we conclude

$$|c|\Im z \le |cz+d| \le 1. \tag{4}$$

If $c = 0$ then $a = d = \pm 1$ and $z' = z \pm b$, which is impossible; therefore $c \ne 0$. Since $\Im z > \sqrt{3}/2$ for all z in F it follows from (4) that $|c| = 1$. Thus, again from (4), we find $|z \pm d| \le 1$. But for any z in F and integer d we have $|z \pm d| > 1$. This contradiction proves that no two distinct points of F are equivalent under Γ.

We must now show that every point in $\mathfrak{H}$ is equivalent to a point in the closure of F. Let z be a point in $\mathfrak{H}$. If $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an element of $\mathrm{SL}(2,\mathbb{Z})$ then $\Im \sigma z = (\Im z)/|cz+d|^2$. Now, the quantity $|cz+d|$ has a minimum as c and d vary over all the integers with $(c,d) \ne (0,0)$. It thus follows that there is a point in the $\mathrm{SL}(2,\mathbb{Z})$-orbit of z with maximum imaginary part; let z' be such a point. Consider the matrix $\gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ in $\mathrm{SL}(2,\mathbb{Z})$. Writing $z' = x' + iy'$ we have

$$\Im \gamma z' = \Im(-1/z') = y'/|z'|^2 \le y'$$

and so it follows that $|z'| \ge 1$. Now consider the matrix $\tau = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ which is also in $\mathrm{SL}(2,\mathbb{Z})$. We have $\tau^n z' = z' + n$. Thus $|\tau^n z'| \ge 1$ since $\tau^n z'$ still has maximal imaginary part in its $\mathrm{SL}(2,\mathbb{Z})$-orbit. We can choose n so that $|\Re \tau^n z'| \le 1/2$. This shows that every point is equivalent to a point in the closure of F.

61 Proposition. *The cusps of* $\mathrm{SL}(2,\mathbb{Z})$ *consist exactly of the points* $\mathbb{QP}^1$*. They are all equivalent.*

Let p/q be a non-infinite element of $\mathbb{QP}^1$ with $(p,q)=1$. We can find r and s such that $rp+sq=1$. The matrix $\begin{bmatrix} r & s \\ -q & p \end{bmatrix}$ is then an element of $\mathrm{SL}(2,\mathbb{Z})$ which takes z to ∞. This shows that all elements in $\mathbb{QP}^1$ are equivalent. The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ belongs to $\mathrm{SL}(2,\mathbb{Z})$, is parabolic and stabilizes ∞. Thus all elements of $\mathbb{QP}^1$ are cusps.

We must now show that a cusp belongs to $\mathbb{QP}^1$. Let s be a cusp, considered as an element of $\mathbb{R}$, and let $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a parabolic element of $\mathrm{SL}(2,\mathbb{Z})$ which stabilizes s. Then s satisfies the equation $cs^2+(d-a)s-b=0$. Since the discriminant of this equation is 0 it follows that s belongs to $\mathbb{Q}$.

62 Proposition. *The points i and $e^{2\pi i/3}$ are elliptic points for* $\mathrm{SL}(2,\mathbb{Z})$ *of orders 2 and 3. All elliptic points are equivalent to one of these two.*

This follows immediately from our classification of the elliptic elements in $\mathrm{SL}(2,\mathbb{Z})$ (*cf.* proposition 57).

63 Proposition. *The quotient* $\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}^*$ *is isomorphic (as a complex manifold) to the Riemann sphere* $\mathbb{CP}^1$*.*

Let F be the fundamental domain for $\mathrm{SL}(2,\mathbb{Z})$ of proposition 60 and let $F' = \overline{F}\cup\{\infty\}$. Then by proposition 61 and the definition of a fundamental domain, every point in $\mathfrak{H}^*$ is equivalent to a point in F'. It is clear that when the correct points are identified F' looks like a sphere.

64 Corollary. *Any subgroup of* $\mathrm{SL}(2,\mathbb{R})$ *which is commensurable with* $\mathrm{SL}(2,\mathbb{Z})$ *is a Fuchsian group of the first kind.*

This follows from propositions 50 and 63.

3.4.3 The genus of $\Gamma\backslash\mathfrak{H}^*$ for subgroups Γ of $\mathrm{SL}(2,\mathbb{Z})$

65 Proposition (Shimura Prop. 1.40). *Let Γ be a finite index subgroup of* $\mathrm{SL}(2,\mathbb{Z})$*. Let μ be the index of $\overline{\Gamma}$ in* $\mathrm{PSL}(2,\mathbb{Z})$*, let ν_2 be the number of Γ inequivalent elliptic points of order 2, let ν_3 be the number of Γ inequivalent elliptic points of order 3, and let ν_∞ be the number of Γ inequivalent cusps. If g is the genus of $\Gamma\backslash\mathfrak{H}^*$ then we have*

$$g = 1+\frac{\mu}{12}-\frac{\nu_2}{4}-\frac{\nu_3}{3}-\frac{\nu_\infty}{2}.$$

Let $\pi:\mathfrak{H}^*\to\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}^*$ and $f:\Gamma\backslash\mathfrak{H}^*\to\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}^*$ be the natural projection maps. The map f is of degree μ (*cf.* proposition 48).

Let $e_1,\ldots,e_r$ be the ramification indices of the points lying above $\pi(e^{2\pi i/3})$. The numbers e_i are either 1 or 3, the number of e_i which are equal to 1 being ν_3. Thus if we write $t=\nu_3+\nu_3'$ then $\mu=\nu_3+3\nu_3'$ (since the sum of the e_i is equal to μ). We then have

$$\sum_{i=1}^{t} e_i - 1 = \mu - t = 2\nu_3' = 2(\mu-\nu_3)/3.$$

Similar reasoning gives

$$\begin{aligned}\sum_{f(P)=\pi(i)} e_P - 1 &= (\mu-\nu_2)/2\\ \sum_{f(P)=\pi(\infty)} e_P - 1 &= \mu-\nu_\infty.\end{aligned}$$

Since $\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}^*$ has genus 0 (*cf.* proposition 63) our result follows by the Hurwitz genus formula (*cf.* proposition 53).

66. Proposition 65 shows that when given a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ the first thing one should do is compute its index, the number of its elliptic points, and the number of its cusps. We will follow this plan in the last part of this section.

3.4.4 Congruence subgroups

67. Let N be a positive integer. Define the following subgroups of $\mathrm{SL}(2,\mathbb{Z})$:

$$\begin{aligned}
\Gamma(N) &= \left\{ g \in \mathrm{SL}(2,\mathbb{Z}) \,\middle|\, g \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod N \right\} \\
\Gamma_0(N) &= \left\{ g \in \mathrm{SL}(2,\mathbb{Z}) \,\middle|\, g \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \pmod N \right\} \\
\Gamma_1(N) &= \left\{ g \in \mathrm{SL}(2,\mathbb{Z}) \,\middle|\, g \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \pmod N \right\}
\end{aligned}$$

The group $\Gamma(N)$ is called the *principal congruence subgroup of level* N. In general, a subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$ is called a *congruence subgroup of level* N if it contains $\Gamma(N)$. Thus both $\Gamma_0(N)$ and $\Gamma_1(N)$ are congruence subgroups of level N. Note that $\Gamma(1)$ is nothing other than $\mathrm{SL}(2,\mathbb{Z})$; we shall often interchange the two names.

68 Proposition (Shimura Lemma 1.38). *The sequence*

$$1 \to \Gamma(N) \to \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z}) \to 1$$

is exact.

The only nontrivial assertion is that reduction modulo N from $\mathrm{SL}(2,\mathbb{Z})$ to $\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$ is surjective. Let A be an element of $\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$. By the elementary divisor theorem we can find matrices X and Y in $\mathrm{SL}(2,\mathbb{Z})$ such that XAY is a diagonal matrix. Thus it suffices to show that a matrix of the form $\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}$ with x in $(\mathbb{Z}/N\mathbb{Z})^*$ comes from a matric in $\mathrm{SL}(2,\mathbb{Z})$. We can find integers $a \equiv x \pmod N$ and $b \equiv x^{-1} \pmod N$ such that $ab \equiv 1 \pmod{N^2}$. Write $ab = 1 + cN^2$. Then the matrix $\begin{bmatrix} a & N \\ cN & b \end{bmatrix}$ lies in $\mathrm{SL}(2,\mathbb{Z})$ and is equivalent to $\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}$ modulo N.

69 Proposition (Shimura Lemma 1.41). *Let x, y, x' and y' be integers such that $(x,y) = 1$ and $(x',y') = 1$. Then $\begin{bmatrix} x \\ y \end{bmatrix} \equiv \begin{bmatrix} x' \\ y' \end{bmatrix}$ (mod N) if and only if there exists σ in $\Gamma(N)$ such that $\begin{bmatrix} x \\ y \end{bmatrix} = \sigma \begin{bmatrix} x' \\ y' \end{bmatrix}$.*

First consider the case when $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Take integers r and s so that $sx - ry = 1$. Let σ be the matrix $\begin{bmatrix} x & r \\ y & s \end{bmatrix}$. Then σ belongs to $\Gamma(N)$ and takes $\begin{bmatrix} x' \\ y' \end{bmatrix}$ to $\begin{bmatrix} x \\ y \end{bmatrix}$.

Now consider the general case. Take integers r and s so that $sx' - ry' = 1$ and let $\tau = \begin{bmatrix} x' & r \\ y' & s \end{bmatrix}$ belong to $\Gamma(1)$. Then $\tau \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \equiv \begin{bmatrix} x \\ y \end{bmatrix} \pmod N$ so that $\tau^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pmod N$. By the first case, we can find σ in $\Gamma(N)$ such that $\sigma \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \tau^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$. Then the matrix $\tau\sigma\tau^{-1}$ belongs to $\Gamma(N)$ and takes $\begin{bmatrix} x' \\ y' \end{bmatrix}$ to $\begin{bmatrix} x \\ y \end{bmatrix}$.

70 Proposition. *Let Γ be a congruence subgroup of level N. Let $s = x/y$ and $s' = x'/y'$ be two cusps of Γ, taken so that $(x,y) = 1$ and $(x',y') = 1$. Then s and s' are Γ-equivalent if and only if there exists σ in Γ such that*

$$\sigma \begin{bmatrix} x \\ y \end{bmatrix} = \pm \begin{bmatrix} x' \\ y' \end{bmatrix} \; (\mathit{mod}\ N).$$

If $\sigma s = s'$ then it is clear that $\sigma \begin{bmatrix} x \\ y \end{bmatrix} = \pm \begin{bmatrix} x' \\ y' \end{bmatrix}$. Now, if $\sigma \begin{bmatrix} x \\ y \end{bmatrix} = \pm \begin{bmatrix} x' \\ y' \end{bmatrix} \pmod N$ then by proposition 69 there exists τ in $\Gamma(N)$ such that $(\tau\sigma) \begin{bmatrix} x \\ y \end{bmatrix} = \pm \begin{bmatrix} x' \\ y' \end{bmatrix}$. It then follows that $\tau\sigma s = s'$ so that s and s' are Γ-equivalent.

71 Proposition. *Let Γ be a congruence subgroup of level N and let X be set set of pairs $(x, y) \in (\mathbb{Z}/N\mathbb{Z})^2$ such that $\gcd(x, y, N) = 1$ (where we use the convention $\gcd(0, x) = x$). Then the equivalence classes of cusps of Γ may be identified with the orbits of $\{\pm 1\} \cdot \Gamma/\Gamma(N)$ on X.*

This is just a restatement of proposition 70

3.4.5 Genus calculations for certain congruence subgroups

72 Proposition (Shimura pg. 22). *Let Γ be the group $\Gamma(N)$ and let other notation be as in proposition 65. We have*

$$\begin{aligned} \mu &= \begin{cases} \frac{1}{2} N^3 \prod_{p|N} (1 - p^{-2}) & N > 2 \\ 6 & N = 2 \end{cases} \\ \nu_\infty &= \mu/N \\ \nu_2 &= 0 \\ \nu_3 &= 0 \\ g &= 1 + \frac{\mu(N-6)}{12N} \end{aligned}$$

Index: If p is a prime number then the cardinality of $\mathrm{SL}(2, \mathbb{Z}/p^e\mathbb{Z})$ is $p^{3e}(1 - p^{-2})$. It follows that the cardinality of $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$, and thus the index of $\Gamma(N)$ in $\Gamma(1)$, is given by $N^3 \prod_{p|N}(1 - p^{-2})$. Now, the group $\Gamma(N)$ for $N > 2$ does not contain -1; thus the index of $\overline{\Gamma}(N)$ in $\overline{\Gamma}(1)$ is half that of $\Gamma(N)$ in $\Gamma(1)$. The group $\Gamma(2)$ does contain -1, so the two indices are equal in this case.

Cusps: This follows immediatel from proposition 71 and an easy count of the cardinality of X.

Elliptic Points: This follows immediately from proposition 57 and the fact that $\Gamma(N)$ is normal.

Genus: This follows immediately from proposition 65 and the above computations.

73. Here is a table of genera for $\Gamma(N)\backslash\mathfrak{H}^*$.

N	g	N	g	N	g
0	0	6	1	12	25
1	0	7	3	13	50
2	0	8	5	14	49
3	0	9	10	15	73
4	0	10	13	16	81
5	0	11	26		

This table contains exactly those N for which $g < 100$.

74 Proposition (Shimura Prop. 1.43). *Let Γ be the group $\Gamma_0(N)$ (with $N > 1$) and let other notation be as in proposition 65. We have*

$$\begin{aligned} \mu &= N \prod_{p|N} (1 + p^{-1}) \\ \nu_\infty &= \sum_{d|N} \phi((d, N/d)) \\ \nu_2 &= \begin{cases} 0 & 4|N \\ \prod_{p|N} \left(1 + \left(\frac{-1}{p}\right)\right) & \text{otherwise} \end{cases} \\ \nu_3 &= \begin{cases} 0 & 9|N \\ \prod_{p|N} \left(1 + \left(\frac{-3}{p}\right)\right) & \text{otherwise} \end{cases} \end{aligned}$$

Index: The group $\Gamma_0(N)/\Gamma(N)$ is isomorphic to the group of upper triangular matrices in $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$; the order of this group is easily seen to be $N^2 \prod_{p|N}(1-p^{-1})$. We thus have (using results from the proof of proposition 72)

$$[\Gamma(1) : \Gamma(N)] = N^3 \prod_{p|N}(1-p^{-2}), \qquad [\Gamma_0(N) : \Gamma(N)] = N^2 \prod_{p|N}(1-p^{-1});$$

the stated result easily follows.

Cusps: By proposition 70 this is reduced to counting the orbits of the Borel subgroup of $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$ in the space X; this is an easy exercise which is left to the reader.

Elliptic points: Do this.

75. Here is a table of genera for $\Gamma_0(N)\backslash\mathfrak{H}^*$.

N	g	N	g	N	g	N	g	N	g	N	g
1	0	30	3	59	5	88	9	117	11	146	17
2	0	31	2	60	7	89	7	118	14	147	11
3	0	32	1	61	4	90	11	119	11	148	17
4	0	33	3	62	7	91	7	120	17	149	12
5	0	34	3	63	5	92	10	121	6	150	19
6	0	35	3	64	3	93	9	122	14	151	12
7	0	36	1	65	5	94	11	123	13	152	17
8	0	37	2	66	9	95	9	124	14	153	15
9	0	38	4	67	5	96	9	125	8	154	21
10	0	39	3	68	7	97	7	126	17	155	15
11	1	40	3	69	7	98	7	127	10	156	23
12	0	41	3	70	9	99	9	128	9	157	12
13	0	42	5	71	6	100	7	129	13	158	19
14	1	43	3	72	5	101	8	130	17	159	17
15	1	44	4	73	5	102	15	131	11	160	17
16	0	45	3	74	8	103	8	132	19	161	15
17	1	46	5	75	5	104	11	133	11	162	16
18	0	47	4	76	8	105	13	134	16	163	13
19	1	48	3	77	7	106	12	135	13	164	19
20	1	49	1	78	11	107	9	136	15	165	21
21	1	50	2	79	6	108	10	137	11	166	20
22	2	51	5	80	7	109	8	138	21	167	14
23	2	52	5	81	4	110	15	139	11	168	25
24	1	53	4	82	9	111	11	140	19	169	8
25	0	54	4	83	7	112	11	141	15	170	23
26	2	55	5	84	11	113	9	142	17	171	17
27	1	56	5	85	7	114	17	143	13	172	20
28	2	57	5	86	10	115	11	144	13	173	14
29	2	58	6	87	9	116	13	145	13	174	27

This table contains all N for which $g \leq 13$ (the only value of N for which $g = 14$, other than those listed above, is 181).

76 Proposition. *Let Γ be the group $\Gamma_1(N)$ (with $N > 1$) and let other notation be as in proposition*

65. We have

$$\mu = \begin{cases} \frac{1}{2}N^2\prod_{p|N}(1-p^{-2}) & N>2 \\ 3 & N=2 \end{cases}$$

$$\nu_\infty = \begin{cases} \frac{1}{2}\sum_{i=1}^{N}\phi(\gcd(i,N)) & N\neq 4 \\ 3 & N=4 \end{cases}$$

$$\nu_2 = \begin{cases} 0 & N\neq 2 \\ 1 & N=2 \end{cases}$$

$$\nu_3 = \begin{cases} 0 & N\neq 3 \\ 1 & N=3 \end{cases}$$

Index: The group $\Gamma_1(N)/\Gamma(N)$ is isomorphic to the group of upper triangular unipotent matrics in $\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$ which is itself isomorphic to $\mathbb{Z}/N\mathbb{Z}$. We thus have

$$[\Gamma(1):\Gamma(N)] = N^3\prod_{p|N}(1-p^{-2}), \qquad [\Gamma_1(N):\Gamma(N)] = N$$

and so it follows that

$$[\Gamma(1):\Gamma_1(N)] = N^2\prod_{p|N}(1-p^{-2}).$$

Since -1 belongs to $\Gamma_1(N)$ if and only if $N=2$, we obtain the stated result.

Cusps: Again, in light of proposition 71 this is a simple counting exercise.

Elliptic points: Do this.

77. Here is a table of genera for $\Gamma_1(N)\backslash\mathfrak{H}^*$.

N	g	N	g	N	g	N	g	N	g	N	g
1	0	13	2	25	12	37	40	49	69	61	126
2	0	14	1	26	10	38	28	50	48	62	91
3	0	15	1	27	13	39	33	51	65	63	97
4	0	16	2	28	10	40	25	52	55	64	93
5	0	17	5	29	22	41	51	53	92	65	121
6	0	18	2	30	9	42	25	54	52	66	81
7	0	19	7	31	26	43	57	55	81	67	155
8	0	20	3	32	17	44	36	56	61	68	105
9	0	21	5	33	21	45	41	57	85	69	133
10	0	22	6	34	21	46	45	58	78	70	97
11	1	23	12	35	25	47	70	59	117	71	176
12	0	24	5	36	17	48	37	60	57	72	97

This table contains all N for which $g\leq 100$.

3.5 Abstact Hecke algebras

3.5.1 The general construction

78. Let G be a group. If Γ and Γ' are two subgroups of G then we write $\Gamma\sim\Gamma'$ if Γ and Γ' are commensurable. For a subgroup Γ of G we put

$$\widetilde{\Gamma} = \{\alpha\in G|\alpha\Gamma\alpha^{-1}\sim\Gamma\}.$$

Since commensurability is an equivalence relation, $\widetilde{\Gamma}$ is a group, called the *commensurator* of Γ (inside of G). Note that if Γ' is commensurable with Γ then their commensurators agree; in particular, Γ' is contained in $\widetilde{\Gamma}$.

79. For the following discussion we will fix a group G, a subgroup Γ and a family $\{\Gamma_\lambda\}$ indexed by λ in Λ of subgroups of G commensurable with Γ.

80. Let $R_{\lambda\mu}$ be the free $\mathbb{Z}$-module on the double cosets of $\Gamma_\lambda \alpha \Gamma_\mu$ as α varies in $\widetilde{\Gamma}_0$, that is to say,

$$R_{\lambda\mu} = \mathbb{Z}[\Gamma_\lambda \backslash \widetilde{\Gamma} / \Gamma_\mu].$$

If x is an element of $R_{\lambda\mu}$ which is just equal to a double coset with a coefficient of 1 then we call x a *double coset* of $R_{\lambda\mu}$

For a double coset X of $R_{\lambda\mu}$ we define the *degree* of X, denoted $\deg X$, to be the cardinality of $\Gamma_\lambda \backslash X$, *i.e.*, the number of right cosets of Γ_λ contained in X. We extended deg by linearity to all of $R_{\lambda\mu}$.

81. We now define a sort of dual module. Let $R^*_{\lambda\mu}$ be the $\mathbb{Z}$-module consisting of compactly supported functions $f : \Gamma_\lambda \backslash \widetilde{\Gamma} / \Gamma_\mu \to \mathbb{Z}$, *i.e.*, integer valued functions f on the double cosets which are zero on all but finitely many double cosets. We will alternatively think of elements of $R^*_{\lambda\mu}$ as functions on the set of double cosets or as functions on $\widetilde{\Gamma}$ which are left invariant under Γ_λ and right invariant under Γ_μ; this should not cause confusion.

For an element f of $R^*_{\lambda\mu}$ we define the *degree* of f by

$$\deg f = \sum_{y \in \Gamma_\lambda \backslash \widetilde{\Gamma}} f(y).$$

Note this sum is actually finite.

82 Proposition. *For a double coset X of $R_{\lambda\mu}$ let δ_X be the point mass at X (either the function on the double cosets $\Gamma_\lambda \backslash \widetilde{\Gamma} / \Gamma_\mu$ whose value on the double coset Y is 1 if $X = Y$ and 0 if $X \neq Y$ or the function on $\widetilde{\Gamma}$ whose value at α is 1 if $\alpha \in X$ and 0 if $\alpha \notin X$). Extend δ linearly to a map*

$$\delta : R_{\lambda\mu} \to R^*_{\lambda\mu}.$$

Then δ is a degree preserving isomorphism of $\mathbb{Z}$-modules.

It is clear that δ is injective. To prove that it is surjective, simply note that if f is any element of $R^*_{\lambda\mu}$ then $f(y) = \sum_X f(X)\delta_X(y)$, the sum being taken over all double cosets X. We now show that for a double coset X we have $\deg X = \deg \delta_X$; this will show that δ preserves degree. Write X as a disjoint union $\coprod_{i=1}^r \Gamma_\lambda \alpha_i$. By definition, the degree of X is r. On the other hand, if y belongs to $\Gamma_\lambda \backslash \widetilde{\Gamma}$ then $\delta_X(y)$ is either 1 or 0 according to whether y is one of the $\Gamma_\lambda \alpha_i$ or not. Thus $\deg \delta_X = r$ as well.

83. We now define a multiplication map

$$R_{\lambda\mu} \otimes R_{\mu\nu} \to R_{\lambda\nu}.$$

It suffices to define the map on double cosets; if X is a double coset of $R_{\lambda\mu}$ and Y is a double coset of $R_{\mu\nu}$ then we define the product of X and Y as

$$XY = \sum_Z m(X, Y; Z) Z$$

where the sum is over the cosets Z of $R_{\lambda\nu}$ and $m(X, Y; Z)$ is an integer, which we now define.

Let X and Y be as above and let Z be a double coset of $R_{\lambda\nu}$. For an element z of $\Gamma_\lambda \backslash Z$ (that is, a right coset of Γ_λ inside Z) let $A(z)$ be the set of y in $\Gamma_\mu \backslash Y$ such that z belongs to $\Gamma_\lambda \backslash Xy$. If z' is another element of $\Gamma_\lambda \backslash Z$ then there exists γ in Γ_ν such that $z' = z\gamma$; right multiplication by γ then induces an isomorphism $A(z) \to A(z')$. It follows that the cardinality of $A(z)$ is independent of z and depends only upon the double cosets X, Y and Z. We define $m(X, Y; Z)$ to be the common value of $\#A(z)$.

84. We also define a product

$$R^*_{\lambda\mu} \otimes R^*_{\mu\nu} \to R^*_{\lambda\nu}.$$

This is nothing other than convolution. To be precise, for f in $R^*_{\lambda\mu}$ and g in $R^*_{\mu\nu}$ we define the product of f and g as

$$(f \star g)(z) = \sum_{y \in \Gamma_\mu \backslash \widetilde{\Gamma}} f(zy^{-1})g(y).$$

Note that $f \star g$ is again bi-invariant and compactly supported.

85 Proposition. *The map δ preserves products, that is, for x in $R_{\lambda\mu}$ and y in $R_{\mu\nu}$ we have*

$$\delta_{\lambda\nu}(xy) = (\delta_{\lambda\mu}x) \star (\delta_{\nu\mu}y).$$

Of course if suffices to consider the case when $x = X$ and $y = Y$ are both double cosets. We then have, for any z in $\widetilde{\Gamma}$,

$$\delta_{XY}(z) = m(X, Y; \Gamma_\lambda z \Gamma_\nu) = \#\{y \in \Gamma_\mu \backslash Y \mid \Gamma_\lambda z \in \Gamma_\lambda \backslash Xy\}.$$

On the other hand,

$$(\delta_X \star \delta_Y)(z) = \sum_{y \in \Gamma_\mu \backslash \widetilde{\Gamma}} \delta_X(zy^{-1})\delta_Y(y) = \#\{y \in \Gamma_\mu \backslash Y \mid zy^{-1} \subset X\}.$$

Since the two conditions $z \in \Gamma_\lambda \backslash Xy$ and $zy^{-1} \subset X$ are clearly equivalent, the proposition is proved.

86. We thus see that there is really no difference between $R_{\lambda\mu}$ and $R^*_{\lambda\mu}$ and, for a given task, we can use whichever is more convenient. Often times, $R^*_{\lambda\mu}$ is easier to prove things with while $R_{\lambda\mu}$ tends to be a little more concrete.

87 Proposition (Shimura Prop. 3.3). *The degree is multiplicative under products, that is if x belongs to $R_{\lambda\mu}$ and let y belongs to $R_{\mu\nu}$ then*

$$\deg(xy) = \deg(x)\deg(y);$$

alternatively, if f belongs to $R^*_{\lambda\mu}$ and g belongs to $R^*_{\mu\nu}$ then

$$\deg(f \star g) = \deg(f)\deg(g).$$

The two phrasings of the statement are, of course, equivalent. We shall nonetheless prove each to demonstrate the flexibility of R^* over R.

We first prove the statement for R. It suffices to prove consider the case where $x = X$ and $y = Y$ are double cosets. Let Z belong to $\Gamma_\lambda \backslash \widetilde{\Gamma}_0 / \Gamma_\nu$. Let $\beta : \Gamma_\mu \backslash Y \to Y$ be a section, *i.e.*, choose coset representatives for Γ_μ in Y. Given y in $\Gamma_\mu \backslash Y$ put

$$B(y, Z) = \{x \in \Gamma_\lambda \backslash X \mid x\beta(y)\Gamma_\nu = Z\}.$$

Let $A(z, Z)$ be as above. We have

$$\begin{aligned} A(Z) &= \coprod_{z \in \Gamma_\lambda \backslash Z} A(z, Z) = \{(z, y) \in (\Gamma_\lambda \backslash Z) \times (\Gamma_\mu \backslash Y) \mid z \in \Gamma_\lambda \backslash Xy\} \\ B(Z) &= \coprod_{y \in \Gamma_\mu \backslash Y} B(y, Z) = \{(x, y) \in (\Gamma_\lambda \backslash X) \times (\Gamma_\mu \backslash Y) \mid x\beta(y)\Gamma_\nu = Z\} \end{aligned}$$

We have maps

$$\begin{aligned} A(Z) &\to B(Z), \qquad (z, y) \mapsto (z\beta(y)^{-1}, y) \\ B(Z) &\to A(Z), \qquad (x, y) \mapsto (x\beta(y), y). \end{aligned}$$

It is clear that these two maps are inverse to each other and so $A(Z)$ and $B(Z)$ are in bijective correspondence.

Now, the cardinality of $A(Z)$ is clearly $m(X,Y;Z)\deg Z$; thus this is the cardinality of $B(Z)$ as well. It follows that the cardinality of $\coprod_Z B(Z)$ is $\deg(XY)$. However,

$$\begin{aligned}\coprod_Z B(Z) &= \{(x,y,Z)\in(\Gamma_\lambda\backslash X)\times(\Gamma_\mu\backslash Y)\times(\Gamma_\lambda\backslash\widetilde{\Gamma}_0/\Gamma_\nu)\mid x\beta(y)\Gamma_\nu = Z\}\\ &\cong (\Gamma_\lambda\backslash X)\times(\Gamma_\mu\backslash Y)\end{aligned}$$

since for each x in $\Gamma_\lambda\backslash X$ and each y in $\Gamma_\mu\backslash Y$ there exists a unique double coset Z such that $x\beta(y)\Gamma_\nu = Z$. Thus the cardinality of $\coprod_Z B(Z)$ is equal to $\deg(X)\deg(Y)$ as well, proving the proposition.

We now prove the statement for R^*. Let f belong to $R^*_{\lambda\mu}$ and let g belong to $R^*_{\mu\nu}$. Then

$$\begin{aligned}\deg(f\star g) &= \sum_{z\in\Gamma_\lambda\backslash\widetilde{\Gamma}}(f\star g)(z) = \sum_{z\in\Gamma_\lambda\backslash\widetilde{\Gamma}}\sum_{y\in\Gamma_\mu\backslash\widetilde{\Gamma}} f(zy^{-1})g(y)\\ &= \sum_{y\in\Gamma_\mu\backslash\widetilde{\Gamma}}\sum_{z\in\Gamma_\lambda\backslash\widetilde{\Gamma}} f(zy^{-1})g(y) = \sum_{y\in\Gamma_\mu\backslash\widetilde{\Gamma}}\sum_{z\in\Gamma_\lambda\backslash\widetilde{\Gamma}} f(z)g(y)\\ &=\Bigg(\sum_{z\in\Gamma_\lambda\backslash\widetilde{\Gamma}} f(z)\Bigg)\Bigg(\sum_{y\in\Gamma_\mu\backslash\widetilde{\Gamma}} g(y)\Bigg) = \deg(f)\deg(g)\end{aligned}$$

and the statement is proved.

88 Proposition (Shimura Prop. 3.4). *The multiplication defined above is associative in the sense that if w belongs to $R_{\kappa\lambda}$, x belongs to $R_{\lambda\mu}$ and y belongs to $R_{\mu\nu}$ then $(wx)y = w(xy)$.*

This statement is much easier to prove for R^* than for R. Thus let f belong to $R^*_{\kappa\lambda}$, g belong to $R^*_{\lambda\mu}$ and h belong to $R^*_{\mu\nu}$. We have

$$((f\star g)\star h)(z) = \sum_{y\in\Gamma_\mu\backslash\widetilde{\Gamma}}(f\star g)(zy^{-1})h(y) = \sum_{y\in\Gamma_\mu\backslash\widetilde{\Gamma}}\sum_{x\in\Gamma_\lambda\backslash\widetilde{\Gamma}} f(zy^{-1}x^{-1})g(x)h(y)$$

while on the other hand

$$(f\star(g\star h))(z) = \sum_{x\in\Gamma_\lambda\backslash\widetilde{\Gamma}} f(zx^{-1})(g\star h)(x) = \sum_{x\in\Gamma_\lambda\backslash\widetilde{\Gamma}}\sum_{y\in\Gamma_\mu\backslash\widetilde{\Gamma}} f(zx^{-1})g(xy^{-1})h(y).$$

The two expressions are easily seen to be equivalent after a change of variables.

89. We now define the *Hecke ring* $R(\Gamma)$ (of Γ relative to G). Taken $\Gamma_\alpha = \Gamma_\lambda = \Gamma$ and put $R(\Gamma) = R_{\alpha\lambda}$, as above. Similarly put $R^*(\Gamma) = R^*_{\alpha\lambda}$. It is clear that $R(\Gamma)$ and $R^*(\Gamma)$ are associative rings.

Now let Δ be a monoid contained in $\widetilde{\Gamma}$ and containing Γ. We let $R(\Gamma,\Delta)$ be the submodule of $R(\Gamma)$ spanned by the double cosets $\Gamma\alpha\Gamma$ with α in Δ. We let $R^*(\Gamma,\Delta)$ be the corresponding submodule of $R^*(\Gamma)$; it consists of those functions f in $R^*(\Gamma)$ whose support is contained in Δ. Since Δ is closed under multiplication, it follows that both $R(\Gamma,\Delta)$ and $R^*(\Gamma,\Delta)$ form associative subrings of $R(\Gamma)$ and $R^*(\Gamma)$.

90 Proposition. *If G has an anti-automorphism $\alpha\mapsto\alpha^*$ such that $\Gamma^* = \Gamma$ and $X^* = X$ for every double coset X in $\Gamma\backslash\Delta/\Gamma$ then the Hecke ring $R(\Gamma,\Delta)$ is commutative.*

For an element f of $R^*(\Gamma,\Delta)$ define f^* to be the element of $R^*(\Gamma,\Delta)$ whose value at z is equal to $f(z^*)$. We then have

$$\begin{aligned}(f\star g)^*(z) &= \sum_{y\in\Gamma\backslash\widetilde{\Gamma}} f(z^*y^{-1})g(y) = \sum_{y\in\Gamma\backslash\widetilde{\Gamma}} f^*((y^*)^{-1}z)g^*(y^*) = \sum_{y\in\widetilde{\Gamma}/\Gamma} f^*(y^{-1}z)g^*(y)\\ &= \sum_{y\in\widetilde{\Gamma}/\Gamma} f^*(y^{-1})g^*(zy) = \sum_{y\in\Gamma\backslash\widetilde{\Gamma}} f^*(y)g^*(zy^{-1}) = (g^*\star f^*)(z)\end{aligned}$$

and so $(f\star g)^* = g^*\star f^*$. On the other hand, it is clear that for a double coset X we have $\delta_X^* = \delta_X$; since the functions of the form δ_X span $R^*(\Gamma,\Delta)$ it follows that $f^* = f$. Thus the identity map $f\mapsto f^*$ is an anti-automorphism and so $R^*(\Gamma,\Delta)$ is commutative.

3.5.2 The Hecke algebra for $\mathrm{SL}(2,\mathbb{Z})$

91. Let $G = \mathrm{GL}_2^+(\mathbb{Q})$ be the subgroup of $\mathrm{GL}_2(\mathbb{Q})$ consisting of those matrices which have positive determinant. Let Γ be the subgroup $\mathrm{SL}(2,\mathbb{Z})$ of G. Let Δ be the sub-monoid of G consisting of those matrices which have integer entries; thus Δ is the monoid of all 2×2 integer matrices with positive determinant.

92 Lemma (Shimura Lemma 2.9). *Let β belong to $M_2(\mathbb{Z})$ be of nonzero determinant b. Then $\Gamma(Nb)\subset\beta^{-1}\Gamma(N)\beta\cap\beta\Gamma(N)\beta^{-1}$.*

Let $\beta' = b\beta^{-1}$; note that β' has integer entries. If γ belongs to $\Gamma(Nb)$ then $\beta'\gamma\beta = \beta'\beta = b\,(\mathrm{mod}\ Nb)$; this shows that $\beta^{-1}\gamma\beta$ has integer entries and also that $\beta^{-1}\gamma\beta = 1\,(\mathrm{mod}\ N)$. We have therefore shown that $\beta^{-1}\gamma\beta$ belongs to $\Gamma(N)$, or in other words, γ belongs to $\beta\Gamma(N)\beta^{-1}$. A similar argument shows that γ belongs to $\beta^{-1}\Gamma(N)\beta^{-1}$. This completes the proof.

93 Lemma (Shimura Lemma 2.10). *We have $\widetilde{\Gamma} = G$; thus Δ is contained in $\widetilde{\Gamma}$ and contains Γ.*

Let α belong to G and write $\alpha = c\beta$ with $c\in\mathbb{Q}$ and $\beta\in\Delta$. We have $\alpha\Gamma\alpha^{-1} = \beta\Gamma\beta^{-1}$. If the determinant of β is b then by lemma 92 the group $\beta\Gamma\beta^{-1}$ contains $\Gamma(b)$. It follows that $[\Gamma:\Gamma\cap\alpha\Gamma\alpha^{-1}]$ is finite; conjugating by α and changing α to α^{-1} shows that $[\alpha\Gamma\alpha^{-1}:\Gamma\cap\alpha\Gamma\alpha^{-1}]$ is finite as well. Thus α belongs to $\widetilde{\Gamma}$, which proves the proposition.

94. Lemma 93 allows us to consider the Hecke algebra $R(\Gamma,\Delta)$, which we call *the* Hecke algebra of $\mathrm{SL}(2,\mathbb{Z})$. For positive integers a and b let $[a,b]$ denote the diagonal matrix $\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}$ and let $T(a,b)$ denote the corresponding double coset, considered as an element of the Hecke algebra. Note that $T(1,1)$ is the identity element of $R(\Gamma,\Delta)$. The main goal of this section is to prove the following theorem:

95 Theorem. *The Hecke algebra $R(\Gamma,\Delta)$ for $\mathrm{SL}(2,Z)$ is the commutative algebra freely generated by the elements $T(1,p)$ and $T(p,p)$ with p prime.*

96 Proposition (Shimura pg. 56). *We have the following:*

1. *Every double coset in $\Gamma\backslash\Delta/\Gamma$ has a representative of the form $[a,b]$ with b dividing a.*
2. *The elements $T(a,b)$ with b dividing a form a basis for the $\mathbb{Z}$-module $R(\Gamma,\Delta)$.*
3. *The ring $R(\Gamma,\Delta)$ is commutative.*

The first statement follows from the theory of elementary divisors. The second follows immediately from the first. The third follows from proposition 90 using transposition as the anti-automorphism.

97 Proposition (Shimura prop 3.17). *For any integers a, b and c we have $T(c,c)T(a,b) = T(ca,cb)$.*

We have

$$(\delta_{T(c,c)}\star\delta_{T(a,b)})(z) = \sum_{y\in\Gamma\backslash\Delta}\delta_{T(a,b)}(zy^{-1})\delta_{T(c,c)}(y) = \sum_{y\in\Gamma\backslash T(c,c)}\delta_{T(a,b)}(zy^{-1}).$$

Now, $T(c,c) = \Gamma[c,c]\Gamma = \Gamma[c,c]$ since $[c,c]$ is in the center of G. Thus the above sum has only one term, which we may take to be $y = [c,c]$. Thus

$$(\delta_{T(c,c)}\star\delta_{T(a,b)})(z) = \delta_{T(a,b)}(c^{-1}z) = \delta_{T(ca,cb)}(z)$$

and the proposition is proved.

98. Let $\mathbb{Q}^2$ be the two dimensional space of row vectors with rational entries. For our purposes, a *lattice* in $\mathbb{Q}^2$ is a free abelian subgroup of rank 2. We let L denote the standard lattice $\mathbb{Z}^2$. Note that if M is any lattice and α belongs to G then $M\alpha$ is also a lattice.

99 Proposition (Shimura Lemma 3.11). *Let M and N be two lattices in $\mathbb{Q}^2$. Then there exists a basis (u, v) of M and positive rational numbers a and b with b dividing a (i.e., $a \in b\mathbb{Z}$) such that (au, bv) is a basis of N.*

This is just the elementary divisors theorem, which can be proved in this case as follows. There exists an integer c_0 such that $c_0 N \subset M$. The abelian group $M/c_0 N$ is finite and has two generators; by the structure theorem for finite abelian groups we can therefore find a basis $(\bar{u}, \bar{v})$ of $M/c_0 N$ such that $\bar{u}$ has order a_0 and $\bar{v}$ has order b_0, with b_0 dividing a_0. Lift $(\bar{u}, \bar{v})$ to a basis (u, v) of M. Then (au, bv) is a basis for N, where $a = a_0/c_0$ and $b = b_0/c_0$.

100. If M and N are two lattices we put $\{M : N\} = (a, b)$ as in the proposition. Note that $N \subset M$ if and only if a and b are integers and in this case we have $[M : N] = ab$. Note that if $\alpha = [a, b]$ with b dividing a then $\{M : M\alpha\} = (a, b)$.

101 Lemma (Shimura Lemma 3.12). *Let M and N be two lattices in $\mathbb{Q}^2$. Then $\{L : M\} = \{L : N\}$ if and only if there exists an element α in Γ such that $M = N\alpha$.*

If $M = N\alpha$ with α in Γ then it is clear that we have $\{L : M\} = \{L : N\}$. We now prove the converse. Let $\{L : M\} = \{L : N\} = (a, b)$. We have bases (u_1, u_2) and (v_1, v_2) of L such that (au_1, bu_2) is a basis for M and (av_1, bv_2) is a basis for N. Define an element α of G by $v_i\alpha = u_i$ (or use $-u_1$ in place of v_1 to ensure that α has positive determinant). Then $L\alpha = L$, so that α belongs to Γ, and we have $M = N\alpha$.

102 Lemma (Shimura Lemma 3.13). *The map $\Gamma\xi \mapsto L\xi$ gives a bijective correspondence between the cosets $\Gamma\xi$ of $T(a, b)$ and the lattices M such that $\{L : M\} = (a, b)$.*

First note that for γ in Γ we have $L\gamma = L$ so that the map $\Gamma\xi \mapsto L\xi$ is well-defined. Now, if $\Gamma\xi$ is a coset of $T(a, b)$ then $L\xi$ clearly satisfies $\{L : L\xi\} = (a, b)$. On the other hand, if M is a lattice such that $\{L : M\} = (a, b)$ then, if $\alpha = [a, b]$, we have $\{L : M\} = \{L : L\alpha\}$ so that, by lemma 101, there exists an element γ of Γ such that $M = L\alpha\gamma$. Thus M comes from the cosets $\Gamma\alpha\gamma$ of $T(a, b)$. Finally we note that, because $\Gamma\alpha = \Gamma\beta$ if and only if $L\alpha = L\beta$ our correspondence is injective. Thus it is bijective, which establishes the proposition.

103 Proposition (Shimura Prop. 3.14). *The degree of $T(a, b)$ is equal to the number of lattices M for which $\{L : M\} = (a, b)$.*

This follows immedaitely from lemma 102.

104 Lemma (Shimura Prop. 3.15). *Let*

$$(\Gamma\alpha\Gamma)(\Gamma\beta\Gamma) = \sum c_\xi(\Gamma\xi\Gamma).$$

Then c_ξ is the number of lattices M for which $\{L : M\} = \{L : L\beta\}$ and $\{M : L\xi\} = \{L : L\alpha\}$.

Let $\Gamma\alpha\Gamma = \coprod \Gamma\alpha_i$ and let $\Gamma\beta\Gamma = \coprod \Gamma\beta_i$. Then

$$c_\xi = \#\{(i, j) \mid \Gamma\alpha_i\beta_j = \Gamma\xi\} = \#\{(i, j) \mid L\alpha_i\beta_j = L\xi\}.$$

Note that i is determined uniquely by j and ξ. Let S be the set of (i, j) for which $L\alpha_i\beta_j = \xi$ and let T be the set of lattices M for which $\{L : M\} = \{L : L\beta\}$ and $\{M : L\xi\} = \{L : L\alpha\}$. We establish a bijection between S and T.

Given (i, j) in S put $M = L\beta_j$. Then $\{L : M\} = \{L : L\beta\}$ and $\{M : L\xi\} = \{L\beta_j : L\alpha_i\beta_j\} = \{L : L\alpha_i\} = \{L : L\alpha\}$. Thus M belongs to T.

Now let a lattice M in T be given. Since $\{L : M\} = \{L : L\beta\}$ lemma 102 implies that there exists a unique j such that $M = L\beta_j$. Since $\{M : L\xi\} = \{L : L\alpha\}$ and $M = L\beta_j$ we see that $\{L : L\xi\beta_j^{-1}\} = \{L : L\alpha\}$. Thus another application of lemma 102 implies that there exists a unique i such that $L\xi\beta_j^{-1} = L\alpha_i$. Therefore $L\xi = L\alpha_i\beta_j$ and (i, j) belongs to S.

The two maps are easily seen to be inverses of each other.

105 Proposition (Shimura Prop. 3.16). *If α and β belong to Δ and* $\det\alpha$ *is coprime to* $\det\beta$ *then $(\Gamma\alpha\Gamma)(\Gamma\beta\Gamma) = \Gamma\alpha\beta\Gamma$ (in the Hecke ring). In other words, we have*

$$T(a, b)T(a', b') = T(aa', bb')$$

if a and a' are coprime (and b is taken to divide a and b' is taken to divide a').

Write

$$(\Gamma\alpha\Gamma)(\Gamma\beta\Gamma) = \sum c_\xi(\Gamma\xi\Gamma).$$

Note that by the original definition of multiplication in the Hecke algebra we know that c_ξ is nonzero if and only if ξ belongs to $\Gamma\alpha\Gamma\beta\Gamma$. Thus assume that ξ belongs to $\Gamma\alpha\Gamma\beta\Gamma$; we will use lemma 104 to determine the integers c_ξ.

Let M and M' be lattices such that $\{L : M\} = \{L : M'\} = \{L : L\beta\}$ and $\{M : L\xi\} = \{M' : L\xi\} = \{L : L\alpha\}$. We have

$$[M + M' : M] = [M' : M \cap M'] \tag{5}$$

Since $M + M' \subset L$, the left side of (5) divides $[L : M]$, which is equal to $\det\beta$. Since $L\xi$ is contained in both M and M', the right side of (5) divides $[M' : L\xi]$, which is equal to $[L : L\alpha]$, which is equal to $\det\alpha$. Thus both terms in (5) divide both $\det\alpha$ and $\det\beta$; since these numbers are coprime, it follows that both sides of (5) are equal to 1. Therefore we have $M + M' = M$ and $M \cap M' = M'$ and so $M = M'$. It now follows by lemma 104 that $c_\xi = 1$.

Now let M satisfy $\{L : M\} = \{L : L\beta\}$ and $\{M : L\xi\} = \{L : L\alpha\}$ (note that such an M exists by lemma 104, since we know $c_\xi = 1$). Since $\{L : M\}$ and $\{M : L\xi\}$ are tuples of integers, it follows that $L\xi \subset M \subset L$. We have an exact sequence

$$1 \longrightarrow M/L\xi \longrightarrow L/L\xi \longrightarrow L/M \longrightarrow 1$$

which splits since the orders of $M/L\xi$ and L/M are coprime (they are $\det\alpha$ and $\det\beta$ respectively). Thus $L/L\xi$ is isomorphic to $M/L\xi \oplus L/M$ which, in turn, is isomorphic to $L/L\alpha \oplus L/L\beta$. Therefore the elementary divisors of $L\xi$, relative to L, are completely determined by α and β.

Let $\Gamma\xi\Gamma$ and $\Gamma\xi'\Gamma$ be two double cosets contained in $\Gamma\alpha\Gamma\beta\Gamma$. By the previous paragraph, we have $\{L : L\xi\} = \{L : L\xi'\}$. Thus, by lemma 101, we have $L\xi = L\xi'\gamma$ for some γ in Γ and so $\xi = \gamma'\xi'\gamma$ for some γ' in Γ; it thus follows that $\Gamma\xi\Gamma = \Gamma\xi'\Gamma$. Therefore $\Gamma\alpha\Gamma\beta\Gamma$ contains only one double coset which clearly must be $\Gamma\alpha\beta\Gamma$. It follows that c_ξ is equal to 1 or 0 according to whether ξ belongs to $\Gamma\alpha\beta\Gamma$ or not.

106. Let R_p denote the subalgebra of $R(\Gamma, \Delta)$ generated by the $T(p^i, p^j)$ where i and j are arbitrary nonnegative integers (and p is a prime number). Proposition 105 implies that $R(\Gamma, \Delta)$ is generated by the R_p. In fact, it is easy to see that $R(\Gamma, \Delta)$ is the restricted tensor product of the R_p (with respect to the identity elements in R_p), that is to say, every element of $R(\Gamma, \Delta)$ can be written in an essentially unique way as a sum of terms of the form $a_1a_2\cdots a_n$ where a_i is an element of R_{p_i} and the p_i are prime numbers. Thus to determine the structure of $R(\Gamma, \Delta)$ it suffices to determine the structure of the R_p.

107 Proposition. *Let p be a prime number.*

1. *If $i > j$ then $T(p^i, 1)T(p^j, 1) = T(p^{i+j}, 1) + p^iT(p^i, p^j)$.*
2. *We have $T(p^i, 1)^2 = T(p^{2i}, 1) + p^i(1 + 1/p)T(p^i, p^i)$.*

Let i and j be integers such that $i \geq j$. Let $\alpha = [p^i, 1]$ and $\beta = [p^j, 1]$ so that $T(p^i, 1) = \Gamma\alpha\Gamma$ and $T(p^j, 1) = \Gamma\beta\Gamma$. Write

$$T(p^i, 1)T(p^j, 1) = \sum c_\xi(\Gamma\xi\Gamma).$$

We will use lemma 104 to determine the integers c_ξ.

Let M be a lattice such that $\{L : M\} = \{L : L\beta\} = (p^j, 1)$ and $\{M : L\xi\} = \{L : L\alpha\} = (p^i, 1)$. As in the proof of proposition 105 we have an exact sequence

$$1 \longrightarrow M/L\xi \longrightarrow L/L\xi \longrightarrow L/M \longrightarrow 1$$

although it is no longer split. Since $M/L\xi$ is isomorphic to $\mathbb{Z}/p^i\mathbb{Z}$ and L/M is isomorphic to $\mathbb{Z}/p^j\mathbb{Z}$ it follows that $L/L\xi$ must be isomorphic to either $\mathbb{Z}/p^{i+j}\mathbb{Z}$ or $\mathbb{Z}/p^i\mathbb{Z} \oplus \mathbb{Z}/p^j\mathbb{Z}$. Thus $\{L : L\xi\}$ is equal to either $(p^{i+j}, 1)$ or (p^i, p^j); thus ξ belongs to either $\Gamma[p^{i+j}, 1]\Gamma$ or $\Gamma[p^i, p^j]\Gamma$.

First consider $\xi = [p^{i+j}, 1]$. The integer c_ξ is the number of lattices M containing $L\xi$ such that $M/L\xi$ is isomorphic to $\mathbb{Z}/p^i\mathbb{Z}$ and L/M is isomorphic to $\mathbb{Z}/p^j\mathbb{Z}$. Clearly, this is the number of subgroups of $L/L\xi \cong \mathbb{Z}/p^{i+j}\mathbb{Z}$ which are isomorphic to $\mathbb{Z}/p^i\mathbb{Z}$. Since there is only one such subgroup we have $c_\xi = 1$.

Now let $\xi = [p^i, p^j]$. As above, the integer c_ξ may be identified with the number of subgroups of $\mathbb{Z}/p^i\mathbb{Z} \oplus \mathbb{Z}/p^j\mathbb{Z}$ which are isomorphic to $\mathbb{Z}/p^i\mathbb{Z}$. This simple counting exercise is left to the reader.

108 Proposition. *The ring R_p is generated by $T(p,1)$ and $T(p,p)$ and these elements are algebraically independent.*

First note that by proposition 97 we have, for $i \geq j$,

$$T(p^i, p^j) = T(p,p)^j T(p^{i-j}, 1)$$

and so R_p is generated by $T(p,p)$ and the $T(p^i, 1)$.

We now introduce a filtratation on R_p. Let $R_p^{(i)}$ be the submodule of R_p generated by the $T(p^j, p^k)$ with $|j-k| \leq i$. Essentially, $T(p,p)$ is given filtration level 0 and $T(p^i, 1)$ is given filtration level i. Proposition 105 implies that R_p, with this filtration, is a filtered ring, *i.e.*, that we have $R_p^{(i)} R_p^{(j)} \subset R_p^{(i+j)}$. In fact, for any i and j we have

$$T(p^i, 1)T(p^j, 1) = T(p^{i+j}, 1) \,(\mathrm{mod}\ R_p^{i+j-1})$$

and so, in the associated graded ring $\operatorname{gr} R_p$, we have

$$T(p,1)^i = T(p^i, 1).$$

This shows that $\operatorname{gr} R_p$, and thus R_p itself, is generated by $T(p,1)$ and $T(p,p)$.

Assume now that $T(p,1)$ and $T(p,p)$ satisfy a relation in R_p, *i.e.*, assume that there exists a polynomial $F(X,Y)$ with integer coefficients such that $F(T(p,1), T(p,p)) = 0$. Write $F(X,Y) = \sum_{i=0}^n F_i(Y) X^i$ with $F_n \neq 0$. In the associated graded ring $\operatorname{gr} R_p$ we then have $F_n(T(p,p))T(p,1)^n = 0$ so that $F_n(T(p,p))T(p^n, 1) = 0$. However, this implies that $F_n(T(p,p))T(p^n,1)$ (as an element of R_p) belongs to $R_p^{(n-1)}$ which is impossible: if we write $F_n(Y) = \sum_{i=0}^m c_i Y^i$ then

$$F_n(T(p,p))T(p^n,1) = \sum_{i=0}^m c_i T(p^{i+n}, p^i)$$

and the sum on the right clearly does not belong to $R_p^{(n-1)}$. This proves the proposition.

109. Proposition 108 together with the comments in article 106 prove theorem 95.

110. For a positive integer m we let $T(m)$ denote the sum of all double cosets of determinant m; more precisely, we define

$$T(m) = \sum_{\substack{ad=m \\ d|a}} T(a,d).$$

It is clear that if m and n are coprime then $T(mn) = T(m)T(n)$.

111 Proposition (Shimura Thm. 3.24). *We have the following:*

1. $\deg T(p^i, 1) = p^i(1+1/p)$*;*
2. $\deg T(p,p) = 1$*;*
3. *The degree of $T(m)$ is equal to the sum of the divisors of m.*

1) By proposition 103 the degree of $T(p^i,1)$ is equal to the number of lattices M such that $\{L : M\} = (p^i, 1)$. This is easily seen to be equal to the number of subgroups of $\mathbb{Z}/p^i\mathbb{Z} \oplus \mathbb{Z}/p^i\mathbb{Z}$ which are isomorphic to $\mathbb{Z}/p^i\mathbb{Z}$. This counting exercise has already been left to the reader.

2) We have $T(p,p) = \Gamma[p,p]\Gamma = \Gamma[p,p]$ since $[p,p]$ belongs to the center of Γ. Thus $T(p,p)$ contains only one right coset and is therefore of degree 1.

3) Since $T(mn) = T(m)T(n)$ if n and m are coprime and the same is true for the sum of the divisors function, it suffices to consider the case where m is a power of a prime, say p^n. We then have

$$T(p^n) = \sum_{0 \leq i \leq n/2} T(p^{n-i}, p^i) = \sum_{0 \leq i \leq n/2} T(p,p)^i T(p^{n-2i}, 1)$$

and so

$$\deg T(p^n) = \sum_{0 \leq i \leq n/2} (\deg T(p,p))^i (\deg T(p^{n-2i}, 1)) = \epsilon_n + \sum_{0 \leq i < n/2} p^{n-2i} + p^{n-2i-1}$$

where ϵ_n is 0 if n is odd and 1 if n is even. This sum is clearly equal to the sum of the divisors of p^n and the proposition is proved.

3.5.3 The Hecke algebra of certain congruence subgroups

112. Again let $G = \mathrm{GL}^+(2, \mathbb{Q})$. Let N be a fixed positive integer. Some definitions:

1. Let $\Delta(N)$ denote the sub-monoid of G consisting of those matrices with integer entries and with determinant coprime to N. Thus $\Delta(1)$ is the monoid Δ of the previous section.

2. Let $\Delta^*(N)$ denote the sub-monoid of $\Delta(N)$ consisting of those matrices which are congruent to $\begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$ modulo N, for some element x of $(\mathbb{Z}/N\mathbb{Z})^*$.

3. If Γ is a group intermediate to $\Gamma(N)$ and $\Gamma(1)$ then let $\Phi = \Phi(\Gamma, N)$ denote the sub-monoid of $\Delta(N)$ consisting of those matrices which normalize Γ modulo N, *i.e.*, matrices α for which $\alpha\Gamma = \Gamma\alpha \pmod{N}$.

113. For the purposes of this section, a *congruence datum* is a triple $(N, t, \mathfrak{h})$ where N is a positive integer, t is a positive divisor of N and $\mathfrak{h}$ is a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$. We usually denote a congruence datum by the letter Θ. Given a congruence datum $\Theta = (N, t, \mathfrak{h})$, define

$$\Delta(\Theta) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Delta(1) \,\middle|\, \begin{array}{ll} a \in \mathfrak{h}, & b = 0 \pmod{t} \\ c = 0 \pmod{N}, & (d, N) = 1 \end{array} \right\}$$
$$\Delta'(\Theta) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Delta(1) \,\middle|\, \begin{array}{ll} a \in \mathfrak{h}, & b = 0 \pmod{t} \\ c = 0 \pmod{N} & \end{array} \right\}$$
$$\Gamma(\Theta) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1) \,\middle|\, \begin{array}{ll} a \in \mathfrak{h}, & b = 0 \pmod{t} \\ c = 0 \pmod{N}, & (d, N) = 1 \end{array} \right\}$$
$$\Phi(\Theta) = \Phi(\Gamma(\Theta), N)$$

Some comments:

1. The condition $(d, N) = 1$ in the definition of $\Gamma(\Theta)$ is superfluous: it is implied by $c = 0 \pmod{N}$.

2. We have
$$\Gamma(N) \subset \Gamma(\Theta) \subset \Gamma(1).$$
The group $\Gamma(\Theta)$ is thus commensurable with $\Gamma(1)$ and its commensurator is therefore G (*cf.* lemma 93).

3. The groups $\Gamma(N)$, $\Gamma_0(N)$ and $\Gamma_1(N)$ are all of the form $\Gamma(\Theta)$ for an appropriate choice of Θ. There are, however, groups intermediate to $\Gamma(N)$ and $\Gamma(1)$ which are not conjugate to any $\Gamma(\Theta)$.

4. We have
$$\Gamma(\Theta) \subset \Delta(\Theta) \subset \Delta'(\Theta) \subset \Delta(1)$$
It therefore makes sense to speak of the Hecke rings $R(\Gamma(\Theta), \Delta(\Theta))$ and $R(\Gamma(\Theta), \Delta'(\Theta))$. These will be our primary objects of study for this section.

5. We have
$$\Delta(\Theta) = \Delta^*(N)\Gamma(\Theta) = \Gamma(\Theta)\Delta^*(N).$$
It thus follows that $R(\Gamma(\Theta), \Delta(\Theta))$ is generated by the $\Gamma(\Theta)\alpha\Gamma(\Theta)$ with α in $\Delta^*(N)$.

6. Note that there is some notational discrepancy between us and Shimura. Where we write $\Delta(\Theta)$ he writes Δ'_N and where we write $\Delta'(\Theta)$ he writes Δ'.

114. The purpose of this section is to determine the structure of the rings $R(\Gamma(\Theta), \Delta(\Theta))$ and $R(\Gamma(\Theta), \Delta'(\Theta))$. We cannot state the full result in a meaningful way at the moment, so we delay the precise statements til later in the section; suffice it to say that boh rings are polynomial rings in an infinite number of variables and subquotients of $R(\Gamma(1), \Delta(1))$.

115 Lemma (Shimura Lemma 3.28). *Let a and b be positive integers with greatest common divisor c. Then $\Gamma(c) = \Gamma(a)\Gamma(b)$.*

The inclusion $\Gamma(a)\Gamma(b) \subset \Gamma(c)$ is immediate; we prove the other inclusion. Thus let α be a given element of $\Gamma(c)$. By the Chinese remainder theorem, there exists an element β of $M_2(\mathbb{Z})$ such that $\beta = 1 \,(\mathrm{mod}\ a)$ and $\beta = \alpha \,(\mathrm{mod}\ b)$. We have $\det \beta = 1 \,(\mathrm{mod}\ ab/c)$. We can thus find an element γ of $\Gamma(1)$ such that $\gamma = \beta \,(\mathrm{mod}\ ab/c)$ (*cf.* proposition 68). Then γ belongs to $\Gamma(a)$ and $\gamma^{-1}\alpha$ belongs to $\Gamma(b)$ so that $\alpha = \gamma \cdot \gamma^{-1}\alpha$ belongs to $\Gamma(a)\Gamma(b)$.

116 Lemma (Shimura Lemma 3.29). *Let $\Gamma = \Gamma(1)$, let N be a positive integer, let Γ' be a group intermediate to $\Gamma(N)$ and $\Gamma(1)$ and let $\Phi = \Phi(\Gamma, N)$. Then we have the following:*

1. *If α belongs to $\Delta(N)$ then $\Gamma\alpha\Gamma = \Gamma\alpha\Gamma' = \Gamma'\alpha\Gamma$.*
2. *If α belongs to Φ then $\Gamma'\alpha\Gamma' = \Gamma\alpha\Gamma \cap \{\xi \in \Delta(1) \,|\, \xi \in \Gamma'\alpha \,(mod\ N)\}$.*
3. *If α belongs to Φ then $\Gamma'\alpha\Gamma' = \Gamma'\alpha\Gamma(N) = \Gamma(N)\alpha\Gamma'$.*
4. *If α and β belong to $\Delta(N)$ then $\Gamma(N)\alpha\Gamma(N) = \Gamma(N)\beta\Gamma(N)$ if and only if $\Gamma\alpha\Gamma = \Gamma\beta\Gamma$ and $\alpha = \beta \,(mod\ N)$.*
5. *If α belongs to Φ then the canonical map*
$$\Gamma'\backslash\Gamma'\alpha\Gamma' \to \Gamma\backslash\Gamma\alpha\Gamma$$
is a bijection. In other words, if α_i are representatives for the right cosets of Γ' in $\Gamma'\alpha\Gamma'$ then they are also representatives for the right cosets of Γ in $\Gamma\alpha\Gamma$.

1) Let a be the determinant of α. By lemma 115 we have $\Gamma = \Gamma(a)\Gamma(N)$ and so $\alpha^{-1}\Gamma\alpha\Gamma = \alpha^{-1}\Gamma\alpha\Gamma(a)\Gamma(N)$. However, by lemma 92, the group $\Gamma(a)$ is contained in $\alpha^{-1}\Gamma\alpha$ so that $\alpha^{-1}\Gamma\alpha\Gamma$ is contained in $\alpha^{-1}\Gamma\alpha\Gamma(N)$. Thus
$$\Gamma\alpha\Gamma \subset \Gamma\alpha\Gamma(N) \subset \Gamma\alpha\Gamma'.$$
The inclusion $\Gamma\alpha\Gamma' \subset \Gamma\alpha\Gamma$ is immediate; the statement is thus proved.

2) Let ξ belong to $\Gamma\alpha\Gamma$ be such that $\xi \in \Gamma'\alpha \,(\mathrm{mod}\ N)$. Thus $\xi = \gamma\alpha \,(\mathrm{mod}\ N)$ with γ in Γ'. On the other hand, the first part of the present proposition, applied with $\Gamma' = \Gamma(N)$, implies that $\Gamma\alpha\Gamma = \Gamma\alpha\Gamma(N)$ so that we can write $\xi = \delta\alpha\epsilon$ with δ in Γ and ϵ in $\Gamma(N)$. Since $\epsilon = 1 \,(\mathrm{mod}\ N)$ and α is invertible modulo N it follows that $\gamma = \delta \,(\mathrm{mod}\ N)$. Since Γ' contains $\Gamma(N)$ and γ belongs to Γ' it thus follows that δ belongs to Γ. We have thus shown that ξ belongs to $\Gamma'\alpha\Gamma(N)$, which, of course, is contained in $\Gamma'\alpha\Gamma'$ (we shall use the stronger statement that ξ belongs to $\Gamma'\alpha\Gamma(N)$ below).

We now prove converse. Let ξ belong to $\Gamma'\alpha\Gamma'$. Clearly ξ belongs to $\Gamma\alpha\Gamma$ and, by the definition of Φ, we see immediatley that $\xi \in \Gamma'\alpha \,(\mathrm{mod}\ N)$. We have thus proved the second statement.

3) In the course of proving the second statement, we showed that $\Gamma'\alpha\Gamma'$ is contained in $\Gamma'\alpha\Gamma(N)$. The opposite inclusion is obvious.

4) This follows from the second assertion upon taking $\Gamma' = \Gamma(N)$ (note then that $\Phi = \Delta(N)$).

5) Let α belong to Φ and let $\Gamma'\alpha\Gamma' = \coprod \Gamma'\alpha_i$. Then $\Gamma\alpha\Gamma = \Gamma\alpha\Gamma' = \cup\Gamma\alpha_i$; it thus suffices to show that this is a disjoint union. Assume $\Gamma\alpha_i = \Gamma\alpha_j$ so that $\alpha_i = \gamma\alpha_j$ with γ in Γ. Since α_i and α_j belong to the same Γ' double coset we have an expression $\alpha_i = \delta_1\alpha_j\delta_2$ with δ_1 and δ_2 in Γ'; reducing this equation modulo N and using the fact that α_j belongs to Φ, and thus normalizes Γ' modulo N we get an expression $\alpha_i = \delta\alpha_j \,(\mathrm{mod}\ N)$. It follows that $\delta = \gamma \,(\mathrm{mod}\ N)$ and so, since Γ' contains $\Gamma(N)$, we find that γ belongs to Γ'. Thus $\Gamma\alpha_i = \Gamma\alpha_j$. This proves the proposition.

