Math 396. Maps of vector bundles and $\mathscr{O}$-modules

1. Introduction

Let $(X, \mathscr{O})$ be a C^p premanifold with corners with $0 \le p \le \infty$. In class we gave a recipe for constructing an $\mathscr{O}$-module $\underline{E}$ associated to any C^p vector bundle $\pi : E \to X$: for any non-empty open set $U \subseteq X$, $\underline{E}(U)$ is the $\mathscr{O}(U)$-module $E(U)$ of C^p sections to $E \to X$ over $U \subseteq X$. (If U is empty, we define $\underline{E}(U) = \{0\}$.) We saw in class that $\underline{E} \simeq \mathscr{O}^{\oplus n}$ as $\mathscr{O}$-modules if and only if $E \simeq X \times \mathbf{R}^n$ as C^p vector bundles over X.

Let us recall how the formation of $\underline{E}$ is well-behaved with respect to restriction to open subsets in X. If $X' \subseteq X$ is an open subset, then we claim that the $\mathscr{O}|_{X'}$-module $\underline{E}|_{X'}$ is exactly the one associated to the vector bundle $E|_{X'} \to X'$. The crux is that open subsets $U \subseteq X'$ are exactly the open sets of X that are contained in X' (as X' is open in X), and for such U we have that the module $E(U)$ over $\mathscr{O}(U) = (\mathscr{O}|_{X'})(U)$ is equal to the set of C^p sections of $E \to X$ over U, which is the same as the set of C^p sections of $E|_{X'} \to X'$ over U.

The passage from E to $\underline{E}$ is much better than merely well-behaved with respect to restriction over open sets in X; it is also well-behaved with respect to *variation in* E. More specifically, if $f : E' \to E$ is a bundle morphism between C^p vector bundles $\pi' : E' \to X$ and $\pi : E \to X$ then we get an $\mathscr{O}$-linear map $\underline{f} : \underline{E}' \to \underline{E}$ as follows. We have to define $\mathscr{O}(U)$-linear maps $\underline{f}_U : \underline{E}'(U) \to \underline{E}(U)$ for all opens $U \subseteq X$ such that the $\underline{f}_U$'s are compatible with shrinking U. In view of how $\underline{E}$ and $\underline{E}'$ are defined, this is a collection of compatible $\mathscr{O}(U)$-linear maps $\underline{f}_U : E'(U) \to E(U)$ between $\mathscr{O}(U)$-modules of C^p sections for non-empty open $U \subseteq X$ (and $\underline{f}_\emptyset$ is taken to be the zero map). The definition of $\underline{f}_U$ for non-empty open U is given by composition: for any C^p-section $s : U \to E'$ we define $\underline{f}_U(s) \in E(U)$ to be $f \circ s : U \to E$. To see that $f \circ s$ really makes sense in $E(U)$, we note that it is a C^p map because f and s are C^p, and it is a section to $\pi : E \to X$ over U because $\pi \circ (f \circ s) = (\pi \circ f) \circ s = \pi' \circ s = 1_U$ due to f being a map of vector bundles (giving $\pi \circ f = \pi'$) and s being a section of E' over U (giving $\pi' \circ s = 1_U$). The following lemma ensures that these set-theoretic maps $\underline{f}_U$ for varying U do define a map of $\mathscr{O}$-modules $\underline{E}' \to \underline{E}$:

Lemma 1.1. *For each non-empty open set $U \subseteq X$, $\underline{f}_U : E'(U) \to E(U)$ is an $\mathscr{O}(U)$-linear map. Moreover, if $U' \subseteq U$ is a non-empty open subset then the diagram*

$$\begin{array}{ccc} E'(U) & \xrightarrow{\underline{f}_U} & E(U) \\ \downarrow & & \downarrow \\ E'(U') & \xrightarrow[\underline{f}_{U'}]{} & E(U') \end{array}$$

commutes, where the vertical maps are restrictions.

We do not need to track the situation with the empty set because there is only one set-theoretic map to the zero module over any ring, namely the zero map.

Proof. To check $\mathscr{O}(U)$-linearity, we must show that for $s_1, s_2 \in E'(U)$ and $a_1, a_2 \in \mathscr{O}(U)$,

$$\underline{f}_U(a_1 s_1 + a_2 s_2) = a_1 \cdot \underline{f}_U(s_1) + a_2 \cdot \underline{f}_U(s_2)$$

in $E(U)$. That is,

$$f \circ (a_1 s_1 + a_2 s_2) \stackrel{?}{=} a_1 \cdot (f \circ s_1) + a_2 \cdot (f \circ s_2)$$

in $E(U)$. Equivalently, for each $u \in U$ we need

$$f|_u((a_1s_1 + a_2s_2)(u)) = a_1(u) \cdot f|_u(s_1(u)) + a_2(u) \cdot f|_u(s_2(u))$$

where $f|_u : E'(u) \to E(u)$ is the $\mathbf{R}$-linear fiber map over u induced by the bundle map f over X. But by *definition* of the $\mathscr{O}(U)$-module structure on $E'(U)$ we have $(a_1s_1 + a_2s_2)(u) = a_1(u)s_1(u) + a_2(u)s_2(u)$ in the $\mathbf{R}$-vector space $E'(u)$, so the desired identity on u-fibers just expresses the $\mathbf{R}$-linearity of $f|_u$. This completes the proof that $\underline{f}_U$ is $\mathscr{O}$-linear.

