Math 395. Compactness review from Math 296

In Math 296, the theory of compact subsets of finite-dimensional normed vector spaces was developed, and some interesting results (such as the extreme value theorem on such subsets, and their characterization as the closed and bounded subsets) were proved. This handout reviews some of those results and proofs, to convince you that the proofs worked much more generally (in either metric or topological spaces, depending on the situation).

1. Relations with continuity

Recall that the crux of the Extreme Value Theorem was the result that a continuous image of a compact set is compact, at least in the context of subsets of finite-dimensional **R**-vector spaces. The same argument works in general:

Theorem 1.1. *Let $f : X \to X'$ be a continuous map between topological spaces. If $K \subseteq X$ is compact (for the subspace topology) then $f(K) \subseteq X'$ is compact.*

Proof. The inclusion $i : K \to X$ is continuous when using the subspace topology on K, so the map $f \circ i : K \to X'$ is continuous. Its image is $f(K)$, so we may replace f with $f \circ i$ to reduce to the case when X is compact. Also, when the subset $f(X) \subseteq X'$ is given the subspace topology then we have seen that in general (having nothing to do with compactness) continuity of f is the same as that of the surjective map $X \to f(X)$. Hence, since our problem concerns subsets of $f(X)$ and the subspace topology is transitive, we can replace X' with $f(X)$ to reduce to the case when f is surjective.

That is, f is now a continuous surjection with compact source, and we want to prove compactness of the target. Let $\{U'_i\}$ be an open cover of X', so $\{f^{-1}(U'_i)\}$ is an open cover of the compact X. By compactness of X, some finite collection of $f^{-1}(U'_i)$'s covers X, say for $i_1, \dots, i_n$, whence the union of their images $f(f^{-1}(U'_{i_j}))$ covers $f(X) = X'$. Since $f(f^{-1}(U'_i)) \subseteq U'_i$ for each i, it follows that the union of $U'_{i_1}, \dots, U'_{i_n}$ fills up all of X'. ■

A map $f : (X, \rho) \to (X', \rho')$ between metric spaces is *uniformly continuous* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(x_1, x_2) < \delta$ implies $\rho'(f(x_1), f(x_2)) < \varepsilon$. This certainly forces f to be continuous (as it imposes a strong condition, namely that "δ is uniform across X for a given ε"), and it is a natural generalization of uniform continuity in the context of maps between subsets of finite-dimensional normed vector spaces. Note that this notion is a metric condition and not topological. Indeed, the definition uses the metric structure in an essential way, and there is no evident way to discuss "uniformity" in general topological spaces without imposing some extra structure on the space to make possible a comparison of what is going on at different points, so this strongly suggests that uniform continuity is a metric notion. To be rigorous (since we have seen that other notions that seem to be metric concepts, such as the "accumulation point" definition for closedness or the "sequential" definition for continuity, do in fact admit topological formulations), we should really exhibit an example of a continuous map between metrizable spaces that is uniformly continuous with respect to one pair of metrics inducing the given topologies, but is not uniform for another such pair of metrics.

Example 1.2. On **R**, define $\rho(x, y) = |e^x - e^y|$. This is readily seen to be a metric, and it induces the usual topology on **R**. Let $f : \mathbf{R} \to \mathbf{R}$ be the map $f(x) = x$. This is certainly continuous (a topological property), and it is uniformly continuous for the usual metric on both source and target (with $\delta = \varepsilon$). However, if we give the source the usual metric and give the target the metric ρ then uniform continuous with $\varepsilon = 1$ would say that for some $\delta > 0$ we have $\rho(f(x), f(y)) < 1$ whenever $|x - y| < \delta$, which is to say $|e^x - e^y| < 1$ for $|x - y| < \delta$. However, taking $y = x + \delta/2$ for any

$\delta > 0$ gives $|e^x - e^y| = e^x|1 - e^{\delta/2}|$ and with any fixed $\delta > 0$ this gets arbitrarily large (and so certainly ≥ 1) if we use x very large. Thus, uniform continuity does not hold for f with respect to this modified choice of metrics to give the usual topology on $\mathbf{R}$.