117 Proposition (Shimura Prop. 3.30). *Keep the same notation as in lemma 116. Then the map $R(\Gamma', \Phi) \to R(\Gamma, \Delta)$ which takes $\Gamma'\alpha\Gamma'$ to $\Gamma\alpha\Gamma$ (and is extended additively) is a ring homomorphism (here we have written $\Delta = \Delta(1)$).*

Let α and β belong to Φ, let $\Gamma'\alpha\Gamma' = \coprod \Gamma'\alpha_i$, let $\Gamma'\beta\Gamma' = \coprod \Gamma'\beta_j$ and let $\Gamma'\alpha\Gamma'\beta\Gamma' = \coprod \Gamma'\xi_k\Gamma'$. We have
$$(\Gamma'\alpha\Gamma')(\Gamma'\beta\Gamma') = \sum c'_k(\Gamma'\xi_k\Gamma'), \qquad c'_k = \{(i,j) \,|\, \Gamma'\alpha_i\beta_j = \Gamma'\xi_k\}.$$
By part 5 of lemma 116 we have that $\Gamma\alpha\Gamma = \coprod \Gamma\alpha_i$ and $\Gamma\beta\Gamma = \coprod \Gamma\beta_j$. Using lemma 116 again, we have
$$\Gamma\alpha\Gamma\beta\Gamma = \Gamma\alpha\Gamma\beta\Gamma' = \Gamma\alpha\Gamma'\beta\Gamma' = \Gamma \cdot \Gamma'\alpha\Gamma'\beta\Gamma' = \Gamma \cdot \coprod \Gamma'\xi_k\Gamma' = \bigcup \Gamma\xi_k\Gamma'.$$

We now show that the final union above is in fact disjoint. Since ξ_k belongs to $\Gamma'\alpha\Gamma'\beta\Gamma'$ and α and β belong to Φ we have $\xi_k \in \Gamma'\alpha\beta \pmod N$. Thus, using lemma 116 once again, we have

$$\Gamma'\xi_k\Gamma' = \{\eta \in \Gamma\xi_k\Gamma \mid \eta \in \Gamma'\xi_k \pmod N\} = \{\eta \in \Gamma\xi_k\Gamma \mid \eta \in \Gamma'\alpha\beta \pmod N\}.$$

It follows that if $\Gamma\xi_k\Gamma = \Gamma\xi_\ell\Gamma$ then $\Gamma'\xi_k\Gamma' = \Gamma'\xi_\ell\Gamma'$ so that $k = \ell$. We thus have shown that the union is disjoint, *i.e.*, that we have

$$\Gamma\alpha\Gamma\beta\Gamma = \coprod \Gamma\xi_k\Gamma.$$

From the above paragraph, it follows that we can write

$$(\Gamma\alpha\Gamma)(\Gamma\beta\Gamma) = \sum c_k(\Gamma\xi\Gamma), \qquad c_k = \{(i,j) \mid \Gamma\alpha_i\beta_j = \Gamma\xi\}.$$

To prove the proposition we must show $c_k = c'_k$. Clearly it suffices to show that $\Gamma'\alpha_i\beta_j = \Gamma'\xi_k$ if and only if $\Gamma\alpha_i\beta_j = \Gamma\xi_k$; to do this it suffices to show that $\Gamma\alpha_i\beta_j = \Gamma\xi_k$ implies that $\Gamma'\alpha_i\beta_j = \Gamma\xi_k$, the other implication being immediate. Thus let $\Gamma\alpha_i\beta_j = \Gamma\xi_k$. Then $\xi_k = \gamma\alpha_i\beta_j$ with γ in Γ. On the other hand, $\xi_k \in \Gamma'\alpha_i\beta_j \pmod N$ so that $\xi_k = \delta\alpha_i\beta_j \pmod N$ with δ in Γ'. It follows that $\gamma = \delta \pmod N$, and thus, since Γ' contains $\Gamma(N)$, we see that γ belongs to Γ'. Therefore $\Gamma'\alpha_i\beta_j = \Gamma'\xi_k$ and the proposition is proved.

118 Proposition (Shimura Prop. 3.31). *Let $\Gamma = \Gamma(1)$ and let $\Gamma' = \Gamma(\Theta)$ where $\Theta = (N, t, \mathfrak{h})$ is a congruence datum. Then the map $R(\Gamma', \Delta(\Theta)) \to R(\Gamma, \Delta(N))$ of proposition 117 is an isomorphism.*

We must show the map is injective and surjective. We begin with surjective. Let η belong to $\Delta(N)$. It suffices to find an element η' of $\Delta(\Theta)$ such that $\Gamma\eta\Gamma = \Gamma\eta'\Gamma$. Thus let $b = \det\eta$, let c be an integer such that $bc = 1 \pmod N$, and let $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$. Then $\det\eta\alpha = 1 \pmod N$ so that there exists an element γ of Γ such that $\eta\alpha = \gamma \pmod N$ (*cf.* proposition 68). We have $\gamma^{-1}\eta = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \pmod N$ so that $\gamma^{-1}\eta \in \Delta^*(N)$. Furthermore, $\Gamma\gamma^{-1}\eta\Gamma = \Gamma\eta\Gamma$, which proves the surjectivity (since $\Delta(\Theta)$ contains $\Delta^*(N)$).

We now prove injectivity. Let α and β belong to $\Delta^*(N)$ and let $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \pmod N$ and let $\beta = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \pmod N$. If $\Gamma\alpha\Gamma = \Gamma\beta\Gamma$ then $a = \det\alpha = \det\beta = b$ so that $\alpha = \beta \pmod N$. It then follows from lemma 116 that $\Gamma'\alpha\Gamma' = \Gamma'\beta\Gamma'$. This proves that the map is injective since $R(\Gamma', \Delta(\Theta))$ is generated by the $\Gamma'\alpha\Gamma'$ with α in $\Delta^*(N)$.

119. Note that $R(\Gamma, \Delta(N))$ is the subring of $R(\Gamma, \Delta(1))$ generated by those double cosets $\Gamma\alpha\Gamma$ for which the elementary divisors of α are coprime to N. Thus it is the polynomial ring on $T(1,p)$ and $T(p,p)$ for primes p not divising N.

120. Let G_p denote the group $\mathrm{GL}(2, \mathbb{Z}/p\mathbb{Z})$. For α in $\Delta(1)$ the double coset $G_p\alpha G_p$ is completely determined by the p-part of the elementary divisors of α and conversely.

121. Following Shimura, for two positive integers m and N we write $m \mid N^\infty$ if all the prime factors of m divide N. Thus any integer can be written uniquely as mq with $m \mid N^\infty$ and $(q, N) = 1$.

122 Proposition (Shimura Prop. 3.32, 3.33). *Let Θ be a congruence datum, let $\Gamma' = \Gamma(\Theta)$ and let $\Delta' = \Delta'(\Theta)$. Let α belong to Δ' and write $\det\alpha = mq$ with $m \mid N^\infty$ and $(q, N) = 1$. Then we have the following:*

1. *We have $\Gamma'\alpha\Gamma' = \{\beta \in \Delta' \mid \det\beta = mq$ and $G_p\beta G_p = G_p\alpha G_p$ for all primes $p \mid q\}$.*
2. *There exists an element ξ of $\Delta^*(N)$ such that $\det\xi = q$ and $G_p\xi G_p = G_p\alpha G_p$ for all prime factors p of q.*
3. *If $q = 1$ then*

$$\Gamma'\alpha\Gamma' = \{\beta \in \Delta' \mid \det\beta = m\} = \coprod_{r=0}^{m-1} \Gamma' \begin{bmatrix} 1 & tr \\ 0 & m \end{bmatrix}.$$

4. If ξ is as above and $\eta = \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix}$ then

$$\Gamma'\alpha\Gamma' = (\Gamma'\xi\Gamma')(\Gamma'\eta\Gamma') = (\Gamma'\eta\Gamma')(\Gamma'\xi\Gamma')$$

holds in $R(\Gamma', \Delta')$.

Let $X(\alpha)$ be the set on the right hand side of the first statement. Let $Y(\alpha)$ be the set elements as in the second statement of the proposition, *i.e.*, the set of ξ in $\Delta^*(N)$ such that $\det \xi = q$ and $G_p\xi G_p = G_p\alpha G_p$ for all primes p dividing q. Throughout η will denote the matrix $\begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix}$. Here is an outline of the proof:

1. For each β in $X(\alpha)$ we will produce an element ξ of $Y(\alpha)$ such that β belongs to $\Gamma'\xi\eta\Gamma'$. This, of course, is a stronger statement than the second part of the proposition which merely asserts that $Y(\alpha)$ is not empty.

2. We show that for any choice of ξ in $Y(\alpha)$ the resulting double coset $\Gamma'\xi\eta\Gamma'$ is the same.

3. We show that this common coset is $\Gamma'\alpha\Gamma'$, *i.e.*, we show that for and ξ in $Y(\alpha)$ we have $\Gamma'\alpha\Gamma' = \Gamma'\xi\eta\Gamma'$. From this we conclude the first part of the proposition.

4. We prove the third part of the proposition.

5. We prove the fourth part of the proposition.

Step 1. Let β be an element of $X(\alpha)$. Let a be the top left entry of β. Since a belongs to $\mathfrak{h}$ it is coprime to mN and we can find an integer e such that $ae = 1 \,(\mathrm{mod}\ mN)$. We can therefore find an element γ of Γ such that $\gamma = \begin{bmatrix} e & 0 \\ 0 & a \end{bmatrix} (\mathrm{mod}\ mN)$ (*cf.* proposition 68). Since e is the inverse of a modulo N it also belongs to $\mathfrak{h}$ so that γ belongs to Γ'. We have $\gamma\beta = \begin{bmatrix} 1 & tb \\ fN & * \end{bmatrix} (\mathrm{mod}\ mN)$ for some integers b and f. Put $\delta = \begin{bmatrix} 1 & 0 \\ -fN & 1 \end{bmatrix}$; this is an element of Γ'. We have $\delta\gamma\beta = \begin{bmatrix} 1 & tb \\ 0 & g \end{bmatrix} (\mathrm{mod}\ N)$ for some integer g. Taking determinants gives $mq = g \,(\mathrm{mod}\ mN)$ so that $\delta\gamma\beta = \begin{bmatrix} 1 & tb \\ 0 & mq \end{bmatrix} (\mathrm{mod}\ mN)$. Put $\epsilon = \begin{bmatrix} 1 & tb \\ 0 & 1 \end{bmatrix}$ and $\xi = \delta\gamma\beta\epsilon^{-1}\eta^{-1}$. Then ξ has integer entries, $\det \xi = q$ and $\xi = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix} (\mathrm{mod}\ N)$ so that ξ belongs to $\Delta^*(N)$. Furthermore, it is clear that β belongs to $\Gamma'\xi\eta\Gamma'$. If p is a prime dividing q, so that η belongs to G_p then we have $G_p\xi G_p = G_p\beta G_p = G_p\alpha G_p$. Therefore ξ belongs to $Y(\alpha)$. This completes the first step.

Step 2. Let ξ and ξ' be two elements of $Y(\alpha)$. Since ξ and ξ' have determinant q and $G_p\xi G_p = G_p\xi' G_p$ for all p dividing q, it follows that ξ and ξ' have the same elementary divisors and so $\Gamma\xi\Gamma = \Gamma\xi'\Gamma$. Since $\xi = \xi' \,(\mathrm{mod}\ N)$ it follows from lemma 116 that $\Gamma(N)\xi\Gamma(N) = \Gamma(N)\xi'\Gamma(N)$ and so we can write $\xi' = \phi\xi\psi$ with ϕ and ψ in $\Gamma(N)$.

By the Chinese remainder theorem we can find a matrix θ' with coefficients taken modulo mqN such that

$$\theta' = 1 \,(\mathrm{mod}\ mN), \qquad \theta' = \eta^{-1}\psi^{-1}\eta \,(\mathrm{mod}\ q).$$

Such a matrix necessarily has determinant 1 and so belongs to $\mathrm{SL}(2, \mathbb{Z}/mqN\mathbb{Z})$. We can therefore lift θ' to an element θ of $\mathrm{SL}(2,\mathbb{Z})$; in fact, θ will belong to $\Gamma(N)$. Put $\omega = \xi\psi\eta\theta\eta^{-1}\xi^{-1}$. We then have

$$\omega = 1 \,(\mathrm{mod}\ N), \qquad \omega = 1 \,(\mathrm{mod}\ q).$$

Since ξ, ψ, η and θ are all integer matrices, the entries of ω are rational; the denominators of its entries must divide $\det(\xi\eta) = mq$. However, any prime dividing mq divides either N or q and thus, by the above, cannot occur in the denominator of any entry of ω. Thus ω is an integer matrix and belongs to $\Gamma(N)$. Since $\xi\psi\eta = \omega\xi\eta\theta^{-1}$ we have

$$\Gamma'\xi'\eta\Gamma' = \Gamma'\xi\psi\eta\Gamma' = \Gamma'\xi\eta\Gamma'.$$

This completes the second step.

Step 3. Let ξ be a fixed element of $Y(\alpha)$. Given any β in $X(\alpha)$ we can, by step 1, find an element ξ' of $Y(\alpha)$ such that β is contained in $\Gamma'\xi'\eta\Gamma'$. However, by step 2 we have $\Gamma'\xi'\eta\Gamma' = \Gamma'\xi\eta\Gamma'$. It follows that β is contained in $\Gamma'\xi\eta\Gamma'$ for any choicde of β. We have therefore shown the second inclusion below (the first inclusion is immediate)

$$\Gamma'\alpha\Gamma' \subset X(\alpha) \subset \Gamma'\xi\eta\Gamma'.$$

The leftmost and rightmost terms are double cosets which intersect; they must therefore be equal. Thus all three terms above are in fact equal. Note that this proves the first part of the proposition. This completes the third step.

Step 4. Under the hypothesis $q = 1$ we must prove

$$\Gamma'\alpha\Gamma' = \{\beta \in \Delta' \mid \det\beta = m\} = \coprod_{r=0}^{m-1} \Gamma' \begin{bmatrix} 1 & tr \\ 0 & m \end{bmatrix}. \tag{6}$$

The first equality follows immediately from the first part of the proposition. It is furthermore clear that the third set is contained in the second one. We now show that the second is contained in the third.

Thus let β be a given element of Δ' of determinant m. Let ξ be the element of $Y(\alpha)$ constructed from β in step 1; recall that ξ is defined as $\delta\gamma\beta\epsilon^{-1}\eta^{-1}$ where δ and γ belong to Γ' and $\epsilon = \begin{bmatrix} 1 & tb \\ 0 & 1 \end{bmatrix}$ for some integer b. It follows that $\delta\gamma\beta = \xi \begin{bmatrix} 1 & tb \\ 0 & m \end{bmatrix}$. Note that since $q = 1$ we have that ξ belongs to $\Gamma(N)$. We have therefore shown that β belongs to $\Gamma' \begin{bmatrix} 1 & tb \\ 0 & m \end{bmatrix}$. Now, we can write $b = mh + r$ with $0 \le r < m$; we then have

$$\begin{bmatrix} 1 & tb \\ 0 & m \end{bmatrix} = \begin{bmatrix} 1 & th \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & tr \\ 0 & m \end{bmatrix}.$$

As the first matrix on the right hand side belongs to Γ' we see that β belongs to $\Gamma' \begin{bmatrix} 1 & tr \\ 0 & m \end{bmatrix}$, where now r is in the correct range. This proves that the second set in (6) is contained in the third set.

We must now prove that the union in (6) is disjoint. Assume the union is not disjoint, *i.e.*, we can find $\gamma = \begin{bmatrix} a & tb \\ c & d \end{bmatrix}$ in Γ' and integers $0 \le r < s \le m-1$ such that $\begin{bmatrix} 1 & tr \\ 0 & m \end{bmatrix} = \gamma \begin{bmatrix} 1 & ts \\ 0 & m \end{bmatrix}$. We then have $\begin{bmatrix} 1 & tr \\ 0 & m \end{bmatrix} = \begin{bmatrix} a & t(as+bm) \\ c & cts+dm \end{bmatrix}$ so that $\gamma = 1$. This completes the proof of the third part of the proposition.

Step 5. Fix an element ξ of $Y(\alpha)$. It follows immediately from the definition of $X(\alpha)$ that both of the sets $\Gamma'\xi\Gamma'\eta\Gamma'$ and $\Gamma'\eta\Gamma'\xi\Gamma'$ are contained in $X(\alpha)$. We therefore have

$$\Gamma'\alpha\Gamma' = \Gamma'\xi\Gamma'\eta\Gamma' = \Gamma'\eta\Gamma'\xi\Gamma'$$

and so it follows that

$$(\Gamma'\xi\Gamma')(\Gamma'\eta\Gamma') = n(\Gamma'\alpha\Gamma'), \qquad (\Gamma'\eta\Gamma')(\Gamma'\xi\Gamma') = n'(\Gamma'\alpha\Gamma') \tag{7}$$

for two positive integers n and n'. We must show $n = n' = 1$.

Let k be large enough so that $m \mid N^k$. We first show that we can find $\zeta = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix} \pmod{tN^k}$ such that $\Gamma'\xi\Gamma' = \Gamma'\zeta\Gamma'$. Consider the congruence datum $\Theta = (M, 1, (\mathbb{Z}/tN^k)^\times)$. Note that every double coset $\Gamma(\Theta)\alpha\Gamma(\theta)$ with α in $\Delta(\Theta)$ has a representative in $\Delta^*(tN^k)$. Proposition 118 asserts that the map $R(\Gamma(\Theta), \Delta(\Theta)) \to R(\Gamma, \Delta(N))$ is an isomorphism (since $\Delta(tN^k) = \Delta(N)$). It thus follows that every double coset $\Gamma\alpha\Gamma$ with α in $\Delta(N)$ has a representative from $\Delta^*(tN^k)$. In particular, we can find ζ in $\Delta^*(tN^k)$ such that $\Gamma\xi\Gamma = \Gamma\zeta\Gamma$. Note that $\det\zeta = \det\xi = q$ and so it follows that $\zeta = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix} \pmod{tN^k}$. In particular, $\zeta = \xi \pmod N$. It thus follows from lemma 116 that $\Gamma(N)\zeta\Gamma(N) = \Gamma(N)\xi\Gamma(N)$. From this we conclude $\Gamma'\zeta\Gamma' = \Gamma'\xi\Gamma'$.

We now show $n = 1$. Write $\Gamma'\zeta\Gamma' = \coprod \Gamma'\zeta\epsilon_i$ with ϵ_i in Γ', taken so that $\epsilon_1 = 1$. Write $\Gamma'\eta\Gamma' = \coprod_{j=0}^{m-1} \Gamma'\eta_j$, where $\eta_j = \begin{bmatrix} 1 & tj \\ 0 & m \end{bmatrix}$, according to part 4 of this proposition. Since $\Gamma'\alpha\Gamma' = \Gamma'\zeta\Gamma'\eta\Gamma' =$

$\Gamma'\zeta\eta\Gamma'$ it follows that n is the multiplicity of $\Gamma'\zeta\eta\Gamma'$ in the product $(\Gamma'\zeta\Gamma')(\Gamma'\eta\Gamma')$. Thus, by definition, we have

$$n = \#\{(i,j) \mid \Gamma'\zeta\eta = \Gamma'\zeta\epsilon_i\eta_j\}.$$

Assume we have i and j such that $\Gamma'\zeta\eta = \Gamma'\zeta\epsilon_i\eta_j$ so that $\gamma\zeta\eta = \zeta\epsilon_i\eta_j$ for some γ in Γ'. Write $\gamma = \begin{bmatrix} * & tb \\ * & * \end{bmatrix}$ and $\epsilon_i = \begin{bmatrix} u & tv \\ * & * \end{bmatrix}$. We then have

$$\gamma\zeta\eta = \begin{bmatrix} * & tb \\ * & * \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} * & tbmq \\ * & * \end{bmatrix} \pmod{tN^k}$$

$$\zeta\epsilon_i\eta_j = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}\begin{bmatrix} u & tv \\ * & * \end{bmatrix}\begin{bmatrix} 1 & tj \\ 0 & m \end{bmatrix} = \begin{bmatrix} u & t(uj+vm) \\ * & * \end{bmatrix} \pmod{tN^k}.$$

Comparing the top right entries gives $tbqm = t(uj+vm) \pmod{tN^k}$, whence $bqm = uj + vm \pmod{N^k}$, whence $uj = 0 \pmod{m}$ since $m \mid N^k$. Since ϵ_i belogns to Γ' the integer u is coprime to m; since j is taken so that $0 \le j \le m-1$ we therefore must have $j = 0$. Note that this forces $i = 1$. Therefore $n = 1$.

We now show that $n' = 1$. Taking dgerees in equation 7 gives

$$n \deg(\Gamma'\alpha\Gamma') = \deg(\Gamma'\xi\Gamma')\deg(\Gamma'\eta\Gamma') = n' \deg(\Gamma'\alpha\Gamma').$$

We thus conclude n' is equal to n, which we already know to be 1. This completes the proof.

123. Keep the same notation as in the statement of proposition 122. We make two definitions:

1. For each positive integer n we define $T'(n)$ (or $T_\Theta(n)$ when the context is not clear) to be the sum of all double cosets $\Gamma'\alpha\Gamma'$ with α in Δ' of determinant n.

2. For two integers a and d coprime to N we let $T'(a,d)$ (or $T_\Theta(a,d)$) be the element of $R(\Gamma',\Delta(\Theta)) \subset R(\Gamma',\Delta')$ which is mapped to $T(a,d)$ in $R(\Gamma,\Delta(N))$ under the isomorphism of proposition 118.

Let $m \mid N^\infty$. Then by part 3 of proposition 122 we see that:

1. We have $T'(m) = \Gamma'\begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix}\Gamma'$.

2. The degree of $T'(m)$ is equal to m.

124 Theorem (Shimura Thm. 3.34, 3.35). *Let notation be as in proposition 122.*

1. *Every double coset of $R(\Gamma',\Delta')$ can be expressed uniquely as a product $T'(m)T'(a,d) = T'(a,d)T'(m)$ with $m \mid N^\infty$, $d \mid a$ and $(a,N) = 1$.*

2. *If m and n divide N^∞ then $T'(mn) = T'(m)T'(n)$.*

3. *If $(m,n) \mid N^\infty$ then $T'(mn) = T'(m)T'(n)$.*

4. *The ring $R(\Gamma',\Delta')$ is a polynomial ring over $\mathbb{Z}$ in the variables*

$$\begin{aligned} T'(p) &\quad \textit{for all primes } p \textit{ dividing } N \\ T'(1,p), T'(p,p) &\quad \textit{for all primes } p \textit{ not dividing } N. \end{aligned}$$

5. *The ring $R(\Gamma',\Delta')$ is the image of $R(\Gamma,\Delta(1))$ under the ring homomorphism defined by*

$$\begin{aligned} T(n) \mapsto T'(n) &\quad \textit{for all } n \\ T(p,p) \mapsto T'(p,p) &\quad \textit{for all primes } p \textit{ not dividing } N \\ T(p,p) \mapsto 0 &\quad \textit{for all primes } p \textit{ dividing } N. \end{aligned}$$

6. *The elements $T'(n)$ generate $R(\Gamma',\Delta') \otimes_{\mathbb{Z}} \mathbb{Q}$.*

1) The existence of such a factorization follows from part 4 of proposition 122. To prove uniqueness, simply note that if X is any double coset in $R(\Gamma', \Delta')$ then $T'(m)X = X \pmod{p}$ if p is any prime which does not divide N. Thus, given $T'(m)X$, one can recover the prime-to-N part of the elementary divisors of X.

2) Let m and n divide N^∞. Using part 3 of proposition 122 we obtain

$$\begin{aligned}
\Gamma' \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \Gamma' \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \Gamma' &= \bigcup_{s=0}^{n-1} \Gamma' \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \Gamma' \begin{bmatrix} 1 & ts \\ 0 & n \end{bmatrix} = \bigcup_{s=0}^{n-1} \bigcup_{r=0}^{m-1} \Gamma' \begin{bmatrix} 1 & tr \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & ts \\ 0 & n \end{bmatrix} \\
&= \bigcup_{s=0}^{n-1} \bigcup_{r=0}^{m-1} \Gamma' \begin{bmatrix} 1 & t(s+rn) \\ 0 & mn \end{bmatrix} = \bigcup_{r=0}^{mn-1} \Gamma' \begin{bmatrix} 1 & tr \\ 0 & mn \end{bmatrix} \\
&= \Gamma' \begin{bmatrix} 1 & 0 \\ 0 & mn \end{bmatrix} \Gamma'.
\end{aligned}$$

It thus follows that $T'(n)T'(m) = cT'(nm)$ for some integer c. Taking degrees gives $nm = cnm$, whence $c = 1$.

3) Let m and n be integers such that $(m, n) \mid N^\infty$. Write $m = m_1 m_2$ and $n = n_1 n_2$ with $n_1, m_1 \mid N^\infty$ and m_2, n_2 coprime to N. We have $T'(m) = T'(m_1)T'(m_2)$ and $T'(n) = T'(n_1)T'(n_2)$ by part 1 so that

$$\begin{aligned}
T'(m)T'(n) &= T'(m_1)T'(m_2)T'(n_1)T'(n_2) \\
&= T'(m_1)T'(n_1)T'(m_2)T'(n_2) && \text{by part 1} \\
&= T'(m_1 n_1)T'(m_2)T'(n_2) && \text{by part 2} \\
&= T'(m_1 n_1)T'(m_2 n_2) && \text{by propositions 118 and 105} \\
&= T'(mn) && \text{by part 1.}
\end{aligned}$$

4) Given a double coset in $R(\Gamma', \Delta')$ we can factor it in the form $T'(m)T'(a, d)$ according to part 1 of the proposition. By proposition 118 and theorem 95 we know that $T'(a, d)$ is expressible as a polynomial in the $T'(1, p)$ and $T'(p, p)$ with p not dividing N. By part 2 of the present proposition we know that $T'(m)$ is expressible as a product of $T'(p)$ with p dividing N. This shows that the stated elements generate the ring. It is not difficult to then prove that they are algebraically independent.

5) This follows easily from what we know about the two rings involved.

6) By propositions 118 and 107 it follows that if p does not divide N then we have

$$pT'(p, p) = T'(p)^2 - T'(p^2).$$

This, together with the fourth part of the proposition, proves the sixth part.

3.6 Automorphic forms

3.6.1 Definitions and first properties

125. For an integer k, an element σ of $\mathrm{GL}(2, \mathbb{R})$ and a complex valued function f on $\mathfrak{H}$ we define another complex valued function on $\mathfrak{H}$, denoted $f \mid [\sigma]_k$, by

$$(f \mid [\sigma]_k)(z) = (\det \sigma)^{k/2} j(\sigma, z)^{-k} f(\sigma z)$$

where, recall, we have defined

$$j(\sigma, z) = c_\sigma z + d_\sigma.$$

The identity

$$f \mid [\sigma\tau]_k = (f \mid [\sigma]_k) \mid [\tau]_k$$

is easily verified. Note that σ and $-\sigma$ induce the same action on $\mathfrak{H}$ but that if k is odd then $f \mid [-\sigma]_k = -f \mid [\sigma]_k$.

126. Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ and let f be a meromorphic function on $\mathfrak{H}$ invariant under the operators $|\,[\sigma]_k$ for σ in Γ (for a fixed value of k). Note that if k is odd and Γ contains -1 then $f = 0$ is the only such function; thus, if k is odd, we assume that Γ does not contain -1. Note that this implies that the stabilizers of cusps are infinite cyclic groups.

We now define the Fourier expansion of f at a cusp s. Let ρ be an element of $\mathrm{SL}(2,\mathbb{R})$ such that $\rho(s) = \infty$. The function $f\,|\,[\rho^{-1}]_k$ is then invariant under $\rho\Gamma\rho^{-1}$. There are three cases:

Case 1: k is even. Let $\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$ be a generator of $\rho\Gamma_s\rho^{-1}$ modulo ± 1 with h positive. Since $f\,|\,[\rho^{-1}]_k$ is invariant under $z \mapsto z+h$ there exists a meromorphic function ϕ defined on a punctured disc about the origin such that $(f\,|\,[\rho^{-1}]_k)(z) = \phi(q)$ where $q = \exp(2\pi i z/h)$. The Fourier expansion of f at s is then defined to be the Laurent expansion of ϕ at 0.

Case 2: k odd and $\rho\Gamma_s\rho^{-1}$ generated by $\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$ with $h > 0$. In this case we call s a *regular cusp*. This case is handled exactly like the case when k is even.

Case 3: k odd and $\rho\Gamma_s\rho^{-1}$ generated by $-\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$ with $h > 0$. In this case we call s an *irregular cusp*. We have $(f\,|\,[\rho^{-1}]_k)(z+h) = -(f\,|\,[\rho^{-1}]_k)(z)$ and thus $(f\,|\,[\rho^{-1}]_k)(z+2h) = (f\,|\,[\rho^{-1}]_k)(z)$. We can therefore find a meromorphic function ϕ defined on a punctured disc about the origin such that $(f\,|\,[\rho^{-1}]_k)(z) = \phi(q)$ where now $q = \exp(\pi i z/h)$; note that ϕ is necessarily an odd function. The Fourier expansion of f at s is then defined to be the Laurent expansion of ϕ at 0.

Note that by our hypotheses we necessarily land in one of these cases. It is easily verified that in all cases the defintion of the Laurent expansion of f at s does not depend on the choice of ρ.

127. Let f be a meromorphic function on $\mathfrak{H}$ invariant under the operators $|\,[\sigma]_k$ for σ in a discrete group Γ. We then say that f is *meromorphic* (resp. *holomorphic*) at a cusp s if the Fourier expansion of f at s has only finitely many terms of negative degree (resp. no terms of negative). Similarly, we say that f *vanishes* at s if its Fourier expansion at s has only positive degree terms.

128 Definition. A meromorphic (resp. holomorphic) *automorphic form* of weight k for the discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ is a complex valued function f on $\mathfrak{H}$ satisfying:

1. $f\,|\,[\sigma]_k = f$ for all σ in Γ;
2. f is meromorphic (resp. holomorphic) on $\mathfrak{H}$;
3. f is meromorphic (resp. holomorphic) at the cusps of Γ.

An automorphic form is called *cuspidal* (or a *cusp form*) if it vanishes at all the cusps of Γ.

129. For a discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ and an integer k we define

$$\begin{aligned} A_k(\Gamma) &= \text{space of meromorphic automorphic forms for } \Gamma \text{ of weight } k \\ G_k(\Gamma) &= \text{space of holomorphic automorphic forms for } \Gamma \text{ of weight } k \\ S_k(\Gamma) &= \text{space of holomorphic cusps forms for } \Gamma \text{ of weight } k \end{aligned}$$

130. If $\Gamma = \Gamma(N)$ is the principal congruence subgroup of level N then an automorphic form for Γ is usually called a *modular form* of level N.

131 Proposition (Shimura Prop 2.4). *Let Γ and Γ' be discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$ and let α be an element of $\mathrm{GL}^+(2,\mathbb{R})$ such that $\alpha\Gamma\alpha^{-1}$ is a subgroup of finite index in Γ'. Then $f \mapsto f\,|\,[\alpha]_k$ gives an injection of $A_k(\Gamma')$ (resp. $G_k(\Gamma')$, $S_k(\Gamma')$) into $A_k(\Gamma)$ (resp. $G_k(\Gamma)$, $S_k(\Gamma)$) which is bijective if $\Gamma' = \alpha\Gamma\alpha^{-1}$.*

Let C (resp. C') denote the set of cusps of Γ (resp. Γ'). Then $C' = \alpha C$; the proposition thus follows immediately from the definitions.

132 Proposition (Shimura Prop 2.6). *Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ and let Γ' be a subgroup of finite index in Γ. Then $A_k(\Gamma)$ (resp. $G_k(\Gamma)$, $S_k(\Gamma)$) is the set of all f in $A_k(\Gamma')$ (resp. $G_k(\Gamma')$, $S_k(\Gamma')$) which are invariant under $[\gamma]_k$ for all γ in Γ.*

The only nontrivial point that must be verified is the condition at the cusps. This is straightforward and left to the reader.

3.6.2 The dimensions of certain spaces of automorphic forms

133 Proposition (Shimura Thm. 2.23, 2.24). *Let Γ be a Fuchsian group of the first kind, let g be the genus of $\Gamma\backslash\mathfrak{H}^*$, let m be the number of inequivalent cusps of Γ and let $e_1,\ldots,e_r$ be the orders of the inequivalent elliptic points of Γ. Then, for an even integer k, we have*

$$\dim G_k(\Gamma) = \begin{cases} (k-1)(g-1) + \frac{1}{2}mk + \sum_{i=1}^r \frac{1}{2}k(1-e_i^{-1}) & (k>2) \\ g+m-1 & k=2, m>0 \\ g & k=2, m=0 \\ 1 & k=0 \\ 0 & k<0 \end{cases}$$

and

$$\dim S_k(\Gamma) = \begin{cases} (k-1)(g-1) + \frac{1}{2}m(k-2) + \sum_{i=1}^r \frac{1}{2}k(1-e_i^{-1}) & k>2 \\ g & k=2 \\ 0 & k=0, m>0 \\ 1 & k=0, m=0 \\ 0 & k<0 \end{cases}$$

134 Proposition (Shimura Thm. 2.25). *Let Γ be a Fuchsian group of the first kind which does not contain -1, let g be the genus of $\Gamma\backslash\mathfrak{H}^*$, let u (resp. u') be the number of inequivalent regular (resp. irregular) cusps of Γ and let $e_1,\ldots,e_r$ be the orders of the inequivalent elliptic points for Γ. Then, for an odd integer k, we have*

$$\dim G_k(\Gamma) = \begin{cases} (k-1)(g-1) + \frac{1}{2}uk + \frac{1}{2}u'(k-1) + \sum_{i=1}^r \frac{1}{2}k(1-e_i^{-1}) & k\geq 3 \\ 0 & k<0 \end{cases}$$

and

$$\dim S_k(\Gamma) = \begin{cases} (k-1)(g-1) + \frac{1}{2}u(k-2) + \frac{1}{2}u'(k-1) + \sum_{i=1}^r \frac{1}{2}k(1-e_i^{-1}) & k\geq 3 \\ 0 & k<0 \end{cases}$$

Furthermore,

$$\dim G_1(\Gamma) \geq u/2.$$

If $u > 2g-2$ then we have equality in the above and also $\dim S_1(\Gamma) = 0$.

3.6.3 The Petersson inner product

135. Let Γ be a discrete group and let f and g be complex valued functions on $\mathfrak{H}$ which are invariant under the operators $|[\sigma]_k$ for all σ in Γ (and a fixed integer k). Note that $z \mapsto f(z)\overline{g(z)}y^k$ is a well defined function on $\Gamma\backslash\mathfrak{H}$ (where $y = y(z)$ is the imaginary part of z). Note also that $y^{-2}dxdy$ is a well-defined 2-form on $\Gamma\backslash\mathfrak{H}$. We may therefore define

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{g(z)}y^{k-2}dxdy.$$

The integral may or may not converge. Note that, when it is defined, $\langle,\rangle$ is positive definite and hermitian.

136 Lemma. *If Γ is a Fuchsian group of the first kind, f and g belong to $G_k(\Gamma)$ and at least one of f and g is a cusp form then the integral defining $\langle f, g\rangle$ converges.*

We assume that f is a cusp form. Since the space $\Gamma\backslash\mathfrak{H}^*$ is compact, it is sufficient to show that the function $f(z)\overline{g(z)}y^k$ on $\Gamma\backslash\mathfrak{H}^*$ is continuous at the cusps (since we know it is continuous away from the cusps). Thus let s be a cusp of Γ, let ρ be an element of $\mathrm{SL}(2,\mathbb{R})$ such that $\rho(s) = \infty$ and let Γ_s be the stabilizer of s in Γ. We have

$$\{\pm 1\}\cdot\rho\Gamma_s\rho^{-1} = \left\{\pm\begin{bmatrix} 1 & h \\ 0 & 1\end{bmatrix}^m \middle| m\in\mathbb{Z}\right\}$$

for some positive real number h. By the definition of "holomorphic automorphic form" there exist holomorphic functions $\Phi(q)$ and $\Psi(q)$, defined on a disc centered at the origin, such that

$$f \,|\, [\rho^{-1}]_k = \Phi(q), \qquad g \,|\, [\rho^{-1}]_k = \Psi(q)$$

where $q = e^{i\pi z/h}$. Furthermore, since f is a cusp form, we have that $\Phi(0) = 0$. Now, letting $w = \rho^{-1}(z)$ we have

$$f(w)\overline{g(w)}\Im(w)^k = f(\rho^{-1}(z))\overline{g(\rho^{-1}(z))}\Im(\rho^{-1}(z))^k = \Phi(q)\overline{\Psi(q)}\Im(z)^k.$$

Now, for z very large (*i.e.*, close to $i\infty$) the function $\Phi(q)$ is approximately equal to $ce^{i\pi z/h}$ and the function $\Psi(q)$ is approximately constant. Since $e^{i\pi z/h}\Im(z)^k \to 0$ as $z \to i\infty$ we see that $f(z)\overline{g(z)}y^k$ is indeed continuous at s.

137. Let Γ be a Fuchsian group of the first kind. Lemma 136 implies that $\langle , \rangle$ gives a positive definite hermitian inner product on the space $S_k(\Gamma)$. This inner product is called the *Petersson inner product*.

3.6.4 The action of Hecke algebras on automorphic forms

138. In this section we explain how the double coset algebras we constructed in the previous section act on various spaces of automorphic forms.

139. To start off with fix a family $\{\Gamma_\lambda\}_{\lambda\in\Lambda}$ of mutually commensurable Fuchsian groups of the first kind. We let $\widetilde{\Gamma}$ be the commensurator of Γ_λ inside of $G = \mathrm{GL}^+(2, \mathbb{R})$ (it obviously does not depend on which λ one chooses). Recall (*cf.* §3.5.1) that we have defined $\mathbb{Z}$-modules $R_{\lambda\mu}$ and multiplication maps $R_{\lambda\mu} \otimes R_{\mu\nu} \to R_{\lambda\nu}$.

140. Let X be a double coset in $R_{\lambda\mu}$ and let f be an element of $A_k(\Gamma_1)$. Write $X = \Gamma_1\alpha\Gamma_2$ with α in G and write

$$\Gamma_1\alpha\Gamma_2 = \coprod \Gamma_1\alpha_i.$$

We then define $f \,|\, [X]_k$ by the formula

$$f \,|\, [X]_k = (\det\alpha)^{k/2-1} \sum f \,|\, [\alpha_i]_k.$$

The definition clearly does not depend on the choices made. We extend the definition of $f \,|\, [X]_k$ to general elements X of $R_{\lambda\mu}$ by linearity.

141 Proposition (Shimura Prop 3.37). *If X belongs to $R_{\lambda\mu}$ then $f \mapsto f \,|\, [X]_k$ takes the spaces $A_k(\Gamma_\lambda)$, $G_k(\Gamma_\lambda)$ and $S_k(\Gamma_\lambda)$ into the spaces $A_k(\Gamma_\mu)$, $G_k(\Gamma_\mu)$ and $S_k(\Gamma_\mu)$ respectively.*

Let f belong to $A_k(\Gamma_\lambda)$, let $X = \Gamma_1\alpha\Gamma_2$ be a double coset in $R_{\lambda\mu}$, and let $g = f \,|\, [X]_k$. Let γ be an arbitrary element of Γ_2. Write $\Gamma_1\alpha\Gamma_2 = \coprod \Gamma_1\alpha_i$. Then the two sets $\{\Gamma_1\alpha_i\}$ and $\{\Gamma_1\alpha_i\gamma\}$ correspond. We therefore have

$$g \,|\, [\gamma]_k = (\det\alpha)^{k/2-1} \sum f \,|\, [\alpha_i\gamma]_k = (\det\alpha)^{k/2-1} \sum f \,|\, [\alpha_i]_k = g$$

so that g is invariant under Γ_2. On the other hand, each term $f \,|\, [\alpha_i]_k$ belongs to $A_k(\alpha_i^{-1}\Gamma_\lambda\alpha_i)$ by proposition 131. Thus, putting

$$\Gamma' = \Gamma_\mu \cap \left(\bigcap \alpha_i^{-1}\Gamma_\lambda\alpha_i \right)$$

we see that Γ' is a subgroup of Γ_μ of finite index such that g belongs to $A_k(\Gamma')$. By proposition 132 it then follows that g belongs to $A_k(\Gamma_\mu)$. The arguments are exactly the same for G_k and S_k.

142 Proposition (Shimura Prop 3.38). *If X belongs to $R_{\lambda\mu}$ and Y belongs to $R_{\mu\nu}$, so that XY belongs to $R_{\lambda\nu}$, then $[XY]_k = [X]_k[Y]_k$ as linear maps from $A_k(\Gamma_\lambda)$ to $A_k(\Gamma_\nu)$.*

It is sufficient to consider the case when X and Y are double cosets; thus let $X = \Gamma_\lambda\alpha\Gamma_\mu$ and $Y = \Gamma_\mu\beta\Gamma_\nu$ with α and β in $\widetilde{\Gamma}$. Let $XY = \sum_k c_k(\Gamma_\lambda\xi_k\Gamma_\nu)$. Write

$$\Gamma_\lambda\alpha\Gamma_\mu = \coprod \Gamma_\lambda\alpha_i, \qquad \Gamma_\mu\beta\Gamma_\nu = \coprod \Gamma_\mu\beta_j, \qquad \Gamma_\lambda\xi_k\Gamma_\nu = \coprod \Gamma_\lambda\xi_{k,\ell}.$$

By the definition of multiplication, we have

$$\sum_{i,j} \Gamma_\lambda\alpha_i\beta_j = \sum_{k,\ell} c_k\Gamma_\lambda\xi_{k,\ell}$$

(taken as formal sums). Thus if f belongs to $A_k(\Gamma_\lambda)$ then

$$\begin{aligned}(f\,|\,[X]_k)\,|\,[Y]_k &= \det(\alpha\beta)^{k/2-1}\sum_{i,j} f\,|\,[\alpha_i\beta_j]_k = \det(\alpha\beta)^{k/2-1}\sum_{k,\ell} c_k f\,|\,[\xi_{k,\ell}] \\ &= \sum_k c_k f\,|\,[\Gamma_\lambda\xi_k\Gamma_\nu] = f\,|\,[XY]_k\end{aligned}$$

(note that $\det\xi_k = \det(\alpha\beta)$ for all k). This proves the proposition.

143 Corollary. *Let Γ be a Fuchsian group of the first kind and let $\widetilde{\Gamma}$ be its commensurator in $\mathrm{GL}^+(2,\mathbb{R})$. Then $(f, X) \mapsto f\,|\,[X]_k$ is a representation of the ring $R(\Gamma,\widetilde{\Gamma})$ on the vector spaces $A_k(\Gamma)$, $G_k(\Gamma)$ and $S_k(\Gamma)$.*

This follows immediately from propositions 142 and 142.

144 Proposition (Shimura Prop 3.39). *Let α be an element of $\widetilde{\Gamma}$. Let f belong to $G_k(\Gamma_\lambda)$, let g belong to $G_k(\Gamma_\mu)$ and assume at least one of f and g is a cusp form. Then*

$$\langle f\,|\,[\Gamma_1\alpha\Gamma_2]_k, g\rangle_\mu = \langle f, g\,|\,[\Gamma_2\alpha^\iota\Gamma_1]_k\rangle_\lambda.$$

Here ι is the "main anti-involution" of $M_2(\mathbb{R})$, given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\iota = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

In particular, the adjoint of the map $[\Gamma_1\alpha\Gamma_2] : S_k(\Gamma_\lambda) \to S_k(\Gamma_\mu)$ is given by $[\Gamma_2\alpha^{-1}\Gamma_1]$.

First note that if c is a positive real number then $[\Gamma_\lambda c\alpha\Gamma_\mu]_k = c^{k/2-1}[\Gamma_\lambda\alpha\Gamma_\mu]$. Therefore, it suffices to consider the case when $\det\alpha = 1$; note that then we have $\alpha^\iota = \alpha^{-1}$. Second, note that if A is any measurable set in $\mathfrak{H}$ then we have

$$\int_{\alpha A} f\overline{g}y^{k-2}dxdy = \int_A (f\,|\,[\alpha]_k)\overline{(g\,|\,[\alpha]_k)}y^{k-2}dxdy.$$

Now, let P be a fundamental domain for that action of Γ_μ on $\mathfrak{H}$. Put

$$\Gamma' = \Gamma_\mu \cap \alpha^{-1}\Gamma_\lambda\alpha, \qquad \Gamma'' = \alpha\Gamma_\mu\alpha^{-1} \cap \Gamma_\lambda$$

and let ϵ_i be right coset representatives for Γ' in Γ_μ, *i.e.*, $\Gamma_\mu = \coprod \Gamma'\epsilon_i$. Note three things: 1) we have $\Gamma_\lambda\alpha\Gamma_\mu = \coprod \Gamma_\lambda\alpha\epsilon_i$; 2) the set $Q = \coprod \epsilon_i P$ is a fundamental domain for Γ'; and 3) the set αQ is a fundamental domain for Γ''. Now, we have

$$\begin{aligned}\int_P (f\,|\,[\Gamma_\lambda\alpha\Gamma_\mu]_k)\overline{g}y^{k-2}dxdy &= \sum \int_P (f\,|\,[\alpha\epsilon_i]_k)\overline{g}y^{k-2}dxdy = \sum \int_{\epsilon_i P} (f\,|\,[\alpha]_k)\overline{g}y^{k-2}dxdy \\ &= \int_Q (f\,|\,[\alpha]_k)\overline{g}y^{k-2}dxdy = \int_{\alpha Q} f\overline{(g\,|\,[\alpha^{-1}]_k)}y^{k-2}dxdy\end{aligned}$$

or in other words, denoting by $\langle,\rangle'$ (resp. $\langle,\rangle''$) the Petersson inner product for Γ' (resp. Γ''), we have

$$\langle f\,|\,[\Gamma_\lambda\alpha\Gamma_\nu]_k, g\rangle_\mu = \langle f\,|\,[\alpha]_k, g\rangle' = \langle f, g\,|\,[\alpha^{-1}]_k\rangle''.$$

Interchanging (λ, f) and (μ, g) and taking α^{-1} in place of α we obtain

$$\langle f, g\,|\,[\Gamma_\mu\alpha^{-1}\Gamma_\lambda] = \langle f, g\,|\,[\alpha^{-1}]_k\rangle''.$$

This proves the proposition.

Chapter 4

Weil Representations

4.1 Notations

1. In this section F will be a local field, either archimedean or not, and K will be an algebra over F of one of the following types:

1. The direct sum $F \oplus F$.
2. A separable quadratic extension of F.
3. A quaternion division algebra over F.
4. The matrix algebra $M(2, F)$.

Note that in all cases F is a subfield of the center of K.

2. The algebra K has a natural anti-involution ι:

1. If $K = F \oplus F$ then $(x, y)^\iota = (y, x)$.
2. If K is a seperable quadratic extension then ι is the nontrivial galois automorphism.
3. If K is a quaternion division algebra then ι is conjugation.
4. If K is a $M(2, F)$ then
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\iota = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

3. The involution ι has the following properties:

1. It is a F-linear map of K and it satisfies $(xy)^\iota = y^\iota x^\iota$ and $\iota^2 = 1$.
2. If x belongs to F then $x^\iota = x$.
3. Both $\tau(x) = x + x^\iota$ and $\nu(x) = xx^\iota$ belong to F.

We call τ and ν the *trace* and *norm* on K.

4. If ψ_F is a nontrivial additive character of F then $\psi_K = \psi_F \tau$ is a nontrivial additive character of K. By means of the pairing
$$\langle x, y \rangle = \psi_K(xy)$$
we may identify K with its Pontrjagin dual.

5. We let $f = \psi_F \nu$. We have
$$f(x + y) f^{-1}(x) f^{-1}(y) = \langle x, y^\iota \rangle.$$

6. We denote by $\mathscr{S}(K)$ the space of Schwartz functions on K. If F is archimedean these are the infinitely differentiable functions which fall off at infinity faster than any polynomial. If F is non-archimedean these are the locally constant compactly supported functions. In the archimedean case we give $\mathscr{S}(K)$ the Schwartz topology; in the non-archimedean case $\mathscr{S}(K)$ is given the discrete topology.

7. There is a unique Haar measure dx on K such that if Φ belongs to $\mathscr{S}(K)$ and Φ' is its Fourier transform with respect to ψ_K and dx, *i.e.*,

$$\Phi'(x) = \int_K \Phi(y)\psi_K(xy)dx,$$

then

$$\Phi(0) = \int_K \Phi'(x)dx.$$

8. If F is non-archimedean and K is a separable quadratic extension of F we let ω be the quadratic character of $F^\times$ associated to K by local class field theory. In all other cases we take ω to be the trivial character of $F^\times$.

4.2 The basic Weil representation

9 Proposition. *There is a constant γ, which depends on ψ_F and K, such that for every Φ in $\mathscr{S}(K)$ the identity*

$$\int_K (\Phi \star f)(y)\psi_K(xy)dy = \gamma f^{-1}(x^\iota)\Phi'(x)$$

holds. Here $\star$ denotes convolution.

10 Proposition. *We have the following values for the constant γ of proposition 9:*

1. *If $K = F \oplus F$ or $K = M(2,F)$ then $\gamma = 1$.*
2. *If K is a quaternion division algebra then $\gamma = -1$.*
3. *If $F = \mathbb{R}$ and $K = \mathbb{C}$ and $\psi_F(x) = e^{2\pi i a x}$ then $\gamma = i \operatorname{sgn} a$.*
4. *Let F be non-archimedean and K be a separable quadratic extension of F. Let $1 + \mathfrak{p}^n$ be the conductor of ω and let $\mathfrak{p}^{-m}$ be the conductor of ψ_F. Then*

$$\gamma = \omega(\varpi^{n+m})\frac{\eta(\omega^{-1}, \varpi^{-n-m})}{|\eta(\omega^{-1}, \varpi^{-n-m})|}$$

   where ϖ is a generator of $\mathfrak{p}$ and η is as in proposition 121.

11 Proposition. *There is a unique representation r of $\mathrm{SL}(2,F)$ on $\mathscr{S}(K)$ such that:*

1. *We have the following formulae:*

   (a) $\left(r\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\Phi\right)(x) = \omega(a)|a|_F\Phi(ax)$

   (b) $\left(r\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}\Phi\right)(x) = \psi_F(z\nu(x))\Phi(x)$

   (c) $\left(r\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\Phi\right)(x) = \gamma\Phi'(x^\iota)$.

2. *The representation r is continuous.*
3. *The representation r can be extended to a unitary representation of $\mathrm{SL}(2,F)$ on $L^2(K)$.*
4. *If F is archimedean and Φ belongs to $\mathscr{S}(K)$ then the function $g \mapsto r(g)\Phi$ is an infinitely differentiable function on $\mathrm{SL}(2,F)$ with values in $\mathscr{S}(K)$.*

4.3 Weil representations for quadratic extensions and quaternion algebras

12. In this section we specialize and assume that K is either a separable quadratic extension of F or a quaternion division algebra over F. We let K' denote the compact subgroup of $K^\times$ consisting of all x with $\nu(x) = 1$. We let G_+ denote the subgroup of $\mathrm{GL}(2, F)$ consisting of those matrices whose determinant belongs to $\nu(K^\times)$.

13 Proposition. *Let (Ω, U) be a finite dimensional irreducible representation of $K^\times$. Let r still denote the representation of $\mathrm{SL}(2, F)$ on $\mathscr{S}(K, U)$ (where $\mathrm{SL}(2, F)$ acts trivially on U).*

1. *Let $\mathscr{S}(K, \Omega)$ be the space of functions Φ in $\mathscr{S}(K, U)$ which satisfy*

$$\Phi(xh) = \Omega^{-1}(h)\Phi(x)$$

for all $x \in K$ and $h \in K'$. Then $\mathscr{S}(K, \Omega)$ is stable under $r(g)$ for g in $\mathrm{SL}(2, F)$.

2. *The representation r of $\mathrm{SL}(2, F)$ on $\mathscr{S}(K, \Omega)$ can be extended to a representation r_Ω of G_+ satisfying*

$$\left(r_\Omega \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \Phi\right)(x) = |h|_K^{1/2}\Omega(h)\Phi(xh)$$

where $a = \nu(h)$.

3. *If η is the central-quasicharacter of Ω then $\omega\eta$ is the central-quasicharacter of r_Ω.*

4. *The representation r_Ω is continuous and all elements of $\mathscr{S}(K, \Omega)$ are smooth.*

5. *If U is a Hilbert space and Ω is unitary then r_Ω can be extended to a unitary representation of G_+ on the closure $L^2(K, \Omega)$ of $\mathscr{S}(K, \Omega)$ inside of $L^2(K, U)$.*

4.4 Weil representations for $F \oplus F$

14. We now assume that $K = F \oplus F$. We regard K as a right module over $M(2, F)$ via matrix multiplication. If g is an element of $M(2, F)$ and Φ an element of $\mathscr{S}(K)$ we let $\rho(g)\Phi$ be the element of $\mathscr{S}(K)$ whose value at x is $\Phi(xg)$.

15 Proposition. *The representation r can be extended to a representation of $\mathrm{GL}(2, F)$ so that:*

1. $r \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = \rho \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$

2. *If $\Phi^\sim$ denotes the partial Fourier transform of Φ, i.e.,*

$$\Phi^\sim(a, b) = \int_F \Phi(a, y)\psi_F(by)dy$$

then

$$(r(g)\Phi)^\sim = \rho(g)\Phi.$$

Chapter 5

Representations of $\mathrm{GL}(2, F)$ in the non-archimedean case

5.1 Preliminaries: representation theory of TDLC groups

[`narch-tdlc`]

5.1.1 Introduction

[`narch-tdlc-intro`]

1. *All topological spaces in §5.1 are assumed to be Hausdorff unless otherwise explicitly mentioned.*

2. In this section we give some basic representation theory for totally disconnected (TD) locally compact Hausdorff (LC) groups. All TDLC groups that we are interested in are of the form $G(R)$ where G is an algebraic group and R is a non-archimedean local field or its ring of integers; of course, $\mathrm{GL}(2, F)$ where F is a non-archimedean local field is such a group.

5.1.2 The topology of totally disconnected groups

[`narch-tdlc-top`]

3. Let X be a topological space.

1. Define an equivalence relation on X by $x \sim y$ if there exists a connected subset of X containing both x and y. The equivalence classes under this relation are called the *components* of X. Equivalently, components are maximal connected subsets of X.
2. Define a different equivalence relation on X by $x \sim y$ if every clopen (*i.e.*, closed and open) set containing x also contains y. The equivalence classes under this relation are called the *quasi-components* of X.
3. Every component of X is contained in a unique quasi-component; in fact, if A is a component of X then the intersection of all clopen sets containing A is the said quasi-component.
4. Every quasi-component of X is the union of the components which it contains.
5. The space X is *totally disconnected* if its components are sets consisting of a single point. This does not imply, in general, that the quasi-components of X consist of a single point.

In what follows, we will take some basic propositions concerning components for granted. They can all be found in elementary point set topology books.

4 Lemma. [narch-tdlc-top-20] *Let X be a compact Hausdorff space, let $\mathcal{F}$ be a family of closed subsets of X closed under finite intersections, let A be the intersection over all sets in $\mathcal{F}$ (which we assume to be non-empty) and let U be an open set containing A. Then there exists an element F of $\mathcal{F}$ which contains A and which is contained in U.*

Assume that the proposition is false. Then for each set F in $\mathcal{F}$ the closed set $F \cap (X \setminus U)$ is nonempty. It thus follows that the family of closed sets $\mathcal{F}' = \{F \cap (X \setminus U) \mid F \in \mathcal{F}\}$ has the finite intersection property. Since X is compact, the intersection of all the sets in $\mathcal{F}'$ is non-empty. However, this intersection is precisely $A \cap (X \setminus U)$, which is, by hypothesis, empty. This is a contradiction.

5 Proposition ([Po] §15, ¶F). [narch-tdlc-top-30] *In a compact Hausdorff space the components and quasi-components coincide.*

Let X be a compact Hausdorff space, let K be a component of X and let L be the unique quasi-component containing K, *i.e.*, the intersection of the clopen sets containing K. It suffices to show that L is connected, for then $K = L$.

Assume that L is not connected; we can then write $L = A \coprod B$ where A and B are closed subsets of L (and thus closed in X since L is closed). Since K is connected it is contained in either A or B; assume $K \subset A$. Since X is compact Hausdorff it is therefore normal and we can separate A and B by disjoint open sets, *i.e.*, there exist disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$. Thus $L \subset U \cup V$. By lemma 4 there exists a clopen set P of X containing L and contained in $U \cup V$. Put $U' = P \cap U$ and $V' = P \cap V$. Then $P = U' \coprod V'$; it follows that U' and V' are clopen in P and thus clopen in X. Therefore U' is a clopen set of X containing K but not containing L. This contradicts the definition of L.

6 Proposition ([Po] §15, ¶G). [narch-tdlc-top-40] *Let X be a locally compact Hausdorff space, let K be a compact component of X and let U be an open set in X containing K. Then there exists a compact open set P of X containing K and contained in U.*

For each x in K pick a neighborhood V_x such that $\overline{V}_x$ is compact and contained in U. The V_x clearly form a cover of K and since K is compact there exists a finite subcover; let V be the union of the sets in this finite subcover. Then $\overline{V}$ is a compact set containing K and contained in U. Note that K is a component of the compact Hausdorff space $\overline{V}$. By proposition 6 it follows that K is the intersection of the clopen sets of $\overline{V}$ containing K; by lemma 5 we may therefore pick a clopen set P of $\overline{V}$ containing K and contained in U. Then P is compact (since it is a closed subset of the compact space $\overline{V}$) and open in X (since it is open in the open subset U of X). This completes the proof.

7 Proposition ([Po] §22, Thm. 16). [narch-tdlc-top-50] *A TDLC group has a neighborhood basis of the identity consisting of compact open subgroups.*

Let G be a TDLC group and let U be a neighborhood of 1. We must produce a compact open subgroup of G contained in U. Since G is totally disconnected the set $\{1\}$ is a component. By proposition 6 there exists a compact open set P of X which contains 1 and is contained in U. Let Q be the set of g in G for which $Pg \subset P$ and put $H = Q \cap Q^{-1}$. We now show that H is the sought after group.

H is a group. It is clear that if h belongs to H then h^{-1} belongs to H. Now, if h_1 and h_2 are two elements of Q then it is clear that h_1h_2 is again an element of Q. Thus if h_1 and h_2 are two elements of H then h_1h_2 belongs to Q (since h_1 and h_2 belong to Q) and $h_2^{-1}h_1^{-1}$ belongs to Q (since h_2^{-1} and h_1^{-1} belong to Q). It thus follows that h_1h_2 and $(h_1h_2)^{-1}$ belong to Q. Therefore h_1h_2 belongs to H.

H is open in G. Fix an element $g \in H$. For each x in P we have that xg belongs to P. Since P is open there exist neighborhoods U_x of x and V_x of g such that $U_xV_x \subset P$. The sets U_x cover P; let $U_{x_1}, \ldots, U_{x_n}$ be a finite subcover. Let V be the intersection $V_{x_1} \cap \cdots \cap V_{x_n}$. Then it is clear that $PV \subset P$ so that $V \subset H$. We have thus produced an open neighborhood of the element g which is contained in H.

H is closed in G. Let g belong to $G \setminus Q$. Thus $Pg \not\subset P$; pick p in P such that pg does not belong to P. Since $G \setminus P$ is open there exists an open neighborhood U of g such that $pU \subset G \setminus P$. It follows that U is contained in $G \setminus Q$. We have thus produced an open neighborhood of the element g which is contained in $G \setminus Q$. The set $G \setminus Q$ is therefore open, from which it follows that Q is closed, from which it follows that H is closed.

H is compact. Since P contains 1 it follows that Q is contained in P and so H is contained in P. Thus H is a closed subset of the compact space P and is therefore itself compact.

8 Proposition. [narch-tdlc-top-60] *Let G be a Hausdorff topological group and let U be an open subgroup. Then the coset space G/U is discrete.*

By the definition of the quotient topology, a subset of G/U is open if and only if its inverse image is open. Since the inverse image of a set in G/U is the union of cosets of U it is open. Thus all sets in G/U are open and therefore G/U has the discrete topology.

9 Proposition ([Po] §22, Thm. 17). [narch-tdlc-top-70] *A compact totally disconnected group has a neighborhood basis of the identity consisting of compact open normal subgroups.*

Let G be a compact totally disconnected group and let U be an open neighborhood of 1. We must produce a compact open normal subgroup contained in U. By proposition 7 the set U contains a compact open subgroup H of G. Let N be the intersection of all the subgroups xHx^{-1} as x varies over all of G. It is clear that N is a closed subgroup of G (and therefore compact). We must prove that N is open.

Since $x1x^{-1} \in H$ and H is open it follows that there exist open neighborhoods U_x of 1 and V_x of x such that $V_x^{-1}U_xV_X$ is contained in H. The sets V_x cover G; let $V_{x_1}, \dots, V_{x_n}$ be a finite subcover. Let U be the intersection $U_{x_1} \cap \cdots \cap U_{x_n}$. Then U is an open set and $x^{-1}Ux$ is contained in H for all x in G. It follows that for any n in N the set Un is contained in N and therefore N is open.

10 Proposition. [narch-tdlc-top-80] *A topological group is a profinite group (i.e., an inverse limit of finite groups with the discrete topology) if and only if it is compact Hausdorff and totally disconnected.*

Say G is a profinite group; let G be the inverse limit of the system $(G_i)_{i\in I}$ where each G_i is finite and I is some index set. Then G is a closed subspace of the product space $P = \prod_{i\in I} G_i$. Since P is compact and totally disconnected it follows that G is as well.

Now say that G is compact Hausdorff and totally disconnected. Let N be a compact open subgroup of G. The space G/N is compact (as it is a continuous image of the compact space G) and discrete (by proposition 8). Thus G/N is a finite group with the discrete topology. Let G' be the inverse limit of the G/N and let π be the canonical map $G \to G'$. We know that π is a continuos homomorphism. We now show that it is in fact a homeomorphism.

First note that if G has only finitely many compact open normal subgroups then it has a unique minimal such subgroup (the intersection) N. By proposition 9 we must have $N = 1$ and so it follows that $G = G'$. We assume hereafter that G has an infinite number of compact open normal subgroups.

π injective. Let g belong to the kernel of π. Then g belongs to every compact open subgroup N. Since these form a neighborhood basis of the identity by proposition 9 and G is Hausdorff, it follows that g must equal the identity. Thus π is injective.

π surjective. Let g' be an element of G'. By definition, for each compact open normal subgroup N of G we are given an element g'_N of G/N such that if $N_1 \subset N_2$ then $g'_{N_1} = g'_{N_2} \pmod{N_2}$. For each N pick an element g_N of G such that $g_N = g'_N \pmod{N}$. By definition the sequence $(\pi(g_N))$ converges to g'. Now, since G is compact there exists a convergent subsequence of (g_N), say $(g_N)_{N\in\mathcal{F}}$, where $\mathcal{F}$ is some infinite family of compact open normal subgroups; let g be the limit of this subsequence. Since π is continuous, $\pi(g)$ is the limit of the sequence $(\pi(g_N))_{N\in\mathcal{F}}$, which we already know converges to g'. Thus $\pi(g) = g'$ and so π is surjective.

π is a homeomorphism. Since π is a continuous bijection between compact Hausdorff spaces it is automatically a homeomorphism.

5.1.3 Smooth and admissible representations

[narch-tdlc-sa]

11. [narch-tdlc-sa-10] Let (π, V) be a representation of the TDLC group G.

1. The representation π is *smooth* if the stabilizer of every vector in V is an open subgroup of G.

2. The representation π is *admissible* if it is smooth and if for any open subgroup U of G the space V^U of all vectors fixed by U is finite dimensional.

12 Proposition. [narch-tdlc-sa-20] *Let (π, V) be a* finite dimensional *representation of the TDLC group G. Then the following are equivalent:*

1. *The representation π is admissible.*

2. *The representation π is smooth.*

3. *The kernel of π is an open subgroup of G.*

4. *The map π is continuous (as a map $G \to \mathrm{GL}(V)$).*

($1 \iff 2$) This is immediate from the definitions.

($2 \iff 3$) If the kernel of π is open then the stabilizer of any element of V is open (since it contains the kernel) and so π is smooth. On the other hand, since the kernel of π is the intersection of the stabilizers of a basis of V, it follows that smooth implies open kernel.

($3 \iff 4$) If the kernel of π is open then the inverse image of any element of $\mathrm{GL}(V)$ is open and so the inverse image of any subset of $\mathrm{GL}(V)$ is an open subset of G. Thus π is continuous.