Next, we have to verify the compatibility with respect to restriction to smaller (non-empty) open sets: this is the commutative diagram in the lemma. We have to show that for $s \in E'(U)$, the restriction $(\underline{f}_U(s))|_{U'} \in E(U')$ is equal to $\underline{f}_{U'}(s|_{U'})$. To check such equality of sections over U' it is the same to check at each point $u' \in U'$, so the problem is to prove $(\underline{f}_U(s))(u') = (\underline{f}_{U'}(s|_{U'}))(u')$ in $E(u')$ for all $u' \in U'$. That is, we want $(f \circ s)(u') = (f \circ s|_{U'})(u')$ for all $u' \in U'$. The map $f \circ s : U \to E$ has restriction to $U' \subseteq U$ that is certainly equal to $f \circ s|_{U'}$, so we are done. ∎

The formation of $\underline{f}$ gives a map of sets

$$\mathrm{Hom}_X(E', E) \to \mathrm{Hom}_{\mathscr{O}}(\underline{E}', \underline{E})$$

from the set of C^p vector bundle morphisms to the set of $\mathscr{O}$-linear maps: we send f to $\underline{f}$. (Note that if $E' = E$ then $\underline{\mathrm{id}_E} = \mathrm{id}_{\underline{E}}$.) Both Hom-sets have an $\mathscr{O}(X)$-module structure (we add maps and multiply by global functions in $\mathscr{O}(X)$ in the evident pointwise manner), and reviewing the definition of $\underline{f}$ shows that this map of Hom-sets is $\mathscr{O}(X)$-linear. Of much greater interest is that it is an *isomorphism*, or equivalently that it is bijective. That is, we claim that any $\mathscr{O}$-linear map $\underline{E}' \to \underline{E}$ has the form $\underline{f}$ for a *unique* C^p vector bundle map $f : E' \to E$ over X. The significance of this is that it ensures we can work with vector bundles via the theory of $\mathscr{O}$-modules without losing touch with C^p vector bundle maps.

Before we take up the task of proving the bijectivity result on Hom-sets, we record that passage from f to $\underline{f}$ is also well-behaved with respect to composition:

Lemma 1.2. *If $g : E'' \to E'$ and $f : E' \to E$ are bundle morphisms between C^p vector bundles, then $\underline{f \circ g} : \underline{E}'' \to \underline{E}$ is equal to $\underline{f} \circ \underline{g}$.*

Proof. By definition of bundle morphisms, we must prove that for each open set $U \subseteq X$, the $\mathscr{O}(U)$-linear map $\underline{f \circ g}_U : \underline{E}''(U) \to \underline{E}(U)$ is the composite of $\underline{g}_U : \underline{E}''(U) \to \underline{E}'(U)$ and $\underline{f}_U : \underline{E}'(U) \to \underline{E}(U)$. The case $U = \emptyset$ is trivial (as everything vanishes in this case), so we may assume U is non-empty. We have to prove that composing the map $E''(U) \to E'(U)$ defined by $s'' \mapsto g \circ s''$ and the map $E'(U) \to E(U)$ defined by $s' \mapsto f \circ s'$ gives the map $E''(U) \to E(U)$ defined by $s'' \mapsto (f \circ g) \circ s''$. Since

$$(f \circ g) \circ s'' = f \circ (g \circ s''),$$

we are done. ∎

2. Bijection of Hom sets

The result is this:

Theorem 2.1. *Let X be a C^p premanifold with corners, $0 \le p \le \infty$. For any two C^p vector bundles E and E' on X the map of sets*

$$\mathrm{Hom}_X(E', E) \to \mathrm{Hom}_{\mathscr{O}}(\underline{E}', \underline{E})$$

defined by $f \mapsto \underline{f}$ is bijective; that is, every $\mathscr{O}$-linear map $\underline{E}' \to \underline{E}$ has the form $\underline{f}$ for a unique C^p bundle mapping $f : E' \to E$ over X.

Before we prove the theorem, we record an important corollary.