Having confirmed that uniform continuity really is a metric notion (or at least that it is definitely not topological), let us prove the natural generalization of the result on uniform continuity from Math 296:

Theorem 1.3. *Let K be a compact metric space and $f : K \to Y$ a continuous map to another metric space. The map f is uniformly continuous.*

Proof. Pick $\varepsilon > 0$. For each $k \in K$ there exists $\delta_k > 0$ such that for any $k' \in K$ satisfying $\rho_K(k', k) < \delta_k$ we have $\rho_Y(f(k'), f(k)) < \varepsilon$. The open balls $B_{\delta_k/2}(k)$ for varying $k \in K$ cover K, so by compactness finitely many do the job, say $B_{\delta_{k_i}/2}(k_i)$ for points $k_1, \ldots, k_n \in K$. Let $\delta = \min_i \delta_{k_i}/2 > 0$. If $k, k' \in K$ satisfy $\rho_K(k, k') < \delta$ then since $k \in B_{\delta_{k_{i_0}}/2}(k_{i_0})$ for some i_0 we have

$$\rho_K(k', k_{i_0}) \leq \rho_K(k', k) + \rho_K(k, k_{i_0}) < \delta + \delta_{k_{i_0}}/2 \leq \delta_{k_{i_0}}/2 + \delta_{k_{i_0}}/2 = \delta_{k_{i_0}},$$

so $k, k' \in B_{\delta_{k_{i_0}}}(k_{i_0})$. Thus, both $f(k)$ and $f(k')$ lie in the open ε-ball around $f(k_{i_0})$, whence by the triangle inequality $\rho_Y(f(k), f(k')) < 2\varepsilon$. To summarize, for $\varepsilon > 0$ we have constructed $\delta > 0$ such that if $k, k' \in K$ satisfy $\rho_K(k, k') < \delta$ then $\rho_Y(f(k), f(k')) < 2\varepsilon$. This is exactly the property of uniform continuity (up to the usual business with universal positive constant multipliers on ε). ■

2. Sequential compactness

A metric space (X, ρ) is *sequentially compact* when every sequence in X has a convergent subsequence. This looks like a metric notion, but it turns out to be purely topological: it only depends on the underlying topological space. For example, in $\mathbf{R}^n$ it was shown that a subset with its subspace topology is compact (a manifestly topological property) if and only if it is sequentially compact. The same result (with suitably generalized proof) works for any metric space:

Theorem 2.1. *Let (X, ρ) be a metric space. It is compact if and only if it is sequentially compact.*

Proof. First assume X is compact, and let $\{x_n\}$ be a sequence in X. If it has no convergent subsequence, then for any $x \in X$ there exists $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x)$ contains x_n for only finitely many n. (Indeed, if for some x no such ε_x exists then each ball $B_{1/N}(x)$ would contain x_n for infinitely many n, and so by taking $N \to \infty$ we could build a subsequence converging to x, contrary to hypothesis.) As an aside, note that we could even shrink ε_x to excise x_n's distinct from x, though it might happen that $x = x_n$ for some (or even finitely many) n and so we certainly cannot shrink ε_x so that $B_{\varepsilon_x}(x)$ is disjoint from the set of x_n's for *all* n.

The open balls $B_{\varepsilon_x}(x)$ for $x \in X$ give an open covering of X, so there is a finite subcover: there exist finitely many x's (which we has better not call $x_1, \ldots, x_m$, as the notation x_i has already been reserved!) whose associated open balls $B_{\varepsilon_x}(x)$ cover X. However, each such ball contains x_n for only finitely many n, and so the union X of the finite collection of such balls likewise contains x_n for only finitely many n. This is absurd.

Conversely, suppose X is sequentially compact. Let $\{U_i\}$ be an open cover of X, and suppose that there is no finite subcover. We seek a contradiction. First of all, X must be totally bounded. Indeed, if for some $\varepsilon_0 > 0$ the space X is not covered by finitely many ε_0-balls then we can recursively make an infinite sequence of points $x_n \in X$ such that x_n is not in the open ε_0-ball around any x_m with $m < n$, and so the sequence $\{x_n\}$ has $\rho(x_n, x_m) \geq \varepsilon_0$ for all $n \neq m$. But such a sequence $\{x_n\}$ clearly cannot have a convergent subsequence, contrary to the assumption that X

is sequentially compact. Hence, for each $N \geq 1$ the space X is covered by finitely balls $B_{1/N}(x)$, say for x in some finite set of points $\Sigma_N \subseteq X$. Since $\{U_i\}$ has no finite subcover yet the finitely many balls $B_{1/N}(x)$ for $x \in \Sigma_N$ do cover X, it follows that for each N there exists some $x_N \in \Sigma_N$ for which the ball $B_{1/N}(x_N)$ is not contained in any U_i. (Indeed, if for some N and each of the finitely many $x \in \Sigma_N$ the ball $B_{1/N}(x)$ is contained in a U_i, then the resulting finitely many such U_i's would cover X, contradicting the assumption that $\{U_i\}$ has no finite subcover.)