Now say that π is continuous. Let U be an open neighborhood of the identity in $\mathrm{GL}(V)$ which contains no nontrivial subgroups. Its inverse image under π is an open neighborhood of 1 in G and therefore, by proposition 7, contains an open subgroup V. The image of V under π is a subgroup of $\mathrm{GL}(V)$ contained in U and is therefore equal to $\{1\}$. Thus V is contained in the kernel of π and thus the kernel of π is open.

5.1.4 The Hecke algebra

[`narch-tdlc-hecke`]

13. Let G be a TDLC group. We define the *Hecke algebra* of G, denoted $\mathscr{H}_G$, to be the vector space of locally constant compactly supported complex valued functions on G with multiplication given by convolution, *i.e.*, if f_1 and f_2 belong to $\mathscr{H}_G$ then

$$(f_1 * f_2)(h) = \int_G f_1(g) f_2(g^{-1}h) dg$$

where dg is a fixed Haar measure on G. For a compact open subgroup U of G let χ_U be the function which is eqaul to $1/(\mathrm{Vol}\, U)$ on U and 0 outside of U. The function χ_U is an idempotent of $\mathscr{H}_G$; we call idempotents of this form *elementary*. The algebra $\mathscr{H}_G$ together with its elementary idempotents forms an idempotented algebra; thus the notions of smooth representation, admissible representations, contragredients, *etc.* are all defined for $\mathscr{H}_G$.

14. Note that G acts on $\mathscr{H}_G$ via right and left translation. To be precise, if g belongs to G and f belongs to $\mathscr{H}_G$ then we put

$$(\rho(g)f)(h) = f(gh), \qquad (\lambda(g)f)(h) = f(g^{-1}h).$$

The identities

$$(\lambda(g)f_1) * f_2 = \lambda(g)(f_1 * f_2), \qquad f_1 * (\rho(g)f_2) = \rho(g)(f_1 * f_2), \qquad f_1 * (\lambda(g)f_2) = (\rho(g)f_1) * f_2$$

are readily verified.

15 Theorem ([JL] pg. 25). *Let G be a TDLC group.*

1. *Let (π, V) be a smooth representation of G. For an element f of $\mathscr{H}_G$ define an operator $\pi(f)$ on V by*
$$\pi(f)v = \int_G f(g)\pi(g)v dg.$$
Then $f \mapsto \pi(f)$ is a smooth representation of $\mathscr{H}_G$ on V.

2. *The above construction gives a bijective correspondence between smooth representations of G and smooth representations of $\mathscr{H}_G$.*

3. *This correspondence takes admissible representations to admissible representations and irreducible representations to irreducible representations (in both directions).*

4. *Let (π, V) be a smooth representation of G or $\mathscr{H}_G$ and let U be a subspace of V. Then U is stable under G if and only if it is stable under $\mathscr{H}_G$.*

1) First note that the integral is well-defined; in fact, since f is locally constant and compactly supported and the map $g \mapsto \pi(g)v$ is locally constant (since the stabilizer of v is open) it follows that the function $f(g)\pi(g)v$ is a locally constant compactly supported function with values in a finite dimensional vector space. The integral therefore reduces to a finite sum.

We now check that $f \mapsto \pi(f)$ is a representation of $\mathscr{H}_F$. It is clearly a linear map. We have

$$\begin{aligned}\pi(f_1 * f_2)v &= \int_G (f_1 * f_2)(g)\pi(g)v dg = \int_G \int_G f_1(h)f_2(h^{-1}g)\pi(g)v dg dh \\ &= \int_G \int_G f_1(h)f_2(g)\pi(hg)dg dh = \pi(f_1)\pi(f_2)v.\end{aligned}$$

Thus π is an algebra homomorphism.

Finally we must check that π is a smooth representation of $\mathscr{H}_G$, *i.e.*, given v in V there exists an elementary idempotent ξ of $\mathscr{H}_G$ such that $\pi(\xi)v = v$. Let U' be the stabilizer of v; it is an open subgroup of G. Since G is totally disconnecte and locally compact it follows that U' contains a compact open subgroup U. The elementary idempotent χ_U will then stabilize v.

2) We now give an inverse construction, *i.e.*, associate to every smooth representation of $\mathscr{H}_G$ a smooth representation of G in a manner inverse to the above. Thus let V be a smooth representation of $\mathscr{H}_G$. Given a vector v in V we can write (by smoothness)

$$v = \sum_{i=1}^{n} \pi(f_i)v_i$$

where f_i belongs to $\mathscr{H}_G$ and v_i belongs to V (in fact, we can accomplish this with $n = 1$ and taking f_1 to be an elementary idempotent). If g belongs to G we then define

$$\pi(g)v = \sum_{i=1}^{n} \pi(\lambda(g)f_i)v_i.$$

We must check that this is well defined, *i.e.*, we must show that if $\sum_{i=1}^{n} \pi(f_i)v_i = 0$ then $w = \sum_{i=1}^{n} \pi(\lambda(g)f_i)v_i$ is also zero. Since w belongs to V we can find an elementary idempotent ξ which stabilizes w. We then have

$$w = \pi(\xi)w = \sum_{i=1}^{n} \pi(\xi * (\lambda(g)f_i))v_i = \sum_{i=1}^{n} \pi((\rho(g)\xi) * f_i)v_i = \pi(\rho(g)\xi) \sum_{i=1}^{n} \pi(f_i)v_i = 0.$$

Thus the action of G is well-defined and we obtain a representation π of G on V.

We now show that the representation of G thus obtained is smooth. Let v be an element of V and let χ_U be an elementary idempotent of $\mathscr{H}_G$ which stabilizes v. If g belongs to U then $\lambda(g)\xi = \xi$ so that

$$\pi(g)v = \pi(\lambda(g)\xi)v = \pi(\xi)v = v.$$

Thus the stabilizer of v contains the open subgroup U and is therefore open. Thus π is smooth.

It is clear that this construction is inverse to the previous construction in both directions. Thus the second statement is established.

3) We now prove that admissible representations correspond to admissible representations. Thus let (π, V) be a representation of G and $\mathscr{H}_G$ which is smooth. If U' is an open subgroup of G then it contains an open compact subgroup U of G. The space stabilized by U' is contained in the space stabilized by U. Furthermore, the space stabilized by U is equal to the image of $\pi(\chi_U)$. Therefore if V is admissible for $\mathscr{H}_G$ (so that the image of $\pi(\chi_U)$ is finite dimensional) then V is admissible for G. Similarly, if V is admissible for G (so that the stabilizer of U is finite dimensional) then V is admissible for $\mathscr{H}_G$. This proves that admissible representations correspond. That irreducible representations correspond follows from the fourth statement of the proposition.

4) This follows immediately from the formulae expressing the representations of G and $\mathscr{H}_G$ in terms of each other.

5.1.5 The Hecke algebra of a compact open subgroup

[narch-tdlc-hecke2]

16. Let G be a TDLC group and let K be a compact open subgroup of G. Let $\mathscr{H}_{G,K}$ be the subalgebra of $\mathscr{H}_G$ consisting of those functions which are K-bi-invariant. The algebra $\mathscr{H}_{G,K}$ is nothing other than the algebra $\mathscr{H}_G[\chi_K]$ in the notation of §1.4.1. In particular, χ_K is the multiplicative identity element of the algebra $\mathscr{H}_{G,K}$.

17. Note that if (π, V) is a representation of G then the subspace V^K of V stabilized by K is stable under the action of $\mathscr{H}_{G,K}$. In fact, in the notation of §1.4.1 the space V^K is nothing other than $V[\chi_K]$.

18 Proposition ([Bu] Prop. 4.2.3). [`narch-tdlc-hecke2-30`] *Let (π, V) be a smooth representation of the TDLC group G. Then π is irreducible if and only for all compact open subgroups K of G the space V^K is irreducible for the aciton of $\mathscr{H}_{G,K}$.*

This is nothing other than §1.4.5, proposition 113 rephrased into the present language.

5.1.6 Schur's lemma and the central quasi-character

[`narch-tdlc-schur`]

19 Proposition. [`narch-tdlc-schur-10`] *Let G be a TDLC group and let (π, V) be an irreducible admissible representation of G. Then any endomorphism of V commuting with the action of G is a scalar.*

Let A be an endomorphism of V commuting with the action of G. Let U be an open subgroup of G which stabilizes a nonzero subspace of V (such a U exists, *e.g.*, take the stabilizer of an element of V). The space V^U is finite dimensional (by admissibility) and taken to itself by A. Thus A has an eigenvector in V^U; say $Av = \lambda v$. The endomorphism $A - \lambda$ of V also commutes with the action of G; its kernel is a nonzero stable subspace and therefore all of V. Thus $A = \lambda$.

20. Let G be a TDLC group, let Z be the center of G and let π be a representation of G. We say that π *admits a central quasi-character* if there exists a quasi-character χ of Z such that $\pi(g) = \chi(g)$ for all g in Z. Proposition 19 implies that irreducible admissible representations admit central quasi-characters.

5.2 First notions and results for $\mathrm{GL}(2, F)$

[`narch-first`]

5.2.1 Notation

[`narch-first-not`]

21. In the present section we give notation that will be in effect for the remainder of chapter 5.

22. We denote by F a fixed local field. We use the following notations:

1. $\mathscr{O}_F$ is ring of integers in F;
2. $\mathfrak{p}$ is the maximal ideal of $\mathscr{O}_F$;
3. ϖ is a generator for $\mathfrak{p}$;
4. U_F is the group of units in $\mathscr{O}_F$;
5. $|\cdot|$ is the absolute value on F; and
6. ψ is a fixed nontrivial additive character of F — this is used to identify F with its Pontrjagin dual.

23. We let G_F denote the topological group $\mathrm{GL}(2,F)$. We also name several subgroups of G_F:

1. Z_F is the center of G_F, consisting of scalar matrices;
2. A_F is the group of diagonal matrices;
3. K_F is the compact open subgroup $\mathrm{GL}(2,\mathscr{O}_F)$;
4. B_F or P_F is the group of upper triangular matrices in G_F;
5. N_F is the group of unipotent matrices in B_F, *i.e.*, matrices of the form $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$.
6. D_F is the group of matrices of the form $\begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix}$.

The word "representation" will typically mean a representation of G_F on a complex vector space.

24. [`narch-first-not-20`] If X is a topological space and V a complex vector space we let $C(X,V)$ denote the functions on X taking values in V; we let $C^\infty(X,V)$ denote the smooth (*i.e.*, locally constant) functions; and we let $C_c^\infty(X,V)$ denote the smooth compactly supported functions on X. The space $C_c^\infty(X,V)$ is also denoted $\mathscr{S}(X,V)$ and is called the *Schwartz space*. For $V=\mathbb{C}$ we simply write $C(X)$, $C^\infty(X)$ and $\mathscr{S}(X)$.

25. [`narch-first-not-30`] We define representations ρ and λ of G_F on $C(G_F)$ by

$$(\rho(g)f)(h) = f(hg) \qquad (\lambda(g)f)(h) = f(g^{-1}h).$$

These are called the *right regular representation* and *left regular representation* respectively. It is clear that both $C^\infty(G_F)$ and $\mathscr{S}(G_F)$ are stable under both ρ and λ.

5.2.2 The Hecke algebra

[`narch-first-hecke`]

26. We denote by $\mathscr{H}_F$ the Hecke algebra of the TDLC group G_F, as discussed in §5.1.4. Moreover, if K is a compact subgroup of G_F then we denote by $\mathscr{H}_{F,K}$ the Hecke algebra $\mathscr{H}_{G_F,K}$ as discussed in §5.1.5. In the special case where $K=K_F$ is the standard maximal compact subgroup of G_F we denote by $\mathscr{H}_F^\circ$ the Hecke algebra $\mathscr{H}_{G_F,K_F}$; it is called the *spherical Hecke algebra*.

27. For an element σ of $\hat{K}_F$, *i.e.*, a finite dimensional irreducible representation σ of K_F, define a function $\xi=\xi_\sigma$ on K_F by $\xi(g)=(\deg\sigma)^{-1}\operatorname{tr}(\sigma g)$. We extend ξ by zero outside of K_F to obtain a function on all of G_F, still denoted ξ. It is clear that ξ belongs to the Hecke algebra $\mathscr{H}_F$ and that in this algebra it is an idempotent. We call ξ the idempotent *corresponding* to the representation σ.

If σ and σ' are distinct elements of $\hat{K}_F$ then it is easily seen that ξ and ξ' are orthogonal idempotents. Thus, given n distinct elements $\sigma_1,\ldots,\sigma_n$ of $\hat{K}_F$ the function $\xi=\xi_1+\cdots+\xi_n$ is an idempotent of $\mathscr{H}_F$.

It is not hard to see that the set of idempotents constructed in the previous paragraph is cofinal with the elementary idempotents of $\mathscr{H}_F$. For this reason, we will also call the idempotents of the previous paragraph "elementary."

5.2.3 The central quasi-character

[`narch-first-cqc`]

28. Note that the center Z_F of G_F can be identified with $F^\times$. In particular, the central quasi-character of a representation of G_F may be identified with a quasi-character of $F^\times$.

5.2.4 Twisting by quasi-characters

[`narch-first-twist`]

29. Let χ be a quasi-character of G_F and let (π, V) be a representation of G_F.

1. We denote again by χ the one dimensional representation of G_F given by $g \mapsto \chi(\det g)$. It is an admissible representation of G_F.

2. We define a representation $\chi \otimes \pi$ of G_F on the space V by the formula

$$(\chi \otimes \pi)(g) = \chi(\det g)\pi(g).$$

   We say that the representation $(\chi \otimes \pi, V)$ is the *twist* of the representation (π, V) by the quasi-character π. The twisted representation is smooth (resp. admissible, irreducible) if and only if the original representation is.

3. If the representation π admits a central quasi-character ω then so does the twist $\chi \otimes \pi$ and its central quasi-character is $\chi^2\omega$.

4. The contradgrediant of $\chi \otimes \pi$ is given by $\chi^{-1} \otimes \widetilde{\pi}$. In particular, the contragrediant of the one dimensional representation afforded by χ is the one dimensional representation afforded by χ^{-1}.

5.2.5 The contragredient representation

[`narch-first-contra`]

30 Proposition. [`narch-first-contra-10`] *Let (π_1, V_1) and (π_2, V_2) be two admissible representations of G_F and let β be a nonzero G_F-invariant bilinear form on $V_1 \times V_2$.*

1. The natural map $V_1 \to V_2^*$ induced by β has its image contained in $\widetilde{V}_2$.

2. If V_1 or V_2 is irreducible then β is nondegenerate.

3. If β is nondegenerate then π_1 is equivalent to the contragrediant of π_2.

This is nothing more than a rephrasing of §1.4.3, proposition 107 into the present language.

5.2.6 Finite dimensional irreducible admissible representations

[`narch-first-fd`]

31 Proposition ([JL] Prop. 2.7). [`narch-first-fd-10`] *Let (π, V) be a finite dimensional irreducible admissible representation of G_F. Then V is one dimensional and there is a quasi-character χ of $F^\times$ such that $\pi(g) = \chi(\det g)$.*

Let H be the kernel of π; since V is finite dimensional it is an open subgroup of G_F. Thus there exists $\epsilon > 0$ such that the matrix

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

belongs to H whenever $|x| < \epsilon$. If x is any element of F there exists an element a of $F^\times$ such that $|ax| < \epsilon$. Thus

$$\begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & ax \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

belongs to H for all $x \in F$. Similarly,

$$\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$$

belongs to H for all $x \in F$. These two types of matrices generate $\mathrm{SL}(2, F)$ and so H contains $\mathrm{SL}(2, F)$. Thus π factors through the determinant, and so $\pi(g) = \chi(\det g)$ for some homomorphism of $F^\times$ into $\mathbb{C}$. To see that χ is continuous observe that

$$\pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = \chi(a)I.$$

Since π is irreducible it follows that V is one dimensional.

32. Becuase of proposition 31, the finite dimensional irreducible admissible representations of G_F are not very interesting. We may thus confine ourselves to the infinite dimensional ones.

5.2.7 The Jacquet functor

[narch-first-jacq]

33. Let ψ_0 be an additive character of F, possibly trivial. Define a character of N_F, also written ψ_0, by

$$\psi_0\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right) = \psi_0(x).$$

By means of ψ_0, we can regard $\mathbb{C}$ as an N_F-module; for clarity, we write $\mathbb{C}_{\psi_0}$ to indicate that $\mathbb{C}$ is thus regarded.

34. The *Jacquet functor* (with respect to ψ_0) is the functor from the category of N_F-modules to the category of complex vector spaces given by $J_{\psi_0}(V) = V \otimes_{N_F} \mathbb{C}_{\psi_0}$. The space $J_{\psi_0}V$ is called the *Jacquet module* of V (with respect to ψ_0).

35. If ψ_0 is the trivial character then we write J in place of J_{ψ_0}. Note that J is simply the covariants functor.

36. [narch-first-jacq-30] Let (π, V) be a representation of N_F. There is a natural surjection $A : V \to J_{\psi_0}V$ given by $v \mapsto v \otimes 1$. It is clear that the kernel V' of A is the submodule of V generated by $\pi(g)v - \psi_0(g)v$ for $v \in V$ and $g \in N_F$.

37 Lemma ([Bu] Prop. 4.4.1). [narch-first-jacq-40] *Let (π, V) be a smooth representation of N_F. Then v is in the kernel of A if and only if*

$$[\textit{narch-first-jacq-40-1}] \int_{\mathfrak{p}^{-n}} \psi_0(-x)\pi(n_x)v dx = 0 \tag{1}$$

for n sufficiently large.

If $v' = \pi(n_\xi)v - \psi_0(\xi)v$ is a typical element of $\ker A$ and n is taken so that $\xi \in \mathfrak{p}^{-n}$ then

$$\int_{\mathfrak{p}^{-n}} \psi_0(-x)\pi(n_x)v' dx = \int_{\mathfrak{p}^{-n}} \psi_0(-x)\pi(n_{x+\xi})v dx - \int_{\mathfrak{p}^{-n}} \psi_0(-x+\xi)\pi(n_x)v dx.$$

The two integrals are seen to be equal after a change of variables. Thus v' satisfies (1).

Now assume that v is given satisfying (1) for some n. Take $m > -n$ sufficiently large so that 1) v is fixed by n_x when $x \in \mathfrak{p}^m$, and 2) ψ_0 is trivial on $\mathfrak{p}^m$. Thus the expression $\psi_0(-x)\pi(n_x)v$ is constant on the cosets of $\mathfrak{p}^m$ and so (1) may be written as

$$\sum_{x \in \mathfrak{p}^{-n}/\mathfrak{p}^m} \psi_0(-x)\pi(n_x)v = 0.$$

Therefore, if c is the reciprocal of the cardinality of $\mathfrak{p}^{-n}/\mathfrak{p}^m$, we have

$$v = v - c \sum_{x \in \mathfrak{p}^{-m}/\mathfrak{p}^n} \psi_0(-x)\pi(n_x)v = c \sum_{x \in \mathfrak{p}^{-m}/\mathfrak{p}^n} \psi_0(-x)\Big(\psi_0(x)v - \pi(n_x)v\Big).$$

which shows that v is an element of $\ker A$. This completes the proof.

38 Proposition. [narch-first-jacq-60] *The functor J_{ψ_0} from the category of smooth N_F-modules to the category of vector spaces is exact.*

Since J_{ψ_0} is defined as a tensor product it is automatically right exact. Thus we need to show that if $V_1 \to V_2$ is an injection of smooth $N(F)$-modules then $J_{\psi_0}V_1 \to J_{\psi_0}V_2$ is an injection. However, if we regard V_1 as a submodule of V_2 then it is clear from lemma 36 that $\ker A_1 = \ker A_2 \cap V_1$, and this implies that $J_{\psi_0}V_1 \to J_{\psi_0}V_2$ is injective.

39. If (π, V) is a representation of some group containing N_F (*e.g.*, G_F or D_F), we write $J_{\psi_0}V$ for the Jacquet module of $(\pi|N_F, V)$.

40 Theorem. [`narch-first-jacq-80`] *Let (π, V) be an irreducible admissible representation of G_F. Then:*

1. *The space $J_\psi V$ is at most one dimensional (where ψ is a nontrivial character).*
2. *The space JV is at most two dimensional.*

41. We will not prove theorem 40 in this section. It is a fairly difficult theorem to prove; indeed, for an infinite dimensional irreducible admissible representation (π, V) the three statements

1. $J_\psi V$ is one dimensional
2. π admits a unique Kirillov model
3. π admits a unique Whittaker model

are essentially equivalent.

42. Note that if (π, V) is a representation of G_F then $J_\psi V$ is a module over the center of G_F and JV is a module over N_F for which N_F acts trivially.

43 Proposition. *If (π, V) is an admissible representation of G_F then JV is an admissible representation of the maximal torus of G_F.*

Proof omitted; see Bump Theorem 4.4.4.

5.3 The Kirillov and Whittaker models

[`narch-kiri`]

5.3.1 The representation ξ_ψ of D_F on certain Schwartz spaces

[`narch-kiri-swz`]

44. [`narch-kiri-swz-10`] Let X be a complex vector space. Define a representation ξ_ψ of D_F on the space $C(F, X)$ and $C(F^\times, X)$)

$$\left(\xi_\psi \begin{bmatrix} a & x \\ 0 & 1 \end{bmatrix} \phi\right)(\beta) = \psi(\beta x)\phi(\beta a).$$

Both $C^\infty(F, X)$ and $\mathscr{S}(F, X)$ are stable subspaces of $C(F, X)$; also, both $C^\infty(F^\times, X)$ and $\mathscr{S}(F^\times, X)$ are stable subspaces of $C(F^\times, X)$.

45. The rest of this section consists of miscellaneous results which we will use later.

46 Proposition ([JL] Lemma 2.9.1). [`narch-kiri-swz-20`] *The representation $(\xi_\psi, \mathscr{S}(F^\times))$ of D_F is irreducible.*

If μ is a character of U_F let ϕ_μ be the element of $\mathscr{S}(F^\times)$ which is equal to μ on U_F and equal to 0 away from U_F. The space $\mathscr{S}(F^\times)$ is spanned by the ϕ_μ and their translates. Thus to prove the proposition it suffices to show that any stable subspace contains all of the ϕ_μ.

Let V be a stable subspace and let ϕ_0 be a nonzero element of V. Let ν be a character of U_F and consider the function

$$\phi = \phi_\nu \star \phi_0 = \int_{F^\times} \phi_\nu(y)\xi \begin{bmatrix} y^{-1} & 0 \\ 0 & 1 \end{bmatrix} \phi_0 dy.$$

The integral may be rewritten as a finite sum and so ϕ lies in V. Clearly,

$$\phi(\epsilon a) = \nu(\epsilon)\phi(a) \tag{2}$$

for all $a \in F^\times$ and $\epsilon \in U_F$. We now show that ν can be chosen so that $\phi \neq 0$. For $a \in F^\times$ we have

$$\phi(a) = \int_{U_F} \phi_0(ay^{-1})\nu(y)dy$$

and so $\phi(a)$ may be regarded as the value of the Fourier transform of the function $y \mapsto \phi_0(ay^{-1})$ on U_F at the character ν. Since there exists a such that this function is not identically zero on U_F it follows that there exists ν such that $\phi(a) \neq 0$. Thus ν can be chosen so that $\phi \neq 0$. We have therefore shown that V contains a nonzero function ϕ satisfying (2). In fact, we may scale and translate ϕ so that $\phi(1) = 1$; note that (2) then implies $\phi(\epsilon) = \nu(\epsilon)$ for $\epsilon \in U_F$.

We are now going to show that for $\mu \neq \nu$ the space V contains the function ϕ_μ. This will establish the proposition because we then replace ϕ by ϕ_μ with $\mu \neq \nu$ (U_F has at least two characters) and run the same argument to conclude that ϕ_ν is in V.

Set

$$\phi' = \int_{U_F} \mu^{-1}(\epsilon)\xi_\psi \begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix} \xi_\psi \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \phi d\epsilon$$

where x is to be determined. Since this integral is really a finite sum, the function ϕ' lies in V. Unravelling the definition of ξ_ψ gives

$$\phi'(a) = \int_{U_F} \mu^{-1}(\epsilon)\phi(\epsilon a)\psi(\epsilon a x)d\epsilon = \phi(a)\int_{U_F} \mu^{-1}(\epsilon)\nu(\epsilon)\psi(\epsilon a x)d\epsilon.$$

If we now take $x = \varpi^{-n-m}$ we find that, by §1.5.1, proposition 121, the integral is equal to $c\phi_{\mu\nu^{-1}}$ (where c is a nonzero constant) and so

$$\phi' = c\phi\phi_{\mu\nu^{-1}} = c\phi_\mu.$$

This proves the proposition.

47 Lemma ([JL] Lemma 2.13.3). [`narch-kiri-swz-30`] *Let ϕ be an element of $\mathscr{S}(F^\times)$. Then there exists a finite subset S of $F^\times$, complex numbers λ_y for $y \in S$, and an element ϕ_0 in $\mathscr{S}(F^\times)$ such that*

$$\phi = \sum_{y \in S} \lambda_y \xi_\psi(n_y)\phi_0.$$

The numbers λ_y satisfy

$$\sum_{y \in S} \lambda_y = 0 \qquad \sum_{y \in S} \lambda_y \psi(y) = \phi(1).$$

Extend ϕ to a function on all of F by setting $\phi(0) = 0$; thus ϕ is a locally constant compactly supported function on F, *i.e.*, an element of the Schwartz space $\mathscr{S}(F)$. Let ϕ' be the Fourier transform of ϕ; it too lies in the Schwartz space. Let Ω be an open compact set of $F^\times$ containing 1 and the support of ϕ. There is an ideal $\mathfrak{a}$ so that for all $x \in \Omega$ the function of y given by $\phi'(-y)\psi(xy)$ is constant on the cosets of $\mathfrak{a}$ in F. Let $\mathfrak{b}$ be an ideal containing $\mathfrak{a}$ and the support of ϕ'. We then have

$$\phi(x) = \int_F \phi'(-y)\psi(xy)dy = c\sum_{y \in \mathfrak{b}/\mathfrak{a}} \phi'(-y)\psi(xy)$$

where c is the measure of $\mathfrak{a}$. If ϕ_0 is the characteristic function of Ω (and thus an element of $\mathscr{S}(F^\times)$), the above relation may be written as

$$\phi = \sum_{y \in S} \lambda_y \xi_\psi(n_y)\phi_0$$

where $\lambda_y = c\phi'(-y)$ and S is a set of representatives for $\mathfrak{b}/\mathfrak{a}$. We have

$$0 = \phi(0) = \sum_{y \in S} \lambda_y \qquad \phi(1) = \sum_{y \in S} \lambda_y \psi(y).$$

The lemma is proved.

48 Lemma ([JL] Lemma 2.13.3). [`narch-kiri-swz-40`] *Let L be a linear functional on the Schwartz space $\mathscr{S}(F^\times)$ such that*

$$L(\xi_\psi(n_x)\phi) = \psi(x)L(\phi)$$

for all ϕ in $\mathscr{S}(F^\times)$ and all x in F. Then there is a scalar λ such that $L(\phi) = \lambda\phi(1)$.

It is sufficient to show $\phi(1) = 0$ implies $L(\phi) = 0$. Thus let ϕ be a function which vanishes at 1. Let S, λ_y and ϕ_0 be as in lemma 47. Since $\sum_{y\in S}\lambda_y\psi(y) = 0$ it follows that

$$\phi = \sum_{y\in S}\lambda_y\Big(\xi_\psi(n_y)\phi_0 - \psi(y)\phi_0\Big)$$

and so $L(\phi) = 0$.

49 Lemma ([JL] Lemma 2.15.2). [`narch-kiri-swz-50`] *The functions of the form $\xi(n_x)\phi - \phi$ with ϕ in $\mathscr{S}(F^\times)$ span $\mathscr{S}(F^\times)$. In other words, $J\mathscr{S}(F^\times) = 0$.*

Let ϕ be an element of $\mathscr{S}(F^\times)$. Let S, λ_y and ϕ_0 be as in lemma 47. Since $\sum_{y\in S}\lambda_y = 0$, we have

$$\phi = \sum_{y\in S}\lambda_y\Big(\xi_\psi(n_y)\phi_0 - \phi_0\Big)$$

and the result is proved.

50 Lemma ([JL] Lemma 2.21.1). [`narch-kiri-swz-60`] *Let T be a linear operator on $\mathscr{S}(F^\times)$ commuting with D_F. Then T is a scalar.*

Since $(\xi_\psi, \mathscr{S}(F^\times))$ is irreducible (*cf.* proposition 46), it suffices to show that T has an eigenvector.

Sublemma A. Let μ be a nontrivial character on U_F with conductor $1+\mathfrak{p}^n$. Let S be the operator on $\mathscr{S}(F^\times)$ defined by

$$S = \int_{U_F}\mu^{-1}(\epsilon)\xi_\psi\left(\begin{bmatrix}\epsilon & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & \varpi^{-m-n}\\ 0 & 1\end{bmatrix}\right)d\epsilon$$

where $\mathfrak{p}^{-m}$ is the conductor of ψ. Clearly T commutes with S.

Sublemma B. Let V be the subsapce of $\mathscr{S}(F^\times)$ consisting of all functions invariant under U_F. Thus ϕ lies in V if and only if

$$\phi = \xi_\psi\begin{bmatrix}\epsilon & 0\\ 0 & 1\end{bmatrix}\phi$$

holds for all ϵ in U_F. From this characterization, it is clear that T maps V into itself.

Sublemma C. If ϕ lies in V then

$$(S\phi)(a) = \int_{U_F}\mu^{-1}(\epsilon)\psi(a\epsilon\varpi^{-n-m})\phi(\epsilon a)d\epsilon = \eta(\mu^{-1}, a\varpi^{-n-m})\phi(a)$$

where η is the Gaussian sum of §1.5.1, proposition 121. From the evaluation of this sum, it follows that $(S\phi)(a) = 0$ if a is not in U_F. If ϵ is in U_F then

$$(S\phi)(\epsilon) = \eta(\mu^{-1}, \epsilon\varpi^{-n-m})\phi(\epsilon) = \mu(\epsilon)\eta(\mu^{-1}, \varpi^{-n-m})\phi(1).$$

Therefore, $S\phi$ is a multiple of the function ϕ_μ, which is defined to be zero outside of U_F and equal to μ on U_F. In fact, if $\phi(1) \neq 0$ then $S\phi$ is a nonzero multiple of ϕ_μ. We have thus shown that $SV = \mathbb{C}\phi_\mu$.

We are now essentially finished. We have

$$T(\mathbb{C}\phi_\mu) = TSV = STV = SV = \mathbb{C}\phi_\mu$$

and therefore ϕ_μ is an eigenvector of T.

5.3.2 The Kirillov model: overview

[`narch-kiri-over`]

51. Let (π, V) be a representation of G_F. A *Kirillov model* of π is a submodule of $(\xi_\psi, C(F^\times))$ (*cf.* article 44) which is isomorphic to the restriction of π to D_F on V.

52 Theorem. [narch-kiri-over-20] *Let (π, V) be an irreducible admissibile infinite dimensional representation of G_F. Then π has a unique Kirillov model.*

53. We first prove the existence of a Kirillov model and then we prove its uniqueness.

The first major step is proposition 56, in which a vector space X is defined in terms of the given representation (π, V) such that V may be realized as a subspace of $C^\infty(F^\times, X)$ with the restriction of π to D_F acting as ξ_ψ. We call this the *pre-Kirillov model.* The remainder of the section is devoted to proving that X is one dimensional, so that the pre-Kirillov model is in fact the Kirillov model.

There are three important classes of matrices in G_F for us: the matrices in D_F, the scalar matrices, and the matrix

$$w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

These three sorts of matrices generate the group G_F. For a representation in the pre-Kirillov form, we know how D_F acts. The aciton of the diagonal matrices is known as well (it acts by the central quasi-character). The only mystery is how the matrix w acts.

Before really starting into the action of w we give proposition 59, which states 1) that V contains the Schwartz space $V_0 = \mathscr{S}(F^\times, X)$, and 2) that V is spanned by V_0 and $\pi(w)V_0$. This gives us much better control on what the space V looks like, although it further underscores the necessity to determine what w does.

To really study w, we introduce a formal Mellin transform in article 60. In proposition 61 we determine how the three classes of matrices interact with the Mellin transform. For the scalar matrices and elements of D_F, this is pretty straightforward. For w the result is a little more complex: there are linear operators $C_n(\nu)$ acting on X (where n is an integer and ν is a character of U_F) such that for elements ϕ of V_0 the Mellin transform of $\pi(w)\phi$ can be expressed in terms of the operators $C_n(\nu)$ acting on the Mellin transform of ϕ. This is the tool that really lets us get at w.

Finally in proposition 64 we show that X is one dimensional by proving that X is irreducible under the action of the $C_n(\nu)$ but that these operators in fact act as scalars. This then completes the proof of the existence of the Kirillov model.

To prove uniqueness, we prove that the space of Whittaker functionals is one dimensional. To do this, we use the Kirillov model we have already constructed (but not, of course, the fact that it is unique). We then deduce the uniqueness of the Kirillov model from the uniqueness of the Whittaker functional.

5.3.3 The Kirillov model: proof of existence

[narch-kiri-ex]

54 Lemma ([JL] Prop. 2.7). [narch-kiri-ex-10] *Let (π, V) be an infinite dimensional irreducible admissible representation of G_F. Then there is no nonzero element of V stabilized by all of N_F.*

Assume there exists such a vector v. Let H be the stabilizer of v. As H is an open subgroup it contains a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $c \neq 0$. It therefore also contains

$$\begin{bmatrix} 1 & -ac^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -dc^{-1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b_0 \\ c & 0 \end{bmatrix} = w_0.$$

Given any $y \in F$, let $x = b_0 y/c$. Then

$$\begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} = w_0 \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} w_0^{-1}$$

is also in H. It now follows that H contains $\mathrm{SL}(2, F)$ (since it contains generators of this group).

The stabilizer of the subspace $\mathbb{C}v$ contains $F^\times H$ and so contains $F^\times \mathrm{SL}(2, F)$. Since this group is of finite index in G_F it follows that v is contained in a finite dimensional stable subspace. This is a contradiction and so no such v can exist.

55 Lemma ([JL] Lemma 2.8.1). [narch-kiri-ex-20] *Let $\mathfrak{p}^{-m}$ be the conductor of ψ is trivial and let f be a locally constant function on $\mathfrak{p}^{-\ell}$ with values in some finite dimensional complex vector space. For any integer $n \leq \ell$ the following two conditions are equivalent:*

1. *f is constant on the cosets of $\mathfrak{p}^{-n}$ in $\mathfrak{p}^{-\ell}$*
2. *The integral*
$$\int_{\mathfrak{p}^{-\ell}} \psi(-ax)f(x)dx$$
is zero for all a outside of $\mathfrak{p}^{-m+n}$.

($1 \implies 2$) Let a be an element of $F^\times$ which is not in $\mathfrak{p}^{-m+n}$. Then $x \mapsto \psi(-ax)$ is a nontrivial character of $\mathfrak{p}^{-n}$. Therefore

$$\int_{\mathfrak{p}^{-\ell}} \psi(-ax)f(x)dx = \sum_{y \in \mathfrak{p}^{-\ell}/\mathfrak{p}^{-n}} \psi(-ay)f(y) \int_{\mathfrak{p}^{-n}} \psi(-ax)dx = 0.$$

($2 \implies 1$) We may think of f as a locally constant function on F with support in $\mathfrak{p}^{-\ell}$. The second condition is essentially the statement that the Fourier transform f' of f has its support in $\mathfrak{p}^{-m+n}$. By the Fourier inversion formula, we thus have

$$f(x) = \int_{\mathfrak{p}^{-m+n}} \psi(-xy)f'(y)dy.$$

For $y \in \mathfrak{p}^{-m+n}$ the function $x \mapsto \psi(-xy)$ is constant on the cosets of $\mathfrak{p}^{-n}$; the proposition follows.

56 Proposition. [narch-kiri-ex-30] *Let (π, V) be an infinite dimensional irreducible admissible representation of G_F. Let A be the projection of V onto its Jacquet module $J_\psi V$ (cf. article 36). Then*

1. *For $v \in V$ let $\phi_v : F^\times \to J_\psi V$ be the function defined by*
$$\phi_v(a) = A\left(\pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} v\right).$$
The map $v \mapsto \phi_v$ is an injection of V into $C^\infty(F^\times, J_\psi V)$.
2. *The map $v \mapsto \phi_v$ is a map of D_F-modules; in other words $\phi_{\pi(d)v} = \xi_\psi(d)\phi_v$ for all $d \in D_F$ and $v \in V$.*

Let V' be the kernel of A. We will use lemma 37 without note.
1) Assume $\phi_v = 0$ identically; we must show that $v = 0$. Let $f(x) = \pi(n_x)v$ and let

$$F_\ell(a) = \int_{\mathfrak{p}^{-\ell}} \psi(-ax)f(x)dx.$$

By lemma 54 it suffices to show that f is constant. We know that f is constant on the cosets of $\mathfrak{p}^{-n_0}$ for some integer n_0. We will now prove by induction that if f is constant on the cosets of $\mathfrak{p}^{-n}$ then it is constant on the cosets of $\mathfrak{p}^{-n-1}$, which will prove the statement. First we establish three sublemmas.

Sublemma A. For $n < m$ we have

$$\begin{aligned} F_m(a) = \int_{\mathfrak{p}^{-m}} \psi(-ax)f(x)dx &= \sum_{y \in \mathfrak{p}^{-m}/\mathfrak{p}^{-n}} \psi(-ay)\pi(n_y) \int_{\mathfrak{p}^{-n}} \psi(-ax)f(x)dx \\ &= \sum_{y \in \mathfrak{p}^{-m}/\mathfrak{p}^{-n}} \psi(-ay)\pi(n_y)F_n(a). \end{aligned}$$

Thus if $F_n(a) = 0$ then $F_m(a) = 0$ for all $m > n$.

Sublemma B. Note that (since $\phi_v = 0$) given any $a \in F^\times$ the vector $\pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} v$ belongs to V'. This means that there exists n such that

$$\begin{aligned} 0 &= \int_{\mathfrak{p}^{-n}} \psi(-x)\pi \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} vdx = \int_{\mathfrak{p}^{-n}} \psi(-x)\pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \pi \begin{bmatrix} 1 & a^{-1}x \\ 0 & 1 \end{bmatrix} vdx \\ &= |a|\pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \int_{a^{-1}\mathfrak{p}^{-n}} \psi(-ax)f(x)dx. \end{aligned}$$

In other words, for all $a \in F^\times$ there exists n such that $F_n(a) = 0$.

Sublemma C. There exists an open subgroup U_1 of U_F such that $\pi \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} v = v$ for $b \in U_1$. For such b we have

$$\pi \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} \int_{\mathfrak{p}^{-\ell}} \psi(-ax) f(x) dx = \int_{\mathfrak{p}^{-\ell}} \psi(-ax) \pi \begin{bmatrix} 1 & bx \\ 0 & 1 \end{bmatrix} \pi \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} v dx = \int_{\mathfrak{p}^{-\ell}} \psi(-(a/b)x) f(x) dx.$$

Thus if $F_\ell(a) = 0$ then $F_\ell(ba) = 0$ for any b in U_1.

We now prove the induction. Assume f is constant on the cosets of $\mathfrak{p}^{-n}$. Let $\mathfrak{p}^{-m}$ be the largest ideal on which ψ is trivial. Let $a_1, \ldots, a_r$ be representatives for the classes of generators of $\mathfrak{p}^{-m+n}$ modulo U_1 (so that any generator of $\mathfrak{p}^{-m+n}$ lies in $U_1 a_i$ for some i). By sublemma B, for each i there exists ℓ_i such that $F_{\ell_i}(a_i) = 0$. Let ℓ' be the maximum of the ℓ_i and $n+1$. By sublemma A we have that $F_\ell(a_i) = 0$ for any $\ell \geq \ell'$. By sublemma C it follows that $F_\ell(a) = 0$ if a is a generator of $\mathfrak{p}^{-m+n}$ and $\ell \geq \ell'$.

Let $\ell \geq \ell'$. Since f is constant on the cosets of $\mathfrak{p}^{-n}$ inside the ideal $\mathfrak{p}^{-\ell}$ and the restriction of f to $\mathfrak{p}^{-\ell}$ takes values in a finite dimensional vector space, lemma 55 implies that $F_\ell(a) = 0$ if a is outside of $\mathfrak{p}^{-m+n}$. However, we just proved above that $F_\ell(a) = 0$ if a is a generator of $\mathfrak{p}^{-m+n}$; thus $F_\ell(a) = 0$ for any a outside of $\mathfrak{p}^{-m+n+1}$. Applying lemma 55 again shows that f is constant on the cosets of $\mathfrak{p}^{-n-1}$ insides of $\mathfrak{p}^{\ell}$. Since this holds for all $\ell \geq \ell'$ it follows that f is constant on the cosets of $\mathfrak{p}^{-n-1}$ in all of F. This proves the first part of the proposition.

2) To prove the second assertion, it is sufficient to show that

$$A\left(\pi \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} v\right) = \psi(y) A(v)$$

for all $v \in V$ and $y \in F$. This is equivalent to showing that $\pi(n_y)v - \psi(y)v$ lies in V', which is true (almost) by definition of J_ψ (*cf.* article 36).

57. Because of proposition 56 we may identify v with ϕ_v. When we do this, we say that π is in *pre-Kirillov form.* In this case, the map A takes the form $\phi \mapsto \phi(1)$. For $d \in D_F$, we know that $\pi(d) = \xi_\psi(d)$. Because of this, the representation is determined entirely by its central quasicharacter and $\pi(w)$ where

$$w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

58 Lemma. `[narch-kiri-ex-50]` *Let (π, V) be an infinite dimensional irreducible admissible representation in pre-Kirillov form. Then*

1. *For all ϕ in V we have $\phi(a) = 0$ for $|a|$ sufficiently large.*
2. *For all ϕ in V and $g \in N_F$ the function $\phi - \pi(g)\phi$ lies in the Schwartz space $\mathscr{S}(F^\times, J_\psi V)$.*

1) Let $\phi \in V$. There exists n so that

$$\pi \begin{bmatrix} a & x \\ 0 & 1 \end{bmatrix} \phi = \phi$$

if x and $a-1$ belong to $\mathfrak{p}^n$. In particular, for $x \in \mathfrak{p}^n$ we have

$$(1 - \psi(ax))\phi(a) = 0.$$

If $\mathfrak{p}^{-m}$ is the conductor of ψ and x is a generator for $\mathfrak{p}^n$ then $\psi(ax) = 1$ only if a is in $\mathfrak{p}^{-m-n}$. Thus $\phi(a) = 0$ unless a is in $\mathfrak{p}^{-m-n}$.

2) Let $\phi \in V$ and let $\phi' = \phi - \pi(n_x)\phi$. Observe that $\phi'(a) = (1 - \psi(ax))\phi(a)$ is identically zero for $x = 0$ and otherwise vanishes at least on $x^{-1}\mathfrak{p}^{-m}$ (where $\mathfrak{p}^{-m}$ is the conductor of ψ). Combining this with part 1 of the lemma proves part 2.

59 Proposition ([JL] Prop. 2.9). [narch-kiri-ex-60] *Let (π, V) be an infinite dimensional admissible irreducible representation in pre-Kirillov form. Then*

1. *V contains $V_0 = \mathscr{S}(F^\times, J_\psi V)$*
2. *V is spanned by V_0 and $\pi(w)V_0$.*

1) Let V_0 be the space of ϕ in V such that $\phi(a)$ vanishes for $|a|$ sufficiently small. By lemma 58 it follows that $V_0 \subset \mathscr{S}(F^\times, J_\psi V)$. We prove the other containment in a series of sublemmas:

Sublemma A. Given $u \in J_\psi V$ there exists ϕ in V such that $\phi(1) = u$ (since the projection map A is by definition surjective). If we now take $x \in F$ such that $\psi(x) \neq 0$ then $\phi' = \phi - \pi(n_x)\phi$ belongs to V_0 by lemma 58 and $\phi'(1) = (1 - \psi(x))u$ is a nonzero multiple of u. Thus for all $u \in J_\psi V$ there exists $\phi \in V_0$ such that $\phi(1) = u$.

Sublemma B. If μ is a character of U_F let $V_0(\mu)$ be the space of functions ϕ in V_0 such that

$$\phi(\epsilon a) = \mu(\epsilon)\phi(a)$$

for all $a \in F^\times$ and $\epsilon \in U_F$. By using the Fourier transform, one easily sees that V_0 is the direct sum of the $V_0(\mu)$. Therefore (applying sublemma A) every vector in u can be written as a finite sum $u = \sum \phi_i(1)$ where ϕ_i belongs to $V_0(\mu_i)$.

Sublemma C. Let ϕ belong to $V(\nu)$ and let $u = \phi(1)$. Let μ be different from ν and let $1 + \mathfrak{p}^n$ be the conductor of $\mu^{-1}\nu$. Define

$$\phi' = \int_{U_F} \mu^{-1}(\epsilon)\xi_\psi \begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix} \xi_\psi \begin{bmatrix} 1 & \varpi^{-n-m} \\ 0 & 1 \end{bmatrix} \phi d\epsilon.$$

The function ϕ' belongs to V. We have

$$\phi'(a) = \int_{U_F} \mu^{-1}(\epsilon)\phi(\epsilon a)\psi(\epsilon a \varpi^{-n-m})d\epsilon = \phi(a)\int_{U_F} \mu^{-1}(\epsilon)\nu(\epsilon)\psi(\epsilon a\varpi^{-n-m})d\epsilon.$$

Applying §1.5.1, proposition 121, we find

$$\phi'(a) = c\phi(a)\phi_{\mu\nu^{-1}}$$

where c is a nonzero constant and $\phi_{\mu\nu^{-1}}$ is the function which is zero outside of U_F and equal to $\mu\nu^{-1}$ on U_F. From this expression, we see that ϕ' belongs to V_0 and that it takes values in the space $\mathbb{C}u$. We have therefore shown that if $u \in J_\psi V$ is of the form $\phi(1)$ where $\phi \in V(\nu)$ then there exists η in $\mathscr{S}(F^\times)$ such that ηu belongs to V_0.

Sublemma D. Proposition 46 can be applied to immediately stengther sublemma C: if u is of the form $\phi(1)$ where $\phi \in V(\nu)$ then V_0 contains *all* functions of the form ηu where η is in $\mathscr{S}(F^\times)$.

Sublemma E. Let u be an element of $J_\psi V$ and let η be in $\mathscr{S}(F^\times)$. By sublemma B we can write $u = \sum u_i$ where each u_i lies in $V(\mu_i)$ for some μ_i. By sublemma D V_0 contains all the functions ηu_i. Therefore we see that V_0 contains ηu. This implies that $V_0 \supset \mathscr{S}(F^\times, J_\psi V)$.

Thus the first statement is proved.

2) Let P_F be the group of upper triangular matrices in G_F. Since A) V_0 is stable under P_F; B) V is irreducible under G_F; and C) G_F is the union of P_F and $N_F w P_F$, it follows that V is spanned by V_0 and the vectors

$$\phi' = \pi \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \pi(w)\phi$$

for $\phi \in V_0$. But

$$\phi' = \left(\pi \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} (\pi(w)\phi) - (\pi(w)\phi)\right) + \pi(w)\phi$$

and the first term is in V_0 by sublemma B. Thus ϕ' belongs to $V_0 + \pi(w)V_0$. This proves the second statement.

60. [narch-kiri-ex-70] Let ϕ be a locally constant function on $F^\times$ with values in X. For a character ν of U_F, put

$$\hat{\phi}_n(\nu) = \int_{U_F} \phi(\epsilon\varpi^n)\nu(\epsilon)d\epsilon$$

where $d\epsilon$ is the normalized Haar measure on U_F Note that $\epsilon \mapsto \phi(\epsilon\varpi^n)$ is a locally constant function on the compact space U_F. This implies 1) that it takes values in a finite dimensional subspace of X and so the above integral is well defined, and 2) for fixed n the function $\hat{\phi}_n(\mu)$ is nonzero for only finitely many values of μ. We let $\hat{\phi}(\nu, t)$ be the formal series

$$\hat{\phi}(\nu, t) = \sum_{n\in\mathbb{Z}} \hat{\phi}_n(\nu).$$

We call $\hat{\phi}(\nu, t)$ the *formal Mellin transform* of ϕ.

If (π, V) is a representation of G_F in pre-Kirillov form and ϕ is an element of V we use the convention

$$\pi(g)\hat{\phi}(\nu, t) = \left(\widehat{\pi(g)\phi}\right)(\nu, t).$$

Note that $\phi(a)$ always vanishes for $|a|$ sufficiently large (*cf.* lemma 58). It follows that $\hat{\phi}(\nu, t)$ will have only finitely many negative terms. If ϕ belongs to the Schwartz space V_0 then $\hat{\phi}(\nu, t)$ is a Laurent polynomial.

61 Proposition ([JL] Prop. 2.10). [narch-kiri-ex-80] *Let (π, V) be an infinite dimensional admissible irreducible representation of G_F in pre-Kirillov form. For a character μ of U_F and an element x of F, recall that (cf. §1.5.1, proposition 121)*

$$\eta(\mu, x) = \int_{U_F} \mu(\epsilon)\psi(\epsilon x)d\epsilon.$$

where $d\epsilon$ is the normalized Haar measure.

1. *If δ belongs to U_F and ℓ is an integer then*

$$\pi\begin{bmatrix} \delta\varpi^\ell & 0 \\ 0 & 1 \end{bmatrix}\hat{\phi}(\nu, t) = t^{-\ell}\nu^{-1}(\delta)\hat{\phi}(\nu, t).$$

2. *If x is an element of F then*

$$\pi\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\hat{\phi}(\nu, t) = \sum_n \left[t^n\Big(\sum_\mu \eta(\mu^{-1}\nu, \varpi^n x)\hat{\phi}_n(\mu)\Big)\right]$$

where the inner sum is over all characters μ of U_F.

3. *Let ω_0 be the central quasicharacter of π. Let ν_0 be its restriction to U_F and let $z_0 = \omega_0(\varpi)$. For each character ν of U_F there is a formal series $C(\nu, t)$ with coefficients in the space of linear operators on $J_\psi X$ so that for every ϕ in V_0 we have*

$$\pi\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\hat{\phi}(\nu, t) = C(\nu, t)\hat{\phi}(\nu^{-1}\nu_0^{-1}, t^{-1}z_0^{-1}).$$

1) Let

$$\phi' = \pi\begin{bmatrix} \delta\varpi^\ell & 0 \\ 0 & 1 \end{bmatrix}\phi.$$

Then

$$\hat{\phi}'_n(\nu) = \int_{U_F} \nu(\epsilon)\phi(\delta\varpi^{n+\ell})\epsilon d\epsilon = \nu^{-1}(\epsilon)\hat{\phi}_n(\nu)$$

and the first part follows.

2) Let

$$\phi' = \pi\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\phi.$$

Then

$$\hat{\phi}'_n = \int_{U_F} \nu(\epsilon)\psi(\epsilon\varpi^n x)\phi(\epsilon\varpi^n)d\epsilon. \tag{3}$$

Now, the Fourier transform of the function $\epsilon \mapsto \nu(\epsilon)\psi(\epsilon\varpi^n x)$ on U_F at the character μ is

$$\int_{U_F} \mu(\epsilon)\nu(\epsilon)\psi(\epsilon\varpi^n x)d\epsilon = \eta(\mu\nu, \varpi^n x)$$

and so by the Fourier inversion formula we have

$$\nu(\epsilon)\psi(\epsilon\varpi^n x) = \int_{\hat{U}_F} \mu(\epsilon)\eta(\mu^{-1}\nu, \varpi^n x)d\mu.$$

Inserting this into (3), and chaning the order of integration, yields

$$\hat{\phi}'_n = \int_{\hat{U}_F} \eta(\mu^{-1}\nu, \varpi^n x)\hat{\phi}_n(\mu)d\mu.$$

Note that this is really a finite sum since $\hat{\phi}_n(\mu)$ is nonzero for only finitely many μ. This proves the second part of the proposition.

3) We proceed by a series of sublemmas.

Sublemma A. Let ν be a character of U_F and suppose $\phi \in V_0$ satisfies $\hat{\phi}(\mu, t) = 0$ unless $\mu = (\nu\nu_0)^{-1}$. This condition is equivalent to the condition

$$\phi(\epsilon a) = (\nu\nu_0)(\epsilon)\phi(a)$$

for all $\epsilon \in U_F$ and $a \in F^\times$, or to the condition

$$\pi \begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix} \phi = (\nu\nu_0)(\epsilon)\phi$$

for all $\epsilon \in U_F$. Now let $\phi' = \pi(w)\phi$. Then

$$\begin{aligned} \pi \begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix} \phi' &= \pi \begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix} \pi(w)\phi = \pi(w)\pi \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} \phi \\ &= \pi(w)\pi \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{bmatrix} \phi = \nu^{-1}(\epsilon)\pi(w)\phi \\ &= \nu^{-1}(\epsilon)\phi'. \end{aligned}$$

Therefore $\hat{\phi}'(\mu, t) = 0$ unless $\mu = \nu$.

Sublemma B. This "sublemma" is not so much a lemma as a defintion: namely the definition of the series $C(\nu, t)$. Let u be in $J_\psi V$, let ν be a character of U_F, and let ϕ be the element of V_0 defined by

$$\phi(\epsilon) = \begin{cases} \nu(\epsilon)\nu_0(\epsilon)u & \epsilon \in U_F \\ 0 & \epsilon \notin U_F \end{cases} \tag{4}$$

Let $\phi' = \pi(w)\phi$. The expression $\hat{\phi}'_n(\nu)$ is a function of n, ν and u and depends linearly on u; we may therefore write

$$\hat{\phi}'_n(\nu) = C_n(\nu)u$$

where $C_n(\nu)$ is a linear operator on $J_\psi V$. We define the formal series

$$C(\nu, t) = \sum_n t^n C_n(\nu).$$

Sublemma C. We now verify the third part of the proposition for functions ϕ of the form (4). We have

$$\pi(w)\hat{\phi}(\mu, t) = \begin{cases} C(\nu, t)u & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

The top case is by the definition given in sublemma B; the bottom case follows from sublemma A. On the other hand, computing the Fourier transform of ϕ gives

$$\hat{\phi}(\mu,t) = \begin{cases} u & \mu = (\nu\nu_0)^{-1} \\ 0 & \mu \neq (\nu\nu_0)^{-1} \end{cases}$$

and so we obtain

$$C(\mu,t)\hat{\phi}(\mu^{-1}\nu_0^{-1}, t^{-1}z_0^{-1}) = \begin{cases} C(\nu,t)u & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

Thus we have shown

$$\pi(w)\hat{\phi}(\mu,t) = C(\mu,t)\hat{\phi}(\mu^{-1}\nu_0^{-1}, t^{-1}z_0^{-1})$$

for all characters μ; this completes the verification.

Sublemma D. We now verify the third part of the proposition for translates of functions of the form (4) by powers of ϖ; that is, we verify it for functions of the form

$$\phi' = \pi \begin{bmatrix} \varpi^\ell & 0 \\ 0 & 1 \end{bmatrix} \phi$$

where ϕ is of the form (4). By part 1 of this proposition, we have

$$C(\mu,t)\hat{\phi}'(\mu^{-1}\nu_0^{-1}, t^{-1}z_0^{-1}) = \left(z_0^\ell t^\ell\right) \times \Big(C(\mu,t)\hat{\phi}(\mu^{-1}\nu_0^{-1}, t^{-1}z_0^{-1})\Big). \tag{5}$$

On the other hand, we have

$$\begin{aligned} \pi(w)\hat{\phi}'(\mu,t) = \pi(w)\begin{bmatrix} \varpi^\ell & 0 \\ 0 & 1 \end{bmatrix}\hat{\phi}(\mu,t) &= \begin{bmatrix} 1 & 0 \\ 0 & \varpi^\ell \end{bmatrix}\pi(w)\hat{\phi}(\mu,t) \\ &= \begin{bmatrix} \varpi^\ell & 0 \\ 0 & \varpi^\ell \end{bmatrix}\begin{bmatrix} \varpi^{-\ell} & 0 \\ 0 & 1 \end{bmatrix}\pi(w)\hat{\phi}(\mu,t) \\ &= \left(z_0^\ell t^\ell\right) \times \left(\pi(w)\hat{\phi}(\mu,t)\right) \end{aligned}$$

where we applied part 1 of this proposition in the last step. Sublemma C now implies that the last line in the above equation is equal to the right hand side of (5). This completes the verification in this case.

Sublemma D in fact completes the proof of part 3: the equation which must be verified is linear in ϕ and the functions considered in sublemma D form a basis of the space V_0. Thus the proposition is proved.

62. [`narch-kiri-ex-90`] Note that for a given $u \in J_\psi V$ and a given character ν of U_F there exists a function ϕ in V_0 (in fact, it is the function given in sublemma B of proposition 61) such that $\hat{\phi}(\nu,t) = \sum_n t^n C_n(\nu)u$. Since $\hat{\phi}(\nu,t)$ is a Laurent series, it follows that there exists n_0 (depending on u and ν) such that for $n < n_0$ we have $C_n(\nu)u = 0$.

63 Proposition ([JL] Prop. 2.11). [`narch-kiri-ex-100`] *Let (π, V) and other notations be as in proposition 61. Let $\mathfrak{p}^{-m}$ be the conductor of ψ. Let ν and ρ be two characters of U_F, let $\chi = \nu\rho\nu_0$ and let*

$$S = S(n,\nu,p,\rho) = \sum_{\sigma \in \hat{U}_F} \eta(\sigma^{-1}\nu, \varpi^n)\eta(\sigma^{-1}\rho, \varpi^p)C_{p+n}(\sigma).$$

1. *If χ is not trivial and has conductor $1 + \mathfrak{p}^k$ then*

$$S = z_0^{m+k}\chi(-1)\eta(\chi^{-1}, \varpi^{-m-k})C_{n-m-k}(\nu)C_{p-m-k}(\rho)$$

for all integers n and p.

2. *If χ is trivial then*

$$S = z_0^p\nu_0(-1)\delta_{n,p} - (1 - |\varpi|)^{-1}z_0^{m+1}C_{n-m-1}(\nu)C_{p-m-1}(\rho) - \sum_{r=-m-2}^{-\infty} z_0^{-r}C_{n+r}(\nu)C_{p+r}(\rho)$$

for all integers n and p (here δ is the Kronecker delta).

Note that these are infinite sums of operators on $J_\psi V$, but when they are applied to any specific element of $J_\psi V$ all but finitely many terms vanish. This is the sense in which these equalities are to be taken.

We begin with four sublemmas.

Sublemma A. The relation

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = - \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

implies that

$$\pi(w)\pi \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \pi(w)\phi = \nu_0(-1)\pi \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \pi(w)\pi \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \phi$$

for all ϕ in V_0. This may be rewritten as

$$\pi(w)\left(\pi \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \pi(w)\phi - \pi(w)\phi\right) + \pi(w)^2\phi = \nu_0(-1)\pi \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \pi(w)\pi \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \phi. \tag{6}$$

Note that the term in parentheses on the left lies in V_0 (*cf.* lemma 58) and that $\pi(w)^2\phi = \nu_0(-1)\phi$.

Sublemma B. We compute the Mellin transform of the right side of (6). We shall use proposition 61 without stopping to say so. We have

$$\pi \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \hat{\phi}(\nu, t) = \sum_n \left[\sum_\rho \eta(\rho^{-1}\nu, -\varpi^n)\hat{\phi}_n(\rho)\right] t^n$$

and

$$\pi(w)\pi \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \hat{\phi}(\nu, t) = \sum_n \left[\sum_{p,\rho} \eta(\rho^{-1}\nu^{-1}\nu_0^{-1}, -\varpi^p) z_0^{-p} C_{p+n}(\nu)\hat{\phi}_p(\rho)\right] t^n$$

and so the Mellin transform is

$$\nu_0(-1)\sum_n \left[\sum_{p,\rho,\sigma} \eta(\sigma^{-1}\nu, -\varpi^n)\eta(\rho^{-1}\sigma^{-1}\nu_0^{-1}, -\varpi^p) z_0^{-p} C_{p+n}(\sigma)\hat{\phi}_p(\rho)\right] t^n.$$

Sublemma C. We now compute the Mellin transform of the left side of (6). We have

$$\pi(w)\hat{\phi}(\nu, t) = \sum_n \left[\sum_p z_0^{-p} C_{p+n}(\nu)\hat{\phi}_p(\nu^{-1}\nu_0^{-1})\right] t^n$$

and

$$\pi \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \pi(w)\hat{\phi}(\nu, t) = \sum_n \left[\sum_{p,\rho} z_0^{-p}\eta(\rho^{-1}\nu, \varpi^n) C_{p+n}(\rho)\hat{\phi}_p(\rho^{-1}\nu_0^{-1})\right] t^n$$

so that

$$\pi \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \pi(w)\hat{\phi}(\nu, t) - \pi(w)\hat{\phi}(\nu, t)$$

is equal to

$$\sum_n \left[\sum_{p,\rho} z_0^{-p}\Big(\eta(\rho\nu\nu_0, \varpi^n) - \delta(\rho\nu\nu_0)\Big) C_{p+n}(\rho^{-1}\nu_0^{-1})\hat{\phi}_p(\rho)\right] t^n.$$

Here $\delta(\chi)$ is 1 if χ is the trivial character and zero otherwise. The Mellin tranform of the left side is therefore

$$\sum_n \left[\nu_0(-1)\hat{\phi}_n(\nu) + \sum_{p,r,\rho} z_0^{-p-r}\Big(\eta(\rho\nu^{-1}, \varpi^r) - \delta(\rho\nu^{-1})\Big) C_{n+r}(\nu) C_{p+r}(\rho^{-1}\nu_0^{-1})\hat{\phi}_p(\rho)\right] t^n.$$

Sublemma D. Note that if

$$\sum_{n,p,\rho} c_{n,p,\rho} t^n \hat{\phi}_p(\rho) = \sum_{n,p,\rho} c'_{n,p,\rho} t^n \hat{\phi}_p(\rho)$$

holds for all $\phi \in V_0$ then $c = c'$. We may therefore equate the coefficients of $t^n \hat{\phi}_p(\rho)$ in the results of sublemmas B and C to obtain the identity

$$\begin{aligned}&\nu_0(-1)\sum_\sigma \eta(\sigma^{-1}\nu, -\varpi^n)\eta(\rho^{-1}\sigma^{-1}\nu_0^{-1}, -\varpi^p) z_0^{-p} C_{p+n}(\sigma)\\ =&\nu_0(-1)\delta_{n,p}\delta(\rho\nu^{-1}) + \sum_r \Big(\eta(\rho\nu^{-1}, \varpi^r) - \delta(\rho\nu^{-1})\Big) z_0^{-p-r} C_{n+r}(\nu) C_{p+r}(\rho^{-1}\nu_0^{-1}),\end{aligned}$$

valid for all n, p, ρ and ν. If we now replace ρ by $\rho^{-1}\nu_0^{-1}$, and use the fact that $\eta(\mu, -x) = \mu(-1)\eta(\mu, x)$, we obtain the identity

$$\chi(-1) z_0^{-p} S = \nu_0(-1)\delta_{n,p}\delta(\chi) + \sum_r \Big(\eta(\chi^{-1}, \varpi^r) - \delta(\chi)\Big) z_0^{-p-r} C_{n+r}(\nu) C_{p+r}(\rho), \tag{7}$$

again, valid for all n, p, ρ and ν.

1) If χ is nontrivial with conductor $1+\mathfrak{p}^k$ then the gaussian sum $\eta(\chi^{-1}, \varpi^r)$ is zero unless $r = -m-k$ (*cf.* §1.5.1, proposition 121). Thus the right side of (7) reduces to

$$z_0^{-p+m+k}\eta(\chi^{-1}, \varpi^{-m-k}) C_{n-m-k}(\nu) C_{p-m-k}(\rho)$$

which establishes the first statement.

2) If χ is trivial then, using the evaluation of $\eta(\chi, \varpi^r)$ given in §1.5.1, proposition 121, we find that the right side of (7) is equal to

$$\nu_0(-1)\delta_{n,p} - (1 - |\varpi|)^{-1} z_0^{-p+m+1} C_{n-m-1}(\nu) C_{n-m-1}(\rho) - \sum_{r=-m-2}^{-\infty} z_0^{-p-r} C_{n+r}(\nu) C_{n+r}(\rho)$$

and the proposition is proved.

64 Proposition ([JL] Prop. 2.12). [narch-kiri-ex-110] *Let (π, V) and other notations be as in proposition 61.*

1. *For all integers n and p and all characters ν and ρ of U_F the operators $C_n(\nu)$ and $C_p(\rho)$ commute.*
2. *There is no nontrivial subspace of $J_\psi V$ stable under all the operators $C_n(\nu)$.*
3. *The space $J_\psi V$ is one dimensional.*

1) Let $\chi = \rho\nu\nu_0$. There are two cases:

Case A: χ nontrivial. Let $1 + \mathfrak{p}^k$ be the conductor of χ. Since $S(n, \nu, p, \rho)$ (*cf.* proposition 63) is symmetric in (n, ν) and (p, ρ) it follows from proposition 63 that

$$\eta(\chi, \varpi^{-m-k}) C_{n-m-k}(\nu) C_{p-m-k}(\rho)$$

is symmetric as well. Since the η coefficient does not vanish, we conclude that the expression $C_n(\nu)C_p(\rho)$ is symmetric, *i.e.*, that $C_n(\nu)$ and $C_p(\rho)$ commute.

