Math 396. Equivalence of bundles and $\mathscr{O}$-modules

Let $(X, \mathscr{O})$ be a C^p premanifold with corners, $0 \le p \le \infty$. In this handout, we establish the important result that "equates" the concepts of C^p vector bundle $\pi : E \to X$ and locally free finite-rank $\mathscr{O}$-module $\mathscr{M}$. (See Theorem 3.1.) This will be a very useful tool in some later considerations with differential equations on vector bundles (especially in the theory of connections and geodesics on Riemannian manifolds).

To any C^p vector bundle E on X we have associated a locally free $\mathscr{O}$-module $\underline{E}$ of finite rank, with $\underline{E}|_U = \underline{E|_U}$ for all open $U \subseteq X$, and to any bundle morphism $f : E' \to E$ we associated an $\mathscr{O}$-linear map $\underline{f} : \underline{E}' \to \underline{E}$. By construction, $\underline{f}|_U = \underline{f|_U}$ for open $U \subseteq X$. It was proved that the resulting $\mathscr{O}(X)$-linear map $\mathrm{Hom}_X(E', E) \to \mathrm{Hom}_{\mathscr{O}}(\underline{E}', \underline{E})$ given by $f \mapsto \underline{f}$ is compatible with composition (for a third C^p vector bundle on X) and is an isomorphism.

Our present goal is to reverse this process: to any locally free $\mathscr{O}$-module of finite rank $\mathscr{M}$, we want to associate a C^p vector bundle $V_{\mathscr{M}}$. For a C^p vector bundle E we will show that $V_{\underline{E}}$ is naturally isomorphic to E and for a locally free $\mathscr{O}$-module of finite rank $\mathscr{M}$ we will show that $\underline{V_{\mathscr{M}}}$ (which we shall denote $\underline{V}_{\mathscr{M}}$ for typographical simplicity) is naturally isomorphic to $\mathscr{M}$. In this sense, the constructions of $\underline{E}$ from E and of $V_{\mathscr{M}}$ from $\mathscr{M}$ are inverse to each other.

1. Some preliminaries with germs and fibers at a point

Let $\mathscr{M}$ be an $\mathscr{O}$-module over X. For a point $x \in X$, just as we defined the local ring $\mathscr{O}_x$ of germs of C^p functions near x (via representative pairs (U, h) for open U around x and $h \in \mathscr{O}(U)$, with equivalence relation via equality near x), we define a set $\mathscr{M}_x$ of *germs of $\mathscr{M}$ at x* as follows. For two pairs (U, s) and (U', s') consisting of opens U, U' around x and $s \in \mathscr{M}(U)$ and $s' \in \mathscr{M}(U')$, we say $(U, s) \sim_x (U', s')$ if $s|_W = s'|_W$ in $\mathscr{M}(W)$ for some open $W \subseteq U \cap U'$ around x; this is an equivalence relation (check!). The equivalence class of (U, s) is denoted $[(U, s)]_x$, or just s_x if we are lazy (since U is not so important). The set $\mathscr{M}_x$ is naturally an $\mathscr{O}_x$-module: we apply linear operations with representatives over a small open around x, the choice of which does not matter up to the equivalence relation.

Recall from the handout on modules and derivations that if M is a module over a ring R and if $J \subseteq R$ is an ideal then we write $JM \subseteq M$ to denote the subset of elements consisting of finite sums $\sum a_i m_i$ with $a_i \in J$ and $m_i \in M$; this is an R-submodule of M precisely because J is an ideal. (Remember that JM is *not* the subset of products am with $a \in J$ and $m \in M$: this subset is not stable under addition in M in general.) The case of relevance to us now is $R = \mathscr{O}_x$ and J the ideal $\mathfrak{m}_x = \ker(\mathscr{O}_x \twoheadrightarrow \mathbf{R})$ of germs of C^p functions around x that vanish at x.

Definition 1.1. The *fiber* of an $\mathscr{O}$-module $\mathscr{M}$ at a point $x \in X$ is the $\mathbf{R}$-vector space $\mathscr{M}_x/\mathfrak{m}_x\mathscr{M}_x$.

In general this fiber may be infinite-dimensional. In most interesting examples $\mathscr{M}_x$ and $\mathfrak{m}_x\mathscr{M}_x$ are infinite-dimensional, but the quotient space $\mathscr{M}(x)$ is actually *finite-dimensional* in many interesting cases:

Example 1.2. Let $E \to X$ be a C^p vector bundle and let $\mathscr{M} = \underline{E}$. In this case I claim that $\mathscr{M}(x)$ is naturally isomorphic to $E(x)$! (This is why we call $\mathscr{M}(x)$ the "fiber at x" in general.) For any open set U around x, there is a natural map $\mathscr{M}(U) = E(U) \to E(x)$ linear over the evaluation map $\mathscr{O}(U) \to \mathbf{R}$ at x via $s \mapsto s(x)$, and if $(U, s) \sim_x (U', s')$ for another open U' around x then $s(x) = s'(x)$. Hence, we get a *well-defined* map $\mathscr{M}_x \to E(x)$ via $[(U, s)]_x \mapsto s(x)$, and this is linear over the evaluation map $\mathscr{O}_x \to \mathbf{R}$. Since evaluation kills $\mathfrak{m}_x$, we see that our map $\mathscr{M}_x \to E(x)$ kills $\mathfrak{m}_x\mathscr{M}_x$, and so it induces an $\mathbf{R}$-linear map $\mathscr{M}(x) \to E(x)$. Our task is to show that this naturally constructed map is an isomorphism. We shall prove that it is both surjective and injective. In each case, the local triviality of $E \to X$ is the key point. To this end, let us fix an open U_0 around x over which there is a trivializing frame of C^p-sections $s_1, \dots, s_n \in E(U_0) = \mathscr{M}(U_0)$.

For surjectivity, pick any $v \in E(x)$. Writing $v = \sum c_i s_i(x)$ with $c_i \in \mathbf{R}$, the section $s = \sum c_i s_i \in E(U_0) = \mathscr{M}(U_0)$ has value $s(x) = v$ at x and so the germ $[(U_0, s)]_x \in \mathscr{M}_x$ maps to v in $E(x)$. Hence, $\mathscr{M}(x) \to E(x)$ carries the residue class s_x mod $\mathfrak{m}_x \mathscr{M}_x$ of the germ s_x to the initial arbitrary $v \in E(x)$. This gives surjectivity. As for injectivity, pick an element in the kernel and let $[(U, s)]_x \in \mathscr{M}_x$ be a representative germ (with $U \subseteq U_0$ without loss of generality). We can write $s = \sum a_i s_i$ for unique C^p functions a_i on U (as the s_i's are a trivializing frame for E over U_0, and $U \subseteq U_0$), and the vanishing of $s(x) = \sum a_i s_i(x)$ in $E(x)$ says that $a_i(x) = 0$ for all i. Hence, each a_i has its germ in $\mathscr{O}_x$ that lies in $\mathfrak{m}_x$, so $s_x \in \mathscr{M}_x$ lies in $\mathfrak{m}_x \mathscr{M}_x$. Thus, the image of s_x in $\mathscr{M}(x)$ is 0, but (U, s) was chosen to that this image in $\mathscr{M}(x)$ was our initial choice of element dying in $E(x)$. This completes the proof of injectivity.

