

MATH 396. EQUIVALENCE BETWEEN C^p -STRUCTURES AND MAXIMAL C^p -ATLASES

Fix $0 \leq p \leq \infty$, and let X be a topological premanifold. We want to explain in a precise sense how the concepts of C^p -structure on X and “maximal” C^p -atlas on X are equivalent notions. This is largely a matter of carefully unwinding definitions, but since each viewpoint is helpful in various situations it is also worthwhile to know that we can use both perspectives to describe the same concept. In this handout we shall assume $X \neq \emptyset$ (as the case $X = \emptyset$ presents no difficulties).

1. DEFINITIONS

Let $\mathcal{A} = \{(\phi_i, U_i)\}$ and $\mathcal{A}' = \{(\phi'_{i'}, U'_{i'})\}$ be two C^p -atlases on X , so $\phi_i : U_i \rightarrow V_i$ and $\phi'_{i'} : U'_{i'} \rightarrow V'_{i'}$ are homeomorphisms onto non-empty open subsets of finite-dimensional \mathbf{R} -vector spaces, and the resulting homeomorphisms

$$\phi_{i_1} \circ \phi_{i_2}^{-1} : \phi_{i_2}(U_{i_1} \cap U_{i_2}) \rightarrow \phi_{i_1}(U_{i_1} \cap U_{i_2}), \quad \phi'_{i'_1} \circ \phi'_{i'_2}{}^{-1} : \phi'_{i'_2}(U'_{i'_1} \cap U'_{i'_2}) \rightarrow \phi'_{i'_1}(U'_{i'_1} \cap U'_{i'_2})$$

between open domains in vector spaces are C^p isomorphisms in the usual sense.

Let us say that a C^p -atlas $\mathcal{A} = \{(\phi_i, U_i)\}_{i \in I}$ is *standardized* if two conditions hold: (i) for each (ϕ_i, U_i) in \mathcal{A} the target vector space for $\phi_i : U_i \rightarrow V_i$ is a Euclidean space \mathbf{R}^{n_i} (with n_i uniquely determined by U_i , as U_i is non-empty), and (ii) \mathcal{A} has no repetitions in the sense that whenever $i \neq j$ we have that either $U_i \neq U_j$ or, when $U_i = U_j$ (so $n_i = n_j$, as $U_i = U_j$ is non-empty) the maps $\phi_i, \phi_j : U_i \rightarrow \mathbf{R}^{n_i}$ do not coincide. Since we are insisting on the lack of repetitions in \mathcal{A} , we may and do drop the indexing set for such atlases: a standardized C^p atlas is a certain kind of subset of the set of pairs (ϕ, U) where $U \subseteq X$ is a non-empty open set and $\phi : U \rightarrow \mathbf{R}^n$ is a homeomorphism onto an open subset of a Euclidean space. (Note that n is permitted to vary, though it is determined by (ϕ, U) since $U \neq \emptyset$.)

If \mathcal{A} and \mathcal{A}' are standardized C^p -atlases on X , then it makes sense to ask if $\mathcal{A} \subseteq \mathcal{A}'$. This means that each $(\phi, U) \in \mathcal{A}$ (with $\phi : U \rightarrow \mathbf{R}^n$ a homeomorphism onto an open subset) is equal to some $(\phi', U') \in \mathcal{A}'$ (with $\phi' : U' \rightarrow \mathbf{R}^{n'}$ a homeomorphism onto an open subset). Here, “equality” means $U = U'$ (so $n = n'$) and the maps $\phi, \phi' : U \rightarrow \mathbf{R}^n$ coincide. We say that a standardized atlas \mathcal{A}' *dominates* a standardized atlas \mathcal{A} if $\mathcal{A} \subseteq \mathcal{A}'$ in the sense just defined. It is clear that if two standardized atlases dominate each other then they are literally equal.