Case B: χ trivial. Fix a vector u in $J_\psi V$. Consider the expression

$$C_{n+r}(\nu) C_{p+r}(\rho) u = C_{p+r}(\rho) C_{n+r}(\nu) u \tag{8}$$

By the comments in article 62, it follows that both sides are equal to 0 if r is large and negative. Using the second part of proposiiton 63 we then conclude, by induction, that (8) holds for all r. Since u was arbitrary, it follows that (8) holds for all u and therefore $C_n(\nu)$ and $C_p(\rho)$ commute.

2) Let X_1 be a nontrivial subspace of $J_\psi V$ stable under all the operator $C_n(\nu)$. Let V_1 be the space of all functions in V_0 which take values in X_1 and let V_1' be the stable subspace generated by V_1. Note that V_1, and therefore V_1' are nonempty.

Consider following three facts: A) the subspace of V taking values in X_1 is stable under P_F; B) if ϕ is a function in V_0 taking values in X_1 then part 3 of proposition 61 and the present assumptions imply that $\pi(w)\phi$ takes values in X_1; C) the Bruhat decomposition: $G_F = P_F \amalg P_F w P_F$. The three statements together imply that all elements of V_1' take values in X_1. Thus V_1 is a proper nontrivial stable subspace of V. This contradicts the fact that V is irreducible and therefore no such space X_1 exists.

3) To prove this statement we will show that the $C_n(\nu)$ all act as scalars; the result will then follow from part 2. We will actually show that any operator commuting with all the $C_n(\nu)$ is a scalar; since the $C_n(\nu)$ are themselves such operators by part 1, the result will follow. Thus let T be an operator on X commuting with all the $C_n(\nu)$.

If ϕ is an element of V let $T\phi$ be the element of $C(F^\times, J_\psi V)$ defined by $(T\phi)(a) = T(\phi(a))$. Clearly, T takes the space V_0 (which, recall, is simply the Schwartz space) to itself. If ϕ_0 is in V_0 we find

$$T\pi(w)\phi_0 = \pi(w)T\phi_0$$

by examining the Mellin transforms of both sides. Since $V = V_0 + \pi(w)V_0$ (*cf.* proposition 59) it follows that T maps V into itself.

We now show that T commutes with the action of G_F on V. Once this is accomplished it will follow that T is a scalar (*cf.* proposition 19) and we will be finished. It is clear that T commutes with the action of P_F. Therefore, we need only show that T commutes with the action of w, *i.e.*, for all ϕ in V that

$$T\pi(w)\phi = \pi(w)T\phi.$$

We have already done this when ϕ lies in V_0; thus it suffices to check it when $\phi = \pi(w)\phi_0$ and ϕ_0 lies in V_0. In this case, the left hand side equals

$$T\pi(w)^2\phi_0 = \nu_0(-1)T\phi_0,$$

while the right hand side equals

$$\pi(w)T\pi(w)\phi_0 = \pi(w)^2T\phi_0 = \nu_0(-1)T\phi_0.$$

Thus the proposition is proved.

65. Note that proposition 64 (together with our knowledge of finite dimensional irreducible admissible representations, *cf.* proposition 31) proves part 1 of theorem 40.

66. Proposition 64 allows us to identify $J_\psi V$ with $\mathbb{C}$. Thus the pre-Kirillov model is really a Kirillov model; the existence part of theorem 52 is thus established.

5.3.4 Uniqueness of the Whittaker functional

[`narch-kiri-unwfn`]

67. Let (π, V) be a representation of G_F. A *Whittaker functional* on V is a linear form L on V which satisfies

$$L\left(\pi\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} v\right) = \psi(x)L(v)$$

for all x in F and v in V. The set of all Whittaker functionals forms a vector space.

68 Proposition. [`narch-kiri-unwfn-20`] *Let (π, V) be an infinite dimensional irreducible admissible representation. Then the space of Whittaker functionals is precisely one dimensional.*

69 Corollary. [`narch-kiri-unwfn-30`] *Let (π, V) be an infinite dimensional irreducible admissible representation taken in Kirillov form. Then the Whittaker functionals on V are precisely the functions of the form*

$$L(\phi) = \lambda\phi(1)$$

where λ is a scalar.

70 Proof of proposition 68 and its corollary. Take V in the Kirillov form which was established in the previous seciton. It is easily seen that $\phi \mapsto \lambda\phi(1)$ is a Whittaker functional on V. We show that there are no others.

Let L be a Whittaker functional on V. We know by lemma 48 that the restriction of L to $V_0 = \mathscr{S}(F^\times)$ is of the desired form, *i.e.*, there exists a scalar λ such that for all ϕ in V_0 we have $L(\phi) = \lambda\phi(1)$.

Now let ϕ be an arbitrary element of V and take x so that $\psi(x) \neq 1$. Then

$$L(\phi) = L\left(\phi - \pi\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\phi\right) + L\left(\pi\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\phi\right). \tag{9}$$

Since

$$\phi - \pi\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\phi$$

belongs to V_0 (*cf.* lemma 62) the right hand side of (9) is equal to

$$\lambda(1-\psi(x))\phi(1) + \psi(x)L(\phi)$$

and so (9) may be rewritten as

$$(1-\psi(x))L(\phi) = \lambda(1-\psi(x))\phi(1).$$

Since $\psi(x) \neq 1$ it follows that $L(\phi) = \lambda\phi(1)$ and the proposition is proved.

5.3.5 The Kirillov model: proof of uniqueness

[`narch-kiri-un`]

71 Proposition. *Let (π, V) be as in proposition 61. Then its Kirillov model is unique.*

Let (π', V') be a representation equivalent to (π, V) such that $V' \subset C(F^\times)$ and the restriction of π' to D_F agrees with ξ_ψ. Let A be an intertwining operator from V to V', *i.e.*, a linear map $V \to V'$ such that $A\pi(g) = \pi'(g)A$ for all $g \in G_F$.

Let L be the linear functional on V defined by

$$L(\phi) = (A\phi)(1).$$

Observe that

$$L\left(\pi\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\phi\right) = \left(\pi'\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}(A\phi)\right)(1) = \psi(x)L(\phi)$$

and so by article 68 it follows that $L(\phi) = \lambda\phi(1)$ for some scalar λ. But

$$(A\phi)(a) = L\left(\pi\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\phi\right) = \lambda\phi(a)$$

and so

$$A\phi = \lambda\phi.$$

Therefore $V = V'$ and $\pi(g) = \pi'(g)$. This proves the proposition.

72. This completes the proof of theorem 52.

5.3.6 The Kirillov model of a twist

[`narch-kiri-twist`]

73 Proposition ([JL] pg. 73). [narch-kiri-twist-10] *Let (π, V) be an infinite dimensional irreducible admissible representation taken in Kirillov form and let ω be a quasi-character of $F^\times$. Then the space of the Kirillov model of $\omega \otimes \pi$ consists of all functions of the form $\omega\phi$ with ϕ in V.*

Let V' be the space consisting of all $\omega\phi$ with ϕ in V. Let G_F act on V' via π', defined by

$$\pi'(g)(\omega\phi) = \omega(\det g)\omega\pi(g)\phi.$$

Then the map $V \to V'$ given by $\phi \mapsto \omega\phi$ gives an equivalence of $(\omega \otimes \pi, V)$ with (π', V').

If

$$d = \begin{bmatrix} a & x \\ 0 & 1 \end{bmatrix}$$

and $\phi' = \omega\phi$ then

$$(\pi'(d)\phi')(\beta) = \omega(a)\omega(\beta)(\pi(d)\phi)(\beta) = \omega(a)\omega(\beta)\psi(\beta x)\phi(\beta a) = \psi(\beta x)\phi'(\beta a)$$

so that $\pi'(d) = \xi_\psi(d)$. Thus, by definition, V' is the space of the Kirillov model of $\omega \otimes \pi$.

5.3.7 The Whittaker model

[narch-kiri-whit]

74. Let $\mathscr{W}(\psi)$ be the subspace of $C(G_F)$ (complex valued functions on G_F; *cf.* article 24) consisting of all functions W satisfying

$$W\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) = \psi(x)W(g)$$

for all $x \in F$ and $g \in G_F$. The space $W(\psi)$ is stable under the right regular representation ρ (*cf.* article 25).

75. Let (π, V) be a representation of G_F. A *Whittaker model* of π is a submodule of $\mathscr{W}(\psi)$ which is isomorphic to (π, V).

76 Theorem ([JL] Thm. 2.14). *Let (π, V) be an infinite dimensional irreducible admissible representation of G_F. Then π has a unique Whittaker model.*

Existence. We take π in the Kirillov form. For an element ϕ of V let W_ϕ be the function on G_F defined by

$$W_\phi(g) = (\pi(g)\phi)(1).$$

We have the following three facts:

1. It is clear that $W_{\pi(g)\phi} = \rho(g)W_\phi$.
2. Since
$$W_\phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right) = \phi(a)$$
the function W_ϕ is zero if and only if ϕ is zero.
3. Since
$$W_\phi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) = \left(\pi \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \pi(g)\phi\right)(1) = \psi(x)(\pi(g)\phi)(1) = \psi(x)W_\phi(g)$$
the function W_ϕ is contained in the space $\mathscr{W}(\psi)$.

The three above facts imply that the map $\phi \mapsto W_\phi$ is an isomorphism of V onto a submodule $\mathscr{W}(\pi, \psi)$ of $\mathscr{W}(\psi)$. This establishes the existence.

Uniqueness. Suppose W is a submodule of $\mathscr{W}(\psi)$ which is isomorphic to V. Let $A : V \to W$ be an isomorhpism, so that

$$A(\pi(g)\phi) = \rho(g)(A\phi).$$

Let L be the linear functional on V defined by $L(\phi) = (A\phi)(1)$. We have

$$L\left(\pi \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \phi\right) = (A\phi)\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right) = \psi(x)(A\phi)(1) = \psi(x)L(\phi).$$

Thus by lemma 68 there exists a scalar λ such that $L(\phi) = \lambda\phi(1)$. But then

$$(A\phi)(g) = (\rho(g)A\phi)(1) = (A\pi(g)\phi)(1) = L(\pi(g)\phi) = \lambda(\pi(g)\phi)(1)$$

and so $A\phi = \lambda W_\phi$. Therefore $W = \mathscr{W}(\pi, \psi)$ and uniqness is proved.

5.3.8 The non-existence of Whittaker models for finite dimensional representations

[narch-kiri-fd]

77 Proposition. *If π is a finite dimensional irreducible admissible representation then π does not have a Whittaker model.*

Let π be the representation associated to the quasi-character χ. Assume π has a Whittaker model and let W be a nonzero member of it. We have $\rho(g)W = \chi(\det g)W$ and so $W(g) = \chi(\det g)W(1)$. Since W is nonzero we have $W(1) \neq 0$. For any x in F we have

$$W(1) = \chi(1)W(1) = W\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \psi(x)W(1).$$

Thus $\psi(x) = 1$ for all x and so ψ is the trivial character; this is a contradiction. Thus π has no Whittaker model.

5.4 Further results

[narch-fur]

5.4.1 The series $C(\nu, t)$ determine π

[narch-fur-x10]

78 Proposition ([JL] Prop. 2.15). [narch-fur-x10-10] *Let (π, V) and (π', V') be two infinite dimensional irreducible admissible representations. If the central quasi-characters of π and π' agree and $C(\nu, t) = C'(\nu, t)$ for all ν then π and π' are equivalent.*

Take both π and π' in the Kirillov form. If ϕ belongs to $\mathscr{S}(F^\times)$ then, by hypothesis,

$$\pi(w)\hat{\phi}(\nu, t) = \pi'(w)\hat{\phi}(\nu, t)$$

and so $\pi(w)\phi = \pi'(w)\phi$. We therefore have

$$V = \mathscr{S}(F^\times) + \pi(w)\mathscr{S}(F^\times) = \mathscr{S}(F^\times) + \pi'(w)\mathscr{S}(F^\times) = V'$$

and the two spaces are the same.

We now must show that π and π' agree. They automatically afree for elements of D_F, by the definition of the Kirillov form. They also agree for scalar matrices, since the central quasi-characters are assumed to be equal. Thus if suffices to show that $\pi(w)\phi = \pi'(w)\phi$ for all ϕ. We have already done this in the case where ϕ is an element of Schwartz space. It therefore suffices to verify this in the case where $\phi = \pi(w)\phi_0$ and ϕ_0 lies in the Schwartz space. But then $\pi(w)\phi = \pi(w)^2\phi_0 = \omega(-1)\phi_0$ and $\pi'(w)\phi = \pi'(w)^2\phi_0 = \omega(-1)\phi_0$ where ω is the central quasi-character. Thus $\pi(w) = \pi'(w)$ and the proposition is proved.

5.4.2 Rationality of the series $C(\nu, t)$

[narch-fur-ratl]

79 Proposition. [narch-fur-ratl-10] *Let (π, V) be an infinite dimensional irreducible admissible representation taken in Kirillov form. Let $C(\nu, t)$ be the series of proposition 61.*

1. *The series $C(\nu, t)$ is a rational function of t.*
2. *For all but finitely many ν the series $C(\nu, t)$ is a negative power of t.*

80. We break the proof of proposition 79 into two lemmas. If ν is a character of U_F we say that the *order* of ν is the integer k such that the conductor of ν is $1+\mathfrak{p}^k$.

81 Lemma ([JL] Lemma 2.16.2). *For any character μ of U_F the formal Laurent series $C(\mu,t)$ is a rational function. Equivalently, there exists an integer n_0 and a family of constants λ_i, $1 \le i \le k$ such that*

$$C_n(\mu) = \sum_{i=1}^{k} \lambda_i C_{n-i}(\mu)$$

for $n > n_0$.

Let $\rho = \nu^{-1}\nu_0^{-1}$. Proposition 63 then states that

$$S = S(n,\nu,p,\rho) = \sum_{\sigma \in \hat{U}_F} \eta(\sigma^{-1}\nu, \varpi^n)\eta(\sigma^{-1}\rho, \varpi^p)C_{p+n}(\sigma).$$

is equal to

$$z_0^p\nu_0(-1)\delta_{n,p} - (1-|\varpi|)^{-1}z_0^{m+1}C_{n-m-1}(\nu)C_{p-m-1}(\rho) - \sum_{r=-m-2}^{-\infty} z_0^{-r}C_{n+r}(\nu)C_{p+r}(\rho)$$

where, recall, $\mathfrak{p}^{-m}$ is the conductor of ψ. We now separate two cases:

Case 1: $\nu = \rho$. Take $p = -m$ and $n > -m$. Then

$$\eta(\sigma^{-1}\nu, \varpi^n)\eta(\sigma^{-1}\nu, \varpi^p) = \delta_{\sigma,\nu}$$

and so $S = C_{n-m}(\nu)$. Since $\delta_{n,p} = 0$ we obtain

$$C_{n-m}(\nu) = -(1-|\varpi|)^{-1}z_0^{m+1}C_{n-m-1}(\nu)C_{-2m-1}(\nu) - \sum_{r=-m-2}^{-\infty} z_0^{-r}C_{n+r}(\nu)C_{-m+r}(\nu).$$

Since almost all of the $C_{-m+r}(\nu)$ are zero in the sum, this yields the required relationship.

Case 2: $\nu \neq \rho$. There exists i so that $C_i(\rho) \neq 0$. Take p strictly greater than $-m-1$ and $m+i$. Take n strictly greater than p. Then

$$\eta(\sigma^{-1}\nu, \varpi^n) = \delta_{\sigma,\nu} \qquad \eta(\sigma^{-1}\rho, \varpi^p) = \delta_{\sigma,\rho}$$

so that $S = 0$. Since $\delta_{n,p} = 0$ as well, we find

$$-(1-|\varpi|)^{-1}z_0^{m+1}C_{n-m-1}(\nu)C_{p-m-1}(\rho) - \sum_{r=-m-2}^{-\infty} z_0^{-r}C_{n+r}(\nu)C_{p+r}(\rho) = 0.$$

This is really a finite sum. From the way we selected p it follows that one of the $C_\bullet(\rho)$ appearing in the above identity is nonzero. Therefore, this identity can be rearranged to yield a recurrence of the required form.

82 Lemma ([JL] Lemma 2.16.6). *Let k_0 be the order of ν_0 and let k_1 be an integer greater than m_0. Write ν_0 in any manner as $\nu_1^{-1}\nu_2^{-1}$ where the orders of ν_1 and ν_2 are strictly less than k_1. If the order k of ρ is sufficiently large then*

$$C(\rho,t) = ct^{-2m-2k}$$

where $\mathfrak{p}^{-m}$ if the conductor of ψ and

$$c = (v_2^{-1}\rho)(-1)z_0^{-m-k}\frac{\eta(\nu_1^{-1}\rho, \varpi^{-m-k})}{\eta(\nu_2\rho^{-1}, \varpi^{-m-k})}.$$

Choose n so that $C_n(\nu_1) \neq 0$. Assume that k is so large that $k > k_1$ and $k > -2m-n$. Then

$$\eta(\sigma^{-1}\nu_1, \varpi^{n+k+m}) = \delta_{\sigma,\nu_1}$$

so that

$$S(\nu_1, n+k+m, \rho, p+k+m) = \eta(\nu_1^{-1}\rho, \varpi^{p+k+m}) C_{p+n+2k+2m}(\nu_1)$$

for any integer p. Since $\chi = \rho\nu_1\nu_0 = \rho\nu_2^{-1}$ is nontrivial of order k, proposition 63 yields

$$\eta(\nu_1^{-1}\rho, \varpi^{p+k+m}) C_{p+n+2k+2m}(\nu_1) = z_0^{k+m}\chi(-1)\eta(\chi^{-1}, \varpi^{-k-m}) C_n(\nu_1) C_p(\rho)$$

again valid for all integers p. Note that the Gaussian sum $\eta(\chi^{-1}, \varpi^{-k-m})$ is nonzero. Now, if $p \neq -2m-2k$ then the Gaussian sum $\eta(\nu_1^{-1}\rho, \varpi^{p+m+k})$ (note the $\nu_1^{-1}\rho$ has order k) vanihses. On the order hand, if $p = -2m-2k$ the terms $C_n(\nu_1)$ in the above identity cancel and we find $C_p(\rho) = c$. This proves the proposition.

5.4.3 The dimension of the Jacquet module

[`narch-fur-dimjacq`]

83 Proposition. [`narch-fur-dimjacq-10`] *Let (π, V) be an infinite dimensional irreducible admissible representation. Write $C(\nu, t) = t^{p_\nu} P(\nu, t)/Q(\nu, t)$ where P and Q are coprime polynomials and coprime to t. Then*

$$\dim JV = \sum_\nu \deg Q(\nu, t).$$

In particular, JV is finite dimensional.

84 Corollary. *If (π, V) is an admissible representation of finite length then JV is finite dimensional.*
This follows from proposition 83 together with the exactness of J (proposition 38).

85 Corollary (J-L Lemma 2.16.1). [`narch-fur-dimjacq-30`] *Let (π, V) be an infinite dimensional irreducible admissible representation in Kirillov form. The $V_0 = \mathscr{S}(F^\times)$ is of finite codimension in V.*
This follows from proposition 83 and lemma 86, which immediately follows.

86 Lemma. [`narch-fur-dimjacq-40`] *Let (π, V) be an infinite dimensional irreducible admissible representation taken in Kirillov form. Then the kernel of the surjection $V \to JV$ is precisely the Schwartz space $\mathscr{S}(F^\times)$.*
The kernel of the map $V \to JV$ is generated by elements of the form $\xi_\psi(n_x)\phi - \phi$. Since these vectors lie in Schwartz space (*cf.* lemma 58) and span Schwartz space (*cf.* lemma 49) it follows that the kernel of the surjection $V \to JV$ is precisely Schwartz space.

87 Proof of proposition 83. Note that we can write $C(\nu, t)$ in the stated form by proposition 79. Also by proposition 79 the degree of $Q(\nu, t)$ is 0 for almost all ν, so the sum in the statement of the proposition is indeed finite.

For a character μ of U_F let ϕ_μ be the function equal to μ on U_F and equal to zero away from U_F. We have

$$\hat{\phi}_\mu(\nu, t) = \delta(\nu\mu\nu_0).$$

The functions of the form

$$\pi \begin{bmatrix} \varpi^n & 0 \\ 0 & 1 \end{bmatrix} \phi_\mu$$

span Schwartz space. We thus see that (with the help of proposition 61) Schwartz space consists exactly of those functions ϕ for which $\hat{\phi}(\nu, t)$ is a Laurent polynomial for all ν.

Let $\eta_\mu = \pi(w)\phi_\mu$. By proposition 61,

$$\hat{\eta}_\mu(\nu, t) = \delta(\nu\mu^{-1}) C(\nu, t).$$

By proposition 79 and the above remarks, it follows that η_μ lies in Schwartz space for all μ outside a finite set S. The functions of the form

$$\eta_{\mu,n} = \pi \begin{bmatrix} \varpi^n & 0 \\ 0 & 1 \end{bmatrix} \eta_\mu$$

with μ in S together with V_0 span V.

Let V_μ denote the subspace of V spanned by the $\eta_{\mu,n}$ and the elements ϕ of Schwartz space satisfying $\phi(\epsilon x) = \mu(\epsilon)\phi(x)$ for $\epsilon \in U_F$. The space V_μ consists precisely of those ϕ for which $\hat{\phi}(\nu, t)$ vanishes if $\nu \neq \mu$. Let $V_{\mu,0} = V_\mu \cap V_0$. Then

$$V = \bigoplus_\mu V_\mu \qquad V_0 = \bigoplus_\mu V_{\mu,0}$$

and so it follows from lemma 86 that

$$\dim JV = \sum_\mu \dim(V_\mu / V_{\mu,0}).$$

Since an element ϕ of V_μ has $\hat{\phi}(\nu, t)$ equal to zero for $\nu \neq \mu$, we may identify ϕ with $\hat{\phi}(\mu, t)$. Under this identification, elements of Schwartz space are identified with Laurent polynomials; the space $V_{\mu,0}$ is identified with $\mathbb{C}[t, t^{-1}]$. The function $\eta_{\mu,n}$ is identified with $t^{p_\mu + n} P(\mu, t)/Q(\mu, t)$; the space V_μ is identified with the $\mathbb{C}[t, t^{-1}]$ module spanned by $Q(\mu, t)^{-1}$ inside $C(t)$. Thus the dimension of $V_\mu / V_{\mu,0}$ is the degree of $Q(\mu, t)$.

5.4.4 The contragrediant of π is $\omega^{-1} \otimes \pi$

[narch-fur-contra]

88 Theorem (J-L Thm. 2.18). [narch-fur-contra-10] *Let (π, V) be an infinite dimensional irreducible admissible represention of G_F with central quasi-character ω. Then the contragrediant representation $\widetilde{\pi}$ is equivalent to $\omega^{-1} \otimes \pi$.*

89. We need several lemmas before proving theorem 88. Throughout this section, (π, V) will be a fixed infinite dimensional irreducible admissible representation with central quasi-character ω, taken in Kirillov form and (π', V') will be the twist of the representation π by the quasi-character ω^{-1}, also taken in Kirillov form. By prosition 73, the elments of V' are of the form $\omega\phi$ with ϕ an element of V, and

$$\pi'(g)(\omega\phi) = \omega(\det g)\omega\pi(g).$$

We let V_0 and V_0' denote the Schwartz spaces inside V and V' (they are the same space, but have different actions of G_F). Our strategy to prove theorem 88 is to construct an invariant bilinear form on $V \times V'$.

90. If ϕ and ϕ' are two elements of $C(F^\times)$, put

$$\langle \phi, \phi' \rangle = \int_{F^\times} \phi(a)\phi'(-a) d^\times a$$

where $d^\times a$ is the Haar measure on the multiplicative group, normalized so that U_F has volume 1. The integral is not defined for all pairs ϕ and ϕ'. It is defined when one lies in Schwartz space and the other is locally constant, and this suffices for our purposes.

91 Lemma. [narch-contra2-40] *Let ϕ and ϕ' be two locally constant functions on $F^\times$, one of which belongs to Schwartz space. Then*

$$\langle \phi, \phi' \rangle = \sum_{n \in \mathbb{Z}} \sum_{\nu \in \hat{U}_F} \nu(-1) \hat{\phi}_n(\nu) \hat{\phi}'_n(\nu^{-1})$$

where $\hat{\phi}_n$ and $\hat{\phi}'_n$ are as defined in article 60.

This follows immediately from the Plancherel formula for U_F.

92 Lemma. [narch-fur-contra-45] *The bilinear form $\langle , \rangle$ is invariant under the action of D_F via ξ_ψ; more precisely, if ϕ and ϕ' are two locally constant functions on $F^\times$, one of which belongs to Schwartz space and d is an element of D_F then*

$$\langle \xi_\psi(d)\phi, \xi_\psi(d)\phi' \rangle = \langle \phi, \phi' \rangle.$$

If

$$d = \begin{bmatrix} b & x \\ 0 & 1 \end{bmatrix}$$

then

$$\langle \xi_\psi(d)\phi, \xi_\psi(d)\phi' \rangle = \int_{F^\times} (\psi(ax)\phi(ab))(\psi(-ax)\phi'(-ab))d^\times a = \int_{F^\times} \phi(ab)\phi'(-ab)d^\times a$$

and the result follows.

93 Lemma (J-L Lemma 2.19.1). [narch-fur-contra-50] *Let ϕ and ϕ' belong to V_0 and V_0'.*

1. *We have*
$$\langle \pi(w)\phi, \phi' \rangle = \nu_0(-1)\langle \phi, \pi'(w)\phi' \rangle.$$
where ν_0 is the restriction of ω to U_F.

2. *If either $\pi(w)\phi$ belongs to V_0 or $\pi'(w)\phi'$ belongs to V_0' then*
$$\langle \pi(w)\phi, \pi'(w)\phi' \rangle = \langle \phi, \phi' \rangle.$$

1) The relation

$$\pi(w)\hat{\phi}(\nu, t) = \sum_n \left[\sum_p z_0^{-p} C_{n+p}(\nu)\hat{\phi}_p(\nu^{-1}\nu_0^{-1}) \right] t^n$$

(*cf.* proposition 61) together with lemma 93 implies that

$$\langle \pi(w)\phi, \phi' \rangle = \sum_{n,p,\nu} \nu(-1) z_0^{-p} C_{n+p}(\nu)\hat{\phi}_p(\nu^{-1}\nu_0^{-1})\hat{\phi}'_n(\nu^{-1}). \tag{10}$$

If we perform the same computation on $\langle \phi, \pi'(w)\phi' \rangle$ then ω is replaced by ω^{-1}, ν_0 by ν_0^{-1}, z_0 by z_0^{-1} and $C(\nu, t)$ by $C(\nu\nu_0^{-1}, z_0^{-1}t)$. Thus

$$\langle \phi, \pi'(w)\phi' \rangle = \sum_{n,p,\nu} \nu(-1) z_0^{-n} C_{n+p}(\nu\nu_0^{-1})\hat{\phi}'_p(\nu^{-1}\nu_0)\hat{\phi}_n(\nu^{-1}).$$

If we replace ν by $\nu\nu_0$ and interchange n and p in the above sum and compare with (10), we obtain the first statement of the proposition.

2) By symmetry it suffices to prove the second part when $\pi(w)\phi$ belongs to V_0. In that case, using the first part of the proposition, we obtain

$$\langle \pi(w)\phi, \pi'(w)\phi' \rangle = \nu_0(-1)\langle \pi(w)^2\phi, \phi' \rangle = \langle \phi, \phi' \rangle$$

since $\pi(w)^2 = \omega(-1) = \nu_0(-1)$.

94. We now define a bilinear form β on $V \times V'$. Let ϕ is a typical element of V and write $\phi = \phi_1 + \pi(w)\phi_2$ with with ϕ_1 and ϕ_2 in V_0. Similarly, let ϕ' be a typical element of V' and write $\phi' = \phi_1' + \pi'(w)\phi_2'$ with ϕ_1' and ϕ_2' in V_0'. We then define

$$\beta(\phi, \phi') = \langle \phi_1, \phi_1' \rangle + \langle \phi_1, \pi'(w)\phi_2' \rangle + \langle \pi(w)\phi_2, \phi_1' \rangle + \langle \phi_2, \phi_2' \rangle.$$

The second part of lemma 93 ensures that β is well defined.

95 Lemma ([JL] pg. 82). [narch-fur-contra-60] *The bilinear form β is G_F-invariant.*

We procede by sublemmas.

Sublemma A. It follows from lemma 93 that β is invariant under the action of w.

Sublemma B. If either ϕ is in V_0 or ϕ' is in V_0' then

$$\beta(\phi, \phi') = \langle \phi, \phi' \rangle.$$

Thus lemma 92 implies that the restriction of β to $V_0 \times V'$ or to $V \times V_0'$ is invariant under P_F.

Sublemma C. If $\phi \in V_0$, $\phi' \in V_0'$ and p is a diagonal matrix then

$$\beta(\pi(p)\pi(w)\phi, \pi'(p)\pi'(w)\phi') = \beta(\pi(w)\pi(p_1)\phi, \pi'(w)\pi'(p_1)\phi')$$

where p_1 is also diagonal. By sublemmas A and B we this is equal to

$$\beta(\pi(p_1)\phi, \pi'(p_1)\phi') = \beta(\phi, \phi') = \beta(\pi(w)\phi, \pi'(w)\phi').$$

Thus if ϕ and ϕ' belong to $\pi(w)V_0$ and $\pi'(w)V_0'$ then

$$\beta(\pi(p)\phi, \pi'(p)\phi') = \beta(\phi, \phi').$$

Together with sublemma B, it follows that β is invariant under all diagonal matrices.

Sublemma D. We now show that β is invariant under N_F. Let ϕ_i, $1 \leq i \leq r$ generate V modulo V_0 let ϕ_i', $1 \leq i \leq r$ generate V' modulo V_0'. That this can be accomplished with a finite number r follows from article 85.

There exists an ideal $\mathfrak{a}$ of F such that

$$\pi(n_x)\phi_i = \phi_i \qquad \pi'(n_x)\phi_i' = \phi_i'$$

for all i and all x in $\mathfrak{a}$. We thus have

$$\beta(\pi(n_x)\phi_i, \pi'(n_x)\phi_j') = \beta(\phi_i, \phi_j') \tag{11}$$

whenever x lies in $\mathfrak{a}$. Given any y in F, there exists x in $\mathfrak{a}$ and $a \in F^\times$ such that

$$\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

Since β is invariant under diagonal matrices by sublemma C, it follows that (11) holds for all x in F. It now follows that β is invariant under N_F (since, by sublemma B, β is invariant under N_F when its arguments lie in Schwartz space).

Sublemmas A, C and D prove the proposition.

96 Proof of theorem 88. By lemma 95 we have a nonzero invariant form β on $V \times V'$. By proposition 30 it follows that (π', V') is the contragrediant of (π, V). Thus theorem 88 is proved.

5.5 Absolutely cuspidal representations

[narch-acsp1]

5.5.1 Definition

[narch-acsp1-def]

97. A representation (π, V) of G_F is called *absolutely cuspidal* if it is infinite dimensional, irreducible, admissible and has $JV = 0$. By lemma 86 the condition $JV = 0$ is equivalent to the space of the Kirillov model being Schwartz space.

5.5.2 Absolutely cuspidal representations are (almost) unitary

[narch-acsp1-un]

98 Lemma ([JL] Prop. 2.20). [narch-acsp1-un-10] *Let (π, V) be an absolutely cuspidal representation. Then for every matrix element Φ of π there is a compact set Ω such that the support of Φ is contained in $Z_F\Omega$.*

Take π and $\widetilde{\pi}$ in Kirillov form and let $\Phi(g) = \langle \pi(g)\phi, \widetilde{\phi}\rangle$ where ϕ is in V and $\widetilde{\phi}$ is in $\widetilde{V}$. Since ϕ and $\widetilde{\phi}$ are invariant under finite index subgroups of K_F and $G_F = K_F A_F K_F$, it is enough to show that the restriction of Φ to A_F has the stated property. Since

$$\langle \pi \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \phi, \widetilde{\phi}\rangle = \omega(a)\langle \phi, \widetilde{\phi}\rangle$$

it is enough to show that the function

$$\langle \pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \phi, \widetilde{\phi}\rangle$$

has compact support on $F^\times$. However, this function is equal to

$$\int_{F^\times} \phi(ax)\widetilde{\phi}(-x)d^\times x$$

and since both ϕ and $\widetilde{\phi}$ have compact support does the integral. This proves the result.

99 Proposition ([JL] Prop. 2.20). [narch-acsp1-un-20] *Let (π, V) be an absolutely cuspidal representation whose central quasi-character is a character. Then π is unitary (and also square integrable).*

Take π and $\widetilde{\pi}$ in Kirillov form. Take $\widetilde{\phi}_0$ in $\widetilde{V}$ and define

$$(\phi_1, \phi_2) = \int_{Z_F\backslash G_F} \langle \pi(g)\phi_1, \widetilde{\phi}_0\rangle \overline{\langle \pi(g)\phi_2, \widetilde{\phi}_0\rangle} dg.$$

By lemma 98 the integral is defined. The hermitian form $(,)$ is clearly G_F-invariant, and so the proposition is proved.

100. Note that if π is any representation then there exists a quasi-character χ of $F^\times$ such that the central quasi-character of $\chi \otimes \pi$ is a character. Thus proposition 99 implies that any absolutely cuspidal representation is not far from being unitary.

101 Proposition ([JL] Prop. 2.21.2). [narch-acsp1-un-30] *Let (π, V) be an absolutely cuspidal representation in Kirillov form whose central quasi-character is a character. Then the hermitian form*

$$\int_{F^\times} \phi_1(a)\overline{\phi_2(a)}d^\times a$$

is G_F-invariant.

Take $\widetilde{\pi}$ in Kirillov form as well. Define a conjugate linear isomorphism $A : V \to \widetilde{V}$ by

$$(\phi_1, \phi_2) = \langle \phi_1, A\phi_2\rangle.$$

Define another conjugate linear isomorphism $A_0 : V \to \widetilde{V}$ by

$$(A_0\phi)(a) = \bar{\phi}(-a)$$

Both A and A_0 commute with the action of ξ_ψ. Therefore $A_0^{-1}A$ is a linear map of V onto itself which commutes with ξ_ψ. Since V is equal to $\mathscr{S}(F^\times)$, lemma 50 implies that $A_0^{-1}A$ is equal to a scalar λ. It thus follows that

$$(\phi_1, \phi_2) = \lambda \int_{F^\times} \phi_1(a)\overline{\phi_2(a)}d^\times a.$$

The invariance of the left side implies the invariance of the right side.

5.5.3 The series $C(\nu, t)$ for absolutely cuspidal representations

[narch-acsp1-cnut]

102 Proposition ([JL] Prop. 2.21.2, 2.23). [narch-acsp1-cnut-10] *Let (π, V) be an absolutely cuspidal representation.*

1. *The series $C(\nu, t)$ is equal to αt^n for some complex number α and integer n.*
2. *If the central quasi-character ω of π is a character then $|\alpha| = 1$.*
3. *If the conductor of ψ is $\mathscr{O}_F$ then $n < -1$.*

1) Take π in Kirillov form. Let ϕ be in V and let $\phi' = \pi(w)\phi$. Both ϕ and $\pi(w)\phi$ are in V_0; we have $\pi(w)\phi' = \omega(-1)\phi$. Two applications of proposition 61 yield

$$\begin{aligned}\hat{\phi}(\nu, t) &= C(\nu, t)\hat{\phi}'((\omega\nu)^{-1}, z_0^{-1}t^{-1}) \\ &= C(\nu, t)C((\omega\nu)^{-1}, z_0^{-1}t^{-1})\omega(-1)\hat{\phi}(\nu, t);\end{aligned}$$

therefore

$$C(\nu, t)C((\omega\nu)^{-1}, z_0^{-1}t^{-1}) = \omega(-1)$$

It thus follows that $C(\nu, t)$ is a multiple of a power of t.

2) Define a function ϕ on $F^\times$ by $\phi(\epsilon\varpi^n) = \delta_{n,m}\nu(\epsilon)\omega(\epsilon)$. Let $\phi' = \pi(w)\phi$. Writing $C(\nu, t) = C_\ell(\nu)t^\ell$, proposition 61 gives

$$\phi'(\epsilon\varpi^n) = \delta_{\ell-n,m}C_\ell(\nu)z_0^{-n}\nu^{-1}(\epsilon).$$

Using proposition 101 and the fact that $|z_0| = 1$, we find

$$1 = \int_{F^\times} |\phi(a)|^2 d^\times a = \int_{F^\times} |\phi'(a)|^2 d^\times a = |C_\ell(\nu)|^2$$

and the proposition is proved.

3) Let ν be a character of U_F and let n_1 be the unique integer such that $C_{n_1}(\nu) \neq 0$. Let $\rho = \nu^{-1}\nu_0^{-1}$ where ν_0 is the restriction of the central quasi-character to U_F. We have seen above that

$$C(\nu, t)C(\rho, t^{-1}z_0^{-1}) = \nu_0(-1);$$

therefore n_1 is also the unique integer so that $C_{n_1}(\rho) \neq 0$. We also have

$$C_{n_1}(\nu)C_{n_1}(\rho) = \nu_0(-1)z_0^{n_1}.$$

Now apply proposition 63 with $n = p = n_1 + 1$ to obtain

$$\begin{aligned}\sum_\sigma \eta(\sigma^{-1}\nu, \varpi^{n_1+1})\eta(\sigma^{-1}\rho, \varpi^{n_1+1})C_{2n_1+2}(\sigma) &= \nu_0(-1)z_0^{n_1+1} - (1 - |\varpi|)^{-1}z_0C_{n_1}(\nu)C_{n_1}(\rho) \\ &= -\nu_0(-1)z_0^{n_1+1}\frac{|\varpi|}{1 - |\varpi|}\end{aligned}$$

Now assume that $n_1 \geq -1$. Then $\eta(\sigma^{-1}\nu, \varpi^{n_1+1})$ is 0 unless $\sigma = \nu$ and $\eta(\sigma^{-1}\rho, \varpi^{n_1+1})$ is 0 unlees $\sigma = \rho$ (*cf.* §1.5.1 proposition 121). Thus if $\nu \neq \rho$ then the left side is 0, which is a contradiction. If $\nu = \rho$ then the left side equals $C_{2n_1+2}(\nu)$; since this cannot vanish we must have $2n_1 + 2 = n_1$ so that $n_1 = -2$, also a contradiction. Thus it must be the case that $n_1 < -1$, proving the proposition.

5.6 The principal series and special representations

[narch-prin]

5.6.1 The representation $\rho(\mu_1, \mu_2)$

[narch-prin-rho]

103. In this section α_F denotes the quasi-character $|\cdot|$ of $F^\times$.

104. [narch-prin-rho-20] Let μ_1 and μ_2 be quasi-characters of $F^\times$. We define a representation $(\rho(\mu_1,\mu_2),\mathscr{B}(\mu_1,\mu_2))$ as follows. The space $\mathscr{B}(\mu_1,\mu_2)$ consists of all locally constant functions f on G_F which satisfy

$$f\left(\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix} g\right) = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} f(g) \tag{12}$$

for all g in G_F. The action $\rho(\mu_1,\mu_2)$ of G_F is simply ρ, right translation.

105. Note that, because of the decomposition $G_F = P_F K_F$, the elements of $\mathscr{B}(\mu_1,\mu_2)$ are determined by their restriction to K_F, and that this restriction can be any function locally constant on K_F satisfying (12) for $a_1, a_2 \in U_F$ and $x \in \mathscr{O}_F$.

5.6.2 The contragrediant of $\rho(\mu_1,\mu_2)$

[narch-prin-contra]

106 Proposition ([Bu] Prop. 4.5.5; [JL] pg. 94). [narch-prin-contra-10] *The contragrediant of $\rho(\mu_1,\mu_2)$ is equivalent to $\rho(\mu_1^{-1},\mu_2^{-1})$.*

107. We need two lemmas before proving this.

108 Lemma ([Bu] Lemma 2.6.1). [narch-prin-contra-30] *Let $\Lambda : \mathscr{S}(G_F) \to C(G_F)$ be given by*

$$(\Lambda\phi)(g) = \int_{P_F} \phi(pg)dp$$

where dp is the left Haar measure. Then Λ is a G_F-equivariant map (under the action of ρ) whose image is precisely $\mathscr{B}(\alpha_F^{1/2},\alpha_F^{-1/2})$.

The G_F-equivariance is clear.

Let $d'p$ be the right Haar measure on P_F. If

$$p = \begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix}$$

then

$$dp = |a_1|^{-1}d^\times a_1 d^\times a_2 dx \qquad d'p = |a_2|^{-1}d^\times a_1 d^\times a_2 dx.$$

It thus follows that

$$\begin{aligned}(\Lambda\phi)(p'g) &= \int_{P_F}\phi(pp'g)dp = \int_{P_F}\phi(pp'g)|a_2/a_1|d'p = \int_{P_F}\phi(pg)|a_2/a_1||a_1'/a_2'|d'p \\ &= |a_1'/a_2'|\int_{P_F}\phi(pg)dp = |a_1'/a_2'|(\Lambda\phi)(g).\end{aligned}$$

Therefore the image of Λ is contained in $\mathscr{B}(\alpha_F^{1/2},\alpha_F^{-1/2})$.

Let V be the space of all locally constant functions f on K which $f(pk) = f(p)$ for $p \in K_F \cap P_F$. The restriction map $\mathscr{B}(\alpha_F^{1/2},\alpha_F^{-1/2}) \to V$ is, as was already mentioned, an isomorphism of vector spaces. Thus if we can show that the composite

$$C_c(G_F) \to \mathscr{B}(\alpha_F^{1/2},\alpha_F^{-1/2}) \to V$$

is surjective, then it will follow that Λ is surjective.

Let ϕ_0 be an element of $\mathscr{S}(G_F)$ whose integral is nonzero. Replacing ϕ_0 by

$$\int_{K_F\cap P_F}\phi_0(gk)dk$$

we may assume $\phi_0(gk) = \phi_0(g)$ for $k \in K_F \cap P_F$; we may also assume ϕ_0 has total integral 1. Given f in V let $\phi(pk) = \phi_0(p)f(k)$; notice that this is well defined. Clearly, the restriction of $\Lambda(\phi)$ to K_F if equal to f. This proves that Λ is surjective.

109 Lemma ([Bu] Lemma 2.6.1; [JL] pg. 93). [narch-prin-contra-40] *The map $I : \mathscr{B}(\alpha_F^{1/2}, \alpha_F^{-1/2}) \to \mathbb{C}$ given by*

$$I(f) = \int_{K_F} f(k)dk$$

is a G_F-invariant linear form.

Let Λ be as in lemma 108. For $f \in \mathscr{S}(G_F)$ we have

$$I(\Lambda f) = \int_{K_F} \int_{P_F} f(pk)dpdk = \int_{G_F} f(g)dg.$$

Thus $I\Lambda$ is a G_F-invariant form on $\mathscr{S}(G_F)$. Since Λ is surjective and G_F-invariant it follows that I is G_F-invariant.

110 Proof of proposition 106. Let I be as in lemma 109. If ϕ_1 is in $\mathscr{B}(\mu_1, \mu_2)$ and ϕ_2 is in $\mathscr{B}(\mu_1^{-1}, \mu_2^{-1})$ then $\phi\phi_2$ is in $\mathscr{B}(\alpha_F^{1/2}, \alpha_F^{-1/2})$. Thus

$$\langle \phi_1, \phi_2 \rangle = I(\phi_1\phi_2)$$

is a nondegenerate G_F-invariant bilinear form on $\mathscr{B}(\mu_1, \mu_2) \times \mathscr{B}(\mu_1^{-1}, \mu_2^{-1})$. The result follows by proposition 30.

5.6.3 The Weil representation for $F \oplus F$

[narch-prin-weil]

111. For a function Φ in $\mathscr{S}(F^2)$, let $\Phi^\sim$ denote its partial Fourier transform:

$$\Phi^\sim(a, b) = \int_F \Phi(a, y)\psi(by)dy.$$

112. Define a representation r of G_F on the space $\mathscr{S}(F^2)$ by

$$\big(r(g)\Phi\big)^\sim = \rho(g)\Phi^\sim.$$

Here F^2 is thought of as row vectors and given a right G_F-module structure via matrix multiplication. $\rho(g)$ is then right translation by g. For quasi-characters μ_1 and μ_2 of $F^\times$ we define another representation r_{μ_1,μ_2} of G_F on $\mathscr{S}(F^2)$ by

$$r_{\mu_1,\mu_2}(g) = \mu_1(\det g)|\det g|^{1/2} r(g).$$

The representations r and r_{μ_1,μ_2} are both examples of *Weil representations.*

113 Proposition. *Let Φ be in $\mathscr{S}(F^2)$.*

1. *For a in $F^\times$ we have*
$$\left(r\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\Phi\right)(a, b) = |\beta|^{-1}\Phi(\alpha a, \beta^{-1}b)$$

2. *For x in F we have*
$$\left(r\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\Phi\right)(a, b) = \psi(abx)\Phi(a, b).$$

3. *We have*
$$\left(r\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\Phi\right)(a, b) = \int_{F^2} \Phi(y, x)\psi(ax + by)dxdy$$

1) We have

$$\left(r\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\Phi\right)^\sim(a, b) = \int_F \Phi(\alpha a, y)\psi(\beta by)dy = |\beta|^{-1}\int_F \Phi(\alpha a, \beta^{-1}y)\psi(by)dy = |\beta|^{-1}\Phi_1^\sim(a, b)$$

where $\Phi_1(a, b) = \Phi(\alpha a, \beta^{-1}b)$. This first statement follows.

2) We have

$$\left(r\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\Phi\right)^{\sim}(a,b)=\int_F \Phi(a,y)\psi((ax+b)y)dy=\int_F \Phi_1(a,y)\psi(by)dy=\Phi_1^{\sim}(a,b)$$

where $\Phi_1(a,b)=\psi(abx)\Phi(a,b)$. The second statement follows.

3) We have

$$\begin{aligned}(r(w)\Phi)(a,b)&=\int_F(\rho(w)\Phi^{\sim})(a,y)\psi(-by)dy=\int_F \Phi^{\sim}(-y,a)\psi(-by)dy\\ &=\int_{F^2}\Phi(-y,x)\psi(ax)\psi(-by)dxdy=\int_{F^2}\Phi(y,x)\psi(ax+by)dxdy\end{aligned}$$

which proves the third part.

5.6.4 The Whittaker and Kirillov models of $\rho(\mu_1,\mu_2)$

[`narch-prin-kiri`]

114. In this section we construct Whittaker and Kirillov models for the representations $\rho(\mu_1,\mu_2)$ for certain μ_1 and μ_2. We construct the Whittaker model first and from it deduce the Kirillov model. We obtain the Whittaker model by defining maps

$$W:\mathscr{S}(F^2)\to W(\psi)\qquad f:\mathscr{S}(F^2)\to\mathscr{B}(\mu_1,\mu_2)$$

and then proving that $W_\Phi\mapsto f_{\Phi^{\sim}}$ is an isomorphism. Unfortunately, W is not injective, so it takes a bit of work to show that this map is well-defined.

115. For Φ in $\mathscr{S}(F^2)$ define

$$\theta(\mu_1,\mu_2;\Phi)=\int_{F^\times}\mu_1(t)\mu_2^{-1}(t)\Phi(t,t^{-1})d^\times t.$$

Define an element W_Φ of $C^\infty(G_F)$ by

$$W_\Phi(g)=\theta(\mu_1,\mu_2;r_{\mu_1,\mu_2}(g)\Phi)$$

and let $\mathscr{W}(\mu_1,\mu_2;\psi)$ be the set of such functions.

116 Lemma. [`narch-prin-kiri-30`] *We have*

1. $W_{r_{\mu_1,\mu_2}(g)\Phi}=\rho(g)W_\Phi$.
2. *The function* W_Φ *belongs to* $\mathscr{W}(\psi)$.

1) This is clear from the definition.

2) By proposition 113 we have $r_{\mu_1,\mu_2}(n_x)\Phi=\Phi_1$ where $\Phi_1(a,b)=\psi(abx)\Phi(a,b)$. Since $\Phi_1(t,t^{-1})=\psi(x)\Phi(t,t^{-1})$, it follows that

$$\theta(\mu_1,\mu_2;r_{\mu_1,\mu_2}(n_x)\Phi)=\psi(x)\theta(\mu_1,\mu_2;\Phi).$$

Therefore we have

$$W_\Phi(n_xg)=\theta(\mu_1,\mu_2;r_{\mu_1,\mu_2}(n_x)r_{\mu_1,\mu_2}(g)\Phi)=\psi(x)\theta(\mu_1,\mu_2;r_{\mu_1,\mu_2}(g)\Phi)=\psi(x)W_\Phi(g)$$

and the proposition is proved.

117. If ω is a quasi-character of $F^\times$ and $\omega(\varpi)=|\varpi|^s$ with $s>0$ then the integral

$$z(\omega,\Phi)=\int_{F^\times}\Phi(0,t)\omega(t)d^\times t$$

is defined for any element Φ of $\mathscr{S}(F^2)$. Thus if μ_1 and μ_2 are quasi-characters of $F^\times$ such that $|(\mu_1\mu_2^{-1})(\varpi)|=|\varpi|^s$ with $s>-1$ then we can define

$$f_\Phi(g)=\mu_1(\det g)|\det g|^{1/2}z(\alpha_F\mu_1\mu_2^{-1},\rho(g)\Phi).$$

118 Lemma ([JL] pg. 95). [narch-prin-kiri-50] *Assume* $|(\mu_1\mu_2^{-1})(\varpi)| = |\varpi|^s$ *with* $s > -1$. *We have*

1. $\rho(g)f_\Phi = f_{\mu_1(\det g)|\det g|^{1/2}\rho(g)\Phi}$.
2. *The function* f_Φ *belongs to* $\mathscr{B}(\mu_1,\mu_2)$.

1) This is immediate from the definition.
2) That f_Φ is locally constant follows from the first part and the fact that the stabilizer of any Φ under the representation $g \mapsto \mu_1(\det g)|\det g|^{1/2}\rho(g)$ is an open subgroup. We have

$$\begin{aligned} f_\Phi\left(\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix} g\right) &= \mu_1(a_1a_2)|a_1a_2|^{1/2}z\left(\alpha_F\mu_1\mu_2^{-1}, \rho\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix}\rho(g)\Phi\right) \\ &= \mu_1(a_1a_2)|a_1a_2|^{1/2}\int_{F^\times}\mu_1(t)\mu_2^{-1}(t)|t|(\rho(g)\Phi)(0,a_2t)d^\times t \\ &= \mu_1(a_1)\mu_2(a_2)|a_1/a_2|^{1/2}f_\Phi(g) \end{aligned}$$

which proves the proposition.

119 Lemma ([JL] Lemma 3.2.1). [narch-prin-kiri-60] *Assume* $|(\mu_1\mu_2^{-1})(\varpi)| = |\varpi|^s$ *with* $s > -1$. *For all* Φ *in* $\mathscr{S}(F^2)$ *the function* q *on* $F^\times$ *given by*

$$q(a) = \mu_2^{-1}(a)|a|^{-1/2}W_\Phi\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

is integrable with respect to the additive Haar measure on F *and*

$$\int_{F^\times} q(a)\psi(ax)da = f_{\Phi^\sim}(-wn_x),$$

where, as always,

$$w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad n_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}.$$

We have (after applying proposition 113)

$$\begin{aligned} q(a) &= \mu_1(a)\mu_2^{-1}(a)\int_{F^\times}\Phi(at,t^{-1})\mu_1(t)\mu_2^{-1}(t)d^\times t \\ &= \int_{F^\times}\Phi(t,at^{-1})\mu_1(t)\mu_2^{-1}(t)d^\times t \end{aligned} \tag{13}$$

Now, note that

$$\int_{F^\times}\int_F \left|\Phi(t,at^{-1})\mu_1(t)\mu_2^{-1}(t)\right| d^\times t da = \int_{F^\times}\int_F |\Phi(t,a)||t|^{s+1}d^\times t da$$

is finite since $s > -1$. Using this fact (to justify the change in order of integration) along with (13) gives

$$\begin{aligned} \int_F \psi(ax)q(a)da &= \int_F \psi(ax)\left[\int_{F^\times}\Phi(t,at^{-1})\mu_1(t)\mu_2^{-1}(t)d^\times t\right]da \\ &= \int_{F^\times}\mu_1(t)\mu_2^{-1}(t)\left[\int_F \Phi(t,at^{-1})\psi(ax)da\right]d^\times t \\ &= \int_{F^\times}\mu_1(t)\mu_2^{-1}(t)|t|\left[\int_F \Phi(t,a)\psi(axt)da\right]d^\times t \\ &= \int_{F^\times}\Phi^\sim(t,xt)\mu_1(t)\mu_2^{-1}(t)|t|d^\times t \\ &= f_{\Phi^\sim}(-wn_x) \end{aligned}$$

and the proposition is proved.

120 Proposition ([JL] Prop. 3.2). [narch-prin-kiri-70] *Assume* $|(\mu_1\mu_2^{-1})(\varpi)| = |\varpi|^s$ *with* $s > -1$.

1. *There is a map* $A : \mathscr{W}(\mu_1, \mu_2; \psi) \to \mathscr{B}(\mu_1, \mu_2)$ *which sends* W_Φ *to* $f_{\Phi^\sim}$.
2. *The map* A *is an isomorphism of* G_F*-modules.*
3. $\mathscr{W}(\mu_1, \mu_2; \psi)$ *is a Whittaker model for* $\mathscr{B}(\mu_1, \mu_2)$.

1) To ensure A is well defined, we must verify that $W_\Phi = 0$ implies $f_{\Phi^\sim} = 0$. If $W_\Phi = 0$ then by lemma 119 (together with the fact that $f_{\Phi^\sim}$ lies in $\mathscr{B}(\mu_1, \mu_2)$) we conclude that $f_{\Phi^\sim}$ vanishes on $P_F w N_F$. Since this is a dense subsets of G_F and $f_{\Phi^\sim}$ is locally constant, it follows that $f_{\Phi^\sim} = 0$. Thus A is well defined.

2) It is clear that A is a map of G_F-modules. We thus need to show that it is a bijection.

Injective. It is enough to show $W_\Phi(1) = 0$ if $f_{\Phi^\sim} = 0$. If $f_{\Phi^\sim} = 0$ then by lemma 119 it follows that

$$W_\Phi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

is zero for almost all a. Since this is a locally constant function, it must therefore be zero. Thus A is injective.

Surjective. We show every element f of $\mathscr{B}(\mu_1, \mu_2)$ is of the form f_Φ for some Φ. Thus let f be given. Define $\Phi(x, y)$ to be 0 if (x, y) is not of the form $(0, 1)g$ for some $g \in K_F$; if (x, y) is of this form put $\Phi(x, y) = \mu_1^{-1}(\det g) f(g)$. It is easy to see that Φ is a well defined function which belongs to $\mathscr{S}(F^2)$. To prove $f = f_\Phi$ it suffices to show that their restrictions to K_F agree.

Thus let g be in K_F. Since $\Phi((0, t)g) = 0$ unless t belongs to U_F we have

$$f_\Phi(g) = \mu_1(\det g) \int_{U_F} \Phi((0, t)g) \mu_1(t) \mu_2^{-1}(t) dt.$$

Since

$$\Phi((0,t)g) = \Phi\left((0,1)\begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} g\right) = \mu_1^{-1}(t)\mu_1^{-1}(\det g) f\left(\begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} g\right) = \mu_1^{-1}(t)\mu_2(t)\mu_1^{-1}(\det g) f(g)$$

we have

$$f_\Phi(g) = \int_{U_F} f(g) dt = f(g)$$

(at least up to a constant) and the result follows.

3) This follows from results we have already proved.

121 Proposition. [narch-prin-kiri-80] *Assume* $|(\mu_1\mu_2^{-1})(\varpi)| = |\varpi|^s$ *with* $s > -1$. *For* W *in* $\mathscr{W}(\mu_1, \mu_2; \psi)$ *let* ϕ_W *be the function on* $F^\times$ *given by*

$$\phi_W(a) = W \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}.$$

Let V *be the space of all the* ϕ_W.

1. *We have* $\phi_{\rho(d)W} = \xi_\psi(d)\phi_W$ *for* d *in* D_F.
2. *The map* $W \mapsto \phi_W$ *is injective.*
3. *If* ϕ *is in* V *then* $\phi(a)$ *vanishes for* $|a|$ *sufficiently large.*
4. V *contains* $\mathscr{S}(V^\times)$.
5. *The space* V *is a Kirillov model for both* $\mathscr{W}(\mu_1, \mu_2; \psi)$ *and* $\mathscr{B}(\mu_1, \mu_2)$.

1) Let

$$d = \begin{bmatrix} a & x \\ 0 & 1 \end{bmatrix}.$$

We have

$$\phi_{\rho(d)W}(\beta) = W\left(\begin{bmatrix} \beta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & x \\ 0 & 1 \end{bmatrix}\right) = W\left(\begin{bmatrix} 1 & \beta x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \beta a & 1 \\ 0 & 1 \end{bmatrix}\right) = \psi(\beta x)\phi_W(\beta a).$$

This proves the first statement.

2) This was essentially proved in the course of proving proposition 120; we reproduce it here. We have $W = W_\Phi$ for some Φ in $\mathscr{S}(F^2)$. If $\phi_W = 0$ then by lemma 119 the function $f_{\Phi^\sim}$ is zero. However, the map $W_\Phi \mapsto f_{\Phi^\sim}$ is injective. Thus $W_\Phi = W = 0$.

3) This follows from equation (13) in the proof of lemma 119 since the function Φ has compact support.

4) From part 3 of the proposition and the same argument as in lemma 58, for any $n \in N_F$ and any $\phi \in V$ the function $\phi - \xi_\psi(n)\phi$ lies in $\mathscr{S}(F^\times)$. Thus the intersection of V with $\mathscr{S}(F^\times)$ is nonempty. Since $\mathscr{S}(F^\times)$ is irreducible under the action of ξ_ψ (proposition 46) it follows that V contains $\mathscr{S}(F^\times)$.

5) This is clear from parts 1 and 2.

122 Proposition ([JL] Prop. 3.4). [narch-prin-kiri-90] *For any quasi-characters μ_1 and μ_2 we have*

$$W(\mu_1, \mu_2; \psi) = W(\mu_2, \mu_1; \psi).$$

If Φ is an element of $\mathscr{S}(F^2)$ let Φ' be the function defined by

$$\Phi'(x, y) = \Phi(y, x).$$

We prove the proposition by establishing the identity

$$\mu_1(\det g)|\det g|^{1/2}\theta(\mu_1, \mu_2; r(g)\Phi') = \mu_2(\det g)|\det g|^{1/2}\theta(\mu_2, \mu_1; r(g)\Phi) \tag{14}$$

for all Φ in $\mathscr{S}(F^2)$. If $g = 1$ then (14) is clear by the definition of θ. If g belongs to $\mathrm{SL}(2, F)$ then by proposition 113 it easily follows that

$$r(g)\Phi' = \big(r(g)\Phi\big)'$$

and so (14) holds for all g in $\mathrm{SL}(2, F)$. Thus it suffices to prove that (14) holds for

$$g = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}.$$

This reduces to the identity

$$\mu_1(a)\int_{F^\times} \Phi'(at, t^{-1})\mu_1(t)\mu_2^{-1}(t)d^\times t = \mu_2(a)\int_{F^\times} \Phi(at, t^{-1})\mu_2(t)\mu_1^{-1}(t)d^\times t$$

which follows after a simple change of variables.

5.6.5 The representations $\pi(\mu_1, \mu_2)$ and $\sigma(\mu_1, \mu_2)$

[narch-prin-pi]

123 Theorem ([JL] Thm. 3.3). [narch-prin-pi-10] *Let μ_1 and μ_2 be two quasi-characters of $F^\times$.*

1. *If neither $\mu_1\mu_2^{-1}$ nor $\mu_2^{-1}\mu_1$ is α_F then $\rho(\mu_1, \mu_2)$ and $\rho(\mu_2, \mu_1)$ are equivalent and irreducible.*
2. *Let $\mu_1\mu_2^{-1} = \alpha_F$ and write $\mu_1 = \chi\alpha_F^{1/2}$ and $\mu_2 = \chi\alpha_F^{-1/2}$.*
   (a) *$\mathscr{B}(\mu_1, \mu_2)$ contains a unique proper stable subspace $\mathscr{B}_s(\mu_1, \mu_2)$ which is irreducible and of codimension 1.*
   (b) *$\mathscr{B}(\mu_2, \mu_1)$ contains a unique proper stable subspace $\mathscr{B}_f(\mu_2, \mu_1)$ which is one dimensional; it is the representation corresponding to χ.*
   (c) *The modules $\mathscr{B}_s(\mu_1, \mu_2)$ and $\mathscr{B}(\mu_2, \mu_1)/\mathscr{B}_f(\mu_2, \mu_1)$ are equivalent.*
   (d) *The modules $\mathscr{B}_f(\mu_2, \mu_1)$ and $\mathscr{B}(\mu_1, \mu_2)/\mathscr{B}_s(\mu_1, \mu_2)$ are equivalent.*

124 Corollary. *If μ_1 and μ_2 are quasi-characters of $F^\times$ such that $\mu_1\mu_2^{-1}$ is not equal to α_F^{-1} then $W(\mu_1,\mu_2;\psi)$ is equivalent to $\rho(\mu_1,\mu_2)$.*

This follows immediately from propositions 120 and 122 and theorem 123.

125. We need two lemmas before proving theorem 123. Before beginning the proof, we make the following definitions.

126. Let μ_1 and μ_2 be quasi-characters of $F^\times$.

1. If $\mu_1\mu_2^{-1}$ is not equal to α_F or α_F^{-1}, we define $\pi(\mu_1,\mu_2)$ to be the representation $\rho(\mu_1,\mu_2)$ on the space $\mathscr{B}(\mu_1,\mu_2)$.
2. If $\mu_1\mu_2^{-1}$ is equal to α_F we define $\pi(\mu_1,\mu_2)$ to be the induced action of $\rho(\mu_1,\mu_2)$ on the one dimensional quotient $\mathscr{B}(\mu_1,\mu_2)/\mathscr{B}_s(\mu_1,\mu_2)$
3. If $\mu_1\mu_2^{-1}$ is equal to α_F^{-1} we define $\pi(\mu_1,\mu_2)$ to be the restriction of $\rho(\mu_1,\mu_2)$ to the one dimensional space $\mathscr{B}_f(\mu_1,\mu_2)$.
4. If $\mu_1\mu_2^{-1}$ is equal to α_F we define $\sigma(\mu_1,\mu_2)$ to be the restriction of $\rho(\mu_1,\mu_2)$ to the space $\mathscr{B}_s(\mu_1,\mu_2)$.
5. If $\mu_1\mu_2^{-1}$ is equal to α_F^{-1} we define $\sigma(\mu_1,\mu_2)$ to be induced action of $\rho(\mu_1,\mu_2)$ on the quotient $\mathscr{B}(\mu_1,\mu_2)/\mathscr{B}_f(\mu_1,\mu_2)$.

The representations $\pi(\mu_1,\mu_2)$ are called the *principal series representations* while the representations $\sigma(\mu_1,\mu_2)$ are called the *special representations.* Some comments:

1. The representation $\pi(\mu_1,\mu_2)$ is defined for all quasi-characters while $\sigma(\mu_1,\mu_2)$ is only defined when $\mu_1\mu_2^{-1}$ is α_F or α_F^{-1}.
2. The representations $\pi(\mu_1,\mu_2)$ and $\sigma(\mu_1,\mu_2)$ are irreducible.
3. The representations $\pi(\mu_1,\mu_2)$ and $\pi(\mu_2,\mu_1)$ are equivalent, as are the representations $\sigma(\mu_1,\mu_2)$ and $\sigma(\mu_2,\mu_1)$.
4. Both $\pi(\mu_1,\mu_2)$ and $\sigma(\mu_1,\mu_2)$ appear both as quotients and subrepresentations of $\rho(\mu_1,\mu_2)$.

127 Lemma ([JL] Lemma 3.3.1). [narch-prin-pi-50] *Suppose there is a nonzero function f in $\mathscr{B}(\mu_1,\mu_2)$ which is invariant under right translation by N_F. Then there is a quasi-character χ such that $\mu_1=\chi\alpha_F^{-1/2}$ and $\mu_2=\chi\alpha_F^{1/2}$ and f is a multiple of χ (i.e., $g\mapsto\chi(\det g)$).*

Let $\omega=\mu_1^{-1}\mu_2\alpha_F^{-1}$. Since for any c in $F^\times$ we have

$$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = \begin{bmatrix} c^{-1} & 1 \\ 0 & c \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & c^{-1} \\ 0 & 1 \end{bmatrix}$$

if follows that

$$f\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = \omega(c) f\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Since f is locally constant, it follows that there exists an ideal $\mathfrak{a}$ of F such that ω is constant on $\mathfrak{a}-0$. Thus ω is the trivial character. This establishes the form of μ_1 and μ_2.