Example 1.3. Now suppose $p \geq 1$ and let $\mathscr{O}'$ be the underlying C^{p-1} structure. To make the theory of the tangent bundle "work", it is crucial to $\mathbf{R}$-linearly identify $\mathrm{T}_x(X)$ with the fiber of the $\mathscr{O}'$-module Vec_X at $x \in X$. We wish to show that the mapping $\phi_x : (\mathrm{Vec}_X)_x \to \mathrm{T}_x(X)$ sending a germ $s = [(U, \vec{v})]_x$ to the tangent vector $\vec{v}(x) \in \mathrm{T}_x(X)$ is well-defined (so we denote it $s(x)$) and satisfies $\phi_x(a_1 s_1 + a_2 s_2) = a_1(x)\phi_x(s_1) + a_2(x)\phi_x(s_2)$ for $a_1, a_2 \in \mathscr{O}'_x$ and $s_1, s_2 \in (\mathrm{Vec}_X)_x$. Using ∂_{t_j}'s for local C^p coordinates t_j around x, we will prove moreover that ϕ_x is surjective with kernel exactly $\mathfrak{m}'_x(\mathrm{Vec}_X)_x$, and so this will show that ϕ_x induces a natural $\mathbf{R}$-linear isomorphism $(\mathrm{Vec}_X)(x) \simeq \mathrm{T}_x(X)$.

To get started, if $\vec{v} \in \mathrm{Vec}_X(U)$ and $\vec{v}' \in \mathrm{Vec}_X(U')$ are C^{p-1} vector fields on opens U and U' around x such that the germs of $(U, \vec{v})$ and $(U', \vec{v}')$ at x agree, then $\vec{v}|_W = \vec{v}'|_W$ as vector fields on some open $W \subseteq U \cap U'$ around x. In particular, $\vec{v}(w) = \vec{v}'(w)$ in $\mathrm{T}_w(X)$ for all $w \in W$, and hence $\vec{v}(x) = \vec{v}'(x)$ in $\mathrm{T}_x(X)$. Thus, ϕ_x is well-defined.

Given germs s_1, s_2 of Vec_X at x and germs a_1, a_2 of $\mathscr{O}'$ at x, we may find a small open U around x on which we may find representatives $\vec{v}_j$ for s_j and h_j for a_j. Hence, by definition of the $\mathscr{O}'_x$-module structure on $(\mathrm{Vec}_X)_x$, $a_1 s_2 + a_2 s_2$ is represented by the vector field $h_1\vec{v}_1 + h_2\vec{v}_2$ over U. By the definition of the $\mathscr{O}'(U)$-module structure on $\mathrm{Vec}_X(U)$, at the point x this vector field specializes to $h_1(x)\vec{v}_1(x) + h_2(x)\vec{v}_2(x)$ in the tangent space at x, and this is exactly $a_1(x)s_1(x) + a_2(x)s_2(x)$ in $\mathrm{T}_x(X)$. This proves the desired "linearity" formula for ϕ_x (over the evaluation at x).

Finally, we show ϕ_x is surjective with kernel $\mathfrak{m}'_x(\mathrm{Vec}_X)_x$. Pick local C^p coordinates $t_1, \dots, t_n$ on an open set U around x. Any tangent vector $\vec{v}_0 \in \mathrm{T}_x(X)$ can be written as $\sum c_j \partial_{t_j}|_x$ with $c_j \in \mathbf{R}$, so consider the "constant" vector field $\vec{v} = \sum \underline{c}_j \partial_{t_j} \in \mathrm{Vec}_X(U)$ where $\underline{c}_j \in \mathscr{O}'(U)$ is the constant function whose value at every point of U is c_j. The germ of $(U, \vec{v})$ at x is sent by ϕ_x to the tangent vector $\sum \underline{c}_j(x)\partial_{t_j}|_x = \sum c_j \partial_{t_j}|_x = \vec{v}_0$. This proves surjectivity. Finally, a germ in $(\mathrm{Vec}_X)_x$ is represented by a vector field of the form $\sum h_j \partial_{t_j}$ on some open around x for some C^{p-1} functions h_j near x, and ϕ_x carries the germ of this at x to $\sum h_j(x)\partial_{t_j}|_x$. Hence, this vanishes if and only if $h_j(x) = 0$ for all j, which is to say the germ of each h_j in $\mathscr{O}'_x$ lies in $\mathfrak{m}'_x$ for all j. That is, the kernel of ϕ_x is the set of $\mathscr{O}'_x$ linear combinations of the germs $(\partial_{t_j})_x \in (\mathrm{Vec}_X)_x$ with coefficients in $\mathfrak{m}'_x$. This is exactly the submodule $\mathfrak{m}'_x(\mathrm{Vec}_X)_x$, as desired.

Let $f : \mathscr{M}' \to \mathscr{M}$ be an $\mathscr{O}$-linear map between $\mathscr{O}$-modules. For any $x \in X$, there is an induced mapping $f_x : \mathscr{M}'_x \to \mathscr{M}_x$ between modules of germs defined as follows:

$$f_x([(U, s')]_x) = [(U, f_U(s'))]_x.$$

This is well-defined: if $U_1, U_2 \subseteq X$ are opens around x and $s'_j \in \mathscr{M}'(U_j)$ with $s'_1|_W = s'_2|_W$ in $\mathscr{M}'(W)$ for an open $W \subseteq U_1 \cap U_2$ around x, then in $\mathscr{M}(W)$ we have

$$f_{U_1}(s'_1)|_W = f_W(s'_1|_W) = f_W(s'_2|_W) = f_{U_2}(s'_2)|_W$$

with the first and last equalities using the compatibility of the f_U's with respect to restriction (as in the *definition* of an $\mathscr{O}$-linear mapping of $\mathscr{O}$-modules). This says exactly that $(U_1, f_{U_1}(s'_1))$ and $(U_2, f_{U_2}(s'_2))$ represent the same germ in $\mathscr{M}_x$. Hence, f_x is well-defined set-theoretically. By chasing representatives, one checks that f_x is $\mathscr{O}_x$-linear (recall how the $\mathscr{O}_x$-module structure on $\mathscr{M}_x$ and $\mathscr{M}'_x$ was *defined* via representatives).

Returning to the general $\mathscr{O}$-linear map $f : \mathscr{M}' \to \mathscr{M}$ and $x \in X$, since $f_x : \mathscr{M}'_x \to \mathscr{M}_x$ is $\mathscr{O}_x$-linear it carries the submodule $\mathfrak{m}_x\mathscr{M}'_x$ into the submodule $\mathfrak{m}_x\mathscr{M}_x$. (Indeed, for elements $m'_j \in \mathscr{M}'_x$ and $a_j \in \mathfrak{m}_x$ we have $f_x(\sum a_j m'_j) = \sum a_j f_x(m'_j) \in \mathfrak{m}_x\mathscr{M}_x$.) Thus, passing to the quotient induces an $\mathbf{R}$-linear *fiber mapping*

$$f(x) : \mathscr{M}'(x) = \mathscr{M}'_x/\mathfrak{m}_x\mathscr{M}'_x \to \mathscr{M}_x/\mathfrak{m}_x\mathscr{M}_x = \mathscr{M}(x);$$

concretely, if an element $\overline{m}' \in \mathscr{M}'(x)$ is represented by some germ $m' \in \mathscr{M}'_x$ then $f(x)(\overline{m}') \in \mathscr{M}(x)$ is represented by the germ $f_x(m') \in \mathscr{M}_x$.

Remark 1.4. It follows from the definitions that if $g : \mathscr{M}'' \to \mathscr{M}'$ is another $\mathscr{O}$-linear mapping, then $(f \circ g)_x = f_x \circ g_x$ as $\mathscr{O}_x$-linear maps from $\mathscr{M}''_x$ to $\mathscr{M}_x$, and so also $(f \circ g)(x) = f(x) \circ g(x)$ as $\mathbf{R}$-linear maps from $\mathscr{M}''(x)$ to $\mathscr{M}(x)$. That is, the passage from an $\mathscr{O}$-module to its fiber at a point is "well-behaved" with respect to composition of mappings.