Consider the sequence $\{x_N\}$. By sequential compactness, some subsequence $\{x_{N_1}, x_{N_2}, \dots\}$ has a limit $x \in X$. We have $x \in U_{i_0}$ for some i_0, and so by openness there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U_{i_0}$. For a large $N > 2/\varepsilon$ we have $x_N \in B_{\varepsilon/2}(x)$, so since $1/N < \varepsilon/2$ the triangle inequality gives

$$B_{1/N}(x_N) \subseteq B_\varepsilon(x) \subseteq U_{i_0},$$

contradicting how the x_N's were chosen. This contradiction ensures that $\{U_i\}$ has to have a finite subcover after all. ■

3. Products and closed sets

Two important properties of compactness were used in the proof that closed and bounded subsets of $\mathbf{R}^n$ (using a norm metric) are necessarily compact: preservation of compactness under formation of products (such as $[a,b]^n \subseteq \mathbf{R}^n$) and compactness of closed subsets of compact sets. We now prove both results in general topological spaces, adapting the proofs from Math 296.

Theorem 3.1. *A closed subset C in a compact topological space X is automatically compact (with its subspace topology), and if $X_1, \dots, X_n$ are compact topological spaces then so is $\prod X_i$ with the product topology.*

Proof. We first prove that a closed set C in a compact space X is necessarily compact when C is given its subspace topology. Since compactness for a subset (with the subspace topology!) can be checked in terms of coverings by opens from the ambient space, let $\{U_i\}$ be a collection of opens in X whose union contains C. We seek a finite subcollection that does the job too. Well, since C is closed in X, the complement $U = X - C$ is open in X and hence U together with the U_i's actually cover all of X (as the union of the U_i's already contains C). Hence, some finite collection of the U_i's, perhaps together with U, covers X since X is compact. We may or may not have included U in this finite subcover, but either way it doesn't touch C and so the finite subcollection of U_i's just found must have union containing C (as points of C sure don't lie in U). This is the desired finite subcover.

Now we prove that a finite product of compact spaces is compact. As in the case of preservation of connectedness under formation of finite products, we may induct on n to reduce to the case $n = 2$. Let X and X' be compact topological spaces; we want to prove that $X \times X'$ is compact. Let $\{U_i\}$ be an open covering, so we seek a finite subcovering. We argue by slices, exactly as in our study of connectivity (and as in the proof used in Math 296). First, we reduce to the case when the opens in the covering are themselves products of opens in the factor spaces. Each open U_i contains a product of opens $V \times V'$ around each of its points, and so the collection of all such opens $V \times V'$ contained in U_i for *varying* i is also an open cover of $X \times X'$. If this covering has a finite subcover, then each open $V \times V'$ in this finite subcover is contained in some U_i, and the resulting finitely many "bigger" U_i's will clearly also cover $X \times X'$. This would provide the desired finite subcover. Thus, we may suppose each U_i is a product, say $U_i = V_i \times V_i'$ for opens $V_i \subseteq X$ and $V_i' \subseteq X'$.

For each $x \in X$, the slice $\{x\} \times X'$ in $X \times X'$ gets as its subspace topology exactly the original topology on X' (as we saw in the discussion of products of connected spaces). Since X' is compact, it follows that finitely many of the opens U_i cover the slice $\{x\} \times X'$. That is, some finite collection

$U_{i_{1,x}}, \dots, U_{i_{n_x,x}}$ has union containing $\{x\} \times X'$. We have $U_i = V_i \times V_i'$ for every i. Let $V_x = V_{i_{1,x}} \cap \dots V'_{i_{n_x,x}}$, so this is an open in X containing x and $V_x \times V'_{i_{j,x}} \subseteq U_{i_{j,x}}$ for each x. Moreover, V_x meets $\{x\} \times X'$ in the subset $V'_{i_{n_x},x}$ of X', so the covering condition on the slice $\{x\} \times X'$ implies that the $V'_{i_{j,x}}$'s cover X' for $1 \leq j \leq n_x$. Hence, the opens $V_x \times V'_{i_{j,x}} \subseteq U_{i_{j,x}}$ for $1 \leq j \leq n_x$ cover $V_x \times X'$ for each $x \in X$, so each subset $V_x \times X'$ in $X \times X'$ is contained in the union of *finitely many* U_i's (namely $U_{i_j,x}$ for $1 \leq j \leq n_x$). By compactness of X, some finite set of V_x's covers X, say $V_{x_1}, \dots, V_{x_m}$. Thus, the finitely many $V_{x_j} \times X'$'s cover $X \times X'$, and since each of these is in turn contained in a union of finitely many U_i's, the finite collection of such U_i's as we run through the finitely many x_j's gives a finite set of U_i's that covers all of $X \times X'$. ■