Recall the Bruhat decomposition: G_F is the disjoint union of P_F and P_FwN_F. It is clear that f is a multiple of χ on each of the sets P_F and P_FwN_F. Since f is continuous it must be the same multiple.

128 Lemma ([JL] Lemma 3.3.2). [narch-prin-pi-60] *Assume $|(\mu_1\mu_2^{-1})(\varpi)|=|\varpi|^s$ with $s>-1$. There is a nonzero minimal stable subspace of $\mathscr{B}(\mu_1,\mu_2)$. For any f in $\mathscr{B}(\mu_1,\mu_2)$ and n in N_F the element $f-\rho(n)f$ belongs to X.*

Let V be the Kirillov model of $\mathscr{B}(\mu_1,\mu_2)$ (*cf.* proposition 121) and let V_0 be the Schwartz space $\mathscr{S}(F^\times)$. For any n in N_F and ϕ in V the function $f-\xi_\psi(n)f$ belongs to V_0. Thus any stable subspace of V meets V_0 and therefore contains V_0. Therefore, the intersection of all nonzero stable subspaces of V is again a nonzero stable subspace.

129 Proof of theorem 123. By taking contragrediants, it suffices to consider the case where $|(\mu_1\mu_2^{-1})(\varpi)| = |\varpi|^s$ with $s > -1$.

We have defined a nondegenerate invariant pairing between $\mathscr{B}(\mu_1,\mu_2)$ and $\mathscr{B}(\mu_1^{-1},\mu_2^{-1})$ (*cf.* proposition 106). Let X be the minimal stable subspace of $\mathscr{B}(\mu_1,\mu_2)$ of lemma 128. Then for v in $\mathscr{B}(\mu_1,\mu_2)$, v' in the orthogonal complement of X and $n \in N_F$ we have

$$\langle v, \rho(n)v' - v'\rangle = \langle \rho(n)v - v, v'\rangle = 0$$

since $\rho(n)v - v$ belongs to X. Thus v' is stabilized under N_F.

1) By lemma 127 any vector stabilized by N_F is zero. Thus the othogonal complement of X is zero and so $\mathscr{B}(\mu_1,\mu_2) = X$. It follows that $\mathscr{B}(\mu_1,\mu_2)$ is irreducible.

The central quasi-character of $\rho(\mu_1,\mu_2)$ is $\omega = \mu_1\mu_2$. It follows (by proposition 106 and theorem 88) that

$$\rho(\mu_1,\mu_2) \cong \omega \otimes \rho(\mu_1^{-1},\mu_2^{-1}) \cong \rho(\mu_2,\mu_1).$$

This proves the first part of the theorem.

2) Write $\mu_1 = \chi\alpha_F^{1/2}$ and $\mu_2 = \chi\alpha_F^{-1/2}$. In this case, lemma 127 shows that X is the space orthogonal to the function χ^{-1} in $\mathscr{B}(\mu_1^{-1},\mu_2^{-1})$. We put $\mathscr{B}_s(\mu_1,\mu_2) = X$; since it has codimension one it is the only proper stable subspace. Thus its orthogonal complement, *i.e.*, the span of χ^{-1}, which we name $\mathscr{B}_f(\mu_1^{-1},\mu_2^{-1})$, is the only proper stable subspace of $\mathscr{B}(\mu_1^{-1},\mu_2^{-1})$.

Let σ be the representation of G_F on $\mathscr{B}_s(\mu_1,\mu_2)$; its central quasi-character is $\omega = \mu_1\mu_2$. By the exactness of the contragrediant (*cf.* §1.4.3, proposition 105) it follows that $\widetilde{\sigma}$ is equivalent to the representation of G_F on $\mathscr{B}(\mu_1^{-1},\mu_2^{-1})/\mathscr{B}_f(\mu_1^{-1},\mu_2^{-1})$. Now, σ is equivalent to $\omega\otimes\widetilde{\sigma}$, which is equivalent to the representation of G_F on $\mathscr{B}(\mu_2,\mu_1)/\mathscr{B}_f(\mu_2,\mu_1)$.

It is clear that both $\mathscr{B}_f(\mu_2,\mu_1)$ and $\mathscr{B}(\mu_1,\mu_2)/\mathscr{B}_s(\mu_1,\mu_2)$ are both equivalent to the representation corresponding to χ. This completes the proof.

5.6.6 The series $C(\nu,t)$ for $\pi(\mu_1,\mu_2)$ and $\sigma(\mu_1,\mu_2)$

[`narch-prin-cnut`]

130 Note. This material does not seem to occur explicitly in either Jacquet-Langlands or in Bump.

131. We first fix some notation which will be in effect for the remainder of the section.

1. If ν is a character of U_F we write ϕ_ν for the function which is equal to ν on U_F and 0 outside of U_F. We will regard ϕ_ν as a function on F and on $F^\times$ at different points.
2. We let μ_1 and μ_2 be quasi-characters of $F^\times$;
3. We let ν_0 for the character $\mu_1\mu_2$ of U_F;
4. We let ν be a fixed character of U_F;
5. We let $\chi_1 = \nu\mu_1$ and $\chi_2 = \nu\mu_2$;
6. We let Φ be the element of $\mathscr{S}(F^2)$ given by $\Phi(x,y) = \chi_2(x)\chi_1(y)$;
7. We let $\mathfrak{p}^{-m}$ be the conductor of ψ;
8. We let $1+\mathfrak{p}^{n_i}$ be the conductor of χ_i when it is nontrivial;
9. We let c_i the nontrivial Gaussian sum $\eta(\chi_i, \varpi^{-m-n_i})$ when χ_i is nontrivial;
10. We let κ_i be $\mu_i(\varpi)$;
11. We let α denote the constant $|\varpi|$;
12. We let β denote the constant $-|\varpi|(1-|\varpi|)^{-1}$.
13. For an integer ℓ and an element ϵ of U_F we put

$$q_\ell(\nu,\epsilon) = W_\Phi\left(\begin{bmatrix} \epsilon\varpi^\ell & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}\right).$$

132 Theorem. [narch-prin-cnut-30] *For the representations $\pi(\mu_1, \mu_2)$ (with $\mu_1\mu_2^{-1}$ equal to neither α_F nor α_F^{-1}) and for the representations $\sigma(\mu_1, \mu_2)$ (with $\mu_1\mu_2^{-1}$ equal to α_F or α_F^{-1}) we have the following:*

1. *If ν is neither μ_1^{-1} nor μ_2^{-1} then*

$$C(\nu, t) = c_1 c_2 \alpha^{\ell/2} \kappa_1^{-m-n_1} \kappa_2^{-m-n_2} t^{\ell}$$

where $\ell = -(2m + n_1 + n_2)$.

2. *If $\mu_1 \neq \mu_2$ on U_F and $\nu = \mu_1^{-1}$ then*

$$C(\nu, t) = c_2 \beta \alpha^{\ell/2} \kappa_1^{-m-1} \kappa_2^{-m-n_2} t^{\ell} \left(\frac{1 - \alpha^{-1/2}\kappa_1 t}{1 - \alpha^{1/2}\kappa_1 t} \right)$$

where $\ell = -(2m + n_2 + 1)$.

3. *If $\mu_1 \neq \mu_2$ on U_F and $\nu = \mu_2^{-1}$ then*

$$C(\nu, t) = c_1 \beta \alpha^{\ell/2} \kappa_1^{-m-n_1} \kappa_2^{-m-1} t^{\ell} \left(\frac{1 - \alpha^{-1/2}\kappa_2 t}{1 - \alpha^{1/2}\kappa_2 t} \right)$$

where $\ell = -(2m + n_1 + 1)$

4. *If $\mu_1 = \mu_2$ on U_F and $\nu = \mu_1^{-1}$ then*

$$C(\nu, t) = \beta^2 (\alpha \kappa_1 \kappa_2 t^2)^{-m-1} \frac{(1 - \alpha^{-1/2}\kappa_1 t)(1 - \alpha^{-1/2}\kappa_2 t)}{(1 - \alpha^{1/2}\kappa_1 t)(1 - \alpha^{1/2}\kappa_2 t)}.$$

133 Corollary. [narch-prin-cnut-40] *We have the following:*

1. *$\pi(\mu_1, \mu_2)$ is never equivalent to $\sigma(\mu_1', \mu_2')$.*
2. *The reresentations $\pi(\mu_1, \mu_2)$ and $\pi(\mu_1', \mu_2')$ are equivalent if and only if $\{\mu_1, \mu_2\} = \{\mu_1', \mu_2'\}$.*
3. *The reresentations $\sigma(\mu_1, \mu_2)$ and $\sigma(\mu_1', \mu_2')$ are equivalent if and only if $\{\mu_1, \mu_2\} = \{\mu_1', \mu_2'\}$.*

We already know, from theorem 123, that $\pi(\mu_1, \mu_2) = \pi(\mu_2, \mu_1)$ and $\sigma(\mu_1, \mu_2) = \sigma(\mu_2, \mu_1)$.

Now, it follows immediately from theorem 132 that the representations $\pi(\mu_1, \mu_2)$ (when $\mu_1\mu_2^{-1}$ is neither α_F nor α_F^{-1}) and $\sigma(\mu_1, \mu_2)$ (when $\mu_1\mu_2^{-1}$ is α_F or α_F^{-1}) determine the datum $\{(\mu_1|_{U_F}, \kappa_1), (\mu_2|_{U_F}, \kappa_2)\}$. This determines $\{\mu_1, \mu_2\}$. Thus the only possible equivalences, besides those already mentioned, are between $\sigma(\mu_1, \mu_2)$ and $\pi(\mu_1, \mu_2)$ when $\mu_1\mu_2^{-1}$ is α_F or α_F^{-1}. However then $\pi(\mu_1, \mu_2)$ is one dimensional and $\sigma(\mu_1, \mu_2)$ is infinite dimensional.

134 Corollary. [narch-prin-cnut-50] *We have the following:*

1. *If $\mu_1\mu_2^{-1}$ is not α_F or α_F^{-1} then the Jacquet module of $\pi(\mu_1, \mu_2)$ is two dimensional.*
2. *If $\mu_1\mu_2^{-1} = \alpha_F$ write $\mu_1 = \chi\alpha_F^{1/2}$ and $\mu_2 = \chi\alpha_F^{-1/2}$. Then the Jacquet module of $\pi(\mu_1, \mu_2)$ is one dimensional if χ is trivial and zero dimensional otherwise.*
3. *The Jacquet module of $\sigma(\mu_1, \mu_2)$ is one dimensional (necessarily $\mu_1\mu_2^{-1}$ is α_F or α_F^{-1}).*

The second assertion is clear. Discounting the case when $\pi(\mu_1, \mu_2)$ is one dimensional, it is clear from theorem 132 and proposition 83 that the Jacuqet module is two dimensional except when $\mu_1 = \mu_2$ on U_F and either $\kappa_1 = \alpha\kappa_2$ or $\kappa_2 = \alpha\kappa_1$, *i.e.*, except when $\mu_1\mu_2^{-1}$ is α_F or α_F^{-1}.

135 Lemma. [narch-prin-cnut-60] *We have*

$$W_\Phi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = \phi_{\nu\nu_0}(a).$$

Working directly from the definition of W_Φ, we have

$$\begin{aligned} W_\Phi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} &= \mu_1(a)|a|^{1/2} \int_{F^\times} \mu_1(t)\mu_2^{-1}(t) \left(r \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \Phi \right) d^\times t \\ &= \mu_1(a)|a|^{1/2} \int_{F^\times} \mu_1(t)\mu_2^{-1}(t)\Phi(at, t^{-1}) d^\times t. \end{aligned}$$

Now, $\Phi(at, t^{-1}) = 0$ unless both t and a are in U_F. If a is in U_F then the above is equal to

$$\begin{aligned} \mu_1(a) \int_{U_F} \mu_1(t)\mu_2^{-1}(t)\chi_1(t^{-1})\chi_2(at) dt &= \mu_1(a)\chi_2(a) \int_{U_F} (\mu_1\mu_2^{-1}\chi_1^{-1}\chi_2)(t) dt \\ &= \nu(a)\mu_1(a)\mu_2(a) \int_{U_F} 1 dt = (\nu\nu_0)(a) \end{aligned}$$

136 Lemma. [narch-prin-cnut-70] *There exists complex numbers $Q_\ell(\nu)$ such that*

$$q_\ell(\nu, \epsilon) = Q_\ell(\nu)\nu^{-1}(\epsilon).$$

The values of Q_ℓ are listed below.

1. *If neither χ_1 nor χ_2 is trivial then*

$$Q_\ell(\nu) = \begin{cases} c_1 c_2 \alpha^{\ell/2} \kappa_1^{-m-n_1} \kappa_2^{-m-n_2} & \ell = -(2m + n_1 + n_2) \\ 0 & \ell \neq -(2m + n_1 + n_2) \end{cases}$$

2. *If χ_1 is trivial and χ_2 is nontrivial then*

$$Q_\ell(\nu) = c_2 \alpha^{\ell/2} \kappa_1^{\ell+m+n_2} \kappa_2^{-m-n_2} \times \begin{cases} 1 & \ell > -2m - n_2 - 1 \\ \beta & \ell = -2m - n_2 - 1 \\ 0 & \ell < -2m - n_2 - 1 \end{cases}$$

3. *If χ_2 is trivial and χ_1 is nontrivial then*

$$Q_\ell(\nu) = c_1 \alpha^{\ell/2} \kappa_1^{-m-n_1} \kappa_2^{\ell+m+n_1} \times \begin{cases} 1 & \ell > -2m - n_1 - 1 \\ \beta & \ell = -2m - n_1 - 1 \\ 0 & \ell < -2m - n_1 - 1 \end{cases}$$

4. *If both χ_1 and χ_2 are trivial then*

$$Q_\ell(\nu) = \alpha^{\ell/2}(A + B + C)$$

where

$$A = \beta\kappa_1^{-m-1}\kappa_2^{\ell+m+1} \times \begin{cases} 1 & \ell > -2m - 2 \\ \frac{1}{2}\beta & \ell = -2m - 2 \\ 0 & \ell < -2m - 2 \end{cases} \qquad B = \beta\kappa_1^{\ell+m+1}\kappa_2^{-m-1} \times \begin{cases} 1 & \ell > -2m - 2 \\ \frac{1}{2}\beta & \ell = -2m - 2 \\ 0 & \ell < -2m - 2 \end{cases}$$

$$C = \begin{cases} \dfrac{\kappa_1^{\ell+m+1}\kappa_2^{-m} - \kappa_1^{-m}\kappa_2^{\ell+m+1}}{\kappa_1 - \kappa_2} & \ell \geq -2m \text{ and } \kappa_1 \neq \kappa_2 \\ (2m + \ell + 1)\kappa_1^\ell & \ell \geq -2m \text{ and } \kappa_1 = \kappa_2 \\ 0 & \ell < -2m \end{cases}$$

By proposition 113 we have

$$\left(r\left(\begin{bmatrix} \epsilon\varpi^\ell & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)\Phi\right)(a,b) = \big(r(w)\Phi\big)(\epsilon\varpi^\ell a, b)$$
$$= \int_{F^2} \Phi(y,x)\psi(\epsilon\varpi^\ell ax + by)dxdy$$
$$= \int_{U_F}\int_{U_F} \chi_1(x)\chi_2(y)\psi(\epsilon\varpi^\ell ax)\psi(by)dxdy$$
$$= \eta(\chi_1, \epsilon\varpi^\ell a)\eta(\chi_2, b)$$

Therefore,

$$\begin{aligned} q_\ell(\nu,\epsilon) &= W_\Phi\left(\begin{bmatrix} \epsilon\varpi^\ell & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) \\ &= \mu_1(\epsilon\varpi^\ell)|\epsilon\varpi^\ell|^{1/2}\int_{F^\times} \mu_1(t)\mu_2^{-1}(t)\left(r\left(\begin{bmatrix} \epsilon\varpi^\ell & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\Phi\right)(t,t^{-1})d^\times t \\ &= \mu_1(\epsilon\varpi^\ell)|\varpi|^{\ell/2}\int_{F^\times} \mu_1(t)\mu_2^{-1}(t)\eta(\chi_1,\epsilon\varpi^\ell t)\eta(\chi_2,t^{-1})d^\times t \\ &= \mu_1(\epsilon\varpi^\ell)|\varpi|^{\ell/2}\sum_k\int_{\varpi^k U_F} \mu_1(t)\mu_2^{-1}(t)\eta(\chi_1,\epsilon\varpi^\ell t)\eta(\chi_2,t^{-1})d^\times t \\ &= \mu_1(\epsilon\varpi^\ell)|\varpi|^{\ell/2}\sum_k \mu_1(\varpi^k)\mu_2(\varpi^{-k})\int_{U_F} \mu_1(t)\mu_2^{-1}(t)\eta(\chi_1,\epsilon\varpi^{\ell+k} t)\eta(\chi_2,\varpi^{-k}t)dt \\ &= \mu_1(\epsilon\varpi^\ell)|\varpi|^{\ell/2}\sum_k \mu_1(\varpi^k)\mu_2(\varpi^{-k})\chi_1^{-1}(\epsilon)\eta(\chi_1,\varpi^{\ell+k})\eta(\chi_2,\varpi^{-k})\int_{U_F}(\mu_1\mu_2^{-1}\chi_1^{-1}\chi_2)(t)dt \end{aligned}$$

and so, with a bit more manipulation, we obtain

$$Q_\ell(\nu) = \alpha^{\ell/2}\sum_k \kappa_1^{\ell+k}\kappa_2^{-k}\eta(\chi_1,\varpi^{\ell+k})\eta(\chi_2,\varpi^{-k}). \tag{15}$$

We now go case by case.

1) The η factors are zero unless $\ell+k=-m-n_1$ and $-k=-m-n_2$. Thus for $Q_\ell(\nu)$ to be nonzero we must have $\ell=-(n_1+n_2+2m)$. In this case, (15) reduces to

$$Q_\ell(\nu) = c_1c_2\alpha^{\ell/2}\kappa_1^{\ell+k}\kappa_2^{-k}$$

which is the stated result.

2) Only the term with $k=m+n_2$ is nonzero. The identity (15) reduces to

$$Q_\ell(\nu) = c_2\alpha^{\ell/2}\kappa_1^{\ell+k}\kappa_2^{-k}\eta(1,\varpi^{\ell+k})$$

and the result follows from the evaluation of $\eta(1,\varpi^p)$.

3) This follows from part 2 and symmetry.

4) The η factors vanish if $k>m+1$ or $k<-\ell-m-1$. Thus

$$Q_\ell(\nu) = \alpha^{\ell/2}\sum_{-\ell-m-1\le k\le m+1} \kappa_1^{\ell+k}\kappa_2^{-k}\eta(\chi_1,\varpi^{\ell+k})\eta(\chi_2,\varpi^{-k}).$$

The sum is zero if $\ell<-2m-2$. We can peel off the first term to get

$$Q_\ell(\nu) = \alpha^{\ell/2}\left(A' + \sum_{-\ell-m\le k\le m+1} \kappa_1^{\ell+k}\kappa_2^{-k}\eta(\chi_1,\varpi^{\ell+k})\eta(\chi_2,\varpi^{-k})\right)$$

where

$$A' = \beta\kappa_1^{-m-1}\kappa_2^{\ell+m+1}\times\begin{cases} 1 & \ell>-2m-2 \\ \beta & \ell=-2m-2 \\ 0 & \ell<-2m-2 \end{cases}$$

The sum is zero if $\ell < -2m-1$. We can peel off the last term to get

$$Q_\ell(\nu) = \alpha^{\ell/2}\left(A' + B' + \sum_{-\ell-m\le k\le m} \kappa_1^{\ell+k}\kappa_2^{-k}\eta(\chi_1,\varpi^{\ell+k})\eta(\chi_2,\varpi^{-k})\right)$$

where

$$B' = \beta\kappa_1^{\ell+m+1}\kappa_2^{-m-1}\times\begin{cases}1 & \ell > -2m-2\\ 0 & \ell < -2m-1\end{cases}$$

Notice two things: 1) all the η factors in the sum are now 1; and 2) $A'+B' = A+B$. Thus

$$Q_\ell(\nu) = \alpha^{\ell/2}\left(A + B + \sum_{-\ell-m\le k\le m} \kappa_1^{\ell+k}\kappa_2^{-k}.\right)$$

The sum is zero if $\ell < -2m$. If $\kappa_1 = \kappa_2$ then each of the $(2m+\ell+1)$ terms in the sum contributes κ_1^ℓ. If $\kappa_1 \ne \kappa_2$ the sum if a geometric series.

137 Proof of theorem 132. Note that since the representations and stated values for the series are symmetric in μ_1 and μ_2 we may interchange μ_1 and μ_2, if necessary, and assume $|(\mu_1\mu_2^{-1})(\varpi)| = |\varpi|^s$ with $s > -1$

Recall how the numbers $C_\ell(\nu)$ are computed: First we take the representation in Kirillov form. Next we take the function ϕ_ν, which is an element of the Kirillov space. Then we have

$$C_\ell(\nu) = \int_{U_F} (\pi(w)\phi_\nu)(\epsilon\varpi^\ell)\nu(\epsilon)d\epsilon.$$

Now, in the present situation, we have demonstrated that

$$W_\Phi\begin{bmatrix} a & 0\\ 0 & 1\end{bmatrix}$$

is equal to ϕ_ν. This function belongs to the Kirillov model (even form the special representation). Thus,

$$\begin{aligned}C_\ell(\nu) &= \int_{U_F} W_\Phi\left(\begin{bmatrix}\epsilon\varpi^\ell & 0\\ 0 & 1\end{bmatrix}cc\right]\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right)\nu(\epsilon)d\epsilon\\ &= \int_{U_F} q_\ell(\nu,\epsilon)\nu(\epsilon)d\epsilon\\ &= Q_\ell(\nu).\end{aligned}$$

Thus lemma 136 gives the values of $C_\ell(\nu)$. The rest is just algebra.

5.7 Classification of irreducible representations

[`narch-class`]

138 Theorem. [`narch-class-10`] *Let (π, V) be an irreducible admissible representation of G_F. Then π is equivalent to one of the following:*

1. *A principal series representation $\pi(\mu_1,\mu_2)$.*
2. *A special representation $\sigma(\mu_1,\mu_2)$.*
3. *An absolutely cuspidal representation.*

Furthermore, these representations are pairwise inequivalent except $\pi(\mu_1,\mu_2) = \pi(\mu_2,\mu_1)$ and $\sigma(\mu_1,\mu_2) = \sigma(\mu_2,\mu_1)$.

139. Note that the representation $\pi(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$ is the one dimensional representation corresponding to χ, so that the finite dimensional representations are not omitted in the above list.

140 Corollary. *Let (π, V) be an infinite dimensional irreducible admissible representation.*

1. *The dimension of JV is zero if and only if π is absolutely cuspidal.*
2. *The dimension of JV is one if and only if π is a special representation $\sigma(\mu_1, \mu_2)$.*
3. *The dimension of JV is two if and only if π is a principal series representation $\pi(\mu_1, \mu_2)$.*

This is just a restatement of already proved results.

141. By theorem 123 and article 133 it follows that to prove theorem 138, we need only show that 1) a representation which is not absolutely cuspidal is a subrepresentation of $\rho(\mu_1, \mu_2)$ for some quasi-characters μ_1 and μ_2; and 2) a representation which is absolutely cuspidal is not a subrepresentation of $\rho(\mu_1, \mu_2)$ for any quasi-characters μ_1 and μ_2. We prove 1 in lemma 143 below and 2 in lemma 145. In fact, we have already proved 2 by our results on that Jacquet modules of the principal series and special representations (*cf.* article 134); lemma 145 offers a different and more direct proof.

142. Note that once theorem 138 is proved we will have completed the proof of theorem 40. The proof that JV is at most two dimensional is quite unsatisfactory: We first show that JV if finite dimensional. We use this fact to prove theorem 138. Then, after we have the classification, we can conclude that the dimension of JV is at most two since we know it to be true for each type of representation in the classification.

143 Lemma ([JL] Prop. 2.17). [`narch-class-20`] *Let (π, V) be an infinite dimensional irreducible admissible representation which is not absolutely cuspidal. Then π is a subrepresentation of $\rho(\mu_1, \mu_2)$ for quasi-characters μ_1 and μ_2 of $F^\times$.*

We know that the contragrediant $\widetilde{\pi}$ is also not absolutely cuspidal. Thus by proposition 83, and our assumptions, the Jacquet module $J\widetilde{V}$ is finite dimensional and nonzero. It is a module over P_F for which N_F acts trivially; let $\widetilde{\sigma}$ be the representation of P_F on $J\widetilde{V}$.

Since P_F/N_F is abelian and $J\widetilde{V}$ is finite dimensional, it follows that there exists L in $J\widetilde{V}$ and quasi-characters μ_1 and μ_2 of $F^\times$ such that

$$\widetilde{\sigma}\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix} L = \mu_1^{-1}(a_1)\mu_2^{-1}(a_2)L$$

for all $a_1, a_2 \in F^\times$ and $x \in F$. By the definition of the contragrediant action, such an L satisfies

$$L\left(\pi\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix} v\right) = \mu_1(a_1)\mu_2(a_2)L(v).$$

The map $A : V \to C(G_F)$ given by $(Av)(g) = L(\pi(g)v)$ is then an injection of V into $\mathscr{B}(\mu_1, \mu_2)$. This completes the proof.

144. Let $\mathscr{B}$ be the subspace of $C(G_F)$ stabilzed by $\lambda(N_F)$, *i.e.*, the space of functions ϕ on G_F which satisfy

$$\phi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) = \phi(g)$$

for all $x \in F$ and $g \in G$. The space $\mathscr{B}$ is stable under the action of ρ and thus $(\rho, \mathscr{B})$ is a representation of G_F. Clearly $\mathscr{B}(\mu_1, \mu_2)$ is a subrepresentation of $\mathscr{B}$ for any pair of quasi-characters, and so to show that a representation is not a subrepresentation of $\mathscr{B}(\mu_1, \mu_2)$ it suffices to show that it is not a subrepresentation of $\mathscr{B}$.

145 Lemma ([JL] Prop. 2.16). [narch-class-40] *Let (π, V) be an infinite dimensional irreducible admissible representation. Then π is absolutely cuspidal if and only if V is not isomorphic to a submodule of $\mathscr{B}$.*

If V is isomorphic to a submodule of $\mathscr{B}$ then there is an injection $A : V \to \mathscr{B}$ such that $A\pi(g)\phi = \rho(g)A\phi$. If $L(\phi) = (A\phi)(1)$ then L is a nonzero linear functional on V satisfying

$$L(\xi_\psi(n_x)\phi) = L(\phi). \tag{16}$$

Conversely, let L be a nonzero linear form on V satisfying (16) and define a map $A : V \to C(G_F)$ by $A(\phi) = L(\pi(g)\phi)$. Clearly A satisfies $A\pi(g) = \rho(g)A$ and maps V into $\mathscr{B}$. Since L is nonzero, A is injective by the usual argument. Thus V is isomorphic to a submodule of $\mathscr{B}$. We conclude that V is isomorphic to a submodule of $\mathscr{B}$ if and only if it admits a nonzero linear form L satisfying (16).

However, it is clear that a linear form L satisfying (16) annihilates vectors of the form $\xi_\psi(n_x)\phi - \phi$, and therefore all of Schwartz space; conversely, a linear form L which annihilates Schwartz space will annihilate the vectors $\xi_\psi(n_x)\phi - \phi$ and therefore satisfy (16). Thus a linear form L satisfies (16) if and only if it annihilates Schwartz space. Therefore, there exists a nonzero linear functional satisfying (16) if and only if Schwartz space does not exhaust V. This completes the proof.

5.8 Local L-functions

[narch-lfn]

5.8.1 The functions $L(s, \pi)$ and $Z(s, \phi, \xi)$

[narch-lfn-x10]

146. Throughout this section, (π, V) will be an infinite dimensional irreducible admissible representation taken in Kirillov form.

147. In the present context, an *Euler factor* is a function of the form $P(q^{-s})^{-1}$ where P is a polynomial and $q = |\varpi|^{-1}$ is the cardinality of the residue field of F.

148. We define the *local L-function* $L(s, \pi)$ of π:

1. If π is absolutely cuspidal then $L(s, \pi) = 1$.
2. If $\pi = \pi(\mu_1, \mu_2)$ then $L(s, \pi) = L(s, \mu_1)L(s, \mu_2)$ where $L(s, \mu_i)$ is the local L-function for GL(1) (*cf.* §2.1.2).
3. If $\pi = \sigma(\mu_1, \mu_2)$ where $\mu_1\mu_2^{-1} = \alpha_F$ then $L(s, \pi) = L(s, \mu_1)$.

Note that although we have defined $L(s, \pi)$ when π is a one dimensional representation many of the theorems in this section do not apply in this case.

149. For ϕ in V and a quasi-character ξ of $F^\times$ define the zeta function $Z(s, \phi, \xi)$ by

$$Z(s, \phi, \xi) = \int_{F^\times} \xi(a)\phi(a)|a|^{s-1/2} d^\times a.$$

If W is an element of the Whittaker model of π and ϕ_W is the corresponding element of the Kirillov model, we write $Z(s, W, \xi)$ in place of $Z(s, \phi_W, \xi)$. If χ is the tirival character we write $Z(s, \phi)$ in place of $Z(s, \phi, \chi)$.

150 Proposition ([JL] Thm 2.18; [Bu] Prop. 4.7.5). [narch-lfn-140] *We have:*

1. *$Z(s, \phi, \xi) = \hat{\phi}(\xi, \xi(\varpi)q^{1/2-s})$.*
2. *For all ϕ in V, the integral defining $Z(s, \phi)$ converges absolutely for $\Re s$ sufficiently large.*

3. *For all ϕ in V the ratio*
$$\frac{Z(s,\phi)}{L(s,\pi)} \tag{17}$$
can be analytically extended to an entire function of s (in fact, it is a Laurent polynomial in q^{-s}).

4. *There exists ϕ such that the ratio (17) is equal to 1.*

5. *The function $L(s,\pi)$ is the unique Euler factor which satisfies 3 and 4.*

1) We have

$$\begin{aligned} Z(s,\phi,\xi) &= \int_{F^\times} \xi(a)\phi(a)|a|^{s-1/2}d^\times a \\ &= \sum_k \int_{U_F} \xi(\varpi^k a)\phi(\varpi^k a)|\varpi^k a|^{s-1/2}d^\times a \\ &= \sum_k \left(\xi(\varpi)|\varpi|^{s-1/2}\right)^k \int_{U_F} \phi(\varpi^k a)\xi(a)da \\ &= \sum_k \left(\xi(\varpi)q^{1/2-s}\right)^k \hat{\phi}_k(\xi) \\ &= \hat{\phi}(\xi, \chi(\varpi)q^{1/2-s}) \end{aligned}$$

2) Since $\hat{\phi}(\xi,t)$ is a rational function of t, part 1 implies that $Z(s,\phi,\xi)$ converges absolutely for $\Re s$ sufficiently large.

3) If ϕ is in Schwartz space then $\hat{\phi}(1,t)$ is a Laurent polynomial and so $Z(s,\phi)$ is already a Laurent polynomial in q^{-s}. If $\phi = \pi(w)\phi_0$ where ϕ_0 is in Schwartz space, then

$$\hat{\phi}(1,t) = C(1,t)\hat{\phi}_0(\omega^{-1}, z_0^{-1}t^{-1})$$

by proposition 61, where ω is the central quasi-character of π and $z_0 = \omega(\varpi)$. Since $\hat{\phi}_0(\chi,t)$ is a Laurent polynomial for all χ, it suffices to show that

$$\frac{C(1,q^{1/2-s})}{L(s,\pi)}$$

has the stated properties. This is clear for absolutely cuspidal representations (since $C(1,t)$ is a Laurent polynomial) and follows for the other representations by theorem 132.

4) If π is absolutely cuspidal then ϕ can be taken to be the characteristic function of U_F.

We now prove statement 4 when $\pi = \pi(\mu_1,\mu_2)$ and μ_1 and μ_2 are unramified (and $\mu_1\mu_2^{-1}$ is not α_F or α_F^{-1}); the other cases may be handled nearly identically. Let ϕ_1 and ϕ_2 be in Schwartz space and let $\phi = \phi_1 + \pi(w)\phi_2$. Write $t = q^{1/2-s}$. Using proposition 61 and theorem 132 we have

$$\begin{aligned} \frac{Z(s,\phi)}{L(s,\pi)} =& \frac{\hat{\phi}_1(1,t) + C(1,t)\hat{\phi}_2(\omega^{-1}, z_0^{-1}t^{-1})}{L(s,\pi)} \\ =&(1-q^{-1/2}\kappa_1 t)(1-q^{-1/2}\kappa_2 t)\hat{\phi}_1(1,t) \\ &+ Kt^{-2m-2}(1-q^{1/2}\kappa_1 t)(1-q^{1/2}\kappa_2 t)\hat{\phi}_2(\omega^{-1}, z_0^{-1}t^{-1}) \end{aligned} \tag{18}$$

where K is a nonzero constant, ω is the central quasi-character, $z_0 = \omega(\varpi)$ and the other notations are as in theorem 132. Now, note two things: 1) given any Laurent polynomials $A(t)$ and $B(t)$ we can find ϕ_1 and ϕ_2 in Schwartz space such that

$$\hat{\phi}_1(1,t) = A(t) \qquad \hat{\phi}_2(\omega^{-1}, z_0^{-1}t^{-1}) = B(t);$$

and 2) the polynomials

$$(1-q^{-1/2}\kappa_1 t)(1-q^{-1/2}\kappa_2 t) \qquad \text{and} \qquad (1-q^{1/2}\kappa_1 t)(1-q^{1/2}\kappa_2 t)$$

are coprime. These two facts, together with (18), imply that ϕ_1 and ϕ_2 may be chosen to make (17) equal to 1.

5) Assume $L'(s,\pi)$ were another such Euler factor. Take ϕ so that $Z(s,\phi)/L(s,\pi)$ is equal to 1. Then

$$\frac{Z(s,\phi)}{L'(s,\pi)} \times \frac{L(s,\pi)}{Z(s,\phi)} = \frac{L(s,\pi)}{L'(s,\pi)}.$$

Since the product on the left is of two entire functions, it follows that $L(s,\pi)/L'(s,\pi)$ is entire. Similarly $L'(s,\pi)/L(s,\pi)$ is entire. Therefore $L'(s,\pi) = L(s,\pi)$.

5.8.2 The local functional equation

[narch-lfn-x20]

151 Theorem ([JL] Thm. 2.18; [Bu] Thm. 4.7.5). [narch-lfn-160] *Let (π, V) be an infinite dimensional irreducible admissible representation taken in Kirillov form with central quasi-character ω.*

1. *There exist ϵ-factors such that*

$$\frac{Z(1-s, \widetilde{\pi}(w)(\omega^{-1}\phi))}{L(1-s,\widetilde{\pi})} = \epsilon(s,\pi,\psi)\frac{Z(s,\phi)}{L(s,\pi)}. \tag{19}$$

   for all ϕ in V and all quasi-characters ξ.

2. *Define γ-factors by*

$$\gamma(s,\pi,\psi) = \frac{L(1-s,\widetilde{\pi})}{L(s,\pi)}\epsilon(s,\pi,\psi).$$

   Then for any quasi-character ξ we have

$$\gamma(s,\xi\otimes\pi,\psi) = C((\omega\xi)^{-1},(\omega\xi)^{-1}(\varpi)q^{s-1/2})$$

   and (19) may be rewritten as

$$Z(1-s,\pi(w)\phi,(\omega\xi)^{-1}) = \gamma(s,\xi\otimes\pi,\psi)Z(s,\phi,\xi). \tag{20}$$

3. *The factors $\epsilon(s,\pi,\psi)$ are of the form ab^s.*

4. *If π is the representation $\pi(\mu_1,\mu_2)$ (with $\mu_1\mu_2^{-1}$ neither α_F nor α_F^{-1}) then*

$$\epsilon(s,\pi,\xi,\psi) = \epsilon(s,\mu_1,\psi)\epsilon(s,\mu_2,\psi).$$

5. *If π is the representation $\sigma(\mu_1,\mu_2)$ with $\mu_1\mu_2^{-1} = \alpha_F$ then*

$$\epsilon(s,\pi,\xi,\psi) = \frac{L(1-s,\mu_1^{-1})}{L(s,\mu_2)}\epsilon(s,\mu_1,\psi)\epsilon(s,\mu_2,\psi).$$

152. The identities (19) and (20) are called the *local functional equation* for GL(2).

153 Corollary ([JL] Cor. 2.19; [Bu] Prop. 4.7.6). [narch-lfn-170] *Let (π, V) and (π', V') be infinite dimensional irreducible admissible representations with the same central quasi-character. Then π is equivalent to π' if and only if*

$$\gamma(s,\xi\otimes\pi,\psi) = \gamma(s,\xi\otimes\pi',\psi)$$

for all quasi-characters ξ.

By part 2 of the theorem, if the factors $\gamma(s,\xi\otimes\pi,\psi)$ are known for all ξ then the series $C(\nu,t)$ are known for all ν. Thus the corollary follows from proposition 78.

154 Proof of part 2 of theorem 151. We assume part 1 of the theorem has been proved. The identity (20) follows from the definition of the γ-factors and what we know of the Kirillov model of a twist (*cf.* proposition 73).

Now, let ϕ be in Schwartz space. On the one hand, by propositions 150 and 61 we have

$$\begin{aligned} Z(1-s,\pi(w)\phi,(\omega\xi)^{-1}) =&\pi(w)\hat{\phi}((\omega\xi)^{-1},(\omega\xi)^{-1}(\varpi)q^{s-1/2}) \\ =&C((\omega\xi)^{-1},(\omega\xi)^{-1}(\varpi)q^{s-1/2})\hat{\phi}(\xi,\xi(\varpi)q^{1/2-s}). \end{aligned}$$

On the other hand applying the local functional equation yields

$$\begin{aligned} Z(1-s,\pi(w)\phi,(\omega\xi)^{-1}) &= \gamma(s,\xi\otimes\pi,\psi)Z(s,\phi,\xi) \\ &= \gamma(s,\xi\otimes\pi,\psi)\hat{\phi}(\xi,\xi(\varpi)q^{1/2-s}). \end{aligned}$$

We thus conclude

$$\gamma(s,\xi\otimes\pi,\psi) = C((\omega\xi)^{-1},(\omega\xi)^{-1}(\varpi)q^{s-1/2}).$$

Part 2 of the theorem is thus proved.

155 Proof of theorem 151 for absolutely cuspidal representations. We have $L(s,\pi) = L(s,\widetilde{\pi}) = 1$ and so by proposition 150 we must prove

$$\pi(w)\hat{\phi}(\omega^{-1},\omega^{-1}(\varpi)q^{s-1/2}) = \epsilon(s,\pi,\psi)\hat{\phi}(1,q^{1/2-s}).$$

Since all ϕ in V are in Schwartz space, the above identity follows immediately from proposition 61 if we take

$$\epsilon(s,\pi,\psi) = C(\omega^{-1},\omega^{-1}(\varpi)q^{s-1/2}).$$

The third statement of the theorem follows immediately from the facts that $C(\nu,t)$ is a multiple of a power of t (*cf.* proposition 102).

156 Lemma. [`narch-lfn-220`] *Let μ_1 and μ_2 be quasi-characters of $F^\times$. Let $\Phi(x,y) = \phi_1(x)\phi_2(y)$ be an element of $\mathscr{S}(F^2)$ where ϕ_i is in $\mathscr{S}(F^1)$. Let W_Φ be the corresponding element of $\mathscr{W}(\mu_1,\mu_2;\Phi)$. Then*

$$Z(s,W_\Phi) = Z(s,\phi_1,\mu_1)Z(s,\phi_2,\mu_2)$$

We have

$$\begin{aligned} W_\Phi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} &= \theta\left(\mu_1,\mu_2;r_{\mu_1,\mu_2}\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\Phi\right) \\ &= \mu_1(a)|a|^{1/2}\int_{F^\times}\mu_1(t)\mu_2^{-1}(t)\Phi(at,t^{-1})d^\times t \end{aligned}$$

and so

$$\begin{aligned} Z(s,W_\Phi) &= \int_{F^\times}\int_{F^\times}\mu_1(t)\mu_2^{-1}(t)\mu_1(a)\Phi(at,t^{-1})|a|^s d^\times a d^\times t \\ &= \int_{F^\times}\int_{F^\times}\mu_1(a)\mu_2^{-1}(t)\Phi(a,t^{-1})|a|^s|t|^{-s}d^\times a d^\times t \\ &= \left(\int_{F^\times}\mu_1(a)\phi_1(a)|a|^s d^\times a\right)\left(\int_{F^\times}\mu_2(t)\phi_2(t)|t|^s d^\times t\right) \\ &= Z(s,\phi_1,\mu_1)Z(s,\phi_2,\mu_2) \end{aligned}$$

and the proposition is proved.

157 Proof of theorem 151 for the principal series representations. Let $\Phi(x,y) = \phi_1(x)\phi_2(y)$ be as in lemma 156. Functions of this form span $\mathscr{S}(F^2)$ so it suffices to prove the theorem for W_Φ.

Note that by proposition 113 we have

$$\rho(w)W_\Phi = W_{r(w)\Phi} = W_{\Phi'}$$

where $\Phi'(x,y) = \hat{\phi}_1(y)\hat{\phi}_2(x)$ and $\hat{\phi}_i$ is the Fourier transform of ϕ_i. Let $\omega = \mu_1\mu_2$ be the central quasi-character of π. We have (using lemma 156 and the local functional equation for GL(1), *cf.* §2.1.3, theorem 7)

$$\begin{aligned}\frac{Z(1-s,\rho(w)W_\Phi,\omega^{-1})}{L(1-s,\widetilde{\pi})} &= \frac{Z(1-s,\hat{\phi}_1,\mu_1^{-1})}{L(1-s,\mu_1^{-1})} \times \frac{Z(1-s,\hat{\phi}_2,\mu_2^{-1})}{L(1-s,\mu_2^{-1})} \\ &= \epsilon(s,\mu_1,\psi)\frac{Z(s,\phi_1,\mu_1)}{L(s,\mu_1)} \times \epsilon(s,\mu_2,\psi)\frac{Z(s,\phi_2,\mu_2)}{L(s,\mu_2)} \\ &= \epsilon(s,\mu_1,\psi)\epsilon(s,\mu_2,\psi)\frac{Z(s,W_\Phi)}{L(s,\pi)}.\end{aligned}$$

This proves parts 1 and 4 of the theorem. To prove part 3 note that

$$\epsilon(s,\pi,\psi) = \epsilon(s,\mu_1,\psi)\epsilon(s,\mu_2,\psi)$$

and use the corresponding result for the GL(1) ϵ-factors (*cf.* §2.1.3, theorem 7).

158 Proof of theorem 151 for the special representations. Take $\mu_1\mu_2^{-1} = \alpha_F$. Note that

$$L(s,\pi) = L(s,\mu_1), \qquad L(s,\widetilde{\pi}) = L(s,\mu_2^{-1}).$$

Again it suffices to prove the theorem for $\Phi(x,y) = \phi_1(x)\phi_2(y)$. We have

$$\begin{aligned}\frac{Z(1-s,\rho(w)W_\Phi,\omega^{-1})}{L(1-s,\widetilde{\pi})} =& L(1-s,\mu_1^{-1}) \times \frac{Z(1-s,\hat{\phi}_1,\mu_1^{-1})}{L(1-s,\mu_1^{-1})} \times \frac{Z(1-s,\hat{\phi}_2,\mu_2^{-1})}{L(1-2,\mu_2^{-1})} \\ =& L(1-s,\mu_1^{-1}) \times \epsilon(s,\mu_1,\psi)\frac{Z(s,\phi_1,\mu_1)}{L(s,\mu_1)} \times \epsilon(s,\mu_2,\psi)\frac{Z(s,\phi_2,\mu_2)}{L(s,\mu_2)} \\ =& \frac{L(1-s,\mu_1^{-1})}{L(s,\mu_2)}\epsilon(s,\mu_1,\psi)\epsilon(s,\mu_2,\psi)\frac{Z(s,W_\Phi)}{L(s,\pi)}\end{aligned}$$

This proves parts 1 and 5 of the theorem. To prove part 3, first note that

$$\epsilon(s,\pi,\psi) = \frac{L(1-s,\mu_1^{-1})}{L(s,\mu_2)}\epsilon(s,\mu_1,\psi)\epsilon(s,\mu_2,\psi).$$

The GL(1) ϵ-factors are of the correct form. If μ_1 is ramified (and thus μ_2 as well) then the L-functions are both 1. If μ_1 is unramified (and thus μ_2 as well) then, since $\mu_1(\varpi) = q^{-1}\mu_2(\varpi)$, we obtain

$$\frac{L(1-s,\mu_1^{-1})}{L(s,\mu_2)} = \frac{1-\mu_2(\varpi)q^{-s}}{1-\mu_1^{-1}(\varpi)q^{s-1}} = \frac{1-\mu_1(\varpi)q^{1-s}}{1-\mu_1^{-1}(\varpi)q^{s-1}} = -\mu_1(\varpi)q^{1-s}$$

5.9 Absolutely cuspidal representations: examples from quaternion algebras

[narch-acsp2]

5.9.1 The Weil representations for a quaternion algebra over F

[narch-acsp2-weil]

159. ADD REFERENCE TO SECTION 2.

160. In this section we quickly review the Weil representation associated to a representation of the multiplicative group of a quaternion algebra K over F. See §2.2.1 for notation regarding K.

161. Let (Ω, U) be a finite dimensional irreducible representation of $K^\times$. The Weil representation r_Ω is a representation of G_F on the space $\mathscr{S}(K, \Omega)$.

162. Let Φ belong to $\mathscr{S}(K, \Omega)$.

1. $\left(r_\Omega \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \Phi\right)(x) = |h|_K^{1/2}\Omega(h)\Phi(xh)$ where $a = \nu(h)$ and h is an arbitrary element of $K^\times$ (note that this is well-defined).
2. $\left(r_\Omega \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \Phi\right)(x) = |a|_F\Phi(ax)$ for all a in $F^\times$.
3. $\left(r_\Omega \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \Phi\right)(x) = \psi_F(z\nu(x))\Phi(x)$ for all z in F.
4. $\left(r_\Omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Phi\right)(x) = -\Phi'(x^\iota)$ where Φ' is the Fourier transform of Φ with respect to Ψ_K (where, recall, $\psi_K(x) = \psi_F(x + x^\iota)$.

163. The central quasi-character of r_Ω is equal to the central quasi-character of Ω.

5.9.2 The representation r_Ω is admissible

[`narch-acsp2-ad`]

164 Proposition ([JL] Thm. 4.2). [`narch-acsp2-ad-10`] *If Ω is a finite dimensional irreducible representation of $K^\times$ then the Weil representation r_Ω is admissible.*

165. The proof is broken into four lemmas. We let G_n denote the subgroup of $\mathrm{GL}(2, \mathscr{O}_F)$ consisting of matrices congruent to 1 modulo $\mathfrak{p}^n$.

166 Lemma. *The group G_n is generated by matrices of the form*

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \qquad w\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}w^{-1}, \qquad w\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}w^{-1}$$

with $a - 1$ and x in $\mathfrak{p}^n$.

If

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

belongs to G_n then

$$g = \begin{bmatrix} 1 & 0 \\ ca^{-1} & 1 \end{bmatrix}\begin{bmatrix} a & b' \\ 0 & d' \end{bmatrix}$$

and both of the matrices on the right belong to G_n. The lemma follows at once.

167 Lemma. *Given Φ in $\mathscr{S}(K, \Omega)$ there exists $n > 0$ such that*

$$r_\Omega \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \Phi = \Phi$$

if $a - 1$ belongs to $\mathfrak{p}^n$.

If $a = \nu(h)$ then

$$\left(r_\Omega \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \Phi\right)(x) = |h|_K^{1/2}\Omega(h)\Phi(xh).$$

Since Φ is locally constant of compact support, there exists a neighborhood U of 1 in $K^\times$ such that

$$|h|_K^{1/2}\Omega(h)\Phi(xh) = \Phi(x)$$

for all h in U and all x in K. The lemma now follows from the fact that ν is an open mapping from $K^\times$ to $F^\times$.

168 Lemma. *Given Φ in $\mathscr{S}(K,\Omega)$ there exists $n > 0$ such that*

$$r_\Omega \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \Phi = \Phi$$

if x belongs to $\mathfrak{p}^n$.

We have

$$\left(r_\Omega \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \Phi \right)(z) = \psi(x\nu(z))\Phi(z).$$

Let $\mathfrak{q}$ be the prime ideal of K and let $\mathfrak{p}^{-m}$ be the conductor of ψ. Since $\nu(\mathfrak{q}^k) = \mathfrak{p}^k$, the assertion of the lemma will be true if we take n so that the support of Φ is contained in $\mathfrak{q}^{-n-m}$.

169 Lemma. *Given $n > 0$ the space of Φ which satisfy*

$$r_\Omega \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \Phi = \Phi, \qquad r_\Omega \left(w^{-1} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} w \right) \Phi = \Phi$$

for all x in $\mathfrak{p}^n$ is finite dimensional.

Let $\mathfrak{q}$ be the ideal of K. Let $\mathfrak{p}^{-m}$ be the conductor of ψ and let k be a natural number such that $\psi_K(y) = 1$ for all y in $\mathfrak{q}^k$.

Let V be the space of Φ satisfying the conditions of the lemma. As in the previous lemma, Φ belongs to V if and only if the support of Φ and of $r_\Omega(w)\Phi$ are contained $\mathfrak{q}^{-n-m}$. Since $(r_\Omega(w)\Phi)(x) = -\Phi'(x^\iota)$ we see that Φ belongs to V if and only if the support of Φ and of Φ' are contained in $\mathfrak{q}^{-n-m}$. Thus if Φ belogns to V we have

$$\Phi(x) = \int_{\mathfrak{p}^{-n-m}} \Phi'(y)\psi_K(-xy)dy$$

and therefore Φ is constant on the cosets of $\mathfrak{q}^{k+m+n}$. Thus V is contained in the space of functions which have support in $\mathfrak{p}^{-n-m}$ and which are constant on the cosets of $\mathfrak{q}^{k+m+n}$. Since this space is finite dimensional so is V and the lemma follows.

170 Proof of proposition 164. We must show 1) that for all Φ in $\mathscr{S}(K,\Omega)$ there exists n such that $r_\Omega(g)\Phi = \Phi$; and 2) that for all n the space of Φ for which $r_\Omega(g)\Phi = \Phi$ for all g in G_n is finite dimensional. This is clearly implied by the four lemmas above.

5.9.3 The representation $\pi(\Omega)$

[narch-acsp2-pi]

171 Proposition ([JL] Thm 4.2). [narch-acsp2-pi-10] *Let Ω be a finite dimensional irreducible representation of $K^\times$ of degree d.*

1. *The representation r_Ω is a direct sum of d copies of an irreducible admissible representation which we denote by $\pi(\Omega)$.*
2. *If $d = 1$, so that Ω is the representation associated to a quasi-character χ of $F^\times$, then $\pi(\Omega)$ is equivalent to $\sigma(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$.*
3. *If $d > 1$ then $\pi(\Omega)$ is absolutely cuspidal.*
4. *The central quasi-characters of $\pi(\Omega)$ and Ω agree.*
5. *We have $\pi(\chi \otimes \Omega) = \chi \otimes \pi(\Omega)$.*

172. We split the proof into two cases: $d = 1$ and $d > 1$. The last two assertions of the proposition will easily seen to be true and we do not give explicit proofs for them.

173 Proof of proposition 171 for $d = 1$. The representation Ω is the representation associated to the quasi-character χ of $F^\times$. The space $\mathscr{S}(K,\Omega)$ is the space of Φ in $\mathscr{S}(K)$ such that $\Phi(xh) = \Phi(x)$ for h in K_1.

For Φ in $\mathscr{S}(K,\Omega)$ we may define a function ϕ_Φ on $F^\times$ by

$$\phi_\Phi(a) = |h|_K^{1/2}\Omega(h)\Phi(h);$$

note that this is well defined. The map $\Phi \mapsto \phi_\Phi$ is clearly injective and satisfies

$$\phi_{r_\Omega(d)\Phi} = \xi_\psi(d)\phi_\Phi$$

for d in D_F. Thus if V is the space of all the ϕ_Φ then V is a Kirillov model of r_Ω.

If ϕ belongs to $\mathscr{S}(F^\times)$ then the function Φ on K defined by

$$\Phi(h) = \begin{cases} |h|_K^{-1/2}\Omega^{-1}(h)\phi(\nu(h)) & h \neq 0 \\ 0 & h = 0 \end{cases}$$

belongs to $\mathscr{S}(K,\Omega)$. Clearly $\phi = \phi_\Phi$. Let $\mathscr{S}_0(K,\Omega)$ be the space of functions in $\mathscr{S}(K,\Omega)$ obtained in this way. It is precisely the space of functions which vanish at zero and is therefore of codimension one. Note that $\mathscr{S}_0(K,\Omega)$ is the space corresponding to Schwartz space in the Kirillov model; since it has codimension one, it follows that r_Ω is not absolutely cuspidal (in fact, it follows that the Jacquet module of r_Ω is one dimensional).

If Φ belongs to $\mathscr{S}_0(K,\Omega)$, is nonnegative and does not vanish identically then

$$\Phi'(0) = \int_K \Phi(x)dx \neq 0.$$

Thus $r_\Omega(w)\Phi$ does not belong to $\mathscr{S}_0(K,\Omega)$ and therefore $\mathscr{S}_0(K,\Omega)$ is not a stable subspace. Since it is of codimension one there is no proper stable subspace containing it.

Let V_1 be a nonzero stable subspace of V. If ϕ belongs to V_1 and is nonzero then

$$\phi - r_\Omega(n_x)\phi$$

vanishes at 0 and thus belongs to V_0; it is clear that there exists x such that it is nonzero. It follows that $V_0 \cap V_1$ is nontrivial. Since V_0 is irreducible under the action of ξ_ψ (proposition 46) it follows that V_1 contains all of V_0 and therefore is all of $\mathscr{S}(K,\Omega)$. Thus $\mathscr{S}(K,\Omega)$ has no proper nonzero stable subspace and the representation r_Ω is irreducible.

Define a linear functional L on $\mathscr{S}(K,\Omega)$ by

$$L(\Phi) = \Phi(0).$$

Note that

$$r_\Omega \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \chi^2(a)I, \qquad \left(r_\Omega \begin{bmatrix} a & x \\ 0 & 1 \end{bmatrix} \Phi\right)(0) = |a|_F\chi(a)\Phi(0)$$

and thus

$$L\left(r_\Omega \begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix} \Phi\right) = \chi(a_1a_2)\left|\frac{a_1}{a_2}\right| L(\Phi).$$

It therefore follows that the map A from $\mathscr{S}(K,\Omega)$ to $C^\infty(G_F)$ given by $(A\Phi)(g) = L(r_\Omega(g)\Phi)$ is an injection of (G_F-modules) into the space $\mathscr{B}(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$. It must therefore be an isomorphism onto the subspace $\mathscr{B}_s(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$ and thus r_Ω is equivalent to $\sigma(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$.

174 Proof of proposition 171 for $d > 1$. Given Φ in $\mathscr{S}(K,\Omega)$ define a function ϕ_Φ on $F^\times$ by

$$\phi_\Phi(a) = |h|_K^{1/2}\Omega(h)\Phi(h)$$

where $a = \nu(h)$. Note that Φ automatically vanishes at the origin and so $\Phi \mapsto \phi_\Phi$ gives a bijection of $\mathscr{S}(K,\Omega)$ with $\mathscr{S}(F^\times, U)$. It is clear that

$$\phi_{r_\Omega(d)\Phi} = \xi_\psi(d)\phi_\Phi$$

for all d in D_F.

Let U_0 be a subspace of U. We now show that the space $\mathscr{S}(F^\times, U_0)$ is a stable subspace of $\mathscr{S}(F^\times, U)$ under the action of G_F. It is clearly stable under Z_F and D_F so it suffices to show that it is stable under w.

For a function ϕ in $\mathscr{S}(F^\times, U)$ define a function $\hat{\phi}$ on the space of quasi-characters of $F^\times$ by

$$\hat{\phi}(\chi) = \int_{F^\times} \chi(a)\phi(a)d^\times a$$

Given Φ in $\mathscr{S}(K, \Omega)$ let $\phi = \phi_\Phi$ and let $\phi' = \phi_{r_\Omega(w)\Phi}$. An easy computation shows

$$\hat{\phi}(\chi) = Z(\tfrac{1}{2}, \Phi, \chi \otimes \Omega), \qquad \hat{\phi}'(\chi^{-1}\omega^{-1}) = -Z(\tfrac{1}{2}, \Phi', \chi^{-1} \otimes \Omega^{-1})$$

where ω is the central quasi-character of Ω and $Z(s, \Phi, \Omega)$ is the zeta function of §2.2.2. It thus follows by the functional equation for Z (*cf.* §2.2.3, proposition 25) that if $\hat{\phi}$ takes its values in U_0 then so does $\hat{\phi}'$. Thus if ϕ takes its values in U_0 then so does ϕ'. Therefore $\mathscr{S}(F^\times, U_0)$ is a stable subspace.

If we take U_0 to be one dimensional then we may identiy $\mathscr{S}(F^\times, U_0)$ with $\mathscr{S}(F^\times)$. Since $\mathscr{S}(F^\times)$ is already irreducible for the action of D_F (*cf.* proposition 46) the space $\mathscr{S}(F^\times, U_0)$ is irreducible. It thus follows that $\mathscr{S}(K, \Omega)$ splits into a direct sum of d irreducible representations.

We must now show that the d irreducible representations are all equivalent. Let A be an arbitrary endomorphism of U. Define an endomorphism A of $\mathscr{S}(K, \Omega)$ by $(A\Phi)(x) = \Omega^{-1}(x)A\Omega(x)\Phi(x)$. It is easily verified that $\phi_{A\Phi} = A\phi_\Phi$. Is is also easily verified that A commutes with the action of N_F and A_F. We now verify that it commutes with the action of w. Let Φ in $\mathscr{S}(K, \Omega)$ be given and put

$$\phi_1 = A\phi_{r_\Omega(w)\Phi}, \qquad \phi_2 = \phi_{r_\Omega(w)A\Phi}.$$

The we have

$$\begin{aligned}\hat{\phi}_2(\chi^{-1}\omega^{-1}) &= -Z(1, (A\Phi)', \chi^{-1} \otimes \Omega^{-1}) = -\epsilon(s, \Omega, \psi)Z(1, A\Phi, \chi \otimes \Omega) \\ &= -A\epsilon(s, \Omega, \psi)Z(1, \Phi, \chi \otimes \Omega) = -AZ(1, \Phi', \chi^{-1} \otimes \Omega^{-1}) \\ &= \hat{\phi}_1(\chi^{-1}\omega^{-1})\end{aligned}$$

and so $\phi_1 = \phi_2$ and A commutes with w.

We have thus shown that every endomorphism of U as a vector space gives rise to an endomorphism of $\mathscr{S}(K, \Omega)$ as a G_F-module. Therefore the dimension of the endomoprihsm ring of $\mathscr{S}(K, \Omega)$ is at least d^2 and so the d irreducible representations must all be equivalent.

5.9.4 The L-function and ϵ-factors of $\pi(\Omega)$

[`narch-acsp2-lfnpi`]

175 Proposition ([JL] Thm. 4.3). [`narch-acsp2-lfnpi-10`] *Let (Ω, U) be an irreducible finite dimensional representation of $K^\times$ and let $\pi = \pi(\Omega)$ be the associated representation of G_F. Then*

$$L(s, \pi) = L(s, \Omega) \qquad \textit{and} \qquad \epsilon(s, \xi \otimes \pi, \psi) = -\epsilon(s, \xi \otimes \Omega, \psi)$$

for any quasi-character ξ of $F^\times$ (for the definition of $L(s, \Omega)$ and $\epsilon(s, \Omega, \psi)$, see §2.2.2 and §2.2.3).

For Φ in $\mathscr{S}(K, \Omega)$ put

$$\phi_\Phi(a) = |h|_K^{1/2}\Omega(h)\Phi(h)$$

where $a = \nu(h)$. Let U_0 be a one dimensional subspace of U, which we will identify with $\mathbb{C}$, and let V be the space consisting of the ϕ_Φ which take values in U_0. We have already seen that V may be identified with the Kirillov model of π. We let ω be the central quasi-character of π.

Let ξ be a quasi-character of $F^\times$. For ϕ_Φ in V we have

$$Z(s, \phi_\Phi, \xi) = \int_{F^\times} |a|_F^{s-1/2}\xi(a)\phi_\Phi(a)d^\times a = \int_{K^\times} |h|^{s/2+1/4}\xi(h)\Omega(h)\Phi(h)d^\times h = Z(s, \Phi, \xi \otimes \Omega).$$

Note that the zeta function on the left is the GL(2) zeta function of §5.8.1 while the zeta function on the right is the GL(1) zeta function of §2.2.2. We also have

$$Z(s,\pi(w)\phi_\Phi,(\xi\omega)^{-1}) = Z(s,r_\Omega(w)\Phi,(\xi\omega)^{-1}\otimes\Omega) = -\int_{K^\times} |h|_K^{s/2+1/4}(\xi\omega)^{-1}(h)\Omega(h)\Phi'(h^\iota)d^\times h$$
$$= -\int_{K^\times} |h|_K^{s/2+1/4}\xi^{-1}(h)\Omega^{-1}(h)\Phi'(h)d^\times h = -Z(s,\Phi',\xi^{-1}\otimes\Omega^{-1})$$

(note that $\Omega(hh^\iota)=\omega(h)$).

Now, the functional equations read

$$Z(1-s,\pi(w)\phi_\Phi,(\xi\omega)^{-1}) = \gamma(s,\xi\otimes\pi,\psi)Z(s,\phi_\Phi,\xi)$$
$$Z(1-s,\Phi',\xi^{-1}\otimes\Omega^{-1}) = \gamma(s,\xi\otimes\Omega,\psi)Z(s,\Phi,\xi\otimes\Omega).$$

From the results of the previous paragraph it is clear that the gamma factors only differ by a sign. The rest follows easily.

176 Corollary ([JL] Cor. 4.4). *If $\pi=\pi(\Omega)$ then $\widetilde{\pi}=\pi(\widetilde{\Omega})$.*

This is clear if Ω is of degree one by the explicit description of $\pi(\Omega)$; thus assume $\deg\Omega>1$. The functional equations of theorem 151 and §2.2.3, theorem 25 give

$$\epsilon(s,\xi\otimes\Omega,\psi)\epsilon(1-s,\xi^{-1}\otimes\widetilde{\Omega},\psi) = \omega(-1)$$
$$\epsilon(s,\xi\otimes\pi,\psi)\epsilon(1-s,\xi^{-1}\otimes\widetilde{\pi},\psi) = \omega(-1)$$

for any quasi-character ξ (here ω is the central quasi-character). Two applications of proposition 175 now show that

$$\epsilon(s,\xi^{-1}\otimes\widetilde{\pi},\psi) = \epsilon(s,\xi^{-1}\otimes\widetilde{\Omega},\psi) = \epsilon(s,\xi^{-1}\otimes\pi(\widetilde{\Omega}),\psi).$$

Thus the ϵ-factors of $\widetilde{\pi}$ and $\pi(\widetilde{\Omega})$ agree; since they are both absolutely cuspidal their γ-factors agree as well. They are therefore equivalent (*cf.* article 153).

5.10 Absolutely cuspidal representations: examples from quadratic extensions

[`narch-acsp3`]

5.10.1 Definitions and notations

[`narch-acsp3-def`]

177. Throughout this section the following definitions and notations will be in effect.

1. We let K be a separable quadratic extension of F.
2. We let ι be the nontrivial automorphism of K which fixes F.
3. We let ν be the "algebraic" norm given by $\nu(x)=xx^\iota$.
4. We let $|\cdot|_K$ be the canonical "analytic" norm of the topological ring K, given by $|a|_K=d(ax)/dx$ where dx is any additive Haar measure on K. Note that for a in F we have $|a|_K=|a|_F^2$.
5. We let F^+ be the image of $K^\times$ under the norm map ν; it is an index two subgroup of $F^\times$.
6. We let G_F^+ be the subgroup of G_F whose determinant lies in F^+; if H is a subgroup of G_F we let H^+ be $H\cap G_F^+$.
7. We let K_1 be the inverse image of 1 under ν.
8. We let η denote the quadratic character of $F^\times$ associated to the extension K/F by local class field theory. It is trivial on F_+.