A standardized C^p -atlas \mathcal{A} on X is *maximal* if it is not strictly contained inside of another standardized C^p -atlas on X . Maximal (standardized) atlases exist in abundance:

Theorem 1.1. *If \mathcal{A} is a standardized C^p -atlas on X , then it is contained in a unique maximal one.*

Proof. Let $U_0 \subseteq X$ be a non-empty open subset and let $\phi_0 : U_0 \rightarrow \mathbf{R}^{n_0}$ be a homeomorphism onto an open subset of a Euclidean space. Say that (ϕ_0, U_0) is *compatible* with \mathcal{A} if for all $(\phi, U) \in \mathcal{A}$ the homeomorphism

$$\phi \circ \phi_0^{-1} : \phi_0(U \cap U_0) \simeq \phi(U \cap U_0) \subseteq \mathbf{R}^{n_0}$$

between open sets in Euclidean spaces is a C^p isomorphism in the usual sense. (This condition is automatically satisfied for those (ϕ, U) such that $U \cap U_0$ is empty.) Define $\widetilde{\mathcal{A}}$ to be the set of pairs (ϕ_0, U_0) that are compatible with \mathcal{A} . Because the C^p property for a continuous map between open sets in Euclidean spaces is a local property on the source and target, and the C^p property is preserved under composition, it is a matter of definition-chasing (using that \mathcal{A} is a C^p -atlas!) to check that $\widetilde{\mathcal{A}}$ is a C^p -atlas and that moreover it is maximal. We leave the (largely mechanical) details to the reader. ■

As the proof shows, the notion of a maximal (standardized) C^p -atlas is rather non-computable and far out. On the other hand, atlases are somewhat clumsy objects since many different atlases seem to want to define the “same” differentiable structure on X . It would definitely be wrong to say that a differentiable structure “is” the data of an atlas (of suitable differentiability type), and so in the old days the notion of maximal (standardized) C^p -atlas was introduced to give a precise condition under which two (standardized) C^p -atlases \mathcal{A} and \mathcal{A}' define the “same” differentiable structure on X : they do so exactly when they lie in the same maximal standardized C^p -atlas. This can be expressed in rather down-to-earth terms, without the intervention of maximal atlases:

Theorem 1.2. *Let \mathcal{A} and \mathcal{A}' be two standardized C^p -atlases on X . They lie in the same maximal standardized C^p -atlas if and only if each $(\phi, U) \in \mathcal{A}$ and $(\phi', U') \in \mathcal{A}'$ are C^p -related in the sense that the homeomorphism*

$$(1) \quad \phi' \circ \phi^{-1} : \phi(U \cap U') \simeq \phi'(U \cap U')$$

between open sets in Euclidean spaces is a C^p isomorphism.

Proof. By the method of construction of the maximal atlas containing a given one, if \mathcal{A} and \mathcal{A}' lie in the same maximal C^p -atlas then the maps in (1) are C^p -isomorphisms. Now suppose that these transition maps are C^p -isomorphisms, and we want to prove that \mathcal{A} and \mathcal{A}' lie in the same maximal C^p -atlas. Consider the union $\mathcal{A} \cup \mathcal{A}'$. This is a C^p -atlas on X precisely because of the assumption that \mathcal{A} and \mathcal{A}' are C^p -atlases and because of the hypothesis on the $\phi' \circ \phi^{-1}$'s. It is also clearly a standardized atlas, and so since it contains \mathcal{A} and \mathcal{A}' we see that the unique maximal C^p -atlas containing this union is also the unique maximal one containing \mathcal{A} and the unique maximal one containing \mathcal{A}' . ■

It is tempting to use the condition in the preceding theorem as a notion of *equivalence* for standardized C^p -atlases on X (it is an equivalence relation, as one sees concretely by thinking about triple overlaps or more simply by using the conclusion in the theorem). Thus, for old-timers a maximal (standardized) C^p -atlas is a single mathematical structure that is a preferred representative of an equivalence class and hence was taken to be the definition of “the” differentiable structure on X arising from some given standardized C^p -atlas. However, the notion of C^p -atlas is sometimes ill-suited to certain clean coordinate-free ways of thinking for which the concept of C^p -structure is much more convenient. It is for this reason that we want to explain how to pass between the two points of view.