Corollary 2.2. *Let $\mathscr{M}$ be a locally free $\mathscr{O}$-module with finite rank. If $E \to X$ and $E' \to X$ are C^p vector bundles and $\theta : \underline{E} \simeq \mathscr{M}$ and $\theta' : \underline{E'} \simeq \mathscr{M}$ are $\mathscr{O}$-module isomorphisms then there is a unique C^p isomorphism of bundles $f : E' \simeq E$ such that $\theta \circ \underline{f} = \theta'$. In other words, up to unique isomorphism there is at most one pair (E, θ) for a given $\mathscr{M}$.*

In a later handout it will be proved that for any $\mathscr{M}$ such a pair (E, θ) always exists, and so we may say that the concepts of C^p vector bundle and locally free $\mathscr{O}$-module of finite rank are "the same".

Proof. The necessary and sufficient condition on f is $\underline{f} = \theta^{-1} \circ \theta'$, and Theorem 2.1 ensures that there do exist unique bundle maps $f : E' \to E$ and $f' : E \to E'$ such that $\underline{f} = \theta^{-1} \circ \theta'$ and $\underline{f'} = \theta'^{-1} \circ \theta$. We have to prove that f is a bundle isomorphism, and we shall actually prove that f' is inverse to f. Using Lemma 1.2 we get

$$\underline{f \circ f'} = \underline{f} \circ \underline{f'} = \theta^{-1} \circ \theta' \circ \theta'^{-1} \circ \theta = \mathrm{id}_{\underline{E}} = \underline{\mathrm{id}_E},$$

so by the injectivity in Theorem 2.1 we must have $f \circ f' = \mathrm{id}_E$. Similarly, $f' \circ f = \mathrm{id}_{E'}$, so f and f' are inverse to each other. In particular, f is an isomorphism of C^p vector bundles. ■

Now we prepare to prove Theorem 2.1. We prove the bijectivity on Hom-sets in two stages: for trivial bundles and then in the general case. First assume that E and E' are trivial, say with trivializing sections $s_1, \dots, s_n$ in $E(X)$ and $s'_1, \dots, s'_{n'}$ in $E'(X)$. To give a map $E' \to E$ is to specify where the s'_j's go, say $s'_j \mapsto \sum a_{ij} s_i$ for $a_{ij} \in \mathscr{O}(X)$ for $1 \le j \le n'$ and $1 \le i \le n$. The trivializations identify $\underline{E}$ with $\mathscr{O}^{\oplus n}$ and $\underline{E'}$ with $\mathscr{O}^{\oplus n'}$, so we have to prove that the only compatible collections of $\mathscr{O}(U)$-linear maps $T_U : \mathscr{O}(U)^{\oplus n'} \to \mathscr{O}(U)^{\oplus n}$ for varying U are those given by

$$(c_1, \dots, c_{n'}) \mapsto \left(\sum_j a_{1j}|_U \cdot c_j, \dots, \sum_j a_{nj}|_U \cdot c_j \right)$$

for unique $a_{ij} \in \mathscr{O}(X)$.

If we are given a compatible collection of T_U's, then by compatibility with restriction from X to U we have

$$T_U((c_1, \dots, c_{n'})) = T_U \left(\sum_j c_j \cdot e_j|_U \right) = \sum_j c_j \cdot T_U(e_j|_U) = \sum_j c_j \cdot T_X(e_j)|_U.$$

Thus, from the expressions $T_X(e_j) = (a_{1j}, \dots, a_{nj}) \in \mathscr{O}(X)^{\oplus n}$ we see that $T = \{T_U\}$ arises from such a_{ij}'s. Moreover, the a_{ij}'s are uniquely determined from the $T_X(e_j)$'s, so this settles the case when E and E' are trivial.

Now we pass to the general case. Let $\{U_i\}$ be an open covering of X on which E and E' become trivial. (To find such a cover, we first find trivializing open covers $\{X_k\}$ for E and $\{X'_{k'}\}$ for E', and we take the U_i's to be the overlaps $X_k \cap X_{k'}$ indexed by ordered pairs $i = (k, k')$. Each $x \in X$ lies in some X_k and some $X'_{k'}$, so x lies in some overlap $X_k \cap X'_{k'}$. Hence, these U_i's do indeed form a trivializing *cover* for E and E'.) Let $E_i = E|_{U_i}$ and $E_{ij} = E|_{U_i \cap U_j}$, and similarly for E', so the bundles E_i and E'_i on U_i are trivial and the bundles E_{ij} and E'_{ij} on U_{ij} are trivial. We will systematically use the settled case of trivial bundles, applied to the restrictions of E and E' over U_i and $U_i \cap U_j$ for all i and j. We will also make frequent use of the observation that for any inclusion of open sets $U' \subseteq U$ in X (such as U_{ij} inside of U_i, or U_i inside of X) the diagram of Hom-sets

$$\begin{array}{ccc} \mathrm{Hom}_U(E', E) & \longrightarrow & \mathrm{Hom}_{\mathscr{O}|_U}(\underline{E}'|_U, \underline{E}|_U) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_U(E'|_{U'}, E|_{U'}) & \longrightarrow & \mathrm{Hom}_{\mathscr{O}|_{U'}}(\underline{E}'|_{U'}, \underline{E}|_{U'}) \end{array}$$

is commutative.