In the special case $\mathscr{M} = \underline{E}$ and $\mathscr{M}' = \underline{E'}$ for C^p vector bundles E and E' over X, we know that every $\mathscr{O}$-linear mapping $\mathscr{M}' \to \mathscr{M}$ arises from a unique C^p vector bundle mapping $E' \to E$. In Example 1.2 we saw that the fibers $\underline{E}'(x)$ and $\underline{E}(x)$ in the sense of $\mathscr{O}$-modules are naturally identified with the fibers $E'(x)$ and $E(x)$ in the sense of vector bundles, and so there arises an important issue of compatibility: if $f : E' \to E$ is a C^p vector bundles mapping and $\underline{f} : \underline{E}' \to \underline{E}$ is the associated $\mathscr{O}$-linear mapping, how do the "fiber mappings"

$$f|_x : E'(x) \to E(x), \;\; \underline{f}(x) : \underline{E}'(x) \to \underline{E}(x)$$

compare via the identifications of $E(x)$ with $\underline{E}(x)$ and of $E'(x)$ with $\underline{E}'(x)$? Fortunately, everything works as it should:

Theorem 1.5. *Fix $x \in X$. The diagram*

$$\begin{tikzcd} \underline{E}'(x) \arrow[r, "\underline{f}(x)"] \arrow[d, "\simeq"'] & \underline{E}(x) \arrow[d, "\simeq"] \\ E'(x) \arrow[r, "f|_x"'] & E(x) \end{tikzcd}$$

commutes, where the columns are the natural isomorphisms made Example 1.2.

Proof. An element in $\underline{E}'(x) = \underline{E}_x/\mathfrak{m}_x\underline{E}'_x$ is represented by a germ $[(U, s')]_x$ with open $U \subseteq X$ around x and $s' \in \underline{E}'(U) = E'(U)$. By definition, $\underline{f}(x)$ is induced by the map $\underline{E}'_x \to \underline{E}_x$ given by "composition with f" on germs at x. This carries the germ $[(U, s')]_x$ of $\underline{E}'$ to the germ $[(U, \underline{f}_U(s'))]_x = [(U, f \circ s')]_x$ of $\underline{E}$. By definition of the columns in our desired commutative square, the images of these germs in $\underline{E}'(x)$ and $\underline{E}(x)$ respectively map to $s'(x) \in E(x)$ and $(f \circ s')(x) \in E(x)$. Hence, we want $f|_x : E'(x) \to E(x)$ to carry $s'(x)$ to $(f \circ s')(x)$. This follows from the very definition of the $\mathscr{O}(U)$-module mapping $f : E'(U) \to E(U)$. ∎

2. Lots of lemmas

We fix $\mathscr{M}$, so for each $x \in X$ we get a finite-dimensional $\mathbf{R}$-vector space $\mathscr{M}(x)$. We define $V_{\mathscr{M}}$ as a set to be the disjoint union

$$V_{\mathscr{M}} = \coprod_{x \in X} \mathscr{M}(x),$$

equipped with the evident projection map of sets $\pi : V_{\mathscr{M}} \to X$ that sends points in $\mathscr{M}(x)$ to $x \in X$. Hence, each fiber $\pi^{-1}(x) = \mathscr{M}(x)$ has a structure of finite-dimensional $\mathbf{R}$-vector space. Our first goal is to give $V_{\mathscr{M}}$ a topology such that $\pi : V_{\mathscr{M}} \to X$ becomes a topological vector bundle (using the linear structure just put on the fiber $\pi^{-1}(x)$ over each $x \in X$). We shall build a topology by gluing topologies on subsets indexed by pairs (U, φ) consisting of non-empty open sets $U \subseteq X$ and $\mathscr{O}|_U$-linear isomorphisms $\varphi : \mathscr{M}|_U \simeq \mathscr{O}|_U^{\oplus n_U}$ with $n_U \geq 0$. Since $\mathscr{M}$ is locally free of finite rank, there exist such pairs (U, φ) with U's covering X. For each such pair (U, φ), we get an $\mathbf{R}$-linear isomorphism $\varphi(u) : \mathscr{M}(u) \simeq \mathbf{R}^{n_U}$ on u-fibers for each $u \in U$. (In particular, n_U only depends on U and not φ: it is the common $\mathbf{R}$-dimension of the fibers $\mathscr{M}(u)$ for all $u \in U$.) Hence, we arrive at a bijection of sets

$$\xi_{U,\varphi} : \pi^{-1}(U) = \coprod_{u \in U} \mathscr{M}(u) \to \coprod_{u \in U} \mathbf{R}^{n_U} = U \times \mathbf{R}^{n_U}$$

that carries $\pi^{-1}(U) \to U$ to the standard projection $U \times \mathbf{R}^{n_U} \to U$ and induces the linear isomorphism $\varphi(u) : \mathscr{M}(u) \simeq \mathbf{R}^{n_U}$ on fibers over each $u \in U$.

Since the U's cover X as we vary (U, φ), the $\pi^{-1}(U)$'s cover $V_{\mathscr{M}}$ set-theoretically. We use the bijection $\xi_{U,\varphi}$ to put a topology on $\pi^{-1}(U)$ via the topology on $U \times \mathbf{R}^{n_U}$ (i.e., we "force" $\xi_{U,\varphi}$ to be a homeomorphism). Letting $S_{U,\varphi}$ denote the subset $\pi^{-1}(U)$ in $V_{\mathscr{M}}$ with the topology just defined via $\xi_{U,\varphi}$, we want to show that these uniquely glue to a topology on $V_{\mathscr{M}}$ with respect to which each $S_{U,\varphi}$ is an open subset of $V_{\mathscr{M}}$ and gets as the subspace topology exactly the topology imposed by forcing $\xi_{U,\varphi}$ to be a homeomorphism. (**Warning**: Each such U admits many φ's, so on the set $\pi^{-1}(U)$ we get many topologies $S_{U,\varphi}$. These will soon be proved to all coincide, but for now we have to keep track of φ and not just U when contemplating topologies on $\pi^{-1}(U)$.) By the criterion for gluing topologies, we have to prove:

Lemma 2.1. *For (U, φ) and (U', φ') as above with $U \cap U' \neq \emptyset$, the subset $\pi^{-1}(U) \cap \pi^{-1}(U') = \pi^{-1}(U \cap U')$ inside $\pi^{-1}(U) = S_{U,\varphi}$ and $\pi^{-1}(U') = S_{U',\varphi}$ is open in each of the topological spaces $S_{U,\varphi}$ and $S_{U',\varphi'}$. It gets the same subspace topology from each.* (*In particular, taking $U = U'$, the topology on $\pi^{-1}(U)$ is independent of φ.*)

If $U \cap U' = \emptyset$ then $\pi^{-1}(U) \cap \pi^{-1}(U') = \emptyset$ and hence $S_{U,\varphi} \cap S_{U',\varphi'} = \emptyset$ in $V_{\mathscr{M}}$. Thus, such pairs do not intervene (or rather, do so trivially) in the gluing criterion.

Proof. Since $U \cap U'$ is non-empty, the common dimensions n_U and $n_{U'}$ of the fibers of $\mathscr{M}$ over points of U and U' respectively must be equal. Let n denote this common dimension. Under the topological identification $\xi_{U,\varphi}$ of $S_{U,\varphi}$ with $U \times \mathbf{R}^n$, the subset $\pi^{-1}(U \cap U')$ goes over to the subset $(U \cap U') \times \mathbf{R}^n$ that is clearly open in $U \times \mathbf{R}^n$. Likewise, $\pi^{-1}(U \cap U')$ is open in $S_{U',\varphi'}$. Hence, it remains to check the equality of subspace topologies. In other words, when we bijectively identify $\pi^{-1}(U \cap U')$ with $(U \cap U') \times \mathbf{R}^n$ using each of $\xi_{U,\varphi}$ and $\xi_{U',\varphi'}$ then we claim that both of these bijections carry the product topology on $(U \cap U') \times \mathbf{R}^n$ over to the same topology on $\pi^{-1}(U \cap U')$. Equivalently, we claim that the bijection

$$\xi_{U',\varphi'} \circ \xi_{U,\varphi}^{-1} : (U \cap U') \times \mathbf{R}^n \to (U \cap U') \times \mathbf{R}^n$$

is a homemorphism.