2. FROM C^p -STRUCTURES TO MAXIMAL C^p -ATLASES

Let \mathcal{O} be a C^p -structure on X . Let \mathcal{A} be the set of all pairs (ϕ, U) where $U \subseteq X$ is a non-empty open set and $\phi : (U, \mathcal{O}|_U) \rightarrow \mathbf{R}^n$ is a C^p -isomorphism onto an open set $\phi(U) \subseteq \mathbf{R}^n$ (with \mathbf{R}^n given its usual C^p -structure). The collection \mathcal{A} is a C^p -atlas because of two facts: a composite of C^p maps is C^p , and for maps between opens in finite-dimensional \mathbf{R} -vector spaces the “old” notion of C^p is the same as the “new” notion (in terms of structured \mathbf{R} -spaces). It is obvious that \mathcal{A} is standardized. We want to prove that the standardized C^p -atlas \mathcal{A} is maximal.

From the construction of the unique maximal C^p atlas containing \mathcal{A} , to prove maximality of \mathcal{A} it suffices to give an affirmative answer to the following problem. Let U_0 be a non-empty open set in X and let $\phi_0 : U_0 \rightarrow \mathbf{R}^{n_0}$ be a homeomorphism onto an open subset. Assume that for each $(\phi, U) \in \mathcal{A}$ the maps $\phi_0 \circ \phi^{-1} : \phi(U \cap U_0) \rightarrow \phi_0(U \cap U_0) \subseteq \mathbf{R}^{n_0}$ are C^p isomorphisms in the usual sense (with $\phi(U \cap U_0)$ open in the target \mathbf{R}^n of ϕ , so $n = n_0$ if $U \cap U_0$ is non-empty). The question is this: is ϕ_0 a C^p isomorphism onto $\phi_0(U_0)$? We must prove that it is. Since ϕ_0 is a homeomorphism, this problem is local on U_0 . Thus, we may work separately over those overlaps

$U \cap U_0$ that are non-empty, so we can assume $U \subseteq U'$ for some $(\phi', U') \in \mathcal{A}$. In other words, we have a C^p -isomorphism $\phi' : U' \simeq \phi'(U') \subseteq \mathbf{R}^{n'}$, so $n' = n$, and by hypothesis $\phi_0 \circ \phi'^{-1}$ is a C^p isomorphism from $\phi'(U_0)$ onto $\phi_0(U_0)$. Composing with ϕ' , it follows that $\phi_0 : U_0 \rightarrow \mathbf{R}^n$ is a C^p isomorphism onto $\phi_0(U_0)$. This completes the proof that \mathcal{A} is maximal.

Let us see that we can recover \mathcal{O} from \mathcal{A} :

Theorem 2.1. *For any non-empty open $U_0 \subseteq X$, $\mathcal{O}(U_0)$ is the set of functions $f : U_0 \rightarrow \mathbf{R}$ such that for each $(\phi, U) \in \mathcal{A}$, the function $f \circ \phi^{-1} : \phi(U \cap U_0) \rightarrow \mathbf{R}$ is C^p on the open subset $\phi(U \cap U_0)$ in the target Euclidean space \mathbf{R}^n for ϕ .*