5.10.2 The Weil representations for a quadratic extension of F

[narch-acsp3-weil]

178. ADD REFERENCE TO SECTION 2.

179. In this section we quickly review the Weil representaiton associated to a quasi-character of $K^{\times}$.

180. Let ω be a quasi-character of $F^{\times}$. The Weil representation r_{ω} associated to ω is a repreantation of G_F^{+} on the space $\mathscr{S}(K,\omega)$ consisting of all functions Φ in $\mathscr{S}(K)$ which satisfy $\Phi(xh) = \omega^{-1}(h)\Phi(x)$ for all h in K_1.

181. Let Φ be an element of $\mathscr{S}(K,\omega)$.

1. $\left(r_{\omega}\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\Phi\right)(x) = |h|_K^{1/2}\omega(h)\Phi(xh)$ where $a = \nu(h)$ and h is an arbitrary element of $K^{\times}$.
2. $\left(r_{\omega}\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\Phi\right)(x) = \eta(a)|a|_K^{1/2}\Phi(ax)$ for all a in $F^{\times}$.
3. $\left(r_{\omega}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\Phi\right)(x) = \psi_F(z\nu(x))\Phi(x)$ for all z in F.
4. $\left(r_{\omega}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\Phi\right)(x) = \lambda(K/F,\psi_F)\Phi'(x^{\iota})$ where Φ' is the Fourier transform of Φ and $\lambda(K/F,\psi_F)$ is the constant defined in SECTION 2 REFERENCE.

182. The central quasi-character of r_{ω} is $\eta\omega$ (where, technically, by ω we mean the restriction of ω to $F^{\times}$).

5.10.3 The representation $\pi(\omega)$

[narch-acsp3-pi]

183. Let $\pi(\omega)$ be the representation of G_F induced from the representation r_{ω} of G_{+}.

184 Proposition ([JL] Thm. 4.6). [narch-acsp3-pi-20] *We have the following.*

1. *The representation r_{ω} is admissible and irreducible.*
2. *The representation $\pi(\omega)$ is admissible and irreducible.*
3. *The representation $\pi(\omega)$ does not depend on the character ψ_F.*
4. *If ω factors as $\chi\nu$ where χ is a quasi-character of $F^{\times}$ and ν is the norm on K then $\pi(\omega)$ is equivalent to $\pi(\chi,\eta\chi)$.*
5. *If ω does not factor through the norm map then $\pi(\omega)$ is absolutely cuspidal.*

185. The proof that r_{ω} is admissible goes much like the proof that r_{Ω} is admissible (proposition 164) and is omitted. We will prove proposition 184 after some discussion.

186. It is worth pointing out that if the characteristic of the residue class field of F is not 2 then every absolutely cuspidal representation is of the form $\pi(\omega)$ for some quasi-character ω of some separable quadratic extension F, as proved by Tunnell in his disseration ([Bu] pg. 549).

187. Note that the group D_F^{+} acts on the space $\mathscr{S}(F^{+})$ via the representation ξ_{ψ} (*cf.* §5.3.1 for the definition of ξ_{ψ}).

188 Lemma. [narch-acsp3-pi-60] *The representation of D_F obtained by induction from the representation ξ_ψ of D_F^+ is equivalent to the representation ξ_ψ of D_F on $\mathscr{S}(F^\times)$. In particular, the representation ξ_ψ of D_F^+ is irreducible.*

Let V be the space of the induced representation. It consists of all functions $\widetilde{\phi}$ on D_F with values in $\mathscr{S}(F^+)$ which satisfy

$$\widetilde{\phi}(d^+d) = \xi_\psi(d^+)\widetilde{d'}$$

for all d^+ in D_F^+. The action of D_F is by right translation ρ.

Let L be the functional on $\mathscr{S}(F^+)$ which associates to a funciton its value at 1. To a function $\widetilde{\phi}$ in V associate the function ϕ on $F^\times$ defined by

$$\phi(b) = L\left(\widetilde{\phi}\left[\begin{array}{cc} b & 0 \\ 0 & 1 \end{array}\right]\right) = L\left(\rho\left[\begin{array}{cc} b & 0 \\ 0 & 1 \end{array}\right]\widetilde{\phi}\right).$$

For $d \in D_F$, a short computation shows that the function associated to $\rho(d)\widetilde{\phi}$ is $\xi_\psi(d)\phi$, so that $\widetilde{\phi} \mapsto \phi$ is a map of D_F-modules. Another easy computation shows that for b in F^+ we have

$$\widetilde{\phi}\left(\left[\begin{array}{cc} a & x \\ 0 & 1 \end{array}\right]\right)(b) = \psi(bx)\phi(ba).$$

The preceeding observations, together with the fact that $F^\times/F^+$ is finite, show that 1) ϕ belongs to $\mathcal{S}(F^\times)$; 2) ϕ can be any function in $\mathcal{S}(F^\times)$; and 3) ϕ is 0 if and only if $\widetilde{\phi}$ is 0. This proves the first statement of the proposition.

The second statement follows from 1) the representation ξ_ψ of D_F on $\mathcal{S}(F^\times)$ is irreducible (proposition 46); and 2) an induced representation can be irreducible only if the initial representation is irreducible.

189 Proof of proposition 184. To every function Φ in $\mathscr{S}(K,\omega)$ we associate the function ϕ_Φ on F^+, defined by

$$\phi_\Phi(a) = \omega(h)|h|_K^{1/2}\Phi(h)$$

where $a = \nu(h)$. It is clear that $\Phi \mapsto \phi_\Phi$ is injective. Let V^+ be the space of all the functions ϕ_Φ. If ϕ belongs to $\mathscr{S}(F^+)$ then the function

$$\Phi(h) = \omega^{-1}(h)|h|_K^{-1/2}\phi(\nu(h))$$

satisfies $\phi = \phi_\Phi$; thus V^+ contains the space $\mathscr{S}(F^+)$.

1) If the restriction of ω to K_1 is nontrivial then every element of $\mathscr{S}(K,\omega)$ vanishes at 0; thus $V^+ = \mathscr{S}(F^+)$ and the first statement of the proposition follows immediately from lemma 188.

If the restriction of ω to K_1 is trivial then $\mathscr{S}(F^+)$ has codimension 1 in V^+. By an argument we have used a number of times it follows that any stable subspace of V^+ contains $\mathscr{S}(F^+)$; thus to prove that V^+ is irreducible it suffices to show that $\mathscr{S}(F^+)$ is not stable. As before, if Φ in $\mathscr{S}(K,\omega) = \mathscr{S}(K)$ taken to vanish at zero, be not identically zero and be non-negative then

$$(r_\omega\Phi)(0) = \lambda(K/F,\psi_F)\int_K \Phi(x)dx \neq 0$$

and so ϕ_Φ belongs to $\mathscr{S}(F^+)$ but $\phi_{r_\omega\Phi}$ does not.

2) The representation $\pi(\omega)$ is the representation obtained by letting G_F act on the right on the space of functions $\widetilde{\phi}$ on G_F with values in V^+ which satisfy

$$\widetilde{\phi}(g^+g) = r_\omega(g^+)\widetilde{\phi}(g)$$

for g^+ in G_F^+. Since $G_F = G_F^+D_F$ such funcitons are determined by their restriction to D_F; the restriciton is a function of the sort considered in lemma 188. By the construction used in that lemma, we can associate to $\widetilde{\phi}$ a function ϕ on $F^\times$. Let V be the space of functions thus obtained; it is clearly a Kirillov model for $\pi(\omega)$. Every function on F^+ can be regarded as a function on $F^\times$ by extension by zero. Since

$$\widetilde{\phi}\left(\left[\begin{array}{cc} a & 0 \\ 0 & 1 \end{array}\right]\right)(b) = \phi(ab)$$

the space V is the space generated by the translates of the functions in V^+. It follows immediately that $\pi(\omega)$ is irreducible and admissible. Note also that the codimension of $\mathscr{S}(F^\times)$ in V is twice that of $\mathscr{S}(F^+)$ in V^+.

3) If we replace the functions $\widetilde{\phi}$ by the functions

$$\widetilde{\phi}'(g) = \widetilde{\phi}\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g\right)$$

we obtain an equivalent representations, that induced from the representation

$$g \mapsto r_\omega\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g \begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix}\right)$$

of G_F^+. This representation is equivalent to the representation r_ω associated to the character $x \mapsto \psi_F(ax)$ REFERENCE. Thus $\pi(\omega)$ is independent of ψ_F.

4) We know that $\mathscr{S}(F^\times)$ has codimension 2 in the space of the Kirillov model V and therefore $\pi(\omega)$ is a principal series representation. We now determine which.

Since ω is trivial on K_1 there is a quasi-character χ of $F^\times$ such that $\omega = \chi\nu$. Any function ϕ in V can be written as

$$\phi = \phi_1 + \pi \begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix} \phi_2$$

where ϕ_1 and ϕ_2 belong to V^+ and ϵ is a fixed element of $F^\times$ which does not belong to F^+. We define a linear functional L on V by

$$L(\phi) = \phi_1(0) + \chi(\epsilon)\phi_2(0).$$

If we can now verify the identity

$$L\left(\pi \begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix} \phi\right) = \chi(a_1a_2)\eta(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} L(\phi) \tag{21}$$

then the fourth statement will follow. Since the central quasi-character of π is $\chi^2\eta$ (21) holds if $a_1 = a_2$. Thus it suffices to prove (21) when $a_2 = 1$. If $\phi = \phi_\Phi$ belongs to V^+ and a belongs to F^+ then

$$L\left(\pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \phi\right) = \left(r_\omega \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \Phi\right)(0) = \chi(a)|a|^{1/2}L(\phi).$$

The identity (21) follows easily.

5) This follows immediately (we know the space of the Kirillov model V is $\mathscr{S}(F^\times)$).

190 Proposition ([JL] Thm. 4.7). [narch-acsp3-80] *Let ω be a quasi-character of $K^\times$.*

1. *The central quasi-character of $\pi(\omega)$ is $\eta\omega$.*
2. *We have $\pi(\omega) = \pi(\omega^\iota)$, where $\omega^\iota(a) = \omega(a^\iota)$.*
3. *If χ is a quasi-character of $F^\times$ then $\chi \otimes \pi(\omega) = \pi(\chi'\omega)$, where $\chi' = \chi\nu$.*
4. *We have $\widetilde{\pi}(\omega) = \pi(\omega^{-1})$.*

1) This is clear.

2) It is clear that $\chi \otimes \pi(\omega)$ is the representation of G_F induced from the representation $\chi \otimes r_\omega$ of G_F^+. However, it is also clear that $\chi \otimes r_\omega$ is equivalent to $r_{\chi'\omega}$.

3) Define a map $\mathscr{S}(K, \omega) \to \mathscr{S}(K, \omega^\iota)$ by associating to Φ the function $x \mapsto \Phi(x^\iota)$. This is clearly an isomorphism of G_F^+-modules and so r_ω is equivalent to r_{ω^ι}. This equivalence is obviously preserved under induction.

4) Since $\eta' = \eta\nu$ is trivial and $\omega\nu = \omega\omega^\iota$ we have

$$\widetilde{\pi}(\omega) = (\eta\omega)^{-1} \otimes \pi(\omega) = \pi(\omega^{-\iota}) = \pi(\omega^{-1}).$$

5.10.4 The L-function and ϵ-factors of $\pi(\omega)$

[narch-acsp3-lfnpi]

191 Proposition ([JL] Thm. 4.7). *Let $\pi = \pi(\omega)$. Then $L(s,\pi) = L(s,\omega)$ and*

$$\epsilon(s,\pi,\psi_F) = \lambda(K/F,\psi_F)\epsilon(s,\omega,\psi_F)$$

where the ϵ-factor on the right is the GL(1) *ϵ-factor of §2.1.3.*

Let $\phi = \phi_\Phi$ belong to V^+. Since ϕ is zero off of $\nu(F^\times)$ we have

$$Z(s,\phi,\xi) = \int_{F^\times} |a|^{s-1/2}\phi(a)d^\times a = \int_{K^\times} |h|^s\omega(h)\xi(\nu(h))\Phi(h)d^\times h = Z(s,\Phi,\omega\xi')$$

for any quasi-character ξ of $F^\times$, where $\xi' = \xi\nu$, the zeta function on the left is a GL(2) zeta function, and the zeta function on the right is a GL(1) zeta function. It thus follows that $Z(s,\phi)/L(s,\omega)$ is an entire function of s.

Let $\omega_0 = \eta\omega$ be the central quasi-character of π. Note that $\omega_0' = \omega\omega^\iota$. If $\phi' = \pi(w)\phi$ and Φ' is the Fourier transform of Φ, so that

$$\phi'(a) = \lambda(K/F,\psi_F)\omega(h)|h|_K^{1/2}\Phi(h^\iota)$$

then

$$Z(s,\phi',\omega_0^{-1}) = \lambda(K/F,\psi_F)Z(s,(\Phi')^\iota,\omega^{-\iota}) = \lambda(K/F,\psi_F)Z(s,\Phi',\omega^{-1}).$$

Thus, using the functional equation on GL(1), we obtain

$$\frac{Z(1-s,\phi',\omega_0^{-1})}{L(1-s,\omega^{-1})} = \lambda(K/F,\psi_F)\epsilon(s,\omega,\psi_K)\frac{Z(s,\phi)}{L(s,\omega)}.$$

We now consider functions ϕ in V of the form

$$\phi = \pi\begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix}\phi_0$$

where ϕ_0 belongs to V^+ and ϵ is an element of $F^\times$ which does not belong to F^+. We have

$$Z(s,\phi,\xi) = \int_{F^\times} |a|^{s-1/2}\xi(a)\phi_0(\epsilon a)d^\times a = |\epsilon|^{1/2-s}\xi^{-1}(\epsilon)Z(s,\phi_0,\xi).$$

Thus $Z(s,\phi)/L(s,\omega)$ is entire; the functional equation follows easily.

Since V is spanned by V^+ and its translation under ϵ (*i.e.*, the functions considered in the previous paragraph) the proposition follows.

5.11 Spherical representations

[`narch-sph`]

5.11.1 Definition

[`narch-sph-def`]

192. A representation (π, V) of G_F is said to be *spherical* if $V^\circ = V^{K_F}$ is nonzero, *i.e.*, if $\mathrm{GL}(2,\mathscr{O}_F)$ stabilizes a nontrivial subspace. An element of V° is called a *spherical vector*.

193. Recall (*cf.* §5.2.2) that the spherical Hecke algebra $\mathscr{H}_F^\circ$ is the Hecke algebra associated to the compact subgroup K_F of G_F. If (π, V) is a representation of G_F then V° is a module over $\mathscr{H}_F^\circ$. If V is irreducible as a representation of $\mathscr{H}_F$ then V° is irreducible as a representation of $\mathscr{H}_F^\circ$ (*cf.* proposition 18).

5.11.2 Classification of spherical representations

[`narch-sph-class`]

194 Proposition ([JL] Lemma 3.9). [`narch-sph-class-10`] *Let (π, V) be an infinite dimensional irreducible admissible representation. If π is the representation $\pi(\mu_1, \mu_2)$ with μ_1 and μ_2 unramified then V° is one dimensional; otherwise V° is zero.*

195 Proof of proposition 194 for absolutely cuspidal representations. We prove that an absolutely cuspidal representation contains no spherical vectors. Thus let (π, V) be an absolutely cuspidal representation; take π in the Kirillov form with respect to a nontrivial additive character ψ of F with conductor $\mathscr{O}_F$. Assume ϕ is a K_F-fixed vector.

Sublemma A. We have

$$\pi \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \phi = \omega(a)\phi = \phi$$

for a in U_F. We thus deduce that ω is unramified.

Sublemma B. We have

$$\pi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \phi = \phi$$

for a in U_F. From this we deduce that $\phi(ax) = \phi(x)$ for a in U_F.

Sublemma C. We have

$$\pi \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \phi = \phi$$

for z in $\mathscr{O}_F$. This implies that $(\psi(xz) - 1)\phi(x) = 0$ for all x and for all z in $\mathscr{O}_F$. Since the conductor of ψ is $\mathscr{O}_F$ we deduce that $\phi(x)$ vanishes if x is in $\varpi^n U_F$ with $n < 0$.

Sublemma D. We now have

$$\hat{\phi}_n(\nu) = \int_{U_F} \phi(\varpi^n \epsilon)\nu(\epsilon)d\epsilon = \phi(\varpi^n) \int_{U_F} \nu(\epsilon)d\epsilon.$$

Thus $\hat{\phi}(\nu, t) = 0$ if ν is nontrivial and $\hat{\phi}(\nu, t)$ has only nonnegative powers of t if ν is trivial.

Sublemma E. We have $\pi(w)\phi = \phi$. Applying proposition 61 (and remembering that the restriction of ω to U_F is trivial) we find

$$\hat{\phi}(1, t) = C(1, t)\hat{\phi}(1, z_0^{-1}t^{-1}).$$

However, the left hand side contains only nonnegative powers of t while the right hand side (by proposition 102) contains only negative powers of t. Thus we must have $\phi = 0$ and the proof is complete.

196 Lemma. [`narch-sph-class-30`] *The space of spherical vectors for $\rho(\mu_1, \mu_2)$ is one dimensional if μ_1 and μ_2 are unramified and zero dimensional otherwise.*

Let f be in $\mathscr{B}(\mu_1, \mu_2)$ be a K_F-fixed vector. Let g be an element of G_F; by the Iwasaawa decomposition $G_F = P_F K_F$ we may write $g = pk$ with k in K_F and

$$p = \begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix}$$

with a_1 and a_2 in $F^\times$ and x in F. Since $f(k) = f(1)$ we have

$$f(pk) = \mu_1(a_1)\mu_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} f(1).$$

This shows that the space of spherical vectors is at most one dimensional.

We now try to construct a spherical vector f by defining

$$f(pk) = \mu_1(a_1)\mu_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2}.$$

For this function to be well defined, we must have $f(p) = 1$ if $p \in P_F \cap K_F$, *i.e.*, if a_1 and a_2 belong to U_F and x belongs to $\mathscr{O}_F$. This condition amounts to

$$\mu_1(a_1)\mu_2(a_2) = 1$$

for all a_1 and a_2 in U_F. This is satisfied if and only if μ_1 and μ_2 are unramfied. Thus the proposition is proved.

197 Proof of proposition 194 for the principal series representations. There is nothing more that needs to be said other than lemma 196.

198 Proof of proposition 194 for the special representations. Consider now a special representation $\sigma(\mu_1, \mu_2)$ with $\mu_1\mu_2^{-1} = \alpha_F$. If μ_1 or μ_2 is ramified then lemma 196 proves the theorem. Thus assume μ_1 and μ_2 are unramified. The space $\mathscr{B}_s(\mu_1, \mu_2)$ of the representation $\sigma(\mu_1, \mu_2)$ is the space orthogonal to $g \mapsto \chi^{-1}(\det g)$, where χ is the trivial character (*cf.* theorem 123). Thus if f is a K_F-fixed vector in $\mathscr{B}(\mu_1, \mu_2)$ then

$$\langle f, \chi^{-1}\rangle = \int_{K_F} f(g)\chi^{-1}(\det g)dg = \int_{K_F} f(1)dg = f(1)$$

and so $\mathscr{B}_s(\mu_1, \mu_2)$ contains no nontrivial K_F-fixed vector. This proves the theorem for the special representations.

5.11.3 The Whittaker model of a spherical representation

[narch-sph-whit]

199 Proposition. [narch-sph-whit-10] *Let π be an infinite dimensional irrreducible admissible spherical representation. If ψ is unramified (i.e., its conductor is $\mathscr{O}_F$) then there exists a unique element W of the Whittaker model $\mathscr{W}(\pi, \psi)$ of π such that W is invariant under K_F and $W(1) = 1$.*

We know that $\pi = \pi(\mu_1, \mu_2)$ with μ_1 and μ_2 unramified. Recall that if Φ is an element of $\mathscr{S}(F^2)$ then

$$W_\Phi(g) = \theta(\mu_1, \mu_2; r_{\mu_1,\mu_2}(g)\Phi)$$

is an element of $\mathscr{W}(\pi, \psi)$ (*cf.* §5.6.3 and §5.6.4).

Let Φ be the characteristic function of $\mathscr{O}_F^2$. Note 1) since ψ is unramified we have $\Phi^\sim = \Phi$; and 2) we have $\rho(g)\Phi = \Phi$ whenever g belongs to K_F. Since $(r(g)\Phi)^\sim = \rho(g)\Phi^\sim$ it thus follows that $r(g)\Phi = \Phi$ whenever g belongs to K_F. Because μ_1 and μ_2 are unramified, we have $r_{\mu_1,\mu_2}(g) = r(g)$ when g belongs to K_F. It thus follows that W_Φ is a spherical vector. We have

$$W_\Phi(1) = \int_{F^\times} \mu_1(t)\mu_2^{-1}(t)\Phi(t, t^{-1})d^\times t = \int_{U_F} dt \neq 0.$$

It therefore follows that we can rescale W_Φ to get a spherical vector W with $W(1) = 1$. Since the space of spherical vectors is one dimensional the uniqueness of W is clear.

5.11.4 The spherical Hecke algebra is commutative

[narch-sph-hecke]

200 Proposition ([Bu] Thm. 4.6.1). [narch-sph-hecke-10] *The spherical Hecke algebra $\mathscr{H}_F^\circ$ is commutative.*

201 Corollary. [narch-sph-hecke-20] *Let (π, V) be an irreducible admissible spherical representation. Then the space V° of spherical vectors is one dimensional.*

The space V° is irreducible under the action of the commutative algebra $\mathscr{H}_F^\circ$ (*cf.* proposition 18) and therefore is one dimensional.

202. Note that the content of article 201 is already contained in proposition 196. We include this second proof because 1) that which is proved is a crucial fact; and 2) the proof given in article 201 is much more accessible (*e.g.*, it does not use the classification theorem) than the proof of proposition 194.

203. We need a lemma before proving proposition 200.

204 Lemma ([Bu] Prop. 4.6.2). [`narch-sph-hecke-50`] *A complete set of double coset representatives for $K_F \backslash G_F / K_F$ consists of the matrices*

$$\begin{bmatrix} \varpi^{n_1} & 0 \\ 0 & \varpi^{n_2} \end{bmatrix}$$

with $n_1 \geq n_2$.

PROVE THIS.

205 Proof of proposition 200. Let ι be the anti-involution of $\mathscr{H}_F^\circ$ given by $\phi^\iota(g) = \phi(g^T)$ where g^T is the transpose of g. The algebra $\mathscr{H}_F^\circ$ has a linear basis consisting of the characteristic functions of double cosets of K_F; by lemma 204 these are stabilized by ι. Thus the anti-involution ι is the identity map and therefore $\mathscr{H}_F^\circ$ is commutative.

5.11.5 The character of $\mathscr{H}_F^\circ$ associated to a spherical representation

[`narch-sph-char`]

206. Let (π, V) be an irreducible admissible spherical representation. Let v be a nonzero spherical vector. Since the space of spherical vectors is one dimensional it follows that for any f in $\mathscr{H}_F^\circ$ there exists a unique complex number $\xi(f)$ such that

$$\pi(f)v = \xi(f)v.$$

The map $\xi : \mathscr{H}_F^\circ \to \mathbb{C}$ is readily verified to be an algebra homomorphism. We call ξ the *character of $\mathscr{H}_F^\circ$ associated to π.*

207 Proposition. [`narch-sph-char-20`] *Let π_1 and π_2 be irreducible admissible spherical representations of $\mathscr{H}_\mathbb{R}$. Then π_1 is equivalent to π_2 if and only if the associated characters of $\mathscr{H}_F^\circ$ are equal.*

This follows immediately from proposition 116.

Chapter 6

Representations of $\mathrm{GL}(2,\mathbb{R})$

6.1 First notions and results

6.1.1 Notations

1. We use the following notations in this section:

1. We let $G_{\mathbb{R}}$ denote the group $\mathrm{GL}(2,\mathbb{R})$;
2. We let $K_{\mathbb{R}}$ denote the group $\mathrm{O}(2,\mathbb{R})$;
3. We let $\mathfrak{g}$ denote the Lie algebra of $G_{\mathbb{R}}$ and we let $\mathfrak{g}_{\mathbb{C}}$ denote its complexification;
4. We let $\mathfrak{k}$ denote the Lie algebra of $K_{\mathbb{R}}$ and we let $\mathfrak{k}_{\mathbb{C}}$ denote its complexification;
5. We let $\mathscr{U}$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$;
6. We let $\mathscr{Z}$ denote the center of the algebra $\mathscr{U}$;
7. We let $\mathscr{U}_K$ denote the universal enveloping algebra of $\mathfrak{k}_{\mathbb{C}}$;
8. We let ψ be a fixed nontrivial additive character of $\mathbb{R}$;
9. We let $\psi_{\mathbb{C}}$ be the complexification of ψ, given by $\psi_{\mathbb{C}}(x) = \psi(x+\bar{x})$.

6.1.2 The Hecke algebra

2. The discussion in this section is based on chapter I of Knapp and Vogan.

3. We define the *Hecke algebra* $\mathscr{H}_{\mathbb{R}}$ to be the algebra of all left $K_{\mathbb{R}}$-finite distributions on $G_{\mathbb{R}}$ with support contained in $K_{\mathbb{R}}$; multiplication in the algebra is given by convolution of distributions. We also define $R(K)$ to be the algebra of left K-finite distributions on $K_{\mathbb{R}}$. (Note that Knapp-Vogan denotes the Hecke algebra by $R(\mathfrak{g},K)$.)

4. Note that this is the Hecke algebra of Flath. In Jacquet-Langlands a different algebra is called the Hecke algebra; we will discuss this alternative Hecke algebra in §6.1.8.

5 Proposition (K-V Prop. 1.80, 1.83). *We have the following.*

1. *The map*
$$\mathscr{U} \otimes_{\mathscr{U}_K} R(K) \to \mathscr{H}_{\mathbb{R}}$$
given by $X \otimes T \mapsto X * T$ is an isomorphism (technically, the map should be $X \otimes T \mapsto X * (i_* T)$ where $i : K_{\mathbb{R}} \to G_{\mathbb{R}}$ is the inclusion; we will usually neglect this detail).
2. *All elements of $\mathscr{H}_{K_{\mathbb{R}}}$ are smooth functions which are K-finite on both sides.*
3. *All elements of $\mathscr{H}_{\mathbb{R}}$ are K-finite on both sides.*

These assertions follows easily from §1.2.6, propositions 45 and 46; see Knapp-Vogan for more details.

6. If ξ is an elementary idempotent of $K_{\mathbb{R}}$ then ξ may be regarded in the obvious way as an element of $\mathscr{H}_{\mathbb{R}}$; we again call such elements *elementary idempotents*. The algebra $\mathscr{H}_{\mathbb{R}}$ together with its elementary idempotents forms an idempotented algebra in the sense of §1.4. Thus we have the notions of smooth and admissible modules (or representations) of $\mathscr{H}_{\mathbb{R}}$ and contragrediants of such representations.

7. In this section we will be primarily studying admissible representations of the Hecke algebra $\mathscr{H}_{\mathbb{R}}$; the phrase "admissible representation" will by default mean an admissible representation of $\mathscr{H}_{\mathbb{R}}$.

6.1.3 Admissible representations of $\mathscr{U}$

8. Let κ_n denote the one dimensional representation of $\mathrm{SO}(2,\mathbb{R})$ given by

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \mapsto e^{in\theta}.$$

We will also use the notion κ_n to denote the corresponding representation of $\mathfrak{k}$.

9. We say a representation π of $\mathscr{U}$ is *admissible* if the restriction of π to $\mathfrak{k}$ decomposes into an algebraic direct sum of the κ_n with each κ_n appearing with finite multiplicity.

10 Proposition. *Let (π, V) be an irreducible admissible representation of $\mathscr{U}$. Then any linear operator on V commuting with $\mathscr{U}$ is a scalar.*

By definition the representation V comes from a representation of $\mathrm{SO}(2,\mathbb{R})$, which we will also denote π. The operator

$$\pi(\xi_n)v = \int_{\mathrm{SO}(2,\mathbb{R})} e^{in\theta}\pi(g)v dg$$

is the projection operator of V onto its κ_n-isotypic component. If A is any operator commuting with $\mathscr{U}$ then clearly A commutes with $\pi(\xi_n)$ and thus A maps the κ_n-isotypic component into itself. Since this is finite dimensional, by admissibility, it follows that A has an eigenvector and is thus a scalar.

6.1.4 Harish-Chandra modules

11. A *Harish-Chandra module* (for $\mathrm{GL}(2,\mathbb{R})$), or a $(\mathfrak{g}, K_{\mathbb{R}})$-module, or a representation of $(\mathfrak{g}, K_{\mathbb{R}})$, is a vector space V together with actions of $\mathfrak{g}$ and $K_{\mathbb{R}}$ (which we denote by π) subject to three conditions:

1. The space V decomposes into an algebraic direct sum of finite dimensional subspaces stable under $K_{\mathbb{R}}$.
2. For any X in $\mathfrak{g}$ and g in $K_{\mathbb{R}}$ we have $\pi((\mathrm{Ad}\, g)X) = \pi(g)\pi(X)\pi(g^{-1})$.
3. For any X in the Lie algebra of K we have

$$\lim_{t\to 0} \frac{\pi(\exp(tX)) - \pi(1)}{t} = \pi(X).$$

The last condition should be explained more precisely. Given v in V there is a finite dimensional subspace U containing v and stable under $K_{\mathbb{R}}$. The third condition states that

$$\lim_{t\to 0} \frac{\pi(\exp(tX))v - v}{t} = \pi(X)v$$

where the limit is now taking place in the finite dimensional vector space U and thus makes sense.

12. A Harish-Chandra module V is *admissible* if its isotypic parts $V(\sigma)$ are finite dimensional (for all σ in $\hat{K}$). Note that this is equivalent to the condition that the range of $\pi(\xi)$ is finite dimensional for all elementary idempotents ξ of K.

13. Note that if V is a Harish-Chandra module then the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ naturally acts on V as does the universal enveloping algebra $\mathscr{U}$ of $\mathfrak{g}_{\mathbb{C}}$. It is clear that if V is an admissible Harish-Chandra module then the associated representation of $\mathscr{U}$ is also admissible.

14 Proposition (J-L pg. 161). *Let V be an irreducible admissible Harish-Chandra module. Then $\mathcal{Z}$ acts as scalars.*

Let X be in the center of $\mathscr{U}$. It suffices to prove that 1) $\pi(X)$ commutes with $\pi(Y)$ for Y in $\mathfrak{g}$; 2) $\pi(X)$ commutes with $\pi(g)$ for g in $K_{\mathbb{R}}$; and 3) $\pi(X)$ has an eigenvector.

1) This is clear.

2) Since X is invariant under the adjoint action of $\mathfrak{g}_{\mathbb{C}}$ on itself, it follows that it is invariant under the adjoint action of $\mathrm{GL}(2,\mathbb{C})$ on $\mathfrak{g}_{\mathbb{C}}$ (since $\mathrm{GL}(2,\mathbb{C})$ is connected). Thus X is invariant under the adjoint action of $K_{\mathbb{R}}$ and so for g in $K_{\mathbb{R}}$ we have

$$\pi(g)\pi(X)\pi(g^{-1}) = \pi((\mathrm{Ad}\, g)X) = \pi(X).$$

Therefore $\pi(X)$ commutes with $\pi(g)$ for g in $K_{\mathbb{R}}$.

3) It follows that $\pi(X)$ commutes with $\pi(\xi)$ for any elementary idempotent ξ. Since the range of $\pi(\xi)$ is finite dimensional it follows that $\pi(X)$ has an eigenvector and therefore acts as a scalar.

15. Note that Jacquet-Langlands uses the notion of a representation of the system $\{\mathscr{U}, \epsilon\}$ where $\mathscr{U}$ is the universal enveloping algebra of $\mathfrak{g}$ and

$$\epsilon = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This notion is not quite the same as that of a Harish-Chandra module because there are representations of the Lie algebra of $K_{\mathbb{R}}$ which do not come from representations of $K_{\mathbb{R}}$. However, they define a representation of $\{\mathscr{U}, \epsilon\}$ to be admissible if the corresponding representation of $\mathscr{U}$ is admissible (in the sense of article 9). Thus admissible representations of the system $\{\mathscr{U}, \epsilon\}$ are the same as admissible Harish-Chandra modules.

6.1.5 Hecke modules versus Harish-Chandra modules

16 Theorem (K-V Thm 1.117). *We have the following:*

1. *Let V be a smooth representation of $\mathscr{H}_{\mathbb{R}}$. Given T in $\mathscr{E}'(K_{\mathbb{R}})$ and v in V pick an elementary idempotent ξ stabilizing v and define*

   $$\pi(T)v = \pi(T * \xi)v.$$

   Then π gives a representation of $\mathscr{E}'(K_{\mathbb{R}})$ on V. In particular, if for g in $K_{\mathbb{R}}$ we define $\pi(g)$ to be $\pi(\delta_g)$ (where δ_g is the Dirac distribution supported at g) then V becomes a Harish-Chandra module.

2. *Let V be a Harish-Chandra module. For T in $\mathscr{E}'(K_{\mathbb{R}})$ and v in V define*

   $$\pi(T)v = \langle T, g \mapsto \pi(g)v \rangle.$$

   Then π gives a representation of $\mathscr{E}'(K_{\mathbb{R}})$ on V in such a way that V becomes a smooth representation of $\mathscr{H}_{\mathbb{R}}$.

3. *The constructions in 1 and 2 are inverse to each other. They yield a bijective correspondence between smooth $\mathscr{H}_{\mathbb{R}}$-module and $(\mathfrak{g}, K_{\mathbb{R}})$-modules. Under this correspondence, the properties admissible and irreducible are preserved.*

4. *If V and V' are Harish-Chandra modules and/or smooth representations of $\mathscr{H}_{\mathbb{R}}$ then a linear map $A : V \to V'$ intertwines for $(\mathfrak{g}, K_{\mathbb{R}})$ if and only if it does so for $\mathscr{H}_{\mathbb{R}}$. Thus*

   $$\mathrm{Hom}_{(\mathfrak{g},K_{\mathbb{R}})}(V, V') = \mathrm{Hom}_{\mathscr{H}_{\mathbb{R}}}(V, V').$$

   Therefore the correspondence in 3 actually gives an equivalence of categories.

5. *Let V is a Harish-Chandra module and/or a smooth representation of $\mathscr{H}_{\mathbb{R}}$. Let f be a measurable function on $K_{\mathbb{R}}$, thought of as an element of $\mathscr{E}'(K_{\mathbb{R}})$. Then we have*

$$\pi(f)v = \int_{K_{\mathbb{R}}} f(g)\pi(g)vdg.$$

1) We first check that $\pi(T)$ is well defined. Let ξ and ξ' be elementary idempotents stabilizing v; we may assume $\xi' \geq \xi$. We then have

$$\pi(T * \xi')v = \pi(T * \xi')\pi(\xi)v = \pi(T * (\xi' * \xi))v = \pi(T * \xi)v$$

and so $\pi(T)$ is well defined. Note that 5 is true in this case by definition.

We must now check that π is an algebra homomorphism, *i.e.*, we must verify $\pi(T * S) = \pi(T)\pi(S)$. Let v be an element of V, let W be a finite dimensional $K_{\mathbb{R}}$-stable subspace containing v and let ξ be the elementary idempotent of W. We have

$$(\xi * S * \xi)(g) = \langle S^\vee * \xi^\vee, \rho(g)\xi\rangle = \langle S^\vee, (\rho(g)\xi) * \xi\rangle$$

but (since ξ is invariant under conjugation, $K_{\mathbb{R}}$ is unimodular, and ξ is idempotent)

$$((\rho(g)\xi) * \xi)(h) = \int_{K_{\mathbb{R}}} \xi(xg)\xi(x^{-1}h)dx = \int_{K_{\mathbb{R}}} \xi(x)\xi(gx^{-1}hgg^{-1})dx = (\xi * \xi)(hg) = (\rho(g)\xi)(h)$$

and so

$$(\xi * S * \xi)(g) = \langle S^\vee, \rho(g)\xi\rangle = (S * \xi)(g);$$

therefore $\xi * S * \xi = S * \xi$. From this, we see that ξ stabilizes $\pi(S)v$ and so

$$\pi(T)\pi(S)v = \pi(T * \xi)\pi(S * \xi)v = \pi(T * \xi * S * \xi)v = \pi((T * S) * \xi)v = \pi(T * S)v$$

which proves that π is an algebra homomorphism. The rest of part 1 is clear.

2) First we must clarify the definition of π somewhat. Let T be an element of $\mathscr{H}_{\mathbb{R}}$, v an element of V and W a finite dimensional $K_{\mathbb{R}}$-stable subspace of V in which v lies. Then the function ϕ_v given by $\pi_v(g) = \pi(g)v$ on $K_{\mathbb{R}}$ takes values in the finite dimensional vector space W, and so the meaning of $\langle T, \phi_v\rangle$ is clear.

We must show that π, thus defined for elements of $\mathscr{E}'(K_{\mathbb{R}})$, is an algebra homomorphism. Note that $\lambda^{-1}(g)\phi_v = \pi(g)\phi_v$ so that

$$(\phi_v * S^\vee)(g) = \langle S, \lambda(g^{-1})\phi_v\rangle = \langle S, \pi(g)\phi_v\rangle = \pi(g)\langle S, \phi_v\rangle = \pi(g)\pi(S)v = \phi_{\pi(S)v}(g).$$

We therefore have

$$\pi(T * S)v = \langle T * S, \phi_v\rangle = \langle T, \phi_v * S^\vee\rangle = \langle T, \phi_{\pi(S)v}\rangle = \pi(T)(\pi(S)v)$$

and so π is an algebra homomorphism.

Note that in the present case, part 5 is clear. From this, it follows that the σ-isotypic components of V are the same when V is regarded as a Harish-Chandra module or as a $\mathscr{H}_{\mathbb{R}}$-module (since the isotypic components are the images of the $\pi(\xi)$). It therefore follows that V is a smooth representation of $\mathscr{H}_{\mathbb{R}}$.

3) It is clear that the constructions are inverse to each other. Since admissibility can be detected by knowing the operators $\pi(\xi)$ it follows that it is preserved under this correspondence. It is clear that irreducible is preserved, for a space stable under one of the actions is clearly stable under the other.

4) This is clear.

5) This has already been remarked upon.

6.1.6 Twisting by a quasi-character

17. Let χ be a quasi-character of $\mathbb{R}^\times$.

1. If (π, V) is a representation of $\mathfrak{g}$ we define a representation $(\chi \otimes \pi, V)$ of $\mathfrak{g}$ by

$$(\chi \otimes \pi)(X) = \tfrac{1}{2} s \operatorname{tr} X + \pi(X)$$

where $\chi(t) = t^s$ for t positive and the trace is evaluated by identifying $\mathfrak{g}$ with the Lie algebra of 2×2 matrices.

2. If (π, V) is a representation of $(\mathfrak{g}, K_\mathbb{R})$ we define a representation $(\chi \otimes \pi, V)$ of $(\mathfrak{g}, K_\mathbb{R})$ by using the previous formula on $\mathfrak{g}$ and by defining

$$(\chi \otimes \pi)(g) = \chi(\det g)\pi(g)$$

for g in $K_\mathbb{R}$.

3. If (π, V) is a representation of $\mathscr{H}_\mathbb{R}$ we define a representation $(\chi \otimes \pi, V)$ of $\mathscr{H}_\mathbb{R}$ by

$$(\chi \otimes \pi)(T) = \pi(\chi T)$$

where χT is the product of the distribution T with the function $g \mapsto \chi(\det g)$.

6.1.7 The central quasi-character

18. Let (π, V) be an admissible representation of $(\mathfrak{g}, K_\mathbb{R})$. Let v be an element of V and let W be a finite dimensional $K_\mathbb{R}$-stable subspace containing v. Define an element z of $\mathfrak{g}$ by

$$z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since z commutes with $K_\mathbb{R}$ (*i.e.*, $(\operatorname{Ad} g)z = z$ for all g in $K_\mathbb{R}$), $\pi(z)$ maps W into itself. If a is a real number we may thus define

$$\pi \begin{bmatrix} e^a & 0 \\ 0 & e^a \end{bmatrix} v = (\exp \pi(az))v = \left[\sum_{n=0}^{\infty} \frac{\pi(az)^n}{n!} \right] v.$$

(The sum on the right side is to take place in $\operatorname{End} W$, where it obviously converges.) Thus we get a representation of the group of positive scalar matrices on V. It is obvious that this representation is compatible with the existing structures on V. Since the matrices

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

belong to $K_\mathbb{R}$, we can in fact build a representation of $Z_\mathbb{R}$, the full group of scalar matrices, on V.

19. Now let (π, V) be an irreducible admissible representation of $(\mathfrak{g}, K_\mathbb{R})$. The action of $Z_\mathbb{R}$ commutes with the actions of $\mathscr{H}_\mathbb{R}$ and $(\mathfrak{g}, K_\mathbb{R})$. Thus by Schur's lemma (*cf.* §1.4.5, proposition 113) for each a in $\mathbb{R}^\times$ there is a complex number $\omega(a)$ such that

$$\pi \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \omega(a) I$$

where I is the identity map on V. The function ω is easily seen to be a quasi-character of $\mathbb{R}^\times$; it is called the *central quasi-character* of π.

6.1.8 The Hecke algebra of Jacquet-Langlands

Definition

20. Let $\mathscr{H}^1_{\mathbb{R}}$ be the subspace of $C_c^\infty(G_{\mathbb{R}})$ consisting of all functions which are $K_{\mathbb{R}}$-finite on both sides. The vector space $\mathscr{H}^1_{\mathbb{R}}$ is made into an algebra by taking convolution for multiplication. If a Haar measure dx has been chosen on $G_{\mathbb{R}}$ then $\mathscr{H}^1_{\mathbb{R}}$ may be regarded as an algebra of measures on $G_{\mathbb{R}}$ by letting the function f correspond to the measure $f dx$.

21. Let $\mathscr{H}^2_{\mathbb{R}}$ be the subspace of functions on $K_{\mathbb{R}}$ spanned by the matrix elements of irreducible representations of $K_{\mathbb{R}}$. We make $\mathscr{H}^2_{\mathbb{R}}$ into an algebra by defining multiplication to be convolution. The algebra $\mathscr{H}^2_{\mathbb{R}}$ may be regarded as an algebra of measures on $K_{\mathbb{R}}$ by letting the function f correspond to $f dx$, where dx is the normalized Haar measure on $K_{\mathbb{R}}$. We may also regard $\mathscr{H}^2_{\mathbb{R}}$ as an algebra of measures on $G_{\mathbb{R}}$ via pushforward, *i.e.*, if μ is an element of $\mathscr{H}^2_{\mathbb{R}}$ (regard as a measure on $K_{\mathbb{R}}$) we define a measure on $G_{\mathbb{R}}$ by letting the measure of the set U be the measure assigned by μ to the set $U \cap K_{\mathbb{R}}$.

22. We define the *Hecke algebra of Jacquet-Langlands* (which we also denote by $\mathscr{H}_{\mathbb{R}}$) to be the algebra $\mathscr{H}^1_{\mathbb{R}} + \mathscr{H}^2_{\mathbb{R}}$ of measures on $G_{\mathbb{R}}$. Multiplication is given by convolution of measures. In particular, if f belongs to $\mathscr{H}^1_{\mathbb{R}}$ and ξ belongs to $\mathscr{H}^2_{\mathbb{R}}$ then

$$(\xi * f)(g) = \int_{K_{\mathbb{R}}} \xi(h) f(h^{-1}g)\,dh, \qquad (f * \xi)(g) = \int_{K_{\mathbb{R}}} f(gh^{-1})\xi(h)\,dh.$$

The algebra $\mathscr{H}_{\mathbb{R}}$ is associative, noncommutative and does not have a unit.

23. As with the Hecke algebra of Flath, we take the elementary idempotents of $K_{\mathbb{R}}$, now regarded as elements of $\mathscr{H}^2_{\mathbb{R}}$, to be the distinguished idempotents of $\mathscr{H}_{\mathbb{R}}$. The Hecke algebra thus becomes an idempotented algebra. However, admissible representations of $\mathscr{H}_{\mathbb{R}}$ are *not* just admissible representations of $\mathscr{H}_{\mathbb{R}}$ as an idempotented algebra.

24. It is easy to see that for any f in $\mathscr{H}^1_{\mathbb{R}}$ there is an elementary idempotent ξ such that

$$\xi * f = f * \xi = f.$$

Moreover, if ξ is any elementary idempotent then

$$\mathscr{H}^1_{\mathbb{R}}[\xi] = \xi * \mathscr{H}^1_{\mathbb{R}} * \xi = \xi * \mathscr{C}_c^\infty(G_{\mathbb{R}}) * \xi$$

is a closed subspace of $C_c^\infty(G_{\mathbb{R}})$ in the Schwartz topology; we give it the induced topology.

25. If we regard elements of $\mathfrak{g}$ as distributions on $G_{\mathbb{R}}$ with support at the identity we may take their convolution with elements of the Hecke algebra. More precisely, for X in $\mathfrak{g}$ and f in $\mathscr{H}_{\mathbb{R}}$ we have

$$(X * f)(g) = \frac{d}{dt} f(\exp(-tX)g)\big|_{t=0} \qquad (f * X)(g) = \frac{d}{dt} f(g\exp(-tX))\big|_{t=0}.$$

If f belongs to $\mathscr{H}^1_{\mathbb{R}}$ then so does $X * f$ and $f * X$. We extend this notation to allow for all X in $\mathscr{U}$.

Admissible representations

26. A representation (π, V) of $\mathscr{H}_{\mathbb{R}}$ is *admissible* if is satisfies the following three conditions:

1. We have $V = \pi(\mathscr{H}^1_{\mathbb{R}})V$.
2. For each elementary idempotent ξ the range of $\pi(\xi)$ is finite dimensional.
3. The map $\mathscr{H}^1_{\mathbb{R}}[\xi] \to \mathrm{GL}(\pi(\xi)V)$ given by $f \mapsto \pi(f)$ is continuous.

27 Proposition. *Let (π, V) be an admissible representation of $\mathscr{H}_{\mathbb{R}}$.*

1. *Every vector in V is stabilized by an elementary idempotent.*
2. *Every vector in V is stabilized by an element of $\mathscr{H}^1_{\mathbb{R}}$.*

Let v be an element of V and write $v = \sum_i \pi(f_i)v_i$.

1) Take ξ such that $\xi * f_i = f_i$. Then $\pi(\xi)v = v$.

2) Let $\{\phi_n\}$ be a sequence in $C_c^\infty(G_\mathbb{R})$ which converges in the space of distributions to the Dirac distribution supported at the identity. Set $\phi'_n = \xi * \phi_n * \xi$ where ξ is as in the first part. For each i the sequence $\{\phi'_n * f_i\}$ converges to f_i in $\mathscr{H}^1_\mathbb{R}[\xi]$. Thus, by the third condition of admissibility, the sequence $\{\pi(\phi'_n)v\}$ converges to v in the finite dimensional space $\pi(\xi)v$. Thus v belongs to the closure of $\pi(\mathscr{H}^1_\mathbb{R}[\xi])$ and therefore belongs to it. Thus there exists f in $\mathscr{H}^1_\mathbb{R}[\xi]$ such that $\pi(f)v = v$.

28. For a smooth function f on $G_\mathbb{R}$ and a compactly supported distribution μ we define $\lambda(\mu)f$ and $\rho(\mu)f$ by

$$(\lambda\mu f)(g) = \mu^\vee(\rho(g)f), \qquad ((\rho(\mu)f)(g) = \mu(\lambda(g^{-1})f).$$

If, for example, μ is a measure then

$$(\lambda(\mu)f)(g) = \int_{G_\mathbb{R}} f(h^{-1}g)d\mu(h), \qquad (\rho(\mu)f)(g) = \int_{G_\mathbb{R}} f(gh)d\mu(h).$$

In any case, the functions $\lambda(\mu)f$ and $\rho(\mu)f$ are smooth functions on $G_\mathbb{R}$.

29. Applying the definitions of article 28 to the elements of $\mathscr{H}_\mathbb{R}$, we obtain representations λ and ρ of $\mathscr{H}_\mathbb{R}$ on the space $C^\infty(G_\mathbb{R})$; these are called the *left regular* and *right regular* representations of $\mathscr{H}_\mathbb{R}$.

The matrix elements of a Hecke module

30 Proposition (J-L pg. 156). *Let (π, V) be an admissible representation of $\mathscr{H}_\mathbb{R}$. Let v be an element of V and $\widetilde{v}$ an element of $\widetilde{V}$.*

1. *There exists a smooth function on $G_\mathbb{R}$, written alternatively as $\phi_{v,\widetilde{v}}$ or $g \mapsto \langle\pi(g)v, \widetilde{v}\rangle$, such that for any f in $\mathscr{H}^1_\mathbb{R}$ we have*
$$\langle\pi(f)v, \widetilde{v}\rangle = \int_{G_\mathbb{R}} f(g)\langle\pi(g)v, \widetilde{v}\rangle dg.$$
2. *The expression $\langle\pi(g)v, \widetilde{v}\rangle$ is bilinear in v and $\widetilde{v}$.*
3. *We have $\langle\pi(g)v, \widetilde{v}\rangle = \langle v, \pi(g^{-1})\widetilde{v}\rangle$.*

Let ξ be an elementary idempotent such that $\pi(\xi)v = v$ and $\widetilde{\pi}(\xi^\vee)\widetilde{v} = \widetilde{v}$. Define a distribution μ on $G_\mathbb{R}$ by assigning to the function f in $C_c^\infty(G_\mathbb{R})$ the number

$$\mu(f) = \langle\pi(\xi * f * \xi)v, \widetilde{v}\rangle.$$

Note that $C_c^\infty(G_\mathbb{R})[\xi] = \mathscr{H}^1_\mathbb{R}[\xi]$ and so this definition makes sense. Note also that for f in $\mathscr{H}^1_\mathbb{R}$ we have

$$\mu(f) = \langle\pi(f)v, \widetilde{v}\rangle.$$

Now select an element ϕ of $\mathscr{H}^1_\mathbb{R}[\xi]$ such that $\pi(\phi)v = v$. Then

$$(\phi^\vee * \mu)(f) = \mu(f\phi) = \mu(\xi f\phi\xi) = \mu(\xi f\xi\phi) = \langle\pi(\xi f\xi\phi)v, \widetilde{v}\rangle = \langle\pi(\xi f\xi)v, \widetilde{v}\rangle = \mu(f).$$

Thus $\mu = \phi^\vee * \mu$. Since μ is the convolution of a test function and a distribution it follows that μ is a smooth function.

31 Proposition (J-L pg. 160). *Let (π, V) be an admissible representation of $\mathscr{H}_\mathbb{R}$, let v be in V, let $\widetilde{v}$ be in $\widetilde{V}$ and let X be in $\mathscr{U}$. Then*

$$\phi_{v,\widetilde{v}} * X = \phi_{\pi(X^\vee)v,\widetilde{v}}.$$

For all f in $\mathscr{H}^1_\mathbb{R}$ we have

$$\begin{aligned}\int_{G_\mathbb{R}} f(g)(\phi_{v,\widetilde{v}} * X)(g)dg = \int_{G_\mathbb{R}} (f * X^\vee)(g)\phi_{v,\widetilde{v}}(g)dg = \langle\pi(f * X^\vee)v, \widetilde{v}\rangle \\ = \langle\pi(f)\pi(X^\vee)v, \widetilde{v}\rangle = \int_{G_\mathbb{R}} f(g)\phi_{\pi(X^\vee)v,\widetilde{v}}(g)dg\end{aligned}$$

The proposition follows.

The associated Harish-Chandra module

32 Proposition. *Let (π, V) be an admissible representation of $\mathscr{H}_{\mathbb{R}}$. Then there is exaclty one representation of $(\mathfrak{g}, K_{\mathbb{R}})$ on V (also written π) such that:*

1. *For all X in $\mathfrak{g}$ and all f in $\mathscr{H}_{\mathbb{R}}^1$ we have*

$$\pi(X)\pi(f) = \pi(X * f), \qquad \pi(f)\pi(X) = \pi(f * X)$$

   (note that both $X * f$ and $f * X$ belong to $\mathscr{H}_{\mathbb{R}}^1$).

2. *For all g in $K_{\mathbb{R}}$ and all f in $\mathscr{H}_{\mathbb{R}}^1$ we have*

$$\pi(g)\pi(f) = \pi(\lambda(g)f), \qquad \pi(f)\pi(g) = \pi(\rho(g^{-1})f)$$

   (note that both $\lambda(g)f$ and $\rho(g^{-1})f$ belong to $\mathscr{H}_{\mathbb{R}}^1$).

We call this action of $(\mathfrak{g}, K_{\mathbb{R}})$ the action compatible *with π or the* associated Harish-Chandra module*. It is admissible.*

That there can be only one such action is clear; thus we need only establish existence.

Let v be an element of V. Since π is admissible, we can write

$$v = \sum_{i=1}^{r} \pi(f_i)v_i$$

with f_i in $\mathscr{H}_{\mathbb{R}}^1$ and v_i in V. We wish to define the action of $\mathfrak{g}$ by

$$\pi(X)v = \sum_{i=1}^{r} \pi(X * f_i)v_i.$$

To check that this is well defined, we must show that if

$$\sum_{i=1}^{r} \pi(f_i)v_i = 0$$

then

$$w = \sum_{i=1}^{r} \pi(X * f_i)v_i$$

is also zero. Now take f in $\mathscr{H}_{\mathbb{R}}^1$ such that $\pi(f)w = w$. Thus

$$\pi(f)w = \sum_{i=1}^{r} \pi(f)\pi(X * f_i)v_i = \sum_{i=1}^{r} \pi(f * (X * f_i))v_i = \sum_{i=1}^{r} \pi((f * X) * f_i)v_i = \pi(f * X)\sum_{i=1}^{r} \pi(f_i)v_i = 0$$

and so our formula for $\pi(X)$ is well defined. The same reasoning shows that, for any element f of $\mathscr{H}_{\mathbb{R}}$, we have

$$\pi(f)\left(\pi(X)\sum_{i=1}^{r} \pi(f_i)v_i\right) = \pi(f)\sum_{i=1}^{r} \pi(X * f_i)v_i = \pi(f * X)\sum_{i=1}^{r} \pi(f_i)v_i$$

and so $\pi(f)\pi(X) = \pi(f * X)$. Thus the action of $\mathfrak{g}$ just defined is compatible with the action of $\mathscr{H}_{\mathbb{R}}$.

Now, note that for f in $\mathscr{H}_1$ and g in $K_{\mathbb{R}}$ we have $\lambda(g)f = \delta_g * f$ and $\rho(g^{-1})f = f * \delta_g$, where δ_g is the Dirac distribution on $G_{\mathbb{R}}$ supported at g. We thus attempt to define an action of $K_{\mathbb{R}}$ on V by

$$\pi(g)v = \sum_{i=1}^{r} \pi(\delta_g * f_i)v_i$$

where v is an arbtirary element of V and v_i and f_i are as above. The same "trick" as above shows that this is well defined and that $\pi(f)\pi(g) = \pi(\rho(g^{-1})f)$. Thus the action of $K_{\mathbb{R}}$ is compatible with the action of $\mathscr{H}_{\mathbb{R}}$.

We must now verify that the actions of $\mathfrak{g}$ and $K_{\mathbb{R}}$ actually yield an admissible Harish-Chandra module. There are three conditions to be verified:

1) We first show that V breaks up into an algebraic direct sum of finite dimensional $K_{\mathbb{R}}$-stable subspaces and that each irreducible representation of $K_{\mathbb{R}}$ appears only finitely many times. This follows immediately from the fact that every vector in V is stabilized by some elementary idempotent ξ and that the range of $\pi(\xi)$ is finite dimensional.

2) Given f in $\mathscr{H}^1_{\mathbb{R}}$, X in $\mathfrak{g}$ and g in $K_{\mathbb{R}}$ we have

$$((\operatorname{Ad} g)X) * f = \delta_g * (X * (\delta_{g^{-1}} * f))$$

and so it follows that

$$\pi((\operatorname{Ad} g)X) = \pi(g)\pi(X)\pi(g^{-1}).$$

This verifies the second condition for Harish-Chandra modules.

3) Now let v be an arbitrary element of V, let f in $\mathscr{H}^1_{\mathbb{R}}$ stabilize v and let X be in the Lie algebra of $K_{\mathbb{R}}$. Since f is $K_{\mathbb{R}}$-finite we can find an elementary idempotent ξ such that

$$\xi * (\lambda(g)f) = (\lambda(g)f) * \xi = \lambda(g)f$$

for any g in $K_{\mathbb{R}}$. Let

$$f_t = \frac{\lambda(\exp(tX))f - f}{t}.$$

We have that f_t belongs to $\xi * \mathscr{H}^1_{\mathbb{R}} * \xi$ for all t and $f_t \to X * f$ in this topology. Thus by the third axiom of admissitibility $\pi(f_t) \to \pi(X * f)$ as operators on the finite dimensional space $\pi(\xi)V$ (to which v belongs). We therefore have

$$\frac{\pi(\exp(tX))v - v}{t} = \frac{\pi(\lambda(\exp(tX))f)v - \pi(f)v}{t} \to \pi(X * f)v = \pi(X)v$$

and this verifies the final condition for Harish-Chandra modules.

33 Proposition (Knapp Thm. 8.7). *Let (π, V) be a representation of $\mathscr{H}_{\mathbb{R}}$. Then the matrix elements $\phi_{v,\widetilde{v}}$ are analytic functions on $G_{\mathbb{R}}$.*

We only sketch a proof; a complete proof is given in Knapp.

Consider the associated Harish-Chandra module structure on V. The space V breaks up into a direct sum of its σ-isotypic parts, each of which are finite dimensional. It suffices to prove the proposition when v lies in one of these isotypic parts, say $V(\sigma)$.

The action of $\mathcal{Z}$ (the center of $\mathscr{U}$) commutes with the action of $K_{\mathbb{R}}$ and so the space $V(\sigma)$ is stable under the aciton of $\mathcal{Z}$. We now consider a specific element of $\mathcal{Z}$, the so-called Casimir operator, and examine how it acts on the finite dimensional space $V(\sigma)$. From the specific form of the Casimir operator, one deduces an elliptic partial differential equation satisfied by the matrix elements, and from this concludes that they are analytic.

34 Proposition (J-L pg. 158). *Let V be a vector space with compatible admissible actions of $(\mathfrak{g}, K_{\mathbb{R}})$ and $\mathscr{H}_{\mathbb{R}}$. Let v be in V and $\widetilde{v}$ be in $\widetilde{V}$. Then for g in $G_{\mathbb{R}}$ and h in $K_{\mathbb{R}}$ we have*

$$\phi_{\pi(h)v,\widetilde{v}}(g) = \phi_{v,\widetilde{v}}(gh), \qquad \phi_{v,\widetilde{\pi^{-1}(h)\widetilde{v}}}(g) = \phi_{v,\widetilde{v}}(hg).$$

In particular, the two definitions of $\langle \pi(h)v, \widetilde{v}\rangle$ agree.

For f in $\mathscr{H}^1_{\mathbb{R}}$ we have

$$\begin{aligned}\int_{G_{\mathbb{R}}} f(g)\phi_{\pi(h)v,\widetilde{v}}(g)dg =& \langle \pi(f)(\pi(h)v), \widetilde{v}\rangle = \langle \pi(\rho(h^{-1})f)v, \widetilde{v}\rangle \\ =& \int_{G_{\mathbb{R}}} f(g)\phi_{v,\widetilde{v}}(gh)dg\end{aligned}$$

and the first identity follows. The second is proved using a similar argument.

35 Proposition. *Let V be a vector space with compatible admissible actions of $(\mathfrak{g}, K_{\mathbb{R}})$ and $\mathscr{H}_{\mathbb{R}}$. Let ξ be in $\mathscr{H}_{\mathbb{R}}^2$ and let v be in V. Then*

$$\pi(\xi)v = \int_{K_{\mathbb{R}}} \xi(g)\pi(g)v dg.$$

Let f be an element of $\mathscr{H}_{\mathbb{R}}^1$ such that $\pi(f)v = v$. Then for any $\widetilde{v}$ in $\widetilde{V}$ we have

$$\begin{aligned}
\langle \pi(\xi)v, \widetilde{v}\rangle = \langle \pi(\xi * f)v, \widetilde{v}\rangle &= \int_{G_{\mathbb{R}}} (\xi * f)(g)\langle \pi(g)v, \widetilde{v}\rangle dg \\
&= \int_{G_{\mathbb{R}}}\int_{K_{\mathbb{R}}} \xi(h)f(h^{-1}g)\langle \pi(g)v, \widetilde{v}\rangle dh dg = \int_{K_{\mathbb{R}}} \xi(h)\langle \pi(\lambda(h)f)v, \widetilde{v}\rangle dh \\
&= \int_{K_{\mathbb{R}}} \xi(h)\langle \pi(h)v, \widetilde{v}\rangle dh
\end{aligned}$$

and the proposition follows.

36 Proposition. *An admissible representation (π, V) of $\mathscr{H}_{\mathbb{R}}$ is irreducible if and only if its associated Harish-Chandra module is irreducible.*

It is clear (by way of definition) that an $\mathscr{H}_{\mathbb{R}}$-stable subspace of V is $(\mathfrak{g}, K_{\mathbb{R}})$-stable as well; thus if the associated Harish-Chandra module is irreducible the original representation of $\mathscr{H}_{\mathbb{R}}$ is irreducible as well.

Now assume V that is irreducible as an $\mathscr{H}_{\mathbb{R}}$-module. Let V_1 be a $(\mathfrak{g}, K_{\mathbb{R}})$-stable subspace of V and let $\widetilde{V}_1$ be its orthogonal complement. Then for any v in V, $\widetilde{v}$ in $\widetilde{V}$, g in $K_{\mathbb{R}}$ and X in $\mathscr{U}$ we have

$$(\phi_{v,\widetilde{v}} * X)(g) = \phi_{\pi(X^\vee)v,\widetilde{v}}(g) = \langle \pi(g)\pi(X^\vee)v, \widetilde{v}\rangle = 0;$$

this follows from propositions 31 and 34 together with the fact that $\pi(g)\pi(X^\vee)v$ belongs to V_1 (since it is stable under the action of $(\mathfrak{g}, K_{\mathbb{R}})$). Thus all the derivatives in every direction of $\phi_{v,\widetilde{v}}$ vanish in $K_{\mathbb{R}}$. Since $K_{\mathbb{R}}$ meets every connected component of $G_{\mathbb{R}}$ and the functions $\phi_{v,\widetilde{v}}$ are analytic (*cf.* proposition 33) it follows that $\phi_{v,\widetilde{v}}$ is identically zero on all of $G_{\mathbb{R}}$. Thus for any f in $\mathscr{H}_{\mathbb{R}}^1$, v in V_1 and $\widetilde{v}$ in $\widetilde{V}_1$ we have

$$\langle \pi(f)v, \widetilde{v}\rangle = \int_{G_{\mathbb{R}}} f(g)\phi_{v,\widetilde{v}}(g)dg = 0.$$

It thus follows that $\pi(f)v$ belongs to V_1. Since V_1 is clearly stable under $\mathscr{H}_{\mathbb{R}}^2$ (*e.g.*, by proposition 35) it follows that V_1 is stable under all of $\mathscr{H}_{\mathbb{R}}$. Therefore V_1 is zero or all of V and we have proved that V is irreducible under the action of $(\mathfrak{g}, K_{\mathbb{R}})$.

37 Proposition. *Let (π_1, V_1) and (π_2, V_2) be representations of $\mathscr{H}_{\mathbb{R}}$ and let $A : V_1 \to V_2$ be a linear map. Then A commutes with the action of $\mathscr{H}_{\mathbb{R}}$ if and only if it commutes with the action of $(\mathfrak{g}, K_{\mathbb{R}})$.*

The proof of this is similar to the proof of proposition 36 and omitted.

38 Proposition. *Two irreducible admissible representations of $\mathscr{H}_{\mathbb{R}}$ are isomorphic if and only if their associated Harish-Chandra modules are isomorphic.*

This follows immediately from proposition 37.

Comparison of the two Hecke algebras

39. There seem to be two differences between the two Hecke algebras. I say "seem" for I am not certain they are actually differences.

1. The definition of admissible for a representation of the Flath Hecke algebra comes directly from the definition of admissible for idempotented algebras. For the Jacquet-Langlands Hecke algebra, the definition of admissible is more complicated (elements have to be stabilized by members of $\mathscr{H}_{\mathbb{R}}^1$ and there is a topological condition).

2. Smooth representations of the Flath Hecke algebra are in obvious bijective correspondence with Harish-Chandra modules. On the other hand, given an admissible representation of the Jacquet-Langland Hecke algebra, there is an associated admissible Harish-Chandra module; there does not seem to be a direct inverse construction. However, it turns out (after we classify irreducible Harish-Chandra modules) that this association is a bijection of the irreducible modules.

For these reasons, I prefer the Flath Hecke algebra.

6.2 Classification of irreducible representations

6.2.1 Restricting representations of $(\mathfrak{g}, K_{\mathbb{R}})$ to $\mathfrak{g}$

40. In this section we gather a few results about the restriction of representations of $(\mathfrak{g}, K_{\mathbb{R}})$ to representations of $\mathfrak{g}$.