We can do better (as we shall require later): this bijection is a C^p isomorphism. We shall compute it explicitly. Let $\{s_j\}$ and $\{s'_i\}$ be the trivializations of $\mathscr{M}|_U$ and $\mathscr{M}|_{U'}$ corresponding to φ and φ', so $\mathscr{M}|_{U\cap U'}$ has two trivializations: $\{s_j|_{U\cap U'}\}$ and $\{s'_i|_{U\cap U'}\}$. Hence, there are unique expressions $s_j|_{U\cap U'} = \sum_{i=1}^n a_{ij} s'_i|_{U\cap U'}$ with $a_{ij} \in \mathscr{O}(U \cap U')$ and $\det(a_{ij}) : U \cap U' \to \mathbf{R}$ a non-vanishing C^p function. In other words, $(a_{ij}) : U \cap U' \to \mathrm{GL}_n(\mathbf{R})$ is a C^p mapping. The mapping $\xi_{U',\varphi'} \circ \xi_{U,\varphi}^{-1}$ is

$$(x, v) \mapsto (x, (a_{ij}(u))(v))$$

for $x \in U \cap U'$ and $v \in \mathbf{R}^n$. This is clearly a C^p mapping, due to the C^p property of the a_{ij}'s on $U \cap U'$ and the formula for evaluating a matrix on a vector in Euclidean space. The same method proves that the inverse map (given by swapping the roles of (U, φ) and (U', φ')) is C^p; explicitly, it is given by the analogous matrix formula using the inverse matrix (that we can also see is C^p via Cramer's formula and the non-vanishing of $\det(a_{ij})$ on $U \cap U'$). ■

We have put a topology on $V_{\mathscr{M}}$, and we claim it makes $\pi : V_{\mathscr{M}} \to X$ continuous. As we vary through (U, φ) as above, the subsets $\pi^{-1}(U)$ give an open covering of X, so by the local nature of continuity it suffices to prove that the restriction $\pi_U : \pi^{-1}(U) \to U$ of π to each $\pi^{-1}(U)$ is a continuous map (taking the target of π_U to be U or X does not affect whether or not it is continuous). The topology on $\pi^{-1}(U)$ is rigged so that we have *homeomorphisms* $\xi_{U,\varphi} : \pi^{-1}(U) \simeq U \times \mathbf{R}^{n_U}$ such that $\pi_U \circ \xi_{U,\varphi}^{-1}$ is the standard projection $U \times \mathbf{R}^{n_U} \to U$ that is certainly continuous. Hence, π_U is continuous. We can do better: the $\xi_{U,\varphi}$'s are $\mathbf{R}$-linear on fibers, and so provide the local topological trivializations that are required to conclude that $\pi : V_{\mathscr{M}} \to X$ with its linear structure on fibers is a topological vector bundle over X.

We now want to enhance our construction to make $V_{\mathscr{M}}$ a C^p vector bundle over X such that the $\xi_{U,\varphi}$'s are C^p trivializations. In the proof of Lemma 2.1, we saw that for pairs (U, φ) and (U', φ') such that $U \cap U'$ is non-empty, the "transition maps" $\xi_{U',\varphi'} \circ \xi_{U,\varphi}^{-1}$ are C^p automorphisms of the C^p premanifold with corners $(U \cap U') \times \mathbf{R}^n$ (with $n = n_U = n_{U'}$). If $U \times \mathbf{R}^n$ and $U' \times \mathbf{R}^n$ were open sets in sectors in finite-dimensional vector spaces then we could say that we have built a C^p atlas on $V_{\mathscr{M}}$ and so we would thereby obtain a C^p structure on $V_{\mathscr{M}}$. However, our situation is slightly more general than the formation of C^p structures from C^p atlases because U might be "big" and so the spaces $U \times \mathbf{R}^n$ are merely C^p premanifolds with corners rather than the open sets in sectors in vector spaces. (We could shrink the U's to bypass this issue, but it would be unnatural to do so here, and more importantly the lemma below by which we will handle this issue will be extremely useful in other settings.) The mild generality of "big" U presents no difficulty, due to:

Lemma 2.2. *Choose $0 \le p \le \infty$. Let Z be a topological space equipped with a covering by open subsets $\{Z_i\}$ and homeomorphisms $\phi_i : Z_i \to Y_i$ to C^p premanifolds with corners $(Y_i, \mathscr{O}_i)$. (In particular, Z is a topological premanifold with corners.) If the transition mappings*

$$\phi_j \circ \phi_i^{-1} : \phi_i(Z_i \cap Z_j) \simeq \phi_j(Z_i \cap Z_j)$$

between open sets in the C^p premanifolds with corners Y_i and Y_j are C^p isomorphisms then there is a unique C^p structure $\mathscr{O}$ on Z with respect to which the ϕ_i's are C^p isomorphisms from $(Z_i, \mathscr{O}|_{Z_i})$ to $(Y_i, \mathscr{O}_i)$ for all i.

This lemma generalizes the construction of C^p-structures from C^p-atlases in the sense that the targets Y_i are now arbitrary C^p premanifolds with corners rather than merely open sets in sectors in finite-dimensional $\mathbf{R}$-vector spaces.

Proof. The proof is *identical* to the construction of a C^p-structure from a C^p-atlas, except that since we now have a global theory of C^p premanifolds with corners (whereas earlier we only had the "local" theory of C^p-structures on open sets in sectors in finite-dimensional $\mathbf{R}$-vector spaces) we may use such global objects in the earlier role of opens in sectors in vector spaces. For each open set $U \subseteq Z$ we define $\mathscr{O}(U)$ to be the set of functions $f : U \to \mathbf{R}$ such that f is "locally C^p with respect to the ϕ_i's", which is to say that for all i the function $f \circ \phi_i^{-1} : \phi_i(U \cap Z_i) \to \mathbf{R}$ on the open set $\phi_i(U \cap Z_i)$ in Y_i is a C^p function. One checks readily that $U \mapsto \mathscr{O}(U)$ is an $\mathbf{R}$-space structure on Z (because the $\mathscr{O}_i$'s are on each Y_i).

To check that $\mathscr{O}$ is a C^p-structure on Z, we want the homeomorphism $\phi_i : Z_i \to Y_i$ to be an isomorphism of structured $\mathbf{R}$-spaces with respect to $\mathscr{O}|_{Z_i}$ on Z_i and $\mathscr{O}_i$ on Y_i. That is, if $U \subseteq Z_i$ is an open subset and $f : U \to \mathbf{R}$ is a function then we claim $f \in \mathscr{O}(U)$ if and only if $f \circ \phi_i^{-1} \in \mathscr{O}_i(\phi_i(U))$. Equivalently, in view of the definition of $\mathscr{O}$, we must prove that if $f \circ \phi_i^{-1}$ is C^p on $\phi_i(U) \subseteq Y_i$ then for all j it is *automatic* that $f \circ \phi_j^{-1}$ is C^p on $\phi_j(U \cap U_j) \subseteq Y_j$. To verify this, note that

$$f \circ \phi_j^{-1} = (f \circ \phi_i^{-1}) \circ (\phi_i \circ \phi_j^{-1}) \tag{1}$$

on $U \cap U_j = U \cap U_i \cap U_j$, and recall the assumption that the transition mapping $\phi_i \circ \phi_j^{-1}$ from $\phi_j(U_i \cap U_j)$ to $\phi_i(U_i \cap U_j)$ is a C^p isomorphism. It follows from (1) that $f \circ \phi_j^{-1}$ is a composite of a C^p function and a C^p isomorphism, so it is a C^p function. This takes care of the existence aspect of the lemma, and uniqueness is clear because the definition given for $U \mapsto \mathscr{O}(U)$ is the only possible one that is consistent with the requirements for $\mathscr{O}$ to be an $\mathbf{R}$-space structure on Z making the ϕ_i's isomorphisms of structured $\mathbf{R}$-spaces. ■

By Lemma 2.2, we obtain a unique C^p-structure on $V_{\mathscr{M}}$ with respect to which the homeomorphisms $\xi_{U,\varphi} : \pi^{-1}(U) \simeq U \times \mathbf{R}^{n_U}$ are C^p isomorphisms. In particular, the restrictions $\pi_U : \pi^{-1}(U) \to U$ of π are C^p because $\xi_{U,\varphi}$ is a C^p isomorphism and the composite $\pi_U \circ \xi_{U,\varphi}^{-1} : U \times \mathbf{R}^{n_U} \to U$ is the standard projection that is certainly a C^p mapping. It follows that $\pi : V_{\mathscr{M}} \to X$ is also a C^p mapping (as we have just checked that it is C^p when restricted to a collection of open subsets that cover $V_{\mathscr{M}}$), so with the linear structure on its fibers the map $\pi : V_{\mathscr{M}} \to X$ is a C^p vector bundle because the $\xi_{U,\varphi}$'s provide C^p trivializations over the opens $U \subseteq X$ that cover X as we vary (U, φ). This completes the construction of $V_{\mathscr{M}}$.