To prove injectivity in the general case, suppose $f, g : E' \rightrightarrows E$ are bundle morphisms such that $\underline{f} = \underline{g}$. This implies the equality $\underline{f}|_{U_i} = \underline{g}|_{U_i}$ of $\mathscr{O}|_{U_i}$-module maps for all i, which is to say that the bundle morphisms $f|_{U_i}, g|_{U_i} : E'|_{U_i} \rightrightarrows E|_{U_i}$ induce the same $\mathscr{O}|_{U_i}$-module maps for all i. Hence, by injectivity in the settled case of trivial bundles (applied over the base space U_i!) it follows that $f|_{U_i} = g|_{U_i}$ for all i, so $f = g$. This proves injectivity in general.

Turning to the case of surjectivity, let $\varphi : \underline{E}' \to \underline{E}$ be a map of $\mathscr{O}$-modules. We seek to construct a bundle morphism $f : E' \to E$ such that $\underline{f} = \varphi$. Let $\varphi_i = \varphi|_{U_i}$ as a map of $\mathscr{O}|_{U_i}$-modules for all i. By the settled case of trivial bundles (applied over the base space U_i!) we have $\varphi_i = \underline{f}_i$ for a unique bundle morphism $f_i : E'|_{U_i} \to E|_{U_i}$ for all i. Consider the two bundle morphisms

$$f_i|_{U_i \cap U_j}, f_j|_{U_i \cap U_j} : E'|_{U_i \cap U_j} \rightrightarrows E|_{U_i \cap U_j}.$$

These give rise to $\mathscr{O}|_{U_i \cap U_j}$-module maps $\underline{f}_i|_{U_i \cap U_j} = \varphi_i|_{U_i \cap U_j}$ and $\underline{f}_j|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ that are equal: they coincide with $\varphi|_{U_i \cap U_j}$. Hence, by injectivity for the settled case of trivial bundles (applied over the base space $U_i \cap U_j$!) we get $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i and j. This says that the f_i's satisfy the hypotheses for gluing of bundle morphisms, so there is a unique bundle morphism $f : E' \to E$ such that $f|_{U_i} = f_i$ for all i. Hence, $\underline{f}|_{U_i} = \underline{f}_i = \varphi_i = \varphi|_{U_i}$ for all i.

To conclude that $\underline{f} = \varphi$, thereby settling surjectivity, it remains to prove that if $\{U_i\}$ is an open covering of X and $\varphi, \psi : \mathscr{M}' \rightrightarrows \mathscr{M}$ is a pair of $\mathscr{O}$-linear maps of $\mathscr{O}$-modules such that $\varphi|_{U_i} = \psi|_{U_i}$ as maps from $\mathscr{M}'|_{U_i}$ to $\mathscr{M}|_{U_i}$ for all i, then $\varphi = \psi$. That is, we want $\varphi_U = \psi_U$ as maps from $\mathscr{M}'(U)$ to $\mathscr{M}(U)$ for all open $U \subseteq X$. Choose $s' \in \mathscr{M}'(U)$, so we want $\varphi_U(s') = \psi_U(s')$ in $\mathscr{M}(U)$. Since $\{U \cap U_i\}$ is an open covering of U, to check equality in $\mathscr{M}(U)$ it suffices to check equality of restrictions in $\mathscr{M}(U \cap U_i)$ for all i. Thus, we pick an i and need to prove $\varphi_U(s')|_{U \cap U_i} = \psi_U(s')|_{U \cap U_i}$ in $\mathscr{M}(U \cap U_i)$. But since φ and ψ are maps of $\mathscr{O}$-modules, we have compatibilities with respect to restriction to smaller opens. In particular,

$$\varphi_U(s')|_{U \cap U_i} = \varphi_{U \cap U_i}(s'|_{U \cap U_i}), \ \ \psi_U(s')|_{U \cap U_i} = \psi_{U \cap U_i}(s'|_{U \cap U_i})$$

in $\mathscr{M}(U \cap U_i)$. Hence, it suffices to prove $\varphi_{U \cap U_i} = \psi_{U \cap U_i}$. But by hypothesis $\varphi|_{U_i} = \psi|_{U_i}$ as maps from $\mathscr{M}'|_{U_i}$ to $\mathscr{M}|_{U_i}$, so in particular these restrictions over U_i induce the same maps from $\mathscr{M}'(U \cap U_i)$ to $\mathscr{M}(U \cap U_i)$. This is exactly the desired equality of maps $\varphi_{U \cap U_i} = \psi_{U \cap U_i}$.