Proof. The condition that $f \circ \phi^{-1}$ be C^p on the open set $\phi(U \cap U_0)$ says exactly that $f \circ \phi^{-1} \in \mathcal{O}_{\mathbf{R}^n}(\phi(U \cap U_0))$, with $\mathcal{O}_{\mathbf{R}^n}$ denoting the usual C^p -structure on \mathbf{R}^n . Thus, since ϕ defines a C^p -isomorphism between $(U, \mathcal{O}|_U)$ and $(\phi(U), \mathcal{O}_{\mathbf{R}^n}|_{\phi(U)})$, by the definition of \mathcal{A} in terms of \mathcal{O} , it follows that composition with ϕ^{-1} carries $\mathcal{O}_{\mathbf{R}^n}(\phi(U'))$ bijectively over to $\mathcal{O}(U')$ for any open subset $U' \subseteq U$. Taking $U' = U \cap U_0$, we conclude that the condition on f with respect to (ϕ, U) in the theorem says exactly that $f \in \mathcal{O}(U \cap U_0)$. Since \mathcal{A} is an atlas, so as we vary $(\phi, U) \in \mathcal{A}$ the opens U cover X , it follows that as we vary $(\phi, U) \in \mathcal{A}$ the opens $U \cap U_0$ cover U_0 . By the *locality axiom* for the \mathbf{R} -space structure \mathcal{O} , it follows that $f : U_0 \rightarrow \mathbf{R}$ lies in $\mathcal{O}(U_0)$ if and only if its restriction to each $U \cap U_0$ lies in $\mathcal{O}(U \cap U_0)$, and hence if and only if $f \circ \phi^{-1} : \phi(U \cap U_0) \rightarrow \mathbf{R}$ is a C^p function on the open set $\phi(U \cap U_0)$ in \mathbf{R}^n . ■

To summarize: we have constructed an injective map from the set of C^p -structures on X into the set of maximal standardized C^p atlases on X . The aim of the next section is to prove that this is surjective.

3. FROM MAXIMAL C^p -ATLASES TO C^p -STRUCTURES

Now let \mathcal{A} be a maximal standardized C^p -atlas on X . We seek to construct a C^p -structure \mathcal{O} on X such that it gives rise to \mathcal{A} by the construction of the preceding section. The definition is quite simple: for any non-empty open set $U_0 \subseteq X$, we define $\mathcal{O}(U_0)$ to be the set of functions $f : U_0 \rightarrow \mathbf{R}$ such that for all $(\phi, U) \in \mathcal{A}$, the composite map

$$f \circ \phi^{-1} : \phi(U \cap U_0) \rightarrow \mathbf{R}$$

is a C^p function on the open subset $\phi(U \cap U_0)$ in the Euclidean space \mathbf{R}^n that is the target of ϕ . Also define $\mathcal{O}(\emptyset) = \{0\}$. We have to prove that \mathcal{O} is a C^p -structure and that it gives rise to the maximal C^p -atlas \mathcal{A} via the construction of the preceding section.

Lemma 3.1. *The correspondence $U_0 \mapsto \mathcal{O}(U_0)$ is an \mathbf{R} -space structure on X . For any $(\phi, U) \in \mathcal{A}$ and open $U_0 \subseteq U$, $\mathcal{O}(U_0)$ is the set of $f : U_0 \rightarrow \mathbf{R}$ such that $f \circ \phi^{-1} : \phi(U_0) \rightarrow \mathbf{R}$ is a C^p function on the open domain $\phi(U_0)$ in a Euclidean space.*

The significance of the second claim in the lemma is that for “sufficiently small” opens U_0 in X , membership in $\mathcal{O}(U_0)$ is determined by using *one* element in the given C^p -atlas \mathcal{A} , and in fact *any* one whose underlying open contains U_0 . This is the essence of the notion of an atlas, as the proof will show.

Proof. The usual notion of C^p function on an open set in a Euclidean space is preserved under restriction to smaller opens and can be checked by working on an open covering. Thus, the first claim in the lemma follows easily from the definition of \mathcal{O} . This step hardly used the “atlas” aspect of \mathcal{A} .