To analyze the properties of this construction, we require some more lemmas. First, we need to glue $\mathscr{O}$-module maps (analogously to how we glue vector bundle morphisms). The following lemma will only be used in the setting of locally free $\mathscr{O}$-modules with finite rank, but such conditions are not relevant in the proof and so we avoid them.

Lemma 2.3. *Let $(X, \mathscr{O})$ be a C^p premanifold with corners, $0 \le p \le \infty$, and let $\mathscr{M}$ and $\mathscr{M}'$ be $\mathscr{O}$-modules. Let $\{U_i\}$ be an open covering of X. Let $\mathscr{M}_i$ denote the $\mathscr{O}|_{U_i}$-module $\mathscr{M}|_{U_i}$, and likewise for $\mathscr{M}_{ij}$ as an $\mathscr{O}|_{U_i \cap U_j}$-module, and similarly for $\mathscr{M}'$.*

If $f_i : \mathscr{M}'_i \to \mathscr{M}_i$ is an $\mathscr{O}|_{U_i}$-linear map for each i and $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ as $\mathscr{O}|_{U_i \cap U_j}$-linear maps from $\mathscr{M}'_{ij}$ to $\mathscr{M}_{ij}$ for all i and j then there exists a unique $\mathscr{O}$-linear map $f : \mathscr{M}' \to \mathscr{M}$ such that the $\mathscr{O}|_{U_i}$-linear restriction $f|_{U_i} : \mathscr{M}'_i \to \mathscr{M}_i$ equals f_i for all i.

Remark 2.4. As an important special case, the isomorphism property for an $\mathscr{O}$-linear map is local. To be precise, suppose we are given an $\mathscr{O}$-linear map $f : \mathscr{M}' \to \mathscr{M}$ and that the restriction f_i of f over each U_i is an $\mathscr{O}|_{U_i}$-linear isomorphism then the unique inverses $f_i^{-1} : \mathscr{M}_i \to \mathscr{M}'_i$ coincide over $U_i \cap U_j$'s (where f_i^{-1} and f_j^{-1} restrict to inverses of $f|_{U_i \cap U_j}$ and hence coincide). Thus, the f_i^{-1}'s globally glue to an $\mathscr{O}$-linear map $f' : \mathscr{M} \to \mathscr{M}'$ that is an inverse to f because the composites $f' \circ f$

and $f \circ f'$ restrict to the identity maps over each U_i and hence (by the uniqueness aspect of the lemma) must be the identity over all of X.

Proof. Let $U \subseteq X$ be an open set, so U is covered by the opens $U \cap U_i$. For $s' \in \mathscr{M}'(U)$, let $s'_i = s'|_{U \cap U_i} \in \mathscr{M}'(U \cap U_i) = \mathscr{M}'_i(U \cap U_i)$. We get elements $(f_i)_{U \cap U_i}(s'_i) \in \mathscr{M}_i(U \cap U_i) = \mathscr{M}(U \cap U_i)$, and on the overlap $(U \cap U_i) \cap (U \cap U_j) = U \cap (U_i \cap U_j)$ we have that $(f_i)_{U \cap U_i}(s'_i)$ and $(f_j)_{U \cap U_j}(s'_j)$ have the same restriction, namely the image of $s'|_{U_i \cap U_j}$ under the common map

$$(f_i)_{U \cap U_i \cap U_j} = (f_j)_{U \cap U_i \cap U_j} : \mathscr{M}'(U \cap (U_i \cap U_j)) \to \mathscr{M}(U \cap (U_i \cap U_j)).$$

Hence, the $(f_i)_{U \cap U_i}(s'_i)$'s are the restrictions of a unique element in $\mathscr{M}(U)$ that we shall denote $f_U(s')$. This defines set-theoretic maps $f_U : \mathscr{M}'(U) \to \mathscr{M}(U)$ for all opens $U \subseteq X$.

To check that f_U is $\mathscr{O}(U)$-linear, pick $s'_1, s'_2 \in \mathscr{M}'(U)$ and $a_1, a_2 \in \mathscr{O}(U)$. We want

$$f_U(a_1 s'_1 + a_2 s'_2) = a_1 f_U(s'_1) + a_2 f_U(s'_2)$$

in $\mathscr{M}(U)$, and to check such equalities it suffices to do so after restriction to $\mathscr{M}(U \cap U_i)$ for all i because the $U \cap U_i$'s are an open cover of U. But by the definition of f_U, restricting to $U \cap U_i$ converts the proposed equalities in the system of equalities

$$(f_i)_{U \cap U_i}((a_1 s'_1 + a_2 s'_2)|_{U \cap U_i}) \stackrel{?}{=} a_1|_{U \cap U_i} \cdot (f_i)_{U \cap U_i}(s'_1|_{U \cap U_i}) + a_2|_{U \cap U_i} \cdot (f_i)_{U \cap U_i}(s'_2|_{U \cap U_i}).$$

This equality follows from the $\mathscr{O}(U \cap U_i)$-linearity of $(f_i)_{U \cap U_i}$ and the equality

$$(a_1 s'_1 + a_2 s'_2)|_{U \cap U_i} = a_1|_{U \cap U_i} \cdot s'_1|_{U \cap U_i} + a_2|_{U \cap U_i} \cdot s'_2|_{U \cap U_i}$$

that expresses the fact that $\mathscr{M}'(U) \to \mathscr{M}'(U \cap U_i)$ is linear over $\mathscr{O}(U) \to \mathscr{O}(U \cap U_i)$.

With the f_U's now known to be $\mathscr{O}(U)$-linear, we next have to check that they fit together to define a map of $\mathscr{O}$-modules $\mathscr{M}' \to \mathscr{M}$, which is to say that if $U' \subseteq U$ is an open subset then $f_U(s')|_{U'} = f_{U'}(s'|_{U'})$ in $\mathscr{M}(U')$ for any $s' \in \mathscr{M}'(U)$. Such equality may be checked in each $\mathscr{M}(U' \cap U_i)$, and so by definition of f_U and $f_{U'}$ we simply invoke the compatibility of $(f_i)_{U \cap U_i}$ and $(f_i)_{U' \cap U_i}$ with respect to restriction maps $\mathscr{M}'(U \cap U_i) \to \mathscr{M}'(U' \cap U_i)$ and $\mathscr{M}(U \cap U_i) \to \mathscr{M}(U' \cap U_i)$. We have now built an $\mathscr{O}$-linear map $f : \mathscr{M}' \to \mathscr{M}$, and by its very construction we see that if $U \subseteq U_i$ for some i then $f_U = (f_i)_U$ as maps from $\mathscr{M}'(U)$ to $\mathscr{M}(U)$. That is, $f|_{U_i} : \mathscr{M}'|_{U_i} \to \mathscr{M}|_{U_i}$ coincides with f_i. Hence, f is the desired "gluing" of the f_i's. An inspection of the definition of f shows that it is the only possibility for a solution to our gluing problem. ■