41. If (π, V) is a representation of $\mathfrak{g}$ we write $((\operatorname{Ad}\epsilon)\pi, V)$ for the representation of $\mathfrak{g}$ on the space V given by $X \mapsto \pi((\operatorname{Ad}\epsilon)X)$.

42 Proposition (J-L Lemma 5.8). *Let (π, V) be an irreducible representation of $(\mathfrak{g}, K_{\mathbb{R}})$. There are two possibilities:*

1. *The restriction of π to $\mathfrak{g}$ is irreducible and the representations (π, V) and $((\operatorname{Ad}\epsilon)\pi, V)$ are equivalent.*
2. *The space V breaks into a direct sum $V_1 \oplus V_2$ of subspace stable under $\mathfrak{g}$. The representations π_1 and π_2 of $\mathfrak{g}$ on V_1 and V_2 are irreducible. The representations (π_1, V_1) and (π_2, V_2) are not equivalent but the representations $((\operatorname{Ad}\epsilon)\pi_1, V_1)$ and (π_2, V_2) are.*

If the restriction of π to $\mathfrak{g}$ is irreducible then π and $(\operatorname{Ad}\epsilon)\pi$ are certainly equivalent; indeed $\pi(\epsilon)$ is an intertwining operator.

Assume now that the restriction of π to $\mathfrak{g}$ is not irreducible, let V_1 be a proper stable subspace and let $V_2 = \pi(\epsilon)V_1$. Both $V_1 + V_2$ and $V_1 \cap V_2$ are stable under $(\mathfrak{g}, K_{\mathbb{R}})$ and thus $V_1 + V_2 = V$ and $V_1 \cap V_2 = 0$; therefore $V = V_1 \oplus V_2$.

If V_1 had a subspace V_1' stable under $\mathfrak{g}$ the same considerations as above would imply that $V_1 = V_1' \oplus V_2'$ where $V_2' = \pi(\epsilon)V_1'$ and thus V_1 would be a proper subspace of V stable under $(\mathfrak{g}, K_{\mathbb{R}})$. Since this is impossible it follows that V_1 and V_2 are irreducible.

If v_1 is in V_1 then

$$\pi_2(X)\pi(\epsilon)v_1 = \pi(\epsilon)((\operatorname{Ad}\epsilon)\pi_1)(X)v_1$$

so that $\pi(\epsilon)$ is an intertwining operator from $(V_1, (\operatorname{Ad}\epsilon)\pi_1)$ to (V_2, π_2). Thus these two representations are equivalent.

Assume now that π_1 and π_2 are equivalent and let A be an intertwining operator, *i.e.*, a bijective linear map $V_1 \to V_2$ such that $A\pi_1(X) = \pi_2(X)A$. For v_1 in V_1 we have

$$A^{-1}\pi(\epsilon)\pi_1(X)v_1 = A^{-1}\pi_2((\operatorname{Ad}\epsilon)X)\pi(\epsilon)v_1 = \pi_2((\operatorname{Ad}\epsilon))X)A^{-1}\pi(\epsilon)v_1.$$

Therefore $(A^{-1}\pi(\epsilon))^2$, regarded as a linear transformation of V_1, commutes with $\mathfrak{g}$ and is therefore a scalar (*cf.* proposition 10). We may take the scalar to be 1. The linear transformation of V given by

$$v_1 + v_2 \mapsto A^{-1}v_2 + Av_1$$

then commutes with the action of $(\mathfrak{g}, K_{\mathbb{R}})$. This is a contradiction; thus π_1 and π_2 are not equivalent.

43 Proposition (J-L Lemma 5.9). *Let (π_0, V) be an irreducible admissible representation of $\mathfrak{g}$ such that π_0 is equivalent to $(\operatorname{Ad}\epsilon)\pi_0$. Let η be the nontrivial quadratic character of $\mathbb{R}^\times$ (i.e., $\eta(t) = \operatorname{sgn} t$).*

1. *There is an irreducible admissible representation π of $(\mathfrak{g}, K_{\mathbb{R}})$ on V whose restriction to $\mathfrak{g}$ is π_0.*
2. *The representations π and $\eta \otimes \pi$ of $(\mathfrak{g}, K_{\mathbb{R}})$ are inequivalent.*
3. *Any representation of $(\mathfrak{g}, K_{\mathbb{R}})$ on V whose restriction to $\mathfrak{g}$ is equivalent to π_0 is equivalent to either π or $\eta \otimes \pi$.*

1) Obviously we take $\pi(X) = \pi_0(X)$ for X in $\mathfrak{g}$. By definition of admissible the representation π_0 comes from a representation π of $\mathrm{SO}(2, \mathbb{R})$. Thus we need only define $\pi(\epsilon)$.

There is an invertible linear transformation A of V into itself such that $A\pi_0(X) = \pi_0((\mathrm{Ad}\,\epsilon)X)A$ for all X. Thus A^2 commutes with $\pi_0(X)$ and is therefore a scalar; we may assume $A^2 = 1$. We define $\pi(\epsilon)$ to be A.

2) If we replace A by $-A$ we obtain the representation $\eta \otimes \pi$. The representation π and $\eta \otimes \pi$ are not equivalent because any intertwining operator would have to commute with the $\pi(X)$ and therefore be a scalar (and thus not intertwine the operators $\pi(\epsilon)$ and $(\eta \otimes \pi)(\epsilon)$).

3) For any such representation ϵ must act as either A or $-A$ and thus the representation is equivalent to π or $\eta \otimes \pi$.

44 Proposition (J-L Lemma 5.10). *Let (π_1, V_1) be an irreducible admissible representation of $\mathfrak{g}$ such that π_1 is not equivalent to $(\mathrm{Ad}\,\epsilon)\pi_1$. Let (π_2, V_2) be the representation given by $V_2 = V_1$ and $\pi_2 = \mathrm{Ad}\,\epsilon\pi_1$. Let $V = V_1 \oplus V_2$.*

1. *There is an irreducible representation π of $(\mathfrak{g}, K_{\mathbb{R}})$ on V whose restriction to $\mathfrak{g}$ is $\pi_1 \oplus \pi_2$.*
2. *Any irreducible admissible representation whose restriction to $\mathfrak{g}$ is equivalent to π_1 is equivalent to π.*
3. *In particular, $\eta \otimes \pi$ is equivalent to π.*

1) We take $\pi(X) = \pi_1(X) \oplus \pi_2(X)$. We obtain for free a representation π of $\mathrm{SO}(2, \mathbb{R})$ on V. We define $\pi(\epsilon)$ by

$$\pi(\epsilon)(v_1 \oplus v_2) = v_2 \oplus v_1.$$

2) This follows from proposition 42.

3) This follows from part 2.

6.2.2 The representations $\rho(\mu_1, \mu_2)$

45. Let μ_1 and μ_2 be quasi-characters of $\mathscr{R}^\times$. Let $\mathscr{B}(\mu_1, \mu_2)$ be the space of functions f on $G_{\mathbb{R}}$ which satify the following two conditions:

1. We have
$$f\left(\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix} g\right) = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} f(g)$$
for all g in $G_{\mathbb{R}}$, a_1 and a_2 in $\mathbb{R}^\times$ and x in $\mathbb{R}$.
2. f is $\mathrm{SO}(2, \mathbb{R})$ finite on the right.

The space $\mathscr{B}(\mu_1, \mu_2)$ is stable under the right regular representation ρ (*cf.* article 29) and thus we get a representation of $\mathscr{H}_{\mathbb{R}}$ on $\mathscr{B}(\mu_1, \mu_2)$. We denote this representation, and the corresponding representation of $(\mathfrak{g}, K_{\mathbb{R}})$ by $\rho(\mu_1, \mu_2)$. They are admissible.

46. Note that by the Iwasawa decomposition $G_{\mathbb{R}} = P_{\mathbb{R}}\mathrm{SO}(2, \mathbb{R})$ (where $P_{\mathbb{R}}$ is group of upper triangular matrices) the elements of $\mathscr{B}(\mu_1, \mu_2)$ are determined by their restiction to $\mathrm{SO}(2, \mathbb{R})$. In particular, since they are $\mathrm{SO}(2, \mathbb{R})$ finite, they are smooth.

47. We now define some notations we will use while studying with the representations $\rho(\mu_1, \mu_2)$.

1. Write
$$\mu_i(t) = (\mathrm{sgn}\, t)^{m_i}|t|^{s_i}$$
where m_i is 0 or 1 and s_i is a complex number.
2. Let $s = s_1 - s_2$ and $m = |m_1 - m_2|$. Note that
$$(\mu_1\mu_2^{-1})(t) = (\mathrm{sgn}\, t)^m|t|^s.$$
3. For integers n of the same parity as m, define an element ϕ_n of $\mathscr{B}(\mu_1, \mu_2)$ by
$$\phi_n\left(\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right) = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} e^{in\theta}.$$
The collection $\{\phi_n\}$ forms a basis for the space $\mathscr{B}(\mu_1, \mu_2)$.

4. We define elements of $K_{\mathbb{R}}$ by

$$\epsilon = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \kappa_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

   Note that $\kappa_{\theta_1}\kappa_{\theta_2} = \kappa_{\theta_1+\theta_2}$.

5. We name several elements of $\mathfrak{g}_{\mathbb{C}}$ (which we identify with the Lie algebra of 2×2 matrices):

$$U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad V_+ = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \quad V_- = \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}$$

$$X_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad X_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

   Note that the definition of V_- given in J-L on pg. 165 is incorrect.

6. We also name an element of the universal enveloping algebra:

$$D = X_+X_- + X_-X_+ + \tfrac{1}{2}Z^2.$$

   In fact, D is the *Casimir operator*, which belongs to $\mathcal{Z}$.

48 Proposition (J-L Lemma 5.6). *We have the following identities:*

1. $\rho(U)\phi_n = in\phi_n$	*2.* $\rho(\epsilon)\phi_n = (-1)^{m_1}\phi_{-n}$
3. $\rho(V_+)\phi_n = (s+1+n)\phi_{n+2}$	*4.* $\rho(V_-)\phi_n = (s+1-n)\phi_{n-2}$
5. $\rho(D)\phi_n = \frac{1}{2}(s^2-1)\phi_n$	*6.* $\rho(J)\phi_n = (s_1+s_2)\phi_n$.

1) Let

$$g = \begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Note that $\exp(tU) = \kappa_t$. We thus have

$$(\rho(U)\phi_n)(g) = \lim_{t\to 0}\frac{\phi_n(g\exp(tU)) - \phi_n(g)}{t} = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2}\lim_{t\to 0}\frac{e^{in(\theta+t)} - e^{in\theta}}{t} = in\phi_n(g).$$

2) Left to the reader.
3) From the identites $(\operatorname{Ad}\kappa_\theta)V_+ = e^{2i\theta}V_+$ and $\rho(\kappa_\theta)\phi_n = e^{in\theta}\phi_n$ we have

$$e^{2i\theta}\rho(V_+)\phi_n = \rho((\operatorname{Ad}\kappa_\theta)V_+)\phi_n = \rho(\kappa_\theta)\rho(V_+)\rho(\kappa_{-\theta})\phi_n = e^{-in\theta}\rho(\kappa_\theta)\rho(V_+)\phi_n$$

and so

$$\rho(\kappa_\theta)\big(\rho(V_+)\phi_n\big) = e^{i(n+2)\theta}\big(\rho(V_+)\phi_n\big).$$

Thus $\rho(V_+)\phi_n$ is a scalar multiple of ϕ_{n+2}. Now, $V_+ = Z - iU + 2iX_+$. For any ϕ in $\mathscr{B}(\mu_1,\mu_2)$ we have

$$(\rho(Z)\phi)(1) = (s+1)\phi(1) \qquad (\rho(X_+)\phi)(1) = 0.$$

It thus follows that the value of $\rho(V_+)\phi_n$ at 1 is $s+1+n$. Therefore $\rho(V_+)\phi_n = (s+1+n)\phi_{n+2}$.
4) Similar to 3, left to the reader.
5) Since D lies in $\mathcal{Z}$ and $D = D^\vee$ we have

$$\rho(D)\phi = \lambda(D^\vee)\phi = \lambda(D)\phi$$

for any ϕ in $\mathscr{B}(\mu_1,\mu_2)$. If we now write D as

$$D = 2X_-X_+ + Z + \tfrac{1}{2}Z^2$$

and observe that

$$\lambda(X_+)\phi = 0 \qquad \lambda(Z)\phi = -(s+1)\phi$$

for any ϕ in $\mathscr{B}(\mu_1,\mu_2)$, we see that

$$\rho(D)\phi_n = \big(-(s+1) + \tfrac{1}{2}(s+1)^2\big)\phi_n = \tfrac{1}{2}(s^2-1)\phi_n$$

6) Left to the reader.

49 Proposition (J-L Lemma 5.7). *We have the following:*

1. *If $s - m$ is not an odd integer then $\mathscr{B}(\mu_1, \mu_2)$ is irreducible under the action of $\mathfrak{g}_{\mathbb{C}}$.*

2. *If $s - m$ is an odd integer and $s \geq 0$ then the only proper subspaces of $\mathscr{B}(\mu_1, \mu_2)$ which are stable under $\mathfrak{g}_{\mathbb{C}}$ are*

$$\mathscr{B}_1(\mu_1, \mu_2) = \bigoplus_{\substack{n \geq s+1 \\ n \equiv s+1 \, (mod\ 2)}} \mathbb{C}\phi_n, \qquad \mathscr{B}_2(\mu_1, \mu_2) = \bigoplus_{\substack{n \geq -s-1 \\ n \equiv s+1 \, (mod\ 2)}} \mathbb{C}\phi_n$$

   and

$$\mathscr{B}_s(\mu_1, \mu_2) = \mathscr{B}_1(\mu_1, \mu_2) + \mathscr{B}_2(\mu_1, \mu_2)$$

   when it is not the entire space, i.e., when $s \neq 0$.

3. *If $s - m$ is an odd integer and $s < 0$ then the only proper subspaces of $\mathscr{B}(\mu_1, \mu_2)$ which are stable under $\mathfrak{g}_{\mathbb{C}}$ are*

$$\mathscr{B}_1(\mu_1, \mu_2) = \bigoplus_{\substack{n \geq s+1 \\ n \equiv s+1 \, (mod\ 2)}} \mathbb{C}\phi_n, \qquad \mathscr{B}_2(\mu_1, \mu_2) = \bigoplus_{\substack{n \geq -s-1 \\ n \equiv s+1 \, (mod\ 2)}} \mathbb{C}\phi_n$$

   and

$$\mathscr{B}_f(\mu_1, \mu_2) = \mathscr{B}_1(\mu_1, \mu_2) \cap \mathscr{B}_2(\mu_1, \mu_2).$$

Any vector v in $\mathscr{B}(\mu_1, \mu_2)$ is a linear combination of the ϕ_n. The operator $\rho(U)$ can be used to recover the ϕ_n which occur in v with a nonzero coefficient (by the formula given in proposition 48). Thus any subspace of $\mathscr{B}(\mu_1, \mu_2)$ which is stable under $\mathfrak{g}_{\mathbb{C}}$ is spanned by the ϕ_n which it contains. The proposition then follows from the identities of proposition 48.

6.2.3 The representations $\pi(\mu_1, \mu_2)$ and $\sigma(\mu_1, \mu_2)$

50 Theorem (J-L Thm. 5.11). *Let μ_1 and μ_2 be quasi-characters of $\mathbb{R}^\times$.*

1. *Let $\mu_1\mu_2^{-1}$ be not of the form $t \mapsto t^p \operatorname{sgn} t$ where p is a nonzero integer.*
   (a) *The space $\mathscr{B}(\mu_1, \mu_2)$ is irreducible under the action of $(\mathfrak{g}, K_{\mathbb{R}})$.*
   (b) *We denote by $\pi(\mu_1, \mu_2)$ any representation in its equivalence class.*

2. *Let $\mu_1\mu_2^{-1}$ be of the form $t \mapsto t^p \operatorname{sgn} t$ where p is a positive integer.*
   (a) *The space $\mathscr{B}(\mu_1, \mu_2)$ contains exactly one proper subspace $\mathscr{B}_s(\mu_1, \mu_2)$ stable under $(\mathfrak{g}, K_{\mathbb{R}})$.*
   (b) *The space $\mathscr{B}_s(\mu_1, \mu_2)$ is infinite dimensional and of finite codimension.*
   (c) *We denote by $\sigma(\mu_1, \mu_2)$ any representation equivalent to the restrcition of $\rho(\mu_1, \mu_2)$ on $\mathscr{B}_s(\mu_1, \mu_2)$.*
   (d) *We denote by $\pi(\mu_1, \mu_2)$ any representation equivalent to the representation on $\mathscr{B}_f(\mu_1, \mu_2) = \mathscr{B}(\mu_1, \mu_2)/\mathscr{B}_s(\mu_1, \mu_2)$ induced by $\rho(\mu_1, \mu_2)$.*

3. *Let $\mu_1\mu_2^{-1}$ be of the form $t \mapsto t^p \operatorname{sgn} t$ where p is a negative integer.*
   (a) *The space $\mathscr{B}(\mu_1, \mu_2)$ contains exactly one proper subspace $\mathscr{B}_f(\mu_1, \mu_2)$ stable under $(\mathfrak{g}, K_{\mathbb{R}})$.*
   (b) *The space $\mathscr{B}_f(\mu_1, \mu_2)$ is finite dimensional and of infinite codimension.*
   (c) *We denote by $\pi(\mu_1, \mu_2)$ any representation equivalent to the restriction of $\rho(\mu_1, \mu_2)$ on $\mathscr{B}_f(\mu_1, \mu_2)$.*
   (d) *We denote by $\sigma(\mu_1, \mu_2)$ any representation equivalent to the representation on $\mathscr{B}_s(\mu_1, \mu_2) = \mathscr{B}(\mu_1, \mu_2)/\mathscr{B}_f(\mu_1, \mu_2)$ induced by $\rho(\mu_1, \mu_2)$.*

4. *The representations $\pi(\mu_1, \mu_2)$ and $\sigma(\mu_1', \mu_2')$ are never equivalent.*

5. *The representations $\pi(\mu_1, \mu_2)$ and $\pi(\mu_1', \mu_2')$ are equivalent if and only if $\{\mu_1, \mu_2\} = \{\mu_1', \mu_2'\}$.*

6. *The representations $\sigma(\mu_1, \mu_2)$ and $\sigma(\mu_1', \mu_2')$ are equivalent if and only if $\{\mu_1, \mu_2\}$ is either $\{\mu_1', \mu_2'\}$ or $\{\eta\mu_1', \eta\mu_2'\}$ (where η is the nontrivial quadratic character of $\mathbb{R}^\times$).*

51. The representations $\pi(\mu_1, \mu_2)$ are called the *principal series representations*; they are defined for all μ_1 and μ_2. The representations $\sigma(\mu_1, \mu_2)$ are called the *special representations*; they are defined only for certain μ_1 and μ_2.

52 Proof of theorem 50. 1, 2, 3) Let $\mu_1\mu_2^{-1} = (\operatorname{sgn} t)^m|t|^s$; then $s - m$ is an odd integer if and only if s is an integer p and $(\mu_1\mu_2^{-1})(t) = t^p \operatorname{sgn} t$. Thus the first three statements of the theorem follow easily from proposition 49 of the previous section, together with general facts about how representations of $(\mathfrak{g}, K_{\mathbb{R}})$ compare with representations of $\mathfrak{g}$ (*i.e.*, propositions 42, 43 and 44).

4) In the representations $\pi(\mu_1, \mu_2)$ which are infinite dimensional, each integer apeears as an eigenvalue of the operator $\rho(U)$ (*cf.* proposition 48). However, in the representations $\sigma(\mu_1', \mu_2')$ there are integers which do not appear. Thus the two representations can never be equivalent.

5, 6) We now prove the equivalences between various representations by constructing an operator $T : \mathscr{B}(\mu_1, \mu_2) \to \mathscr{B}(\mu_2, \mu_1)$ which commutes with the action of $(\mathfrak{g}, K_{\mathbb{R}})$ (and thus with the action of $\mathscr{H}_{\mathbb{R}}$ as well). We let ϕ_n denote the basis for $\mathscr{B}(\mu_1, \mu_2)$ we have previously mentioned and we let ϕ_n' denote the corresponding basis of $\mathscr{B}(\mu_2, \mu_1)$.

We first assume $s - m$ is not an odd integer (where $\mu_1\mu_2^{-1})(t) = (\operatorname{sgn} t)^m t^s$). Since ϕ_n and ϕ_n' are the unique eigenvectors of $\rho(U)$ of eigenvalue in it follows that T must take ϕ_n to $a_n\phi_n'$ for some constant a_n. In fact, by proposition 48 we see that T commutes with $(\mathfrak{g}, K_{\mathbb{R}})$ if and only if

$$(s + 1 - n)a_{n+2} = (-s + 1 + n)a_n, \qquad (s + 1 - n)a_{n-2} = (-s + 1 - n)a_n, \qquad a_n = (-1)^m a_{-n}.$$

These relations are satisfied if we define

$$a_n = a_n(s) = \frac{\Gamma(\frac{1}{2}(-s + 1 + n))}{\Gamma(\frac{1}{2}(s + 1 + n))}.$$

Since n has the same parity of m and $s - m - 1$ is not an even integer, the arguments stay away from the poles of the gamme function and all quantites are defined and nonzero. Thus T defines an equivalence between $\pi(\mu_1, \mu_2)$ and $\pi(\mu_2, \mu_1)$.

If $s \le 0$ and $s - m$ is an odd integer we let

$$a_n = a_n(s) = \lim_{z \to s} a_n(z).$$

The numbers a_n are defined and finite although some are now zero. The associated operator of T is still a map of Harish-Chandra modules. If $s = 0$ the map T is a bijection. If $s < 0$ then the kernel of T is $\mathscr{B}_f(\mu_1, \mu_2)$ and T induces a bijection between $\mathscr{B}_s(\mu_1, \mu_2)$ and $\mathscr{B}_s(\mu_2, \mu_1)$.

If $s > 0$ and $s - m$ is an odd integer then functions $a_n(z)$ have at most simple poles; we let

$$b_n = b_n(s) = \lim_{z \to s}(z - s)a_n(z).$$

The sequence b_n still satisfies the necessary conditions so that the associated operator T commutes with the action of $(\mathfrak{g}, K_{\mathbb{R}})$. Its kernel is $\mathscr{B}_s(\mu_1, \mu_2)$ so that it defines an equivalence between $\mathscr{B}_f(\mu_1, \mu_2)$ and $\mathscr{B}_f(\mu_2, \mu_1)$.

Finally proposition 44 shows that $\sigma(\mu_1, \mu_2)$ is equivalent to $\sigma(\eta\mu_1, \eta\mu_2)$. Thus all the stated equivalences have been established.

Now assume $\pi = \pi(\mu_1, \mu_2)$ and $\pi' = \pi(\mu_1', \mu_2')$ or $\pi = \sigma(\mu_1, \mu_2)$ and $\pi' = \sigma(\mu_1', \mu_2')$ are equivalent. Let

$$\mu_i(t) = (\operatorname{sgn} t)^{m_i}|t|^{s_i}, \qquad \mu_i'(t) = (\operatorname{sgn} t)^{m_i'}|t|^{s_i'}$$

and

$$s = s_1 - s_2, \quad m = |m_1 - m_2|, \qquad s' = s_1' - s_2', \quad m' = |m_1' - m_2'|.$$

By examining the actions of $\pi(\epsilon)$ and $\pi'(\epsilon)$ (*cf.* proposition 48) we see that $m = m'$. Similarly, looking at $\pi(D)$ and $\pi'(D)$ shows that $s = \pm s'$. Similarly, looking at $\pi(J)$ and $\pi'(J)$ we conclude $s_1 + s_2 = s_1' + s_2'$. Thus we see that $\{\mu_1, \mu_2\}$ must be either $\{\mu_1', \mu_2'\}$ or $\{\eta\mu_1', \eta\mu_2'\}$. Proposition 43 shows that $\pi(\mu_1, \mu_2)$ is not equivalent to $\pi(\eta\mu_1, \eta\mu_2)$. This proves all the stated inequivalences and finishes the proof of the theorem.

6.2.4 Classification of irreducible representations

53 Theorem (J-L Thm. 5.11; Bump Thm. 2.5.5). *Every irreducible admissible representation of* $(\mathfrak{g}, K_{\mathbb{R}})$ *or* $\mathscr{H}_{\mathbb{R}}$ *is equivalent to one of the representations* $\pi(\mu_1, \mu_2)$ *or* $\sigma(\mu_1, \mu_2)$*.*

54 Corollary. *The process of taking the associated Harish-Chandra module gives a bijective correspondence between irreducible admissible representations of* $\mathscr{H}_{\mathbb{R}}$ *and irreducible admissible representations of* $(\mathfrak{g}, K_{\mathbb{R}})$*.*

The only part of this statement that we have not yet established by prior results is that every irreducible admissible Harish-Chandra module comes from an irreducible admissible representation of $\mathscr{H}_{\mathbb{R}}$. Theorem 53 implies this.

55 Proof of theorem 53. Let (π, V) be an irreducible admissible representation of $(\mathfrak{g}, K_{\mathbb{R}})$. By the definition of admissible the space V breaks up into a direct sum of spaces V_n where V_n is the in eigenspace of $\rho(U)$ and n is an integer. From the relations

$$[U, V_+] = 2iV_+, \qquad [U, V_-] = -2iV_-, \qquad (\operatorname{Ad}\epsilon)U = -U$$

it follows that $\rho(V_+)$ takes V_n into V_{n+2}, $\rho(V_-)$ takes V_n into V_{n-2} and $\rho(\epsilon)$ takes V_n into V_{-n}.

An easy calculation shows that

$$D = -\tfrac{1}{2}U^2 + \tfrac{1}{4}(V_+V_- + V_-V_+), \qquad (V_+V_- - V_-V_+) = -4iU$$

from which we conclude

$$V_+V_- = 2D + U^2 - 2iU, \qquad V_-V_+ = 2D + U^2 + 2iU.$$

Since D is in the center of $\mathfrak{U}$ it acts as a scalar under ρ; call this scalar α. Since U acts as a scalar on the spaces V_n, it follows that V_+V_- and V_-V_+ both act as scalars on V_n. Note that we have

$$(\operatorname{Ad}\epsilon)V_+ = V_-, \qquad (\operatorname{Ad}\epsilon)V_- = V_+.$$

Let v be in V_n. Let V' be the smallest space containing v and stable under the actions of V_+, V_-, U and ϵ. From the above considerations it follows that $V' \cap V_n = \mathbb{C}v$. However, since V is irreducible we have $V = V'$. Thus all the spaces V_n are one dimensional.

We now show that we can pick a basis v_n of V such that v_n spans V_n and

$$\rho(V_+)v_n = (s+1+n)v_{n+2}, \qquad \rho(V_-)v_n = (s+1-n)v_{n-2} \tag{1}$$

where $\frac{1}{2}(s^2-1) = \alpha$ (note that s is determined only up to sign). To see this we examine the representation of $\mathfrak{g}$ on V; according to proposition 42 there are two cases: V remains irreducible, or V is a direct sum $V_1 \oplus V_2$ of irreducible representations.

We consider the case where V remains irreducible first. Pick some n_0 such that V_{n_0} is nonzero and pick some nonzero v_{n_0} in V_{n_0}. If v_n is defined, simply define v_{n+2} and v_{n-2} by

$$v_{n+2} = \frac{1}{s+1+n}\rho(V_+)v_n, \qquad v_{n-2} = \frac{1}{s+1-n}\rho(V_-)v_n$$

(so long as the denominators are nonzero). Note that

$$\begin{aligned}\rho(V_-)v_{n+2} =& \frac{1}{s+1+n}\rho(V_-)\rho(V_+)v_n = \frac{1}{s+1+n}(2D + U^2 + 2iU)v_n \\ =& \frac{s^2 - 1 - n^2 - 2n}{s+1+n}v_n = (s-1-n)v_n\end{aligned}$$

so that (1) is indeed satisfied. The v_n span a subspace of V stable under $\mathfrak{g}$, and thus form a basis for V.

Now consider the case where V splits as $V_1 \oplus V_2$. Again, pick some n_0 and v_{n_0} (say in V_1) and carry out the process in the previous paragraph. This will yield a basis for V_1. Now define $v_{-n_0} = \rho(\epsilon)v_{n_0}$ and repeat the process, yielding a basis for V_2. We thus have obtain a basis satisfying (53).

We now examine how $\rho(\epsilon)$ acts. Since $\rho(\epsilon)$ takes V_n to V_{-n} we can write $\rho(\epsilon)v_n = a(n)v_n$ for some constants $a(n)$. We clearly have $a(n)a(-n) = 1$. Since $(\operatorname{Ad}\epsilon)V_+ = V_-$, we obtain

$$a(n+2)(s+1+n)v_{-n-2} = \rho(\epsilon)\rho(V_+)v_n = \rho(V_-)\rho(\epsilon)v_n = a(n)(s+1+n)v_{-n-2}.$$

Thus (so long as V_{n+2} is nonzero) we find $a(n) = a(n+2)$. If the action of $\mathfrak{g}$ on V is irreducible then $a(n)$ must be equal some constant a and $a^2 = a(n)a(-n) = 1$; thus $a = \pm 1$. If under the action of $\mathfrak{g}$ the space V breaks up into $V_1 \oplus V_2$, we find that $a(n)$ is constant on V_1 and V_2. However, by definition $\rho(\epsilon)v_{n_0} = v_{-n_0}$ and therefore $a(n_0) = a(-n_0) = 1$. It thus follows that $a(n) = 1$ for all n. We have thus shown that $\rho(\epsilon)v_n = \pm v_{-n}$ where the sign is independent of n.

Finally note that since J is in the center of $\mathscr{U}$ it acts as a constant. Comparing our results so far with proposition 48, we see that π is a subrepresentation of $\rho(\mu_1, \mu_2)$ for some μ_1 and μ_2. It now follows from theorem 50 that π is equivalent to $\pi(\mu_1, \mu_2)$ or $\sigma(\mu_1, \mu_2)$.

6.3 The Whittaker model

6.3.1 The Whittaker model: overview

56. Let ψ be a nontrivial additive character of $\mathbb{R}$. Let $\mathscr{W}(\psi)$ be the space of complex valued functions W on $G_{\mathbb{R}}$ which satisfy the following conditions:

1. We have
$$W\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) = \psi(x)W(g)$$
for all x in $\mathbb{R}$ and all g in $G_{\mathbb{R}}$.

2. W is smooth and K-finite on the right.

3. For all T in $\mathscr{H}_{\mathbb{R}}$ there exists a positive real number N such that
$$(\rho(T)W)\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} = O(|t|^N)$$
as $|t| \to \infty$.

Clearly $\mathscr{H}_{\mathbb{R}}$ acts on the space $\mathscr{W}(\psi)$ via the right regular representation ρ and so $\mathscr{W}(\psi)$ may be regarded as either an $\mathscr{H}_{\mathbb{R}}$-module or as a $(\mathfrak{g}, K_{\mathbb{R}})$-module.

57. We say a function W on $G_{\mathbb{R}}$ is *rapidly decreasing* if for all g in $G_{\mathbb{R}}$ and all positive real numbers N we have
$$|t|^N W\left(\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} g\right) \to 0$$
as $|t| \to \infty$.

58. Let (π, V) be an admissible representation of $\mathscr{H}_{\mathbb{R}}$. A *Whittaker model* of π is a $\mathscr{H}_{\mathbb{R}}$-submodule of $\mathscr{W}(\psi)$ which is isomorphic to π as $\mathscr{H}_{\mathbb{R}}$-modules.

59 Theorem (J-L Thm 5.13). *Let (π, V) be an infinite dimensional irreducible admissible representation of $\mathscr{H}_{\mathbb{R}}$. Then π admits a unique Whittaker model. Furthermore, all members of the Whittaker model are rapidly decreasing and analytic.*

60. The proof of theorem 59 will consume this section. The existence proof is handled separately for the special representations and the principal series representations. In both proofs, we first establish an isomorphism of the representation in question with a certain Weil representation (for special representations it will be a Weil representation corresponding to the extension $\mathbb{C}$ of $\mathbb{R}$ while for the principal series representations it will be a Weil representation corresponding to the algebra $\mathbb{R} \oplus \mathbb{R}$). We then use the functions in the space of the Weil representation to obtain a space of functions on $G_{\mathbb{R}}$.

61. Let $\mathscr{W}'(\psi)$ be the space of all smooth functions W on $G_{\mathbb{R}}$ which satisfy

$$W\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) = \psi(x)W(g)$$

for all x in $\mathbb{R}$ and all g in $G_{\mathbb{R}}$. It is clear that $\mathscr{W}'(\psi)$ is a $(\mathfrak{g}, K_{\mathbb{R}})$-module (under ρ) which contains $\mathscr{W}(\psi)$ as a submodule.

We are sometimes in the position that we have a map $V \to \mathscr{W}'(\psi)$ of $\mathscr{H}_{\mathbb{R}}$-modules and we would like to know that the image belongs to $\mathscr{W}(\psi)$. This would involve checking $K_{\mathbb{R}}$-finiteness and the growth condition. The following simple lemma, whose proof is clear, eases the burden. We will use it implicitly in the sequel.

62 Lemma. *Let (π, V) be an admissible $\mathscr{H}_{\mathbb{R}}$-module. Let $f : V \to \mathscr{W}'(\psi)$ be a map of $\mathscr{H}_{\mathbb{R}}$-modules such that for all W in the image of f there exists N such that*

$$W\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} = O(|t|^N)$$

as $|t| \to \infty$. Then the image of f is contained in $\mathscr{W}(\psi)$.

6.3.2 The Weil representations for $\mathbb{C}/\mathbb{R}$

63. ADD REFERENCE TO SECTION 2.

64. We quickly review the Weil representations associated to the seperable quadratic extension $\mathbb{C}$ of $\mathbb{R}$.

65. Let ω be a quasi-character of $\mathbb{C}^\times$. The Weil representation r_ω is a representation of the subgroup G_+ of $G_{\mathbb{R}}$, consisting of those matrices with positive determinant, on the space $\mathscr{S}(\mathbb{C}, \omega)$, which is the subspace of $\mathscr{S}(\mathbb{C})$ consisting of those functions Φ for which

$$\Phi(xh) = \omega^{-1}(h)\Phi(x)$$

holds for all h of modulus 1.

66. As always, ψ is a fixed nontrivial additive character of $\mathbb{R}$ and $\psi_{\mathbb{C}}$ is its complexification. We let u be the real number such that $\psi(x) = e^{2\pi i\alpha x}$ and we let γ be the constant $i \operatorname{sgn} \alpha$. The self dual Haar measure on $\mathbb{R}$ with repsect to ψ is $|\alpha|^{1/2}dx$; the self dual Haar measure on $\mathbb{C}$ with respect to $\psi_{\mathbb{C}}$ is $|\alpha|dxdy$.

67. For Φ in $\mathscr{S}(\mathbb{C}, \omega)$ we have the following:

1. $\left(r_\omega \begin{bmatrix} a^2 & 0 \\ 0 & 1 \end{bmatrix} \Phi\right)(x) = \omega(a)|a|\Phi(ax)$ for all a in $R^\times$ (note that this is consistent).
2. $\left(r_\omega \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \Phi\right)(x) = \operatorname{sgn}(a)|a|\Phi(ax)$ for all a in $R^\times$.
3. $\left(r_\omega \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \Phi\right)(x) = \psi(z|x|^2)\Phi(x)$ for all z in $\mathbb{R}$.
4. $\left(r_\omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Phi\right)(x) = \gamma\Phi'(\bar{x})$ where Φ' is the Fourier transform of Φ with respect to the character $\psi_{\mathbb{C}}$.

6.3.3 The representation $\pi(\omega)$

68. As in the previous section, ω denotes a quasi-character of $\mathbb{C}^\times$. We write

$$\omega(z) = (z\bar{z})^r \frac{z^m \bar{z}^n}{(z\bar{z})^{\frac{1}{2}(m+n)}}$$

where r is a complex number and m and n are nonnegative integers, one of which is zero.

69. Note that since the elements of $\mathscr{S}(\mathbb{C},\omega)$ are smooth, the representation r_ω of G_+ may be differentiated to obtain a representation of $\mathfrak{g}$ on $\mathscr{S}(\mathbb{C},\omega)$ (which we still denote by r_ω). The next proposition describes this action.

70 Proposition. *Let Φ be an element of $\mathscr{S}(\mathbb{C},\omega)$.*

1. $(r_\omega(X_+)\Phi)(z) = 2\pi i\alpha|z|^2\Phi(z)$.
2. $r_\omega(X_-)\Phi = -\dfrac{1}{2\pi i\alpha}\dfrac{\partial}{\partial z}\dfrac{\partial}{\partial \bar{z}}\Phi$.
3. $r_\omega(J)\Phi = (2r-n-m)\Phi$.
4. $(r_\omega(Z)\Phi)(z) = \Phi(z) + z\dfrac{\partial}{\partial z}\Phi(z) + \bar{z}\dfrac{\partial}{\partial \bar{z}}\Phi(z)$.

1) By defintion,

$$(r_\omega(X_+)\Phi)(z) = \lim_{t\to 0}\frac{1}{t}\left(r_\omega\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\Phi - \Phi\right) = \Phi(z)\lim_{t\to 0}\frac{e^{2\pi i\alpha t|z|^2}-1}{t} = 2\pi i\alpha|z|^2\Phi(z)$$

and the first statement is proved.

2) We have $X_- = -(\operatorname{Ad} w)X_+$ where, as always,

$$w = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

Since $w^{-1} = -w$ we have

$$\begin{aligned} r_\omega(X_-)\Phi &= -\omega(-1)r_\omega(w)r_\omega(X_+)r_\omega(w)\Phi \\ &= -\gamma\omega(-1)r_\omega(w)r_\omega(X_+)\Phi_1 && \Phi_1(x+iy) = \Phi'(x-iy) \\ &= -2\pi i\alpha\gamma\omega(-1)r_\omega(w)\Phi_2 && \Phi_2(x+iy) = (x^2+y^2)\Phi_1(x+iy) \\ &= -2\pi i\alpha\gamma^2\omega(-1)\Phi_3 && \Phi_3(x+iy) = \Phi_2'(x-iy) \end{aligned}$$

where throughout we use a prime to denote Fourier transform. Now, $\Phi_1'(x+iy) = -\Phi(-x+iy)$ and so (by standard properties of the Fourier transform)

$$\Phi_2'(x+iy) = -\frac{1}{(2\pi\alpha)^2}(\Delta\Phi_1')(x+iy) = \frac{1}{(2\pi\alpha)^2}(\Delta\Phi)(-x+iy)$$

where

$$\Delta = \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}.$$

We now have

$$\Phi_3(x+iy) = \frac{1}{(2\pi\alpha)^2}(\Delta\Phi)(-x-iy) = \frac{\omega(-1)}{(2\pi\alpha)^2}(\Delta\Phi)(x+iy)$$

and so, since $\gamma^2 = -1$, we have

$$(r_\omega(X_-)\Phi)(x) = -\frac{1}{2\pi i\alpha}(\Delta\Phi)(x).$$

3) By definition,

$$(r_\omega(J)\Phi)(z) = \lim_{t\to 0}\frac{1}{t}\left(r_\omega\begin{bmatrix}e^t & 0\\ 0 & e^t\end{bmatrix}\Phi - \Phi\right)(z) = \Phi(z)\lim_{t\to 0}\frac{e^{(2r-n-m)t}-1}{t} = (2r-n-m)\Phi(z)$$

and the third statement is proved.

4) By definition,

$$(r_\omega(Z)\Phi)(z) = \lim_{t\to 0}\frac{1}{t}\left(r_\omega\begin{bmatrix}e^t & 0\\ 0 & e^{-t}\end{bmatrix}\Phi - \Phi\right)(z) = \lim_{t\to 0}\frac{e^t\Phi(e^tz)-\Phi(z)}{t} = \frac{d}{dt}e^t\Phi(e^tz)\Big|_{t=0}$$

and the result follows from rules of differentiation.

71 Proposition (J-L Lemma 5.12). *Let $\mathscr{S}_0(\mathbb{C}, \omega)$ be the space of functions Φ in $\mathscr{S}(\mathbb{C}, \omega)$ of the form*

$$\Phi(z) = e^{-2\pi|\alpha|z\bar{z}} P(z, \bar{z})$$

where P is a bivariate polynomial.

1. *The space $\mathscr{S}_0(\mathbb{C}, \omega)$ is stable under the action of $\mathfrak{g}$. Let π_1 be the representation of $\mathfrak{g}$ on $\mathscr{S}_0(\mathbb{C}, \omega)$.*
2. *π_1 is admissible.*
3. *π_1 is irreducible.*
4. *π_1 is not equivalent to $(\operatorname{Ad} \epsilon)\pi_1$.*

1) It is clear from proposition 70 that $\mathscr{S}_0(\mathbb{C}, \omega)$ is stable under X_+, X_- and J; since these generate $\mathfrak{g}$ as a Lie algebra the first statement follows.

2) The functions

$$\Phi_p(z) = e^{-2\pi|\alpha|z\bar{z}} z^{n+p} \bar{z}^{m+p}$$

with p a nonnegative integer form a basis for $\mathscr{S}_0(\mathbb{C}, \omega)$. Applying proposition 70 yields

$$\begin{aligned}
r_\omega(X_+)\Phi_p &= (2\pi i\alpha)\Phi_{p+1} \\
r_\omega(X_-)\Phi_p &= (2\pi i\alpha)\Phi_{p+1} - (i \operatorname{sgn} \alpha)(2p + n + m + 1)\Phi_p - \frac{(n+p)(m+p)}{2\pi i\alpha}\Phi_{p-1} \\
r_\omega(J)\Phi_p &= (2r - n - m)\Phi_p \\
r_\omega(Z)\Phi_p &= (2p + n + m + 1)\Phi_p - 4\pi|\alpha|\Phi_{p+1}.
\end{aligned}$$

Since $U = X_+ - X_-$ we thus have

$$r_\omega(U)\Phi_p = (i \operatorname{sgn} \alpha)(2p + n + m + 1)\Phi_p + \frac{(n+p)(m+p)}{2\pi i\alpha}\Phi_{p-1}.$$

Therefore, we can pick a basis Ψ_p of the form

$$\Psi_p = \Phi_p + \sum_{q=0}^{p-1} a_{pq}\Phi_p$$

such that

$$r_\omega(U)\Psi_p = (i \operatorname{sgn} \alpha)(2p + n + m + 1)\Psi_p$$

and it follows that the representation is admissible.

3) Note that

$$r_\omega(Z - (2i \operatorname{sgn} \alpha)X_+)\Phi_p = (2p + n + m + 1)\Phi_p.$$

Thus if $\Phi = \sum_p a_p\Phi_p$ is any function in $\mathscr{S}_0(\mathbb{C}, \omega)$ it follows that $a_p\Phi_p$ lies in the smallest $\mathfrak{g}$-stable subspace containing Φ. Thus any stable subspace of $\mathscr{S}_0(\mathbb{C}, \omega)$ is spanned by the Φ_p it contains. From the equations which give the action of $\mathfrak{g}$ on Φ_p it therefore follows that $\mathscr{S}_0(\mathbb{C}, \omega)$ is irreducible.

4) Note that the eigenvalues of U under π_1 are $(i \operatorname{sgn} \alpha)(2p + n + m + 1)$ with n, m and p positive, while the eigenvalues of U under $(\operatorname{Ad} \epsilon)\pi_1$ are $-(i \operatorname{sgn} \alpha)(2p + n + m + 1)$ with the same restrictions on n, m and p. It is thsu clear that the two representations are inequivalent.

72. It follows from proposition 71 and proposition 44 that there is a unique irreducible admissible representation of $(\mathfrak{g}, K_\mathbb{R})$ whose restriction to $\mathfrak{g}$ is a direct sum of the representations π_1 and $(\operatorname{Ad} \epsilon)\pi_1$. We denote this representation by $\pi(\omega)$.

73 Proposition. *We have the following:*

1. *If both n and m are zero then $\pi(\omega)$ is equivalent to $\pi(\mu_1, \eta\mu_1)$ with $s_1 = r$.*

2. *If either n or m is nonzero then $\pi(\omega)$ is equivalent to $\sigma(\mu_1, \mu_2)$ with $s_1 = r + \frac{1}{2}(n+m)$ and $s_2 = r - \frac{1}{2}(n+m)$.*

3. *The representations $\pi(\omega)$ and $\pi(\omega')$ are equivalent if and only if ω and ω' are conjugate.*

Here, as always, we write $\mu_i(t) = (\operatorname{sgn} t)^{m_i}|t|^{s_i}$.

These statements follow at one from the detailed description of the action of $\mathfrak{g}$ given in the proof of proposition 71, together with what we know about the representations $\pi(\mu_1, \eta\mu_1)$ and $\sigma(\mu_1, \mu_2)$.

6.3.4 The Whittaker model for $\pi(\omega)$

74. For a function Φ in $\mathscr{S}(\mathbb{C}, \omega)$ let W_Φ be the function on $G_\mathbb{R}$ given by

$$W_\Phi(g) = \begin{cases} (r_\omega(g)\Phi)(1) & g \in G_+ \\ 0 & g \notin G_+ \end{cases}$$

Let $\mathscr{W}_1(\omega, \psi)$ be the space of all the W_Φ for Φ in $\mathscr{S}_0(\mathbb{C}, \omega)$. Define

$$\mathscr{W}(\omega, \psi) = \mathscr{W}_1(\omega, \psi) + \rho(\epsilon)\mathscr{W}_1(\omega, \psi).$$

75 Proposition. *The space $\mathscr{W}(\omega, \psi)$ is a Whittaker model for $\pi(\omega)$.*

This is fairly clear; we make some comments.

1) The map from the space of $\pi(\omega)$ to $\mathscr{W}(\omega, \psi)$ is given by

$$\Phi \mapsto W_\Phi, \qquad \pi(\epsilon)\Phi \mapsto \rho(\epsilon)W_\Phi$$

where Φ lies in $\mathscr{S}_0(\mathbb{C}, \omega)$. This is clearly a map of $(\mathfrak{g}, K_\mathbb{R})$-modules.

2) The space $\mathscr{W}(\omega, \psi)$ is stable under the action of $\mathscr{H}_\mathbb{R}$; for this it is enough to show that $\mathscr{S}_0(\mathbb{C}, \omega)$ is stable under $\mathscr{H}_\mathbb{R}$. If ϕ is any function in $\mathscr{S}(\mathbb{C}, \omega)$ and f is in $\mathscr{H}_\mathbb{R}$ then $\pi(f)\phi$ is $K_\mathbb{R}$-finite. Since $\mathscr{S}_0(\mathbb{C}, \omega)$ is precisely the space of $K_\mathbb{R}$-finite vectors in $\mathscr{S}(\mathbb{C}, \omega)$, the statement follows.

3) For any Φ in $\mathscr{S}(\mathbb{C}, \omega)$ we have

$$W_\Phi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = \begin{cases} \omega(a^{1/2})a^{1/2}\Phi(a^{1/2}) & a > 0 \\ 0 & a < 0 \end{cases}$$

Thus the functions in $\mathscr{W}(\omega, \psi)$ satisfy the necessary growth restrictions.

76. Note that since any $\sigma(\mu_1, \mu_2)$ is a $\pi(\omega)$ for some ω, proposition 75 establishes the existence of a Whittaker model for the special representations.

6.3.5 The Weil representation for $\mathbb{R} \oplus \mathbb{R}$

77. ADD REFERENCE TO SECTION 2.

78. We quickly recall the Weil representations associated to the algebra $\mathbb{R} \oplus \mathbb{R}$ over $\mathbb{R}$.

79. The Weil representation is a representation r of $G_\mathbb{R}$ on the space $\mathscr{S}(\mathbb{R}^2)$. It can be described briefly by

$$(r(g)\Phi)^\sim = \rho(g)\Phi^\sim$$

where $(\rho(g)\Phi)(x) = \Phi(xg)$ and $G_\mathbb{R}$ acts on $\mathbb{R}^2$ by matrix multiplication; $\Phi^\sim$ is the partial Fourier transform of Φ given by

$$\Phi^\sim(a, b) = \int_\mathbb{R} \Phi(a, y)\psi(by)dy.$$

80. We have the following:

1. $\left(r\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\Phi\right)(x,y) = \Phi(ax,y)$ for a in $\mathbb{R}^\times$.

2. $\left(r\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\Phi\right)(x,y) = |a|\Phi(ax,ay)$ for a in $\mathbb{R}^\times$.

3. $\left(r\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\Phi\right)(x,y) = \psi(zxy)\Phi(x,y)$ for z in $\mathbb{R}$.

4. $\left(r\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\Phi\right)(x,y) = \int_{\mathbb{R}^2}\Phi(u,v)\psi(vx+uy)dudv.$

81. If μ_1 and μ_2 are quasi-characters of $\mathbb{R}^\times$ we define a representation r_{μ_1,μ_2}, also on the space $\mathscr{S}(\mathbb{R}^2)$ by

$$r_{\mu_1,\mu_2}(g) = \mu_1(\det g)|\det g|^{1/2}r(g).$$

6.3.6 The Whittaker model for $\rho(\mu_1,\mu_2)$

82. In this section we construct the Whittaker model for $\rho(\mu_1,\mu_2)$. This construction is almost identitical to the analagous construction in the non-archimedean case.

83. For Φ in $\mathscr{S}(\mathbb{R}^2)$ define

$$\theta(\mu_1,\mu_2;\Phi) = \int_{\mathbb{R}^\times}\mu_1(t)\mu_2^{-1}(t)\Phi(t,t^{-1})d^\times t.$$

Define an element W_Φ of $C^\infty(G_\mathbb{R})$ by

$$W_\Phi(g) = \theta(\mu_1,\mu_2;r_{\mu_1,\mu_2}(g)\Phi).$$

Let $\mathscr{W}(\mu_1,\mu_2;\psi)$ be the space of all W_Φ corresponding to $K_\mathbb{R}$-finite functions Φ.

84 Lemma. *We have the following*

1. $W_{r_{\mu_1,\mu_2}(g)\Phi} = \rho(g)W_\Phi$.
2. *If Φ is $K_\mathbb{R}$-finite then the function W_Φ belongs to $\mathscr{W}(\psi)$.*

The only nontrivial point is to establish the growth conditions on W_Φ in part 2.

The $K_\mathbb{R}$-finite functions in $\mathscr{S}(\mathbb{R}^2)$ are linear combinations of functions of the form $f(x^2+y^2)x^n$ and $f(x^2+y^2)y^n$ where f is a rapidly decreasing function and n is a nonnegative integer. By symmetry, it thus suffices to establish the growth condition when

$$\Phi(x,y) = f(x^2+y^2)x^n$$

where f is rapidly decreasing.

To ease notation, we will be sloppy and write t^p for any power function of t and drop constants. We have

$$W_\Phi\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = a^p\int_{\mathbb{R}^\times}t^pf(a^2t^2+t^{-2})dt.$$

We need only consider the positive half of the integral, the other half can be handled in the same way. We have

$$\begin{aligned}
\int_0^\infty t^pf(a^2t^2+t^{-2})dt &= \int_0^\infty t^pf(a^2t+t^{-1})dt \\
&= a^p\int_0^\infty t^pf(a(t+t^{-1}))dt \\
&= a^p\int_0^1 t^pf(a(t+t^{-1}))dt + a^p\int_1^\infty t^pf(a(t+t^{-1}))dt \\
&\le Ca^p\int_0^\infty u^pf(au)du \\
&\le Ca^p
\end{aligned}$$

In the second to last step we made the change of variables $u = t+t^{-1}$ and applied some simple inequalites. This proves the requisite condition.

85. If ω is a quasi-character of $\mathbb{R}^\times$ and $\omega(t) = (\operatorname{sgn} t)^m |t|^s$ with $s > 0$ then the integral

$$z(\omega, \Phi) = \int_{\mathbb{R}^\times} \Phi(0,t)\omega(t)d^\times t$$

is defined for any element Φ of $\mathscr{S}(\mathbb{R}^2)$. Thus if μ_1 and μ_2 are quasi-characters such that $(\mu_1\mu_2^{-1})(t) = (\operatorname{sgn} t)^m |t|^s$ with $s > -1$ then we can define

$$f_\Phi(g) = \mu_1(\det g)|\det g|^{1/2} z(\mu_1\mu_2^{-1}\alpha_{\mathbb{R}}, \rho(g)\Phi)$$

where $\alpha_{\mathbb{R}}$ is the quasi-character of $\mathbb{R}^\times$ given by $t \mapsto |t|$.

86 Lemma. *Assume $(\mu_1\mu_2^{-1})(t) = (\operatorname{sgn} t)^m |t|^s$ with $s > -1$.*

1. *$\rho(g) f_\Phi = f_{\mu_1(\det g)|\det g|^{1/2}\rho(g)\Phi}$.*
2. *If Φ is $K_{\mathbb{R}}$-finite then f_Φ belongs to $\mathscr{B}(\mu_1, \mu_2)$.*

These again are easy calculations left to the reader.

87 Lemma. *Assume $(\mu_1\mu_2^{-1})(t) = (\operatorname{sgn} t)^m |t|^s$ with $s > -1$. For all Φ in $\mathscr{S}(\mathbb{R}^2)$ the function q on $\mathbb{R}^\times$ given by*

$$q(a) = \mu_2^{-1}(a)|a|^{-1/2} W_\Phi \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

is integrable with respect to the additive Haar measure on $\mathbb{R}$ and

$$\int_{\mathbb{R}^\times} q(a)\psi(ax)da = f_{\Phi^\sim}(-wn_x).$$

The proof is exactly the same as the proof of §5.6.4, lemma 119 and not worth repeating.

88 Proposition (J-L Lemma 5.13.1). *Assume $(\mu_1\mu_2^{-1})(t) = (\operatorname{sgn} t)^m |t|^s$ with $s > -1$.*

1. *There is a map $A : \mathscr{W}(\mu_1, \mu_2; \psi) \to \mathscr{B}(\mu_1, \mu_2)$ which sends W_Φ to $f_{\Phi^\sim}$ (where Φ is $K_{\mathbb{R}}$)-finite).*
2. *The map A is an isomorphism of $\mathscr{H}_{\mathbb{R}}$-modules.*
3. *$\mathscr{W}(\mu_1, \mu_2; \psi)$ is the Whittaker model for $\mathscr{B}(\mu_1, \mu_2)$.*

The proof is exactly the same as that of §5.6.4, proposition 120 except for the proof that A is surjective.

We must thus show that given f in $\mathscr{B}(\mu_1, \mu_2)$ there exists a $K_{\mathbb{R}}$-finite function Φ in $\mathscr{S}(\mathbb{R}^2)$ such that $f = f_{\Phi^\sim}$. In fact, if Φ is any element of $\mathscr{S}(\mathbb{R}^2)$ such that $f = f_{\Phi^\sim}$ then we can find an elementary idempotent ξ which stabilizes f and thus

$$f = \rho(\xi) f_{\Phi^\sim} = f_{\Phi_1^\sim}$$

where $\Phi_1 = r_{\mu_1,\mu_2}(\xi)\Phi$ is now $K_{\mathbb{R}}$-finite. Since $\mathscr{S}(\mathbb{R}^2)$ is self dual under the Fourier transform, it thus suffices to find a function Φ in $\mathscr{S}(\mathbb{R}^2)$ such that $f = f_\Phi$. In fact (by linearity) we need only exhibit such Φ when f is one of the basis elements ϕ_n of $\mathscr{B}(\mu_1, \mu_2)$ where n has the same parity as m (*cf.* article 47).

Consider the function

$$\Phi(x, y) = e^{-\pi(x^2+y^2)}(x + (i \operatorname{sgn} n)y)^{|n|}.$$

We have $\rho(\kappa_\theta)\Phi = e^{in\theta}\Phi$ and (since $\det \kappa_\theta = 1$) we also have

$$\rho(\kappa_\theta) f_\Phi = f_{\rho(\kappa_\theta)\Phi} = e^{in\theta} f_\Phi.$$

It thus follows that f_Φ is a multiple of ϕ_n. Since

$$f_\Phi(1) = i^{|n|} \int_{\mathbb{R}} e^{-\pi t^2} t^{|n|+s+1} d^\times t = \tfrac{1}{2} i^{|n|} \pi^{-\frac{1}{2}(|n|+s+1)} \Gamma(\tfrac{1}{2}(|n|+s+1))$$

is nonzero the function f_Φ is a nonzero multiple of ϕ_n. This completes the proof.

89 Proposition. *For any quasi-characters μ_1 and μ_2 we have*

$$\mathscr{W}(\mu_1, \mu_2; \psi) = \mathscr{W}(\mu_2, \mu_1; \psi).$$

The proof is the same as that of §5.6.4, proposition 122 and thus omitted.

90. Note that proposition 88 establishes the existence of a Whittaker model for the infinite dimensional principal series representations. We have thus completed the existence part of theorem 59.

6.3.7 Uniqueness of the Whittaker model

91 Lemma (J-L pg. 188). *Let $\mathscr{W}(\pi, \psi)$ be a Whittaker model for the irreducible admissible representation (π, V). Let κ_n be a representation of $\mathfrak{k}$ appearing in π and let W be an element of $\mathscr{W}(\pi, \psi)$ which satisfies*

$$W(g\kappa_\theta) = e^{in\theta} W(g).$$

Define a function ϕ on $\mathbb{R}^\times$ by

$$\phi(t) = W \begin{bmatrix} (\operatorname{sgn} t)|t|^{1/2} & 0 \\ 0 & |t|^{-1/2} \end{bmatrix}.$$

We say ϕ corresponds *to W.*

1. *ϕ is smooth.*
2. *ϕ determines W.*
3. *$\rho(V_+)W$ corresponds to $2t\dfrac{d\phi}{dt} - (2\alpha t - n)\phi$.*
4. *$\rho(V_-)W$ corresponds to $2t\dfrac{d\phi}{dt} + (2\alpha t - n)\phi$.*
5. *$\rho(D)W$ corresponds to $2t^2\dfrac{d^2\phi}{dt^2} + (2\alpha n t - 2\alpha^2 t^2)\phi$.*
6. *$\rho(\epsilon)W$ corresponds to $\phi(-t)$.*

1) This follows immediately since, by definition, elements of the Whittaker model are smooth.
2) We have

$$W\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (\operatorname{sgn} t)t^{1/2} & 0 \\ 0 & t^{-1/2} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \right) = \omega(a)\psi(x)e^{in\theta}\phi(t)$$

where ω is the central quasi-character of π. Since $G_\mathbb{R} = P_\mathbb{R} K_\mathbb{R}$ (the Iwasawa decomposition) it follows that W is determined by ϕ.
3, 4, 5, 6) Left to the reader.

92 Lemma. *Consider the following differential equation*

$$\frac{d^2\phi}{dt^2} + \left(-\alpha^2 + \frac{n\alpha}{t} + \frac{\lambda}{t^2} \right)\phi = 0 \tag{2}$$

where α is a nonzero constant and λ is an arbitrary constant.

1. *There exist two solutions ϕ_1 and ϕ_2 of (2) on the interval $(0, \infty)$ such that if ϕ is any solution of (2) on $\mathbb{R}^\times$ then there exist constants α_i and β_i such that*

$$\phi(t) = \begin{cases} \alpha_1\phi_1(t) + \alpha_2\phi_2(t) & t > 0 \\ \beta_1\phi_1(-t) + \beta_2\phi_2(-t) & t < 0 \end{cases}$$

2. *The functions ϕ_1 and ϕ_2 are analytic.*

3. As t tends to infinity $\phi_1(t) \sim t^{N_1} e^{ct}$ while $\phi_2(t) \sim t^{N_2} e^{-ct}$ where N_1 and N_2 are real numbers and c is a positive real number.

If we replace $\phi(t)$ by $\phi(t/2\alpha)$ then (2) takes the form

$$\frac{d^2\phi}{dt^2} + \left(-\frac{1}{4} + \frac{n}{2t} + \frac{\lambda}{t^2} \right)\phi = 0.$$

This is the *confluent hypergeometric equation* the solutions of which are called *Whittaker functions* (not to be confused with Whittaker functionals). The stated properties of the ϕ_i follow easily from the corresponding properties of Whittaker functions, developped in *e.g.* Watson and Whittaker Chapter 16.

93 Proposition (J-L pg. 189). *Let (π, V) be an infinite dimensional irreducible admissible representation of $\mathscr{H}_\mathbb{R}$. Then π admits at most one Whittaker model.*

Take π to be a $\pi(\mu_1, \mu_2)$ or a $\sigma(\mu_1, \mu_2)$ and write $(\mu_1\mu_2^{-1})(t) = (\operatorname{sgn} t)^m |t|^s$. Let $\mathscr{W}$ be the Whittaker model we have already constructed and let $\mathscr{W}'$ be another Whittaker model. We consider two cases.

Case 1: $s - m$ is an odd integer. Let $n = s + 1$. Then the representation κ_n of $\mathfrak{k}$ appears in π (or, more precisely, the restriction of the associated action of $(\mathfrak{g}, K_\mathbb{R})$ to $\mathfrak{k}$. Let W and W' belong to the nth weigth space of $\mathscr{W}$ and $\mathscr{W}'$ (*i.e.*, they satisfy $W(g\kappa_\theta) = e^{in\theta}$). Let ϕ and ϕ' be the corresponding functions on $\mathbb{R}^\times$. Now, by proposition 48 we have $\rho(V_-)W = 0$ and $\rho(V_-)W' = 0$. Thus by lemma 91 both ϕ and ϕ' satisfy the equation

$$2t\frac{d\phi}{dt} + (2\alpha t - n)\phi = 0.$$

However,

$$\phi(t) = \begin{cases} |t|^{n/2} e^{-\alpha t} & \alpha t > 0 \\ 0 & \alpha t < 0 \end{cases}$$

is the only solution (up to a constant) which satisfies the growth condition. Thus $\phi = C\phi'$ for some constant C and this proves that $\mathscr{W} = \mathscr{W}'$.

Case 2: $s - m$ is not an odd integer. Let n be some integer such that κ_n appears in π. Let W and W' be elements of the nth weight space of the two Whittaker models and let ϕ and ϕ' be the correspodning functions on $R^\times$. By proposition 48 we have $\rho(D) = \frac{1}{2}(s^2 - 1)$ and so lemma 91 implies that both ϕ and ϕ' satisfy

$$\frac{d^2\phi}{dt^2} + \left(-\alpha^2 + \frac{n\alpha}{t} + \frac{(s-1)^2}{4t^2} \right)\phi = 0. \tag{3}$$

Since ϕ and ϕ' are $O(t^N)$ for some N, it follows from lemma 92 that ϕ and ϕ' are scalar multiples of ϕ_1 on $(0, \infty)$ and scalar multiples of $t \mapsto \phi_1(-t)$ on $(-\infty, 0)$. Therefore $\phi(t) = \alpha\phi'(t)$ for $t > 0$ and $\phi(t) = \beta\phi'(t)$ for $t < 0$. If we can pick n so that $\alpha = \beta$ then the uniqueness will follow. There are now two subcases.

Subcase A: $m = 0$. Take $n = 0$. Proposition 48 gives

$$\pi(\epsilon)W = (-1)^{m_1} W$$

and similarly for W' (where m_1 is such that $\mu_1(t) = (\operatorname{sgn} t)^{m_1} |t|^{s_1}$). Thus $\phi(-t) = (-1)^{m_1}\phi(t)$ and similarly for ϕ'. Therefore $\alpha = \beta$.

Subcase B: $m = 1$. Take $n = 1$. Then by proposition 48 we have

$$\rho(V_-)W = (-1)^{m_1} s\rho(\epsilon)W$$

and similarly for W'. Thus, applying lemma 91, we see that both ϕ and ϕ' satisfy the equation

$$2t\frac{d\phi}{dt}(t) + (2\alpha t - 1)\phi(t) = (-1)^{m_1} s\phi(-t)$$

and therefore $\alpha = \beta$.

94 Proposition. *Let π be an infinite dimensional irreducible admissible representation of $\mathscr{H}_{\mathbb{R}}$. Then the members of the Whittaker model of π are rapidly decreasing and analytic.*

It suffices to prove the proposition for members W of the nth weight space of π (for all n). Let g be an arbitrary element of $G_{\mathbb{R}}$ and write

$$g = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \kappa_\theta.$$

First note that if ω is the central quasi-character of π and ϕ is the function on $R^\times$ associated to W then

$$W \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} = \omega(|t|^{1/2})\phi(t).$$

A simple computation now shows

$$W\left(\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} g \right) = \psi(tx)e^{in\theta}\omega((\operatorname{sgn} a_2)|a_1 a_2 t|^{1/2})\phi(a_1 a_2^{-1} t).$$

Since

$$(g,t) \mapsto \psi(tx)e^{in\theta}\omega((\operatorname{sgn} a_2)|a_1 a_2 t|^{1/2})$$

is an analytic function of (g,t) which is $O(t^N)$ as t tends to infinity with g fixed, it suffices to show that ϕ is analytic and rapidly decreasing. As in the proof of proposition 93, case 2, the function ϕ satisfies the differential equation of lemma 92. Thus ϕ, when restricted to either $(0,\infty)$ or $(-\infty,0)$, is a multiple of ϕ_1 (ϕ_2 is excluded for growth reasons) and therefore analytic and rapidly decreasing.