Now pick $(\phi, U) \in \mathcal{A}$ and choose an open subset $U_0 \subseteq U$. By definition, $\mathcal{O}(U_0)$ is the set of functions $f : U_0 \rightarrow \mathbf{R}$ such that $f \circ \phi'^{-1} : \phi'(U' \cap U_0) \rightarrow \mathbf{R}$ is a C^p function for *every* $(\phi', U') \in \mathcal{A}$. Thus, the problem is to show the sufficiency of using just the single element (ϕ, U) . Since $U_0 \subseteq U$, for any $(\phi', U') \in \mathcal{A}$ we have $U' \cap U_0 \subseteq U' \cap U$. By the definition of a C^p -atlas, the homeomorphism

$$\phi' \circ \phi^{-1} : \phi(U \cap U') \simeq \phi'(U \cap U')$$

between open domains in Euclidean spaces is a C^p isomorphism. Hence, composition with this map induces a bijection between sets of C^p functions on corresponding open subsets. In particular, for the open subsets $\phi(U_0 \cap U')$ and $\phi'(U_0 \cap U')$ we conclude that composition with $\phi' \circ \phi^{-1}$ induces a bijection between the sets of C^p functions on these opens. Hence, $f \circ \phi'^{-1} : \phi'(U' \cap U_0) \rightarrow \mathbf{R}$ is a C^p function if and only if

$$(f \circ \phi'^{-1}) \circ (\phi' \circ \phi^{-1}) = f \circ \phi^{-1} : \phi(U_0 \cap U') \rightarrow \mathbf{R}$$

is a C^p function. Note that $\phi(U_0 \cap U')$ is an open subset of $\phi(U_0)$. Thus, since the property of being a C^p function on an open domain in a Euclidean space is preserved under restriction to an open subset, the assumption that $f \circ \phi^{-1}$ is a C^p function on $\phi(U_0 \cap U) = \phi(U_0)$ (!) implies that the restriction of $f \circ \phi^{-1}$ to $\phi(U_0 \cap U')$ is also a C^p function. ■

Lemma 3.2. *The \mathbf{R} -space structure \mathcal{O} on X is a C^p -structure.*

Proof. As we vary $(\phi, U) \in \mathcal{A}$, the opens U cover X . Hence, it suffices to show that each \mathbf{R} -structure space $(U, \mathcal{O}|_U)$ is isomorphic (as an \mathbf{R} -structured space!) to an open domain in a Euclidean space (with its usual \mathbf{R} -space structure). Consider the homeomorphism $\phi : U \simeq \phi(U)$ onto an open subset in some \mathbf{R}^n . We claim that this is an isomorphism of structured \mathbf{R} -spaces. It suffices to show that for each open set $U_0 \subseteq U$, composition with ϕ defines a bijection between elements of $\mathcal{O}(U_0)$ and C^p functions on $\phi(U_0)$. This property is exactly the content of the second claim in the preceding lemma! ■

Now that we have proved \mathcal{O} is a C^p -structure on X , it remains to check the maximal standardized C^p -atlas \mathcal{A}' arising from \mathcal{O} under the construction of the preceding section is in fact \mathcal{A} . Since \mathcal{A} is maximal, it suffices to prove $\mathcal{A} \subseteq \mathcal{A}'$. That is, for each $(\phi, U) \in \mathcal{A}$ we want $(\phi, U) \in \mathcal{A}'$. By the construction of \mathcal{A}' in terms of \mathcal{O} , this means that the homeomorphism $\phi : U \simeq \phi(U) \subseteq \mathbf{R}^n$ is a C^p -isomorphism with respect to the C^p -structure $\mathcal{O}|_U$ on U and the C^p -structure on $\phi(U)$ as an open subset of \mathbf{R}^n (endowed with its usual C^p -structure). That is, for each open subset $U_0 \subseteq U$ we have to prove that composition with ϕ defines a bijection between $\mathcal{O}(U_0)$ and the set of C^p -functions on $\phi(U_0)$. This is precisely the condition that we verified in the proof of Lemma 3.2.