Lemma 2.5. *Let $\mathscr{M}'$ and $\mathscr{M}$ be locally free $\mathscr{O}$-modules with finite rank.*

(1) *If two $\mathscr{O}$-linear maps $f, g : \mathscr{M}' \rightrightarrows \mathscr{M}$ satisfy $f(x) = g(x)$ as $\mathbf{R}$-linear maps from $\mathscr{M}'(x)$ to $\mathscr{M}(x)$ for all $x \in X$ then $f = g$.*

(2) *If $f : \mathscr{M}' \to \mathscr{M}$ is an $\mathscr{O}$-linear map then the set-theoretic map $V(f) : V_{\mathscr{M}'} \to V_{\mathscr{M}}$ over X given on fibers by the linear map $f(x) : \mathscr{M}'(x) \to \mathscr{M}(x)$ is a C^p vector bundle morphism. Also, f is an isomorphism if and only if all $\mathbf{R}$-linear maps $f(x) : \mathscr{M}'(x) \to \mathscr{M}(x)$ are linear isomorphisms.*

The first part of the lemma says that maps between locally free $\mathscr{O}$-modules of finite rank are uniquely determined by their effect on fibers (taken in the sense of fibers for $\mathscr{O}$-modules), and this is analogous to (but not as trivial as!) the physically obvious fact that a bundle morphism between C^p vector bundles over X is uniquely determined by its induced linear maps on fibers (taken in the sense of vector bundles). The end of the second part of the lemma is analogous to the result proved earlier for vector bundles, namely that a bundle morphism that is a linear isomorphism on fibers (taken in the sense of vector bundles) is an isomorphism of vector bundles.

Proof. We first prove (1). Since $f(x) - g(x) = (f - g)(x)$ (why?), we may focus our attention on $f - g$. That is, we just have to show that if $\theta : \mathscr{M}' \to \mathscr{M}$ is $\mathscr{O}$-linear and $\theta(x) = 0$ for all $x \in X$ then $\theta = 0$ (i.e., $\theta_U : \mathscr{M}'(U) \to \mathscr{M}(U)$ vanishes for all opens $U \subseteq X$). This problem is local on X, and since X can be covered by opens X_i such that $\mathscr{M}|_{X_i}$ and $\mathscr{M}'|_{X_i}$ are both free $\mathscr{O}|_{X_i}$-modules (say with respective ranks n_i and n_i') then we may replace X with X_i and so we may assume that there exist $\mathscr{O}$-linear isomorphisms $\mathscr{M} \simeq \mathscr{O}^{\oplus n}$ and $\mathscr{M}' \simeq \mathscr{O}^{\oplus n'}$. Composing θ with isomorphisms on its source and target is harmless for the purpose of proving vanishing and it preserves the assumption of vanishing on fibers, so we may assume $\mathscr{M} = \mathscr{O}^{\oplus n}$ and $\mathscr{M}' = \mathscr{O}^{\oplus n'}$. Thus, θ is given by a matrix (a_{ij}) with $a_{ij} \in \mathscr{O}(X)$ in the sense that $\theta_U((c_1, \dots, c_n)) = (\sum_j a_{1j}|_U \cdot c_j, \dots, \sum_i a_{n'j}|_U \cdot c_j)$ for $c_1, \dots, c_n \in \mathscr{O}(U)$. On x-fibers, $\theta(x) : \mathbf{R}^n \to \mathbf{R}^{n'}$ is given by the matrix $(a_{ij}(x))$, and the vanishing assumption on fibers therefore implies $a_{ij}(x) = 0$ for all $x \in X$ and all i, j. Hence, $a_{ij} = 0$ for all i, j, so $\theta_U = 0$ for all U; i.e., $\theta = 0$. This completes the proof of (1).

For (2), since $V(f)$ is a map over X and is linear on fibers, it is a morphism of C^p vector bundles if and only if it is a C^p mapping. This problem is local over X (why?), so we may assume $\mathscr{M}$ and $\mathscr{M}'$ are free of respective finite ranks n and n' as $\mathscr{O}$-modules. Fix choices of trivializations $\varphi : \mathscr{M} \simeq \mathscr{O}^{\oplus n}$ and $\varphi' : \mathscr{M}' \simeq \mathscr{O}^{\oplus n'}$. Let $\{s_j\}$ and $\{s_i'\}$ be the corresponding trivializating sections in $\mathscr{M}(X)$ and $\mathscr{M}'(X)$, so $f_X(s_j) = \sum a_{ij} s_i'$ for unique $a_{ij} \in \mathscr{O}(X)$. By the construction of the C^p structure on $V_{\mathscr{M}}$ and $V_{\mathscr{M}'}$, we have C^p vector bundle isomorphisms

$$\xi_{X,\varphi} : V_{\mathscr{M}} \simeq X \times \mathbf{R}^n, \;\; \xi_{X,\varphi'} : V_{\mathscr{M}'} \simeq X \times \mathbf{R}^{n'}.$$

Thus, the set-theoretic map $V(f)$ is C^p if and only if the composite map

$$\xi_{X,\varphi'} \circ V(f) \circ \xi_{X,\varphi}^{-1} : X \times \mathbf{R}^n \to X \times \mathbf{R}^{n'}$$

is C^p. But unwinding the definitions of $V(f)$ and the ξ's shows that this map is exactly

$$(x, v) \mapsto (x, (a_{ij}(x))(v))$$

for $x \in X$ and $v \in \mathbf{R}^n$, so the C^p property of the a_{ij}'s and the formula for evaluating a matrix on a vector in Euclidean space gives the desired C^p result for $V(f)$.

To complete the proof of (2), we have to prove that f is an isomorphism of $\mathscr{O}$-modules if and only if $f(x)$ is an $\mathbf{R}$-linear isomorphism for all $x \in X$. Since passage to the fiber map is compatible with composition, if f is an isomorphism (with inverse f^{-1}) then $f(x)$ is an $\mathbf{R}$-linear isomorphism (with inverse $f^{-1}(x)$) for all $x \in X$. For the converse, by Remark 2.4 we may work locally over X. Thus, we can assume $\mathscr{M}$ and $\mathscr{M}'$ are trivial. Fix $\mathscr{O}$-linear isomorphisms $\varphi : \mathscr{M} \simeq \mathscr{O}^{\oplus n}$ and $\varphi' : \mathscr{M}' \simeq \mathscr{O}^{\oplus n'}$. Since $f(x)$ is an isomorphism for each $x \in X$ and we can assume X is non-empty, we must have $n' = n$. The map f is an isomorphism if and only if $\varphi' \circ f \circ \varphi^{-1}$ is an isomorphism, and the x-fiber map for $\varphi' \circ f \circ \varphi^{-1}$ is the map $\varphi'(x) \circ f(x) \circ \varphi^{-1}(x)$ that is certainly an isomorphism for all $x \in X$. Hence, we may assume $\mathscr{M} = \mathscr{O}^{\oplus n}$ and $\mathscr{M}' = \mathscr{O}^{\oplus n}$. The map f is given by an $n \times n$ matrix (a_{ij}) with $a_{ij} \in \mathscr{O}(X)$ in the sense that $f_U(c_1, \dots, c_n) = (\sum_j a_{1j}|_U \cdot c_j, \dots, \sum_j a_{nj}|_U \cdot c_j)$ for all open $U \subseteq X$ and $c_1, \dots, c_n \in \mathscr{O}(U)$. The fibral map $f(x) : \mathbf{R}^n \to \mathbf{R}^n$ is given by the matrix $(a_{ij}(x))$, so the fibral isomorphism condition implies $\det(a_{ij}) : X \to \mathbf{R}$ is nowhere-vanishing. Hence, we can use Cramer's formula to write down an $\mathscr{O}$-linear inverse map $\mathscr{M} \to \mathscr{M}'$ using a matrix whose entries are C^p functions on X. ∎

3. Main Theorem and applications

The following theorem makes precise the sense in which the construction of the C^p vector bundle $V_{\mathscr{M}}$ from a locally free and finite rank $\mathscr{O}$-module $\mathscr{M}$ is inverse to the construction of the locally free and finite rank $\mathscr{O}$-module $\underline{E}$ from a C^p vector bundle E. In particular, it shows that every

$\mathscr{O}$-module $\mathscr{M}$ that is locally free with finite rank is naturally isomorphic to $\underline{E}$ for a C^p vector bundle E that is naturally assocated to $\mathscr{M}$.