6.3.8 Comparison of $\mathscr{W}(\omega;\psi)$ and $\mathscr{W}(\mu_1,\mu_2;\psi)$

95. We have defined two Whittaker spaces: the space $\mathscr{W}(\omega;\psi)$ associated to a quasi-character ω of $\mathbb{C}^\times$ and the space $\mathscr{W}(\mu_1,\mu_2;\psi)$ associated to a pair of quasi-characters of $\mathbb{R}^\times$. There is some overlap, however. If, for instance, $\pi(\omega) = \pi(\mu_1, \eta\mu_1)$ then the two spaces must be equal. If $\pi(\omega)$ is a special representation, then $\mathscr{W}(\omega;\psi)$ must appear as a constiuent of $\mathscr{W}(\mu_1,\mu_2;\psi)$. The following proposition gives more detail.

96 Proposition (J-L Cor. 5.14). *Let ω be a quasi-character of $\mathbb{C}^\times$ given by*

$$\omega(z) = (z\bar{z})^r \frac{z^n \bar{z}^m}{(z\bar{z})^{\frac{1}{2}(n+m)}}$$

where one of m and n is positive and the other is zero. Let μ_1 and μ_2 be quasi-characters of $\mathbb{R}^\times$ such that

$$(\mu_1\mu_2)(t) = |t|^{2r}(\operatorname{sgn} t)^{m+n+1}, \qquad (\mu_1\mu_2^{-1})(t) = t^{m+n}\operatorname{sgn} t.$$

1. *We have $\pi(\omega) = \sigma(\mu_1,\mu_2)$.*
2. *We have a diagram*

$$\begin{array}{ccc} W(\omega;\psi) & \longrightarrow & W(\mu_1,\mu_2;\psi) \\ \uparrow & & \uparrow \\ \mathscr{B}_s(\mu_1,\mu_2) & \longrightarrow & \mathscr{B}(\mu_1,\mu_2) \end{array}$$

   where the vertical arrows are isomorphisms.

3. *Let $\mathscr{W}_s(\mu_1,\mu_2;\psi)$ be the image of $\mathscr{B}_s(\mu_1,\mu_2)$ in the space $\mathscr{W}(\mu_1,\mu_2;\psi)$. If Φ belongs to $\mathscr{S}(\mathbb{R}^2)$ and W_Φ belongs to $\mathscr{W}(\mu_1,\mu_2;\psi)$ then W_Φ belongs to $\mathscr{W}_s(\mu_1,\mu_2;\psi)$ if and only if*

$$\int_{\mathbb{R}} x^i \frac{\partial^j \Phi}{\partial y^j}(x,0)dx = 0$$

   for two nonnegative integers i and j satisfying $i+j = m+n-1$.

PROVE THIS.

6.3.9 The non-existence of Whittaker models for finite dimensional representations

97 Proposition. *If π is a finite dimensional representation then π does not posess a Whittaker model.*

Since π is finite dimensional its contragrediant is equal to its dual and is finite dimensional. Thus $\widetilde{\pi}(X_+)$ is nilpotent. If π had a Whittaker model then it would have a Whittaker functional λ (*e.g.*, $W \mapsto W(1)$ for W in the Whittaker model) and neccessarily $\widetilde{\pi}(X_+)\lambda = -2\pi i\alpha\lambda$ (where $\psi(x) = e^{2\pi i\alpha}$). But this contradicts $\widetilde{\pi}(X_+)$ being nilpotent. Therefore π cannot have a Whittaker model.

6.4 Local L-functions

6.4.1 The functions $L(s,\pi)$ and $Z(s,\phi,\xi)$

98. Let π be an irreducible admissible representation. We define the *local L-function* $L(s,\pi)$ of π:

1. If $\pi = \pi(\mu_1,\mu_2)$ then $L(s,\pi) = L(s,\mu_1)L(s,\mu_2)$ where $L(s,\mu_i)$ is the local L-function for $\mathrm{GL}(1,\mathbb{R})$ (*cf.* §2.1.2).
2. If $\pi = \pi(\omega)$ then $L(s,\pi) = L(s,\omega)$ where $L(s,\omega)$ is the local L-function for $\mathrm{GL}(1,\mathbb{C})$.

Note that we have defined $L(s,\pi)$ even when π is finite dimensional. Note also that we have given two definitions of $L(s,\pi)$ when $\pi = \pi(\mu_1,\eta\mu_1)$; it is a simple consequence of the duplication formula for the gamma function that they agree.

99. Let π be an infinite dimensional irreducible admissible representation. For a quasi-character ξ of $\mathbb{R}^\times$ and a function W in the Whittaker model of π define the zeta function

$$Z(s,W,\xi) = \int_{\mathbb{R}^\times} \xi(a)|a|^{s-1/2} W \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} d^\times a.$$

If ξ is the trivial character we write $Z(s,W)$ in place of $Z(s,W,\xi)$.

100 Proposition (J-L Thm. 5.15). *Let π be an infinite dimensional irreducible admissible representation*

1. *For any W in the Whittaker model and g in $G_{\mathbb{R}}$ the integral definining $Z(s,\rho(g)W,\xi)$ converges in some half plane $\Re s > s_0$.*
2. *For all g and W the ratio*
$$\frac{Z(s,\rho(g)W)}{L(s,\pi)} \tag{4}$$
can be analytically continued to an entire function of s.
3. *There exists W such that the quotient 4 is equal to 1 (with $g = 1$).*
4. *If W is fixed then $Z(s,\rho(g)W)$ remains bounded as g varies in a compact set and s varies in a vertical strip of finite width with discs removed about the poles of $L(s,\pi)$.*

101. We delay the proof of proposition 100 and prove it simultaneously with the local functional equation.

6.4.2 The local functional equation

102 Theorem (J-L Thm. 5.15). *Let π be an infinite dimensional irreducible admissible representation of $\mathscr{H}_{\mathbb{R}}$ with central quasi-character ω.*

1. *There exist ϵ-factors such that for any W and g*
$$\frac{Z(1-s,\rho(wg)W,(\omega\xi)^{-1})}{L(1-s,\widetilde{\pi})} = \epsilon(s,\pi,\xi,\psi)\frac{Z(s,\rho(g)W,\xi)}{L(s,\pi)}. \tag{5}$$
If ξ is trivial we write $\epsilon(s,\pi,\psi)$ in place of $\epsilon(s,\pi,\xi,\psi)$.

2. *The factors* $\epsilon(s, \pi, \xi)$ *are of the form* ab^s.

3. *If* $\pi = \pi(\mu_1, \mu_2)$ *then*
$$\epsilon(s, \pi, \psi) = \epsilon(s, \mu_1, \psi)\epsilon(s, \mu_2, \psi)$$
where $\epsilon(s, \mu_i, \psi)$ *are the epsilon factors for* $\mathrm{GL}(1, \mathbb{R})$.

4. *If* $\pi = \pi(\omega')$ *and* $\psi(x) = e^{2\pi i\alpha x}$ *then*
$$\epsilon(s, \pi, \psi) = (i \operatorname{sgn} \alpha)\epsilon(s, \omega', \psi_{\mathbb{C}}).$$

103. We prove this theorem in the next two sections.

104. The identity (5) is called the *local functional equation* for $\mathrm{GL}(2, \mathbb{R})$.

105. As before, we define γ-factors by
$$\gamma(s, \pi, \xi, \psi) = \frac{L(1 - s, \widetilde{\pi})}{L(s, \pi)}\epsilon(s, \pi, \xi, \psi)$$
so that the local functional equation takes the form
$$Z(1 - s, \rho(wg)W, \xi) = \gamma(s, \pi, \xi, \psi)Z(s, \rho(g)W, \xi).$$

6.4.3 Proofs for $\pi(\mu_1, \mu_2)$

106. In this section we prove proposition 100 and theorem 102 for the representations $\pi(\mu_1, \mu_2)$. It suffcies to prove most of the statements for $g = 1$ and $W = W_\Phi$ with Φ an arbitrary element of $\mathscr{S}(\mathbb{R}^2)$ (since $\rho(g)W_\Phi = W_{\rho(g)\Phi}$).

107 Lemma. *For* Φ *in* $\mathscr{S}(R^2)$ *we have* $Z(s, W_\Phi, \xi) = Z(s, \xi\mu_1; s, \xi\mu_2; \Phi)$ *where*
$$Z(s_1, \mu_1; s_2, \mu_2; \Phi) = \int_{(\mathbb{R}^\times)^2} \mu_1(x)|x|^{s_1}\mu_2(y)|y|^{s_2}\Phi(x, y)d^\times x d^\times y.$$
In particular, if $\Phi(x, y) = \phi_1(x)\phi_2(y)$ *where* ϕ_1 *and* ϕ_2 *belong to* $\mathscr{S}(\mathbb{R})$ *then*
$$Z(s_1, \mu_1; s_2, \mu_2; \Phi) = Z(s_1, \Phi, \mu_1)Z(s_2, \Phi, \mu_2)$$
so that the $\mathrm{GL}(2)$ *zeta function factors into* $\mathrm{GL}(1)$ *zeta functions.*

This is a simple computation which has already been performed in §5.8.2, lemma 156.

108. Since the integral defining $Z(s, \xi\mu_1; s, \xi\mu_2; \Phi)$ clearly converges in some half plane, part 1 of proposition 100 follows. Also if we take ϕ_1 and ϕ_2 so that $Z(s, \phi_1)/L(s, \mu_1) = 1$ (possible by §2.1.2, proposition 6) and let $\Phi(x, y) = \phi_1(x)\phi_2(y)$ then it follows that (4) is equal to 1. Thus part 2 of proposition 100 is proved.

109. Note that the effect of changing μ_i to $\mu_i\alpha_{\mathbb{R}}^{r_i}$ is the same as changing s to $s + r_1 + r_2$, and since the statements of the proposition and theorem are stable under translations in s, we may assume that μ_1 and μ_2 are characters. We make this assumption for the rest of the section (to ease some notation).

110 Lemma (J-L Lemma 5.15.1). *Assume* μ_1 *and* μ_2 *are characters. Then for all* Φ *and* Ψ *in* $\mathscr{S}(\mathbb{R}^2)$ *and complex numbers* s_1 *and* s_2 *with real parts in* $(0, 1)$ *we have*
$$Z(s_1, \mu_1; s_2, \mu_2; \Phi)Z(1 - s_1, \mu_1^{-1}; 1 - s_2, \mu_2^{-1}; \Psi') = Z(1 - s_1, \mu_1^{-1}; 1 - s_2, \mu_2^{-1}; \Phi')Z(s_1, \mu_1; s_2, \mu_2; \Psi)$$
where prime denotes Fourier transform.

First note that since μ_1 and μ_2 are characters the integrals $Z(s_1, \mu_1; s_2, \mu_2; \Phi)$ converge for $\Re s_1, \Re s_2 > 0$; thus the statement makes sense.

The left hand side equals

$$\int \Phi(x,y)\Psi'(u,v)\mu_1(xu^{-1})\mu_2(yv^{-1})|xu^{-1}|^{s_1}|yv^{-1}|^{s_2}d^\times x d^\times y du dv$$

where the integral is over $(\mathbb{R}^\times)^4$ and we have used $d^\times x = |x|^{-1}dx$. A change of variables gives

$$\int \mu_1(x)\mu_2(y)|x|^{s_1}|y|^{s_2}\left[\int \Phi(xu,yv)\Phi'(u,v)du dv\right]d^\times x d^\times y$$

In the same way, the right hand side is equal to

$$\int \mu_1^{-1}(x)\mu_2^{-1}(y)|x|^{1-s_1}|y|^{1-s_2}\left[\int \Phi'(xu,yv)\Psi(u,v)du dv\right]d^\times x d^\times y$$
$$= \int \mu_1(x)\mu_2(y)|x|^{s_1}|y|^{s_2}\left[|xy|^{-1}\int \Phi'(x^{-1}u,y^{-1}v)\Psi(u,v)du dv\right]d^\times x d^\times y.$$

Now, the Fourier transform of the function $(u,v)\mapsto \Phi(xu,yv)$ is the function $|xy|^{-1}\Phi'(x^{-1}u,y^{-1}v)$ and so the Plancherel formula gives

$$\int \Phi(xu,yv)\Phi'(u,v)du dv = |xy|^{-1}\int \Phi'(x^{-1}u,y^{-1}v)\Psi(u,v)du dv$$

which proves the lemma.

111. Now pick ψ_1 and ψ_2 in $\mathscr{S}(\mathbb{R})$ so that $L(s,\mu_i) = Z(s,\psi_i,\mu_i)$. By the local functional equation for GL(1) we have

$$Z(1-s,\psi_i',\mu_i^{-1}) = \epsilon(s,\mu_i,\psi)L(1-s,\mu_i^{-1})$$

Now let $\Psi(x,y) = \psi_1(x)\psi_2(y)$. Lemma 110 then gives

$$\epsilon(s_1,\mu_1,\psi)\epsilon(s_2,\mu_2,\psi)\frac{Z(s_1,\mu_1;s_2,\mu_2;\Phi)}{L(s_1,\mu_1)L(s_2,\mu_2)} = \frac{Z(1-s_1,\mu_1^{-1};1-s_2,\mu_2^{-1};\Phi')}{L(1-s_1,\mu_1^{-1})L(1-s_2,\mu_2^{-1})}.$$

when $\Re s_1$ and $\Re s_2$ lie in $(0,1)$. However, the left hand side is defined and holomorphic for $\Re s_1,\Re s_2 > 0$ while the right hand side is defined and holomorphic for $\Re s_1,\Re s_2 < 1$. It follows that both sides are in fact entire functions of s_1 and s_2. Letting $s_1 = s_2 = s$ we obtain the functional equation.

112. The only thing left to prove is the final statement of proposition 100, which we take care of in the following lemma.

113 Lemma. *Let Ω be a compact subset of $\mathscr{S}(\mathbb{R}^2)$ and C a domain in $\mathbb{C}^2$ obtained by removing balls about the poles of $L(s_1,\mu_1)L(s_2,\mu_2)$ from the region $a_1 \le \Re s_1 \le b_1$, $a_2 \le \Re s_2 \le b_2$. Then $Z(s_1,\mu_1;s_2,\mu_2;\Phi)$ remains bounded as Φ varies in Ω and (s_1,s_2) varies in C.*

It suffices to prove this when both a_1 and a_2 are greater than 0 or both b_1 and b_2 are less than 1. On a region of the first type the function $Z(\mu_1,s_1;\mu_2,s_2;\Phi)$ is defined by a definite integral; integrating by parts gives

$$Z(\mu_1,\sigma_1+i\tau_1;\mu_2,\sigma_2+i\tau_2;\Phi) = O((\tau_1^2+\tau_2^2)^{-n})$$

as $\tau_1^2+\tau_2^2\to\infty$ uniformly for Φ in Ω and $a_1\le\sigma_1\le b_1$, $a_2\le\sigma_2\le b_2$, which is much stronger than required. For a region of the second type, combine the result just obtained with the functional equation and known facts about the gamma function.

6.4.4 Proofs for $\pi(\omega)$

114. In this section we prove proposition 100 and theorem 102 for the representations $\pi(\omega)$. We assume ω does not factor through $|\cdot|_{\mathbb{C}}$; such representations were handled in the previous section.

115. The space $\mathscr{W}(\omega;\psi)$ is the sum of $\mathscr{W}_1(\omega,\psi)$ and its right translate by ϵ. Since

$$Z(s,\rho(g\epsilon)W) = \omega(-1)Z(s,\rho(\epsilon^{-1}g\epsilon)W)$$

it suffices to prove the statements for W in $\mathscr{W}_1(\omega;\psi)$. As in the previous section, we prove the statements for $g=1$ and $W=W_\Phi$ with Φ an arbitrary element of $\mathscr{S}(\mathbb{C},\omega)$.

116 Lemma. *Let Φ belong to $\mathscr{S}(\mathbb{C},\omega)$. Then*

$$Z(s,W_\Phi) = cZ(s,\Phi,\omega)$$

where the left side is a GL(2) *zeta function, the right side is a* GL(1) *zeta function and c is some absolute constant.*

We have

$$\begin{aligned} Z(s,W_\Phi) &= \int_0^\infty a^{s-1/2}W_\Phi\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}d^\times a = \int_0^\infty a^{s-1/2}\omega(a^{1/2})a^{1/2}\Phi(a^{1/2})d^\times a \\ &= 2\int_0^\infty a^{2s}\omega(a)\Phi(a)d^\times a \end{aligned}$$

Now if x is a complex number of modulus 1 then $\Phi(ax) = \omega^{-1}(x)\Phi(a)$. Thus, continuing the above derivation, we have

$$Z(s,W_\Phi) = \frac{1}{\pi}\int_0^\infty\int_0^{2\pi}|ae^{i\theta}|^{2s}\omega(ae^{i\theta})\Phi(ae^{i\theta})d\theta d^\times a = c\int_{\mathbb{C}^\times}|z|_{\mathbb{C}}^s\omega(z)\Phi(z)d^\times z.$$

This proves the lemma.

117. Most of proposition 100 and theorem 102 now follow from facts about GL(1) (*cf.* §2.1.2 and §2.1.3). The last part of proposition 100 follows from a lemma similar to lemma 113, which we do not bother to state or prove.

The only statement left to prove is that there exists Φ in $\mathscr{S}_0(\mathbb{C},\omega)$ such that $Z(s,\Phi,\omega)/L(s,\omega)=1$ (note that the GL(1) theory asserts that the existence of Φ in the larger space $\mathscr{S}(\mathbb{C})$). Let $\omega(z) = (z\bar{z})^r z^n\bar{z}^m$. The function

$$\Phi(z) = e^{-2\pi z\bar{z}}z^m\bar{z}^n$$

belongs to $\mathscr{S}_0(\mathbb{C},\omega)$. We have

$$Z(s,\Phi,\omega) = 2\pi\int_0^\infty x^{2(s+r+n+m)-1}e^{-2\pi x^2}dx = \pi(2\pi)^{-(s+r+n+m)}\Gamma(s+r+n+m) = cL(s,\omega)$$

where c is some nonzero constant. This finishes the proof.

6.4.5 The γ-factors determine π

118 Proposition (J-L Prop. 5.18). *Let π and π' be infinite dimensional irreducible admissible representations of $\mathscr{H}_{\mathbb{R}}$. Then π and π' are equivalent if and only if they have the same central quasi-character and*

$$\gamma(s,\pi,\xi,\psi) = \gamma(s,\pi',\xi,\psi)$$

for all quasi-characters ξ of $\mathbb{R}^\times$.

6.5 Representations associated to $\mathbb{H}$

DO THIS.

Chapter 7

Representations of $\mathrm{GL}(2, \mathbb{C})$

7.1 Notations

1. Throughout this section we will use the following notations:

1. We let $G_{\mathbb{C}}$ denote the Lie group $\mathrm{GL}(2, \mathbb{C})$ (thought of as a *real* Lie group);
2. We let $K_{\mathbb{C}}$ be the standard maximal compact subgroup $\mathrm{U}(2, \mathbb{C})$ of $G_{\mathbb{C}}$;
3. We let $\mathfrak{g}$ denote the Lie algebra of $G_{\mathbb{C}}$ (thought of as a *real* Lie algebra);
4. We let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$;
5. We let $\mathscr{U}$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$;
6. We let $\mathscr{Z}$ be the center of $\mathscr{U}$;
7. We let ψ be a fixed nontrivial additive character of $\mathbb{C}$;
8. We let ρ_n denote the unique irreducible representation of $\mathrm{SU}(2, \mathbb{C})$ of degree $n + 1$.

7.2 Basic constructs and their properties

2. Everything in §6.1 carries over to the present case. We give some brief comments.

3. We define the *Hecke algebra* $\mathscr{H}_{\mathbb{C}}$ to be the algebra of $K_{\mathbb{C}}$-finite distributions of compact support on $G_{\mathbb{C}}$ with support contained in $K_{\mathbb{C}}$ under convolution. It is an idempotented algebra; we thus have the notions of smooth representations, admissible representations and contragrediants.

4. Again, this Hecke algebra (of Flath) differs from the Hecke algebra of Jacquet-Langlands. The situation is exactly analagous to the real case and we do not both to expound.

5. A representation of $\mathscr{U}$ is defined to be admissible if its restriction to the Lie algebra of $K_{\mathbb{C}}$ decomposes into a direct sum of finite dimensional irreducible representations each occurring with finite multiplicity.

6. We again have the notion of a Harish-Chandra module or a representaiton of $(\mathfrak{g}, K_{\mathbb{C}})$. However, since $K_{\mathbb{C}}$ is now connected an admissible represnetation of $(\mathfrak{g}, K_{\mathbb{C}})$ is the same thing as an admissible representation of $\mathfrak{g}$. Thus we do not use Harish-Chandra modules in this section.

7. Once again, there is an equivalence of categories between smooth $\mathscr{H}_{\mathbb{C}}$-modules and Harish-Chandra modules (and thus $\mathscr{U}$-modules as well, in the present case).

8. Given an admissible representation π of $\mathscr{H}_{\mathbb{C}}$ one can construct a representation of $Z_{\mathbb{C}}$ on the same space. If π is irreducible then Schur's lemma is satisfied and the elements of $Z_{\mathbb{C}}$ act as scalars. Thus we have a central quasi-character associated to an irreducible admissible representation.

9. One can again twist a representation of $\mathscr{H}_\mathbb{C}$ or $\mathscr{U}$ by a qausi-character of $\mathbb{C}^\times$.

7.3 Classification of irreducible representations

10. Let μ_1 and μ_2 be quasi-characters of $\mathbb{C}^\times$. Let $\mu = \mu_1\mu_2^{-1}$. As before, write

$$\mu_i(z) = (z\bar{z})^{s_i} \frac{z^{a_i}\bar{z}^{b_i}}{(z\bar{z})^{\frac{1}{2}(a_i+b_i)}} \qquad \mu(z) = (z\bar{z})^{s} \frac{z^{a}\bar{z}^{b}}{(z\bar{z})^{\frac{1}{2}(a+b)}}$$

where a_i, b_i, a and b are nonnegative integers and one from each pair is zero.

11. We define the space $\mathscr{B}(\mu_1,\mu_2)$ to be the space of all complex valued functions f on $G_\mathbb{C}$ which are $K_\mathbb{C}$-finite on the right and which satisfy

$$f\left(\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix} g\right) = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} f(g).$$

Both $\mathscr{H}_\mathbb{C}$ and $\mathscr{U}$ act on $\mathscr{B}(\mu_1,\mu_2)$ via ρ and we denote the resulting representation as $\rho(\mu_1,\mu_2)$.

12. We identify $\mathfrak{g}_\mathbb{C}$ with $\mathfrak{gl}(2,\mathbb{C})\oplus\mathfrak{gl}(2,\mathbb{C})$ in such a way that the elements of $\mathfrak{g}$ correspond to elements of the form $X \oplus \bar{X}$. If $\mathscr{U}_1$ is the universal enveloping algebra of $\mathfrak{gl}(2,\mathbb{C})$ then $\mathscr{U}$ is identified with $\mathscr{U}_1 \otimes \mathscr{U}_1$.

13. In section §6.2 we introduced the elements D and J of the complexification of the universal enveloping algebra of $G_\mathbb{R}$. We derive four elements of $\mathscr{U}$ from them:

$$D_1 = D\otimes 1, \qquad D_2 = 1\otimes D, \qquad J_1 = J\otimes 1, \qquad J_2 = 1\otimes J.$$

All four of these lie in $\mathscr{Z}$.

14 Proposition (J-L Lemma 6.1). *For the representation $\rho(\mu_1,\mu_2)$ we have the following:*

$$\rho(D_1) = \tfrac{1}{2}(s + \tfrac{1}{2}(a-b))^2 - \tfrac{1}{2}, \qquad \rho(D_2) = \tfrac{1}{2}(s + \tfrac{1}{2}(b-a))^2 - \tfrac{1}{2},$$
$$\rho(J_1) = s_1 + s_2 + \tfrac{1}{2}(a_1 - b_1 + a_2 - b_2), \qquad \rho(J_2) = s_1 + s_2 + \tfrac{1}{2}(b_1 - a_1 + b_2 - a_2).$$

15 Proposition (J-L Lemma 6.1). *We have:*

1. *The representation $\rho(\mu_1,\mu_2)$ is admissible.*
2. *The restriction of $\rho(\mu_1,\mu_2)$ to the Lie algebra of $\mathrm{SU}(2,\mathbb{C})$ contains the representation ρ_n if and only if $n \ge a+b$ and n has the same parity as $a+b$, in which case it occurs with multiplicity one.*

16. We let $\mathscr{B}(\mu_1,\mu_2;\rho_n)$ be the space of functions in $\mathscr{B}(\mu_1,\mu_2)$ which transform according to ρ_n.

17 Theorem (J-L Thm. 6.2). *We have the following:*

1. *Let μ not be of the form $z\mapsto z^p\bar{z}^q$ or $z\mapsto z^{-p}\bar{z}^{-q}$ with $p\ge 1$ and $q \ge 1$.*
   (a) *The representation $\rho(\mu_1,\mu_2)$ is irreducible.*
   (b) *We denote by $\pi(\mu_1,\mu_2)$ any representation in its equivalence class.*
2. *Let μ be of the form $z\mapsto z^p\bar{z}^q$ with $p \ge 1$ and $q\ge 1$.*
   (a) *The space*
   $$\mathscr{B}_s(\mu_1,\mu_2) = \bigoplus_{\substack{n\ge p+q \\ n\equiv p+q\ (mod\ 2)}} \mathscr{B}(\mu_1,\mu_2;\rho_n)$$
   is the unique proper stable subspace of $\mathscr{B}(\mu_1,\mu_2)$.

(b) *We denote by $\sigma(\mu_1, \mu_2)$ any representation equivalent to the restriction of $\rho(\mu_1, \mu_2)$ to $\mathscr{B}_s(\mu_1, \mu_2)$.*

(c) *We denote by $\pi(\mu_1, \mu_2)$ any representation equivalent to the representation on $\mathscr{B}_f(\mu_1, \mu_2) = \mathscr{B}(\mu_1, \mu_2)/\mathscr{B}_s(\mu_1, \mu_2)$ induced by $\rho(\mu_1, \mu_2)$.*

3. *Let μ be of the form $z \mapsto z^{-p}\bar{z}^{-q}$ with $p \geq 1$ and $q \geq 1$.*

   (a) *The space*

   $$\mathscr{B}_f(\mu_1, \mu_2) = \bigoplus_{\substack{|p-q| \leq n \leq p+q \\ n \equiv p+q \, (mod\ 2)}} \mathscr{B}(\mu_1, \mu_2; \rho_n)$$

   is the unique proper stable subspace of $\mathscr{B}(\mu_1, \mu_2)$.

   (b) *We denote by $\pi(\mu_1, \mu_2)$ any representation equivalent to the restriction of $\rho(\mu_1, \mu_2)$ to $\mathscr{B}_f(\mu_1, \mu_2)$.*

   (c) *We denote by $\sigma(\mu_1, \mu_2)$ any representation equivalent to the representation on $\mathscr{B}_s(\mu_1, \mu_2) = \mathscr{B}(\mu_1, \mu_2)/\mathscr{B}_f(\mu_1, \mu_2)$ induced by $\rho(\mu_1, \mu_2)$.*

4. *The representations $\pi(\mu_1, \mu_2)$ and $\pi(\mu_1', \mu_2')$ are equivalent if and only if $\{\mu_1, \mu_2\} = \{\mu_1', \mu_2'\}$.*

5. *The representations $\sigma(\mu_1, \mu_2)$ and $\sigma(\mu_1', \mu_2')$ are equivalent if and only if $\{\mu_1, \mu_2\} = \{\mu_1', \mu_2'\}$.*

6. *If μ is of the form $z \mapsto z^p\bar{z}^q$ with $p \geq 1$ and $q \geq 1$ then there is a pair of characters ν_1, ν_2 such that*

   (a) *$\mu_1\mu_2 = \nu_1\nu_2$;*

   (b) *$\nu_1\nu_2^{-1}$ is of the form $z \mapsto z^p\bar{z}^{-q}$;*

   (c) *the representation $\sigma(\mu_1, \mu_2)$ is equivalent to $\pi(\nu_1, \nu_2)$.*

7. *Every irreducible admissible representation of $\mathscr{H}_{\mathbb{C}}$ or $\mathscr{U}$ is equivalent to a $\pi(\mu_1, \mu_2)$.*

7.4 The Whittaker model

18. Let $\mathscr{W}(\psi)$ be the space of all complex valued functions W on $G_{\mathbb{C}}$ which satisfy the following conditions.

1. W is K-finite on the right and smooth.

2. We have

   $$W\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) = \psi(x)W(g)$$

   for all x in $\mathbb{C}$ and all g in $G_{\mathbb{C}}$.

3. For all g in $G_{\mathbb{R}}$ there exists a positive real number N (depending on g and W) such that

   $$W\left(\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} g\right) = O(|t|^N)$$

   as $|t| \to \infty$.

There is an action of $(\mathfrak{g}, K_{\mathbb{C}})$ and $\mathscr{H}_{\mathbb{C}}$ on $\mathscr{W}(\psi)$.

19. A *Whittaker model* of a representation π of $\mathscr{H}_{\mathbb{C}}$ is an $\mathscr{H}_{\mathbb{C}}$-submodule of $\mathscr{W}(\psi)$ which is equivalent to π.

20 Theorem (J-L Thm 6.3). *Let π be an infinite dimensional irreducible admissible representation of $\mathscr{H}_{\mathbb{C}}$. Then π has a unique Whittaker model. All members of the Whittaker model are analytic and rapidly decreasing (remember, we treat $G_{\mathbb{C}}$ as a* real *analytic manifold.)*

7.5 The functions $L(s,\pi)$ and $Z(s,\phi,\xi)$ and the local functional equation

21. Let π be an irreducible admissible representation. By theorem 17 π is equivalent to $\pi(\mu_1,\mu_2)$ for some quasi-characters μ_1 and μ_2 of $\mathbb{C}^\times$. We define the *local L-function* of π to be

$$L(s,\pi) = L(s,\mu_1)L(s,\mu_2)$$

where $L(s,\mu_i)$ is the local L-function for $\mathrm{GL}(1,\mathbb{C})$ (*cf.* §2.1.2).

22. Now let π be an infinite dimensional irreducible admissible representation. For a quasi-character ξ of $\mathbb{C}^\times$ and a function W in the Whittaker model of π define the zeta function

$$Z(s,W,\xi) = \int_{\mathbb{C}^\times} \xi(a)|a|_{\mathbb{C}}^{s-1/2} W \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} d^\times a.$$

If ξ is the trivial character we write $Z(s,W)$ in place of $Z(s,W,\xi)$.

23 Proposition. *Let π be an infinite dimensional irreducible admissible representation.*

1. *For any W in the Whittaker model and g in $G_{\mathbb{C}}$ the integral defining $Z(s,\rho(g)W,\xi)$ converges in some half plane $\Re s > s_0$.*
2. *For all W and g the ratio*
$$\frac{Z(s,W)}{L(s,\pi)} \tag{1}$$
can be analytically continued to an entire function of s.
3. *There exists W such that the quotient (1) is equal to 1 (with $g=1$).*
4. *If W is fixed then $Z(s,\rho(g)W)$ remains bounded as g varies in a compact set and s varies in a vertical strip of finite width with discs removed about the poles of $L(s,\pi)$.*

24 Theorem. *Let π be an infinite dimensional irreducible admissible representation with central quasi-character ω.*

1. *There exist ϵ-factors such that for all W and g*
$$\frac{Z(1-s,\rho(wg)W,(\omega\xi)^{-1})}{L(1-s,\tilde{\pi})} = \epsilon(s,\pi,\xi,\psi)\frac{Z(s,\rho(g)W,\xi)}{L(s,\pi)} \tag{2}$$
If ξ is the trivial character we write $\epsilon(s,\pi,\psi)$ in place of $\epsilon(s,\pi,\xi,\psi)$.
2. *The factors $\epsilon(s,\pi,\xi,\psi)$ are of the form ab^s.*
3. *If π is equivalent to $\pi(\mu_1,\mu_2)$ then*
$$\epsilon(s,\pi,\psi) = \epsilon(s,\mu_1,\psi)\epsilon(s,\mu_2,\psi)$$

25. The identity (2) is called the *local functional equation* for $\mathrm{GL}(2,\mathbb{C})$.

26. As before, we define γ-factors by

$$\gamma(s,\pi,\xi,\psi) = \frac{L(1-s,\tilde{\pi})}{L(s,\pi)}\epsilon(s,\pi,\xi,\psi)$$

so that the local functional equation takes the form

$$Z(1-s,\rho(wg)W,\xi) = \gamma(s,\pi,\xi,\psi)Z(s,\rho(g)W,\xi).$$

27 Proposition (J-L Lemma 6.6). *Let π and π' be infinite dimensional irreducible admissible representations. Then π and π' are equivalent if and only if they have the same central quasi-character and*

$$\gamma(s, \pi, \xi, \psi) = \gamma(s, \pi', \xi, \psi)$$

for all quasi-characters ξ of $\mathbb{C}^\times$.

Chapter 8

Representations of $\mathrm{GL}(2, A)$

8.1 Notations

1. Throughout this section we will use the following notations:

1. We let F denote a global field;
2. We let A denote the adele ring of F;
3. We let I denote the idele group of F;
4. We let Σ denote the set of all places of F; we let Σ_f denote the set of finite places and Σ_∞ the set of infinite places;
5. If X is some adelic object and S is a finite subset of Σ, we let X_S denote the S-part of X and X^S the complement of the S-part of X. We also let X_∞ and X_f denote the infinite and finite parts of X. When $S = \{v\}$ we write X_v in place of X_S. For example, F_v is just the local field of F at v; A_S is the product $\prod_{v \in S} F_v$; I_S is the restricted direct product $\prod_{v \notin S}(F_v^\times : U_v)$, *etc.*;
6. We let ψ denote a nontrivial additive character of A which is trivial on F; we let ψ_v denote the corresponding character of F_v; note that ψ_v is also nontrivial and for almost all v it is unramified.

8.2 General representation theory

8.2.1 The Hecke algebra

2. Let v be a place of F. Define ϵ_v to be the characteristic function of K_v on G_v. Then (when regarded in the appropriate manner) ϵ_v is an element of $\mathscr{H}_v$; in fact, it is the elementary idempotent corresponding to the trivial representation of K_v.

3. We define the *global Hecke algebra* $\mathscr{H}$ to be the restrictred direct product of the Hecke algebras $\mathscr{H}_v$ with respect to the idempotents ϵ_v. It is again an idempotented algebra. We thus have for free the notions of smooth representations, admissible representations and contragrediants.

4. Recall (§1.4.2) that an *admissible family* of representations (π_v) consists of, for each place v, an admissible representation π_v of $\mathscr{H}_v$ such that for almost all v the image of $\pi_v(\epsilon_v)$ is one dimensional. Note that this condition is equivalent to π_v being spherical (*cf.* §5.11.1) for almost all v.

Let (π_v) be an admissible family. Pick an element x_v of the space of π_v such that for almost all v x_v spans the image of $\pi(\epsilon_v)$ (*i.e.*, x_v is spherical for almost all v). We define the tensor product of the family (π_v), denoted $\otimes\pi_v$, to be the restricted direct product of the π_v with respect to the x_v. It does not depend on the choice of x_v (up to isomorphism).

5 Theorem. *We have the following.*

1. *Let (π_v) be an admissible family of irreducible representations of $\mathscr{H}_v$. Then $\otimes\pi_v$ is an admissible irreducible representation of $\mathscr{H}_v$.*
2. *Let π be an irreducible admissible representation of $\mathscr{H}$. Then there exists an admissible family of representations (π_v) such that π is equivalent to $\otimes\pi_v$.*

This is precisely the assertion of §1.4.6, theorem 119.

6. Note that if $\pi = \otimes\pi_v$ is irreducible and admissible then for almost all v the representation π_v is a principal series representation corresponding to unramified quasi-characters of $F_v^\times$ (*cf.* §5.11.2, proposition 194).

8.2.2 The Whittaker model

7. Let ψ be a nontrivial additive character of A. Let $\mathscr{W}(\psi)$ be the space of complex valued functions W on G_A which satisfy the following conditions:

1. We have
$$W\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) = \psi(x)W(g)$$
for all x in A and all g in G_A.
2. For all g in G_f the restriction of $\rho(g)W$ to G_∞ is smooth and K_∞-finite on the right.
3. For all g in G_f, all T in $\mathscr{H}_\infty$ and all non-archimedean places v, the restriction of $\rho(T)\rho(g)W$ to G_v is locally constant.
4. For all g in G_f, all T in $\mathscr{H}_\infty$ and all archimedean places v there exists a positive real number N such that
$$(\rho(T)\rho(g)W)\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = O(|a|^N)$$
where a belongs to $F_v^\times$ and $|a| \to \infty$.

It is clear that $\mathscr{H}_\infty$ acts smoothly on $\mathscr{W}(\psi)$ via ρ. It is also clear that G_f (and thus $\mathscr{H}_f$) acts smoothly on $\mathscr{W}(\psi)$. Therefore, there is a smooth action of $\mathscr{H}$ on $\mathscr{W}(\psi)$.

8. A *Whittaker model* of a representation π of $\mathscr{H}$ is a submodule of $\mathscr{W}(\psi)$ which is isomorphic to π.

9 Theorem (J-L Prop. 9.2, 9.3). *Let $\pi = \otimes\pi_v$ be an irreducible admissible representation of $\mathscr{H}$.*

1. *If each π_v is infinite dimensional for all v then π admits a unique Whittaker model $\mathscr{W}(\pi,\psi)$.*
2. *If π_v is finite dimensional for any v then π does not have a Whittaker model.*

1) For each place v we have a Whittaker model $\mathscr{W}_v = \mathscr{W}(\pi_v, \psi_v)$. For almost all v, π_v is spherical and ψ_v is unramified; for such v there exists a unique element W_v° of $\mathscr{W}_v$ such that W_v° is invariant under K_v and $W_v^\circ(1) = 1$ (*cf.* §5.11.3, proposition 199). Define $\mathscr{W}(\pi,\psi)$ to be the restricted tensor product of the $\mathscr{W}_v$ with respect to the elements W_v°. We interpret elements of $\mathscr{W}(\pi,\psi)$ as functions on G_A as follows: if $W = \otimes W_v$ and $g = (g_v)$ then $W(g) = \prod W_v(g_v)$. Since for almost all v we have $W_v = W_v^\circ$ and $g_v = 1$ it follows that almost all factors in the product are 1 and so the definition of $W(g)$ makes sense. The requisite properties of W follow easily from the corresponding properties of the W_v.

We must now show that the Whittaker model just constructed is the only Whittaker model of π. Let $\mathscr{W}'$ be another Whittaker model and let $T : \mathscr{W}(\pi,\psi) \to \mathscr{W}'$ be an isomorphism of $\mathscr{H}$-modules. We will show that T is given by multiplication by a scalar, which will establish the uniqueness.

For a subset S of Σ put
$$\mathscr{W}_S = \bigotimes_{v\in S}(\mathscr{W}_v : W_v^\circ).$$

We first show that if S is a finite subset of Σ with complement $\widetilde{S}$ then there exists a complex valued function c_S on $G_{\widetilde{S}} \times \mathscr{W}_{\widetilde{S}}$ such that if $f = T(\phi \otimes \widetilde{\phi})$ with ϕ in $\mathscr{W}_S$ and $\widetilde{\phi}$ in $\mathscr{W}_{\widetilde{S}}$ then

$$f(g\widetilde{g}) = c_S(\widetilde{g}, \widetilde{\phi})\phi(g)$$

for g in G_S and $\widetilde{g}$ in $G_{\widetilde{S}}$ (note that $\phi(g) = \prod_{v \in S} \phi_v(g_v)$). It is clear that the number $c_S(\widetilde{g}, \widetilde{\phi})$, if it exists, is unique.

First consider the case when S contains a single element v. Let $\widetilde{\phi}$ belong to $\mathscr{W}_{\widetilde{S}}$ and let $\widetilde{g}$ belong to $G_{\widetilde{S}}$. Given ϕ_v in $\mathscr{W}_v$ let ϕ_v' be the function on G_v given by

$$\phi_v'(g_v) = f(g_v\widetilde{g})$$

where $f = T(\phi \otimes \widetilde{\phi})$. It is easily verified that A) ϕ_v' belongs to $\mathscr{W}_v$; and B) if ϕ_v is replaced by $\rho(f_v)\phi_v$ (with f_v in $\mathscr{H}_v$) then ϕ_v' is replaced with $\rho(f_v)\phi_v'$. It thus follows that $\phi_v \mapsto \phi_v'$ is an endomorphism of the irreducible $\mathscr{H}_v$-module $\mathscr{W}_v$ and is therefore equal to a constant $c_S(\widetilde{g}, \widetilde{\phi})$.

Assume we we have proved the statement for the finite set S, *i.e.*, we have proved the existence of c_S. Let S' be obtained from S by adjoining a single place w. Let $\widetilde{\phi}$ belong to $\mathscr{W}_{\widetilde{S}'}$ and let $\widetilde{g}$ belong to $G_{\widetilde{S}'}$. Given ϕ in $\mathscr{W}_S$, ϕ_w in $\mathscr{W}_w$, g in G_S and g_w in G_w, we have (by the inductive hypothesis)

$$f(gg_w\widetilde{g}) = c_S(g_w\widetilde{g}, \phi_w \otimes \widetilde{\phi})\phi(g)$$

where $f = T(\phi \otimes \phi_w \otimes \widetilde{\phi})$. The argument used in the previous paragraph now shows that the function

$$g_w \mapsto c_S(g_w\widetilde{g}, \phi_w \otimes \widetilde{\phi})$$

is a multiple $c_{S'}(\widetilde{g}, \widetilde{\phi})$ of ϕ_w.

We have thus proved the existence of c_S for all finite subsets S. Before continuing, we make one observation. Suppose S is the disjoint union of S_1 and S_2. Given h_1 in $G_{\widetilde{S}_1}$ we may write $h_1 = (\prod_{v \in S_2} h_v)h$ where h belongs to $G_{\widetilde{S}}$. Similarly, given ϕ_1 in $\mathscr{W}_{\widetilde{S}_1}$ we may write $\phi_1 = (\bigotimes_{v \in S_2} \phi_v) \otimes \phi$ where ϕ belongs to $\mathscr{W}_{\widetilde{S}}$. We then have

$$c_{S_1}(h_1, \phi_1) = \left(\prod_{v \in S_2} \phi_v(h_v)\right) c_S(h, \phi) \tag{1}$$

since the right side satisfies the defining properties of the left.

We now prove that T is a scalar. Take S_1 so large that $\widetilde{S}_1$ contains only spherical places. Then, by its definition, $c_{S_1}(h, \bigotimes_{v \in \widetilde{S}_1} W_v^\circ)$ is equal to a constant $c(S_1)$ as h varies in $K_{\widetilde{S}_1}$. The identitiy (1) shows that if S contains S_1 then $c(S) = c(S_1)$. Let c be the common value of all these constants.

Now, given $\phi = \otimes\phi_v$ in $\mathscr{W}(\pi, \psi)$ and $g = (g_v)$ take S containing S_1 large enough such that for v not in S_1 we have $\phi_v = W_v^\circ$ and g_v belongs to K_v. Then

$$(T\phi)(g) = c\left(\prod_{v \in \widetilde{S}} g_v, \bigotimes_{v \in \widetilde{S}} \phi_v\right) \prod_{v \in S} \phi_v(g_v) = c \prod_{v \in S} \phi_v(g_v) = c\phi(g),$$

and we have proved uniqueness.

2) As we have seen above, if ϕ belongs to the Whittaker model of π then the restriction of ϕ to G_v belongs to the Whittaker model of π_v. Thus if π has a Whittaker model so do all the π_v. Since we know that finite dimensional representations do not have Whittaker models, the second statement follows.

8.3 First properties of automorphic representations

8.3.1 Automorphic representations

10. Let ϕ be a function on $G_F \backslash G_A$. We may think of ϕ as a function on G_A which satisfies $\phi(gh) = \phi(h)$ for all g in G_F. We make some definitions:

1. We say ϕ is *smooth* if for every g in G_A there exists a neighborhood U and a smooth function f_g on G_∞ such that $f(h) = f_g(h_\infty)$ for all h in U. Note that if F is a function field this is equivalent to ϕ being locally constant.

2. We say ϕ is *slowly increasing* if for any compact set Ω of G_A there exist constants M_1 and M_2 such that for all g in Ω and all a in I with $|a| \gg 0$ we have
$$\left|\phi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g\right)\right| \leq M_1 |a|^{M_2}. \tag{2}$$

3. We say ϕ is *rapidly decreasing* if for any compact set Ω and any M_2 (positive or negative) there exists a constant M_1 such that (2) holds for all g in Ω and all a in I with $|a| \gg 0$.

4. We say ϕ is *cuspidal* if for all g in G_A we have
$$\int_{F\backslash A} \phi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) dx = 0.$$

5. We say ϕ has central quasi-character η (where η is a quasi-character of $F^\times \backslash I$, also regarded as a quasi-character of $Z_F \backslash Z_A$) if for all a in Z_A and all g in G_A we have $\phi(ag) = \eta(a)\phi(g)$.

11. A smooth function ϕ on $G_F \backslash G_A$ is an *automorphic form* if:

1. ϕ is K-finite on the right.
2. For every elementary idempotent ξ the space $\rho(\xi)\rho(\mathscr{H})\phi$ is finite dimensional.
3. ϕ is slowly increasing.

We say that ϕ is a *cusp form* if, in addition to the above conditions, it is cuspidal.

12. We let $\mathscr{A}$ denote the vector space of all automorphic forms; we let $\mathscr{A}_0$ denote the subspace of cusp forms. For a quasi-character η of $F^\times \backslash I$ we let $\mathscr{A}(\eta)$ and $\mathscr{A}_0(\eta)$ denote the subspaces consisting of functions with central quasi-character η. All of these spaces are stable under $\mathscr{H}$ acting via the right regular representation ρ.

13. An irreducible admissible representation of $\mathscr{H}$ is called an *automorphic representation* if it is isomorphic to a constiuent (*i.e.*, subquotient) of $\mathscr{A}$. An irreducible admissible representation of $\mathscr{H}$ is called an *automorphic cuspidal representation* if it is isomorphic to a constiuent of $\mathscr{A}_0$.

14 Proposition. *Automorphic forms are Z_A-finite (Z_A is the center of G_A).*

Let f be an automorphic form. Since f is K-finite there exists a finite set of place S (which we take to contain all the archimedean places) such that f is invariant under K^S. Thus, in particular, f is invariant under $U^S = \prod_{v \notin S} U_v$. If S is sufficiently large then we will have $I = F^\times U^S I_S$ and so to show that f is Z_A-finite it suffices to show that it is Z_S-finite (since f is fixed by $F^\times U^S$). Thus, in fact, it suffices to show that f is Z_v-finite for each place v.

Consider first the case when v is non-archimedean. Let ξ be an elementary idempotent stabilizing f. Since $\rho(\xi)$ and $\rho(Z_v)$ commute and $\rho(G_v) \subset \rho(\mathscr{H})$ for v non-archimedean, we have
$$\rho(Z_v)f = \rho(Z_v)\rho(\xi)f = \rho(\xi)\rho(Z_v)f \subset \rho(\xi)\rho(\mathscr{H})f.$$
By the definition of an automorphic form, the rightmost space is finite dimensional. Thus $\rho(Z_v)f$ is finite dimensional as well.

Now consider the case when v is archimedean. Note that Z_v is isomorphic to $F_v^\times$ which looks like $\mathbb{R}^+$ times some compact group which is contained in K. Since f is K-finite, we can ignore the compact part of $F_v^\times$. Thus it suffices to show that f is $\mathbb{R}^+$-finite. In fact, since the space of automorphic forms is closed under right translation, it suffices to show that the restriction of f to $\mathbb{R}^+ \subset F_v^\times$ is finite.

We now introduce some notation that will make the argument a bit more simple to state. If f is any cusp form, let $\hat{f}$ be the function on $\mathbb{R}$ given by

$$\hat{f}(a) = f\left(\begin{bmatrix} e^a & 0 \\ 0 & e^a \end{bmatrix}\right).$$

The above argument shows that it suffices to prove that $\hat{f}$ is finite for each automorphic form f.

Let Z be the element of the Lie algebra $\mathfrak{g}_v$ given by the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

As in the previous case, $\rho(Z)$ and $\rho(\xi)$ commute and $\rho(Z)^n \in \rho(\mathscr{H}$, so the span of the $\rho(Z)^n f$ is finite dimensional. In particular, we have a reltationship of the form

$$\sum_{i=0}^{p} a_i \rho(Z)^i f = 0.$$

If we apply $\hat{\cdot}$ to the above relation, and use the fact that it is linear and transforms $\rho(Z)$ into differentiation, we deduce the relation

$$\sum_{i=0}^{p} a_i \hat{f}^{(i)} = 0,$$

where $\hat{f}^{(i)}$ denotes the ith derivative of $\hat{f}$. The solutions to this differential equation are sums of polynomials times exponentials, and therefore finite. Thus $\hat{f}$ is finite, which finishes the proof.

8.3.2 An automorphic cuspidal representation is a constituent of $\mathscr{A}_0(\eta)$

15 Proposition (J-L Prop. 10.11). *Let π be an automorphic cuspidal representation. Then there exists a quasi-character η of $I/F^\times$ such that π is a constituent of $\mathscr{A}_0(\eta)$.*

Let f be a cusp form. Since f is Z_A-finite (*cf.*, proposition 14), it follows from §1.6, proposition 126 that there is an expansion of the form

$$f\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} g\right) = \sum_{i=0}^{\infty} \sum_{\eta} \eta(a)(\log|a|)^i f_{\eta,i}(g).$$

where the sum is over all quasi-characters η of $I/F^\times$, $f_{\eta,i}$ is a uniquely determined function, and $f_{\eta,i} = 0$ for i sufficiently large and for η outside of some finite set. Note that all the $f_{\eta,i}$ are cusp forms.

For a finite set S of quasi-characters of $I/F^\times$ and an integer M, let $\mathscr{A}_0(S, M)$ denote the set of cusp forms f for which $f_{i,\eta} = 0$ for $i > M$ or for $\eta \notin S$. This forms a vector and is stable under the action of $\mathscr{H}$. We have

$$\mathscr{A}_0 = \bigcup \mathscr{A}_0(S, M).$$

By §1.7, proposition 134, it follows that any irreducible constituent of $\mathscr{A}_0$ is a constituent of some $\mathscr{A}_0(S, M)$. Furthermore, from the easily seen decomposition

$$\mathscr{A}_0(S, M) = \bigoplus_{\eta \in S} \mathscr{A}_0(\eta, M)$$

and §1.7, proposition 135, it follows that any irreducible constituent of $\mathscr{A}_0(S, M)$ is a constituent of some $\mathscr{A}_0(\eta, M)$. Therefore, we are reduced to showing that any irreducible constituent of $\mathscr{A}_0(\eta, M)$ is a consistutent of $\mathscr{A}_0(\eta)$.

Let f be an element of $\mathscr{A}_0(\eta)$. We have

$$f\left(\begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix} g\right) = f\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\left(\begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} g\right)\right)$$

from which we obtain the identity

$$\sum_{i=0}^{M} \eta(ab)(\log|ab|)^i f_i(g) = \sum_{i=0}^{M} \eta(a)(\log|a|)^i f\left(\begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} g\right).$$

By the uniqueness of the functions f_i we may equate like coefficients and deduce that f_M belongs to $\mathscr{A}_0(\eta)$. This gives us a map $\mathscr{A}_0(\eta, M) \to \mathscr{A}_0(\eta)$ (namely, $f \mapsto f_M$), there kernel of which is obviously $\mathscr{A}_0(\eta, M-1)$; in other words we have an exact sequence

$$0 \longrightarrow \mathscr{A}_0(\eta, M-1) \longrightarrow \mathscr{A}_0(\eta, M) \longrightarrow \mathscr{A}_0(\eta).$$

By §1.7, proposition 133, and an easy induction argument, we are finished.

8.3.3 The admissibility and complete reducibility of $\mathscr{A}_0(\eta)$

16 Proposition (J-L Prop. 10.5, 10.9). *Let η be a quasi-character of $F^\times \backslash I$. The representation $\mathscr{A}_0(\eta)$ is admissible and decomposes as a direct sum of irreducible admissible representations, each occuring with finite multiplicity.*

17. Proposition 16 will take the remained of the section to prove. Note that, together with proposition 15, it shows that any automorphic cuspidal representation occurs both as a subrepresentation and as a quotient of $\mathscr{A}_0(\eta)$ for some η.

8.3.4 The Fourier expansion of a cusp form

18 Proposition (Bump Thm. 3.5.5). *Let ϕ be a cusp form. Define a function W_ϕ on G_A by*

$$W_\phi(g) = \int_{A/F} \phi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) \psi(-x)dx.$$

Then

$$\phi(g) = \sum_{\alpha \in F^\times} W_\phi\left(\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} g\right).$$

The sum is absolutely convergent and uniformly convergent as g varies in compact sets.

Let g be a fixed element of G_A. Define a function $A/F \to \mathbb{C}$ by

$$x \mapsto \phi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right). \tag{3}$$

Since this is a continuous function on the compact group A/F it has a Fourier expansion. Note that the characters of A/F are of the form $x \mapsto \psi(ax)$ where a belongs to F. We thus have

$$\phi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) = \sum_{\alpha \in F} C(\alpha)\psi(\alpha x) \tag{4}$$

where

$$C(\alpha) = \int_{A/F} \phi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) \psi(-\alpha x)dx.$$

Note that since ϕ is cuspidal we have $C(0) = 0$. Since ϕ is automorphic, and thus invariant under G_F, we have

$$\begin{aligned} C(\alpha) &= \int_{A/F} \phi\left(\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) \psi(-\alpha x)dx \\ &= \int_{A/F} \phi\left(\begin{bmatrix} 1 & \alpha x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} g\right) \psi(-\alpha x)dx. \end{aligned}$$

Upon making the change of variables $x \mapsto \alpha^{-1}x$ (which is unimodular), we find

$$C(\alpha) = W_\phi\left(\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} g\right).$$

Writing this in (4), dropping the $\alpha = 0$ term, and setting $x = 0$ gives the stated formula for $\phi(g)$. Since the function (3) belongs to $L^2(A/F)$ the coefficients $C(\alpha)$ are square-summable, and therefore the series for $\phi(g)$ converges absolutely.

8.3.5 Automorphic cuspidal representations have Whittaker models

19 Proposition. *Let (π, V) be a subrepresentation of $\mathscr{A}_0$ (note that this applies to any automorphic cuspidal representation, cf., the remarks following theorem 16). Then π has a Whittaker model.*

Let the function W_ϕ be as in proposition 18. It is clear that W_ϕ satisfies

$$W_\phi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) = \psi(x)W_\phi(g).$$

The necessary smoothness and growth conditions that W_ϕ must satisfy to belong to $\mathscr{W}(\psi)$ can be deduced from the smoothness and growth conditions on ϕ; thus W_ϕ belongs to $\mathscr{W}(\psi)$. The map $\phi \mapsto W_\phi$ therefore gives a map $V \to \mathscr{W}(\psi)$ of $\mathscr{H}$-modules, which is injective by proposition 18. The image of this map is therefore a Whittaker model for V.

8.3.6 The multiplicity one theorem

20 Theorem (J-L Prop. 11.1.1). *If an irreducible representation is contained in $\mathscr{A}_0(\eta)$ then it is contained with multiplicity one.*

This is a logical consequence of the uniquess of Whittaker models (*cf.*, proposition 9) and the Fourier expansion of a cusp form (*cf.*, proposition 18). We spell out the argument nonetheless.

Let U and V be irreducible submodules of $\mathscr{A}_0(\eta)$ which are abstractly isomorphic. Let $\mathscr{W}(U)$ (resp. $\mathscr{W}(V)$) be the collection of all the functions W_ϕ with ϕ in U (resp. ϕ in V). Then $\mathscr{W}(U)$ is literally *equal* to $\mathscr{W}(V)$, since both are Whittaker models for the same isomorphism class of irreducible representations. Since a cusp form ϕ is determined by W_ϕ, it therefore follows that $U = V$.

8.3.7 Automorphic representations which are not cuspidal

21 Proposition (J-L Thm. 10.10). *Let $\pi = \otimes\pi_v$ be an automorphic representation which is not cuspidal, i.e., not a constituent of $\mathscr{A}_0$. Then there are two quasi-characters μ and ν of $I/F^\times$ such that for each place v the representation π_v is a constituent of $\rho(\mu_v, \nu_v)$.*

22. The proof of proposition 21 will take the rest of the section. The strategy of proof is as follows: we first introduce a space $\mathscr{B}$ and show that a non-cuspidal representation occurs as a constituent of $\mathscr{B}$. We then show that the constituents of $\mathscr{B}$ have the required properties.

23. Let $\mathscr{B}$ be the space of all smooth functions f on G_A satisfying the following four properties:

1. f is invariant under N_A on the left, *i.e.*, we have
$$f\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) = f(g)$$
for all x in A and g in G_A.
2. f is invariant under A_F on the left (where A_F is the diagonal subgroup of G_F), *i.e.*, we have
$$f\left(\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} g\right) = f(g)$$
for all α and β in F and g in G_A.
3. f is K-finite on the right.
4. For every elementary idempotent ξ of $\mathscr{H}$ the space $\rho(\xi)\rho(\mathscr{H})f$ is finite dimensional.

Note that $\mathscr{H}$ acts on $\mathscr{B}$ via ρ.

If f belongs to $\mathscr{B}$ and is A_A-finite on the left then, by §1.6, proposition 126, we have an expansion

$$f\left(\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} g\right) = \sum_{i,j,\mu,\nu} \mu(\alpha)\nu(\beta)(\log|\alpha|)^i(\log|\beta|)^j f_{i,j,\mu,\nu}(g) \tag{5}$$

where the sum is extended over all nonnegative integers i and j and all quasi-characters μ and ν of $I/F^\times$. The functions $f_{i,j,\mu,\nu}$ are uniquely defined and only a finite number of them are nonzero. For a finite set

S of pairs of quasi-characters of $I/F^\times$ and an integer M we let $\mathscr{B}(S,M)$ be the space of all left A_A-finite functions f in $\mathscr{B}$ for which $f_{i,j,\mu,\nu}=0$ if $i+j>M$ or (μ,ν) does not belong to S. We write $\mathscr{B}(S)$ in place of $\mathscr{B}(S,0)$ and $\mathscr{B}(\mu,\nu,M)$ in place of $\mathscr{B}(S,M)$ when $S=\{(\mu,\nu)\}$.

24 Proposition (J-L Lemma 10.10.2). *An automorphic representation which is not cuspidal is a constituent of $\mathscr{B}$.*

If ϕ belongs to $\mathscr{A}$ then the function ϕ_0 on G_A defined by

$$\phi_0(g)=\int_{A/F}\phi\left(\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}g\right)dx$$

clearly belongs to $\mathscr{B}$. Furthermore, the map $\phi\mapsto\phi_0$ commutes with $\mathscr{H}$. In this way we obtain a map of $\mathscr{H}$-modules $\mathscr{A}\to\mathscr{B}$, the kernel of which, by definition, is $\mathscr{A}_0$. Thus we have an exact sequence

$$0\longrightarrow\mathscr{A}_0\longrightarrow\mathscr{A}\longrightarrow\mathscr{B}.$$

Therefore by §1.7, proposition 133, any irreducible constituent of $\mathscr{A}$ which is not a constituent of $\mathscr{A}_0$ is necessarily a constituent of $\mathscr{B}$.

25 Lemma (J-L Lemma 10.10.1). *Let f be a function on G_A which is invariant under N_A and A_F on the left and is K-finite on the right. The the two conditions*

1. *for every elementary idempotent ξ of $\mathscr{H}$ the space $\rho(\xi)\rho(\mathscr{H})f$ is finite dimensional;*
2. *f is A_A finite on the left;*

are equivalent. In particular, we have

$$\mathscr{B}=\bigcup\mathscr{B}(S,M).$$

PROVE THIS.

26 Lemma (J-L Lemmas 10.10.3, 10.10.4). *An irreducible constituent of $\mathscr{B}$ is a constituent of $\mathscr{B}(\mu,\nu)$ for some pair of quasi-characters (μ,ν) of $I/F^\times$.*

The proof of this proposition proceeds much like that of proposition 15.

Since $\mathscr{B}$ is the union of the $\mathscr{B}(S,M)$, §1.7, proposition 134 implies that any irreducible constituent of $\mathscr{B}$ is a constituent of some $\mathscr{B}(S,M)$. From the direct sum decomposition

$$\mathscr{B}(S,M)=\bigoplus_{(\mu,\nu)\in S}\mathscr{B}(\mu,\nu,M)$$

we have, by §1.7, proposition 135, that any irreducible constituent of $\mathscr{B}(S,M)$ is a constituent of $\mathscr{B}(\mu,\nu,M)$ for some pair of quasi-characters (μ,ν).

Now, let f belong to $\mathscr{B}(\mu,\nu,M)$ and write

$$f\left(\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}g\right)=\sum_{i+j\le M}\mu(\alpha)\nu(\beta)(\log|\alpha|)^i(\log|\beta|)^j f_{i,j}(g).$$

From the equality

$$f\left(\begin{bmatrix}\alpha_1\alpha_2 & 0\\ 0 & \beta_1\beta_2\end{bmatrix}g\right)=f\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{bmatrix}\left(\begin{bmatrix}\alpha_2 & 0\\ 0 & \beta_2\end{bmatrix}g\right)\right)$$

we obtain the identity

$$\begin{aligned}&\sum_{i+j\le M}\mu(\alpha_1\alpha_2)\nu(\beta_1\beta_2)(\log|\alpha_1\alpha_2|)^i(\log|\beta_1\beta_2|)^j f_{i,j}(g)\\ &=\sum_{i+j\le M}\mu(\alpha_1)\nu(\beta_1)(\log|\alpha_1|)^i(\log|\beta_1|)^j f\left(\begin{bmatrix}\alpha_2 & 0\\ 0 & \beta_2\end{bmatrix}g\right).\end{aligned}$$

Since the functions $f_{i,j}$ are uniquely determined, we may equate like coefficients to deduce that $f_{i,j}$ belongs to $\mathscr{B}(\mu,\nu)$ whenever $i+j=M$. It is also clear that $f_{i,j}=0$ for all $i+j=M$ if and only if f belongs to $\mathscr{B}(\mu,\nu,M-1)$. We thus obtain an exact sequence

$$0 \longrightarrow \mathscr{B}(\mu,\nu,M-1) \longrightarrow \mathscr{B}(\mu,\nu,M) \longrightarrow \bigoplus_{i+j=M} \mathscr{B}(\mu,\nu)$$

where the first map is inclusion and the second map sends f to the tuple $(f_{i,j})$ with $i+j=M$. Using §1.7 propositions 133 and 135 and an inductive argument completes the proof.

27 Lemma (J-L pg. 345). *Let the automorphic representation $\pi=\otimes\pi_v$ be a constituent of $\mathscr{B}(\mu,\nu)$. Then for each place v the representation π_v is a constituent of $\rho(\mu_v,\nu_v)$.*

For almost all places v, both μ_v and ν_v are unramified; at such places $\mathscr{B}(\mu_v,\nu_v)$ contians a unique function ϕ_v° which is right invariant under K_v and has $\phi_v^\circ(1)=1$ (*cf.*, §5.11.2, lemma 196). We may thus form the restricted tensor product

$$\bigotimes_v (\mathscr{B}(\mu_v,\nu_v) : \phi_v^\circ).$$

There is a natural map

$$\bigotimes_v (\mathscr{B}(\mu_v,\nu_v) : \phi_v^\circ) \to \mathscr{B}$$

which is easily seen to be surjective (and is in fact an isomorphism). Thus π is a constituent of $\otimes\rho(\mu_v,\nu_v)$ and so by §1.7, proposition 136, each π_v is a constituent of $\rho(\mu_v,\nu_v)$.

28. This completes the proof of proposition 21.

Bibliography

[Bu] Bump, *Automorphic forms and representations*, Cambridge studies in advanced math, no. 55, Cambridge University Press, 1997.

[JL] H. Jacquet and R. P. Langlands, *Automorphic forms on* GL(2), Lecture notes in mathematics, no. 114, Springer-Verlag, 1970.

[Kn] A. W. Knapp, *Representation theory of semisimple groups*, Princeton University Press, 1986.

[KV] A. W. Knapp and D. A. Vogan, *Cohomological induction and unitary representations*, Princeton University Press, 1995.

[Po] L. S. Pontryagin, *Topological groups*, 2nd ed. (translated by A. Brown), Gordon and Breach, Science Publishers Inc., 1966.

[Sh] G. Shimura, *Introduction to the arithmetic theory of automorphic forms*, Iwanami Shoten, Publishers and Princeton University Press, 1971.