Theorem 3.1. *For any locally free $\mathscr{O}$-module of finite rank $\mathscr{M}$, there is a unique isomorphism of $\mathscr{O}$-modules $\theta_{\mathscr{M}} : \underline{V}_{\mathscr{M}} \simeq \mathscr{M}$ such that on x-fibers for each $x \in X$ it is the identity map on $\mathscr{M}(x)$. Also, for any C^p vector bundle $\pi : E \to X$ there is a unique C^p vector bundle morphism $\theta_E : V_{\underline{E}} \simeq E$ such that on fibers over each $x \in X$ it is the identity map on $E(x)$.*

In the fibral descriptions, we are implicitly using the $\mathbf{R}$-linear isomorphisms $\underline{E}(x) \simeq E(x)$ for any C^p vector bundle $E \to X$ (this was discussed in Example 1.2) and $V_{\mathscr{M}}(x) \simeq \mathscr{M}(x)$ for any locally free $\mathscr{O}$-module of finite rank $\mathscr{M}$ (this follows from how $V_{\mathscr{M}}$ was defined). There are nice relations between the two θ's: $V(\theta_{\mathscr{M}}) : V_{\underline{V}_{\mathscr{M}}} \simeq V_{\mathscr{M}}$ is equal to $\theta_{V_{\mathscr{M}}}$, and $\underline{\theta}_E : \underline{V}_{\underline{E}} \simeq \underline{E}$ is equal to $\theta_{\underline{E}}$. These equalities are easily checked by calculation on fibers.

Proof. The uniqueness of θ_E and $\theta_{\mathscr{M}}$ is due to the fact that maps of C^p vector bundles and locally free $\mathscr{O}$-modules of finite rank are uniquely determined by their effect on fibers over the base space. We need to prove existence for θ_E and $\theta_{\mathscr{M}}$. Once we construct θ_E as a C^p map of vector bundles, it must be a bundle isomorphism since it is an isomorphism on fibers. Likewise, since $\theta_{\mathscr{M}}$ has been specified to be an isomorphism on fibers, by Lemma 2.5(2) it must be an $\mathscr{O}$-linear isomorphism once it is merely constructed as an $\mathscr{O}$-linear map. The problem of constructing θ_E (resp. $\theta_{\mathscr{M}}$) is local over X. Indeed, if $\{U_i\}$ is an open covering of X such that we can solve the construction problem over each of the U_i's, then we get maps $\theta_{E|_{U_i}}$ (resp. $\theta_{\mathscr{M}|_{U_i}}$) for each i and over $U_i \cap U_j$ the solutions over U_i and U_j restriction to solutions to the same construction problem. Hence, by the *uniqueness* that has already been established, these restrictions over $U_i \cap U_j$ must agree; i.e.,

$$\theta_{E|_{U_i}}|_{U_i \cap U_j} = \theta_{E|_{U_j}}|_{U_i \cap U_j}, \;\; \theta_{\mathscr{M}|_{U_i}}|_{U_i \cap U_j} = \theta_{\mathscr{M}|_{U_j}}|_{U_i \cap U_j}.$$

The lemma on gluing for vector bundle morphisms (resp. Lemma 2.3) then ensures that there is a unique vector bundle morphism $V_{\underline{E}} \to E$ (resp. $\mathscr{O}$-linear map $\underline{V}_{\mathscr{M}} \to \mathscr{M}$) that restricts to $\theta_{E|_{U_i}}$ (resp. $\theta_{\mathscr{M}|_{U_i}}$) over U_i for each i, and this is clearly the desired map θ_E (resp. $\theta_{\mathscr{M}}$).

Since our construction problem is now proved to be local over X, we may assume that E and $\mathscr{M}$ are trivial for the assertions at issue. That is, we can assume that there exists a C^p vector bundle isomorphism $\varphi : E \simeq X \times \mathbf{R}^n$ for some n and an $\mathscr{O}$-linear isomorphism $\varphi' : \mathscr{M} \simeq \mathscr{O}^{\oplus n'}$ for some n'. The bundle isomorphism $E \simeq X \times \mathbf{R}^n$ corresponds to trivializing sections $\{s_i\}$ in $E(X)$ and so defines an $\mathscr{O}$-linear isomorphism $\underline{E} \simeq \mathscr{O}^{\oplus n}$ by expressing elements in $E(U)$ as unique $\mathscr{O}(U)$-linear combinations of the $s_i|_U$'s for all open $U \subseteq X$. This $\mathscr{O}$-linear isomorphism yields a C^p vector bundle isomorphism $V_{\underline{E}} \simeq V_{\mathscr{O}^{\oplus n}}$ (by Lemma 2.5(2)), and by construction of $V_{\mathscr{M}}$ we have a C^p vector bundle isomorphism $V_{\mathscr{O}^{\oplus n}} \simeq X \times \mathbf{R}^n$ given on fibers by the identity map $\mathbf{R}^n \simeq \mathbf{R}^n$. The composite bundle isomorphism $V_{\underline{E}} \simeq V_{\mathscr{O}^{\oplus n}} \simeq X \times \mathbf{R}^n \overset{\varphi^{-1}}{\simeq} E$ is checked to induced the identity on $E(x)$ on x-fibers for each $x \in X$, so we have constructed θ_E.

The construction of $\theta_{\mathscr{M}}$ in the trivial case goes essentially the same as for θ_E. The choice of $\mathscr{O}$-linear isomorphism $\varphi' : \mathscr{M} \simeq \mathscr{O}^{\oplus n'}$ corresponds to trivializing sections $\{s_i\}$ in $\mathscr{M}(X)$ and (using Lemma 2.5(2)) it gives rise to an isomorphism $V_{\mathscr{M}} \simeq V_{\mathscr{O}^{\oplus n'}} \simeq X \times \mathbf{R}^{n'}$. Passing to the associated locally free $\mathscr{O}$-modules of finite rank gives an $\mathscr{O}$-linear isomorphism

$$\underline{V}_{\mathscr{M}} \simeq \underline{X \times \mathbf{R}^{n'}} \simeq \mathscr{O}^{\oplus n'} \overset{\varphi'^{-1}}{\simeq} \mathscr{M}$$

in which the middle isomorphism is the standard one (through the identification of C^p sections $U \to X \times \mathbf{R}^{n'}$ with maps $u \mapsto (u; c_1(u), \dots, c_{n'}(u))$ for $c_j \in \mathscr{O}(U)$). This induces the identity on $\mathscr{M}(x)$ on x-fibers over each $x \in X$, so it gives the desired $\mathscr{O}$-linear isomorphism $\theta_{\mathscr{M}}$. ■

As a fundamental application of Theorem 3.1, we may now construct the tangent bundle. Suppose X is a C^p premanifold with corners, with $1 \le p \le \infty$. Let $\mathscr{O}'$ be the $\mathbf{R}$-space structure on X associated to the "underlying" C^{p-1} premanifold with corners. Of special interest to us is the $\mathscr{O}'$-module Vec_X whose value on a non-empty open set U is the $\mathscr{O}'(U)$-module of C^{p-1} vector fields on U.

Definition 3.2. The *tangent bundle* $TX \to X$ is the C^{p-1} vector bundle V_{Vec_X}.

We have natural $\mathbf{R}$-linear isomorphism $TX(x) \simeq \mathrm{Vec}_X(x) \simeq \mathrm{T}_x(X)$, where the first step is the composite isomorphism $V_{\mathscr{M}}(x) \simeq \underline{V}_{\mathscr{M}}(x) \overset{\theta_{\mathscr{M}}(x)}{\simeq} \mathscr{M}(x)$ for $\mathscr{M} = \mathrm{Vec}_X$ and the second step is induced by $[(U, \vec{v})]_x \mapsto \vec{v}(x)$ for $\vec{v} \in \mathrm{Vec}_X(U)$ with open U around x (this second map was studied in Exampe 1.3). The natural $\mathscr{O}'$-linear isomorphism $\theta_{\mathrm{Vec}_X} : \underline{TX} \simeq \mathrm{Vec}_X$ provides the precise relationship between the tangent bundle and C^{p-1} vector fields on opens in X: a set-theoretic vector field $\vec{v} : u \mapsto \vec{v}(u) \in \mathrm{T}_u(X) \simeq (TX)(u)$ over an open $U \subseteq X$ is C^{p-1} if and only if as a set-theoretic U-section of the C^{p-1} vector bundle $TX \to X$ it is a C^{p-1} mapping.

Example 3.3. If $\{x_1, \dots, x_n\}$ is a C^p coordinate system on an open U containing x, then $\{\partial_{x_j}\}$ trivializes $\mathrm{Vec}_X|_U$ and hence trivializes $TX = V_{\mathrm{Vec}_X}$ over U via sections whose values in each fiber $TX(u) = \mathrm{Vec}_X(u) \simeq \mathrm{T}_u(X)$ over $u \in U$ are exactly the $\partial_{x_j}|_u$'s. In other words, when we identify the ∂_{x_j}'s with trivializing sections for $TX|_U$ the natural isomorphism $TX(x) \simeq \mathrm{T}_x(X)$ for $x \in U$ carries their fiber-values over to the tangent vectors $\partial_{x_j}|_x$ at x. Thus, there is no risk of confusion with viewing the ∂_{x_j}'s as elements of $TX(U)$ and identifying $TX(x)$ with $\mathrm{T}_x(X)$ for $x \in U$.

Example 3.4. Let us push the preceding calculation further in the special case that X is open in a finite-dimensional vector space V. In this case I claim that there is a canonical isomorphism $X \times V \simeq TX$ of C^∞ vector bundles over X, given on fibers by the mapping

$$V \simeq (TX)(x) \simeq \mathrm{T}_x(X)$$

sending $v \in V$ to the directional derivative operator $D_{v,x} \in \mathrm{T}_x(X)$ at x. Let us choose a basis $\{v_1, \dots, v_n\}$ of V and let $t_1, \dots, t_n$ be the dual functionals (basis of $V^\vee$). We then have a bundle isomorphism $X \times \mathbf{R}^n \simeq TX$ via the preceding example, inducing the fiber mapping $\mathbf{R}^n \simeq (TX)(x) \simeq \mathrm{T}_x(X)$ that sends e_j to $\partial_{t_j}|_x = D_{v_j,x}$ for all j and all $x \in X$. Thus, the composite bundle isomorphism

$$X \times V \simeq X \times \mathbf{R}^n \simeq TX$$

induces the fiber mapping $V \simeq \mathbf{R}^n \simeq (TX)(x) \simeq \mathrm{T}_x(X)$ that sends v_j to $D_{v_j,x}$ for all j. But the mapping $V \to \mathrm{T}_x(X)$ defined by $v \mapsto D_{v,x}$ is linear and has the same effect on the basis vectors v_j, so these mappings coincide.

An important property of the tangent bundle is that it globalizes the theory of the tangent map at points. More precisely:

Theorem 3.5. *Let $f : X' \to X$ be a C^p mapping between C^p premanifolds with corners, $1 \le p \le \infty$. There is a unique morphism of C^{p-1} vector bundles*

$$\begin{tikzcd} TX' \arrow[r, "\mathrm{d}f"] \arrow[d] & TX \arrow[d] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$

such that for each $x' \in X'$ the induced **R***-linear map* $TX'(x') \to TX(f(x'))$ *is exactly the old tangent map* $\mathrm{d}f(x') : \mathrm{T}_{x'}(X') \to \mathrm{T}_{f(x')}(X)$*. Moreover, if* $g : X'' \to X'$ *is a second such map, then* $\mathrm{d}(f \circ g) = \mathrm{d}f \circ \mathrm{d}g$*.*

As a special case, if $X' = X$ and f is the identity map then $\mathrm{d}f$ is the identity map on TX because this holds on fibers over each $x \in X$ (due to the tangent map of the identity of X being the identity on each tangent space of X).

Proof. Once the global $\mathrm{d}f$ is constructed, the identity $\mathrm{d}(f \circ g) = \mathrm{d}f \circ \mathrm{d}g$ may be checked by working on fibers where (via the isomorphisms $TX(x) \simeq \mathrm{T}_x(X)$ for each $x \in X$ and analogues for X' and X'') it is just the Chain Rule. Also, the uniqueness of $\mathrm{d}f$ is due to the fact that we are specifying it on fibers. Our problem is therefore one of constructing $\mathrm{d}f$ as a map of C^{p-1} bundles over the C^{p-1} mapping $f : X' \to X$. By gluing for bundle morphisms and the uniqueness of what we are trying to construct, it suffices to work locally on X and X' since solutions locally over the base must agree over overlaps and hence glue to give a global mapping that is (check!) a solution to our global problem. We may therefore work locally over X and then over X' to reduce to the special case when X admits C^p coordinates $\{x_1, \dots, x_n\}$ and X' admits C^p coordinates $\{x'_1, \dots, x'_{n'}\}$. Let $f_i = x_i \circ f : X' \to \mathbf{R}$ be the component functions of f.

By Example 3.3 we have global trivializing sections $\{\partial_{x_i}\}$ in $TX(X)$ and $\{\partial_{x'_j}\}$ in $TX'(X')$ that induce the $\partial x_i|_x$'s in $TX(x) \simeq \mathrm{T}_x(X)$ for each $x \in X$ and the $\partial_{x'_j}|_{x'}$'s in $TX'(x') \simeq \mathrm{T}_{x'}(X')$ for each $x' \in X'$. These give C^{p-1} bundle isomorphisms $TX \simeq X \times \mathbf{R}^n$ over X and $TX' \simeq X' \times \mathbf{R}^{n'}$ over X'. Consider the C^{p-1} mapping $TX' \simeq X' \times \mathbf{R}^n \to X \times \mathbf{R}^n \simeq TX$ in which the middle mapping is given by

$$(x', v) \mapsto (f(x'), ((\partial_{x'_j} f_i)(x'))(v))$$

for $x' \in X'$ and $v \in \mathbf{R}^n$. This middle mapping is certainly a C^{p-1} mapping since f is C^p and the functions $\partial_{x'_j} f_i : X' \to \mathbf{R}$ are C^{p-1}. Thus, if we call this composite C^{p-1} mapping $\mathrm{d}f$, then it is a mapping over $f : X' \to X$ inducing the fibral map $TX'(x') \to TX(f(x'))$ that is exactly the Jacobian matrix $((\partial_{x'_j} f_i)(x'))$ when $TX'(x') \simeq \mathrm{T}_{x'}(X')$ is given the ordered basis $\mathbf{e}' = \{\partial_{x'_j}|_{x'}\}$ and $TX(f(x')) \simeq \mathrm{T}_{f(x')}(X)$ is given the ordered basis $\mathbf{e} = \{\partial_{x_i}|_{f(x')}\}$. This Jacobian matrix calculates the old tangent mapping $\mathrm{d}f(x') : \mathrm{T}_{x'}(X') \to \mathrm{T}_{f(x')}(X)$ with respect to the ordered bases $\mathbf{e}'$ and $\mathbf{e}$, so our construction $\mathrm{d}f : TX' \to TX$ has the desired fibral properties. ∎