Math 396. Linear algebra operations on vector bundles

1. Motivation

Let $(X, \mathscr{O})$ be a C^p premanifold with corners, $0 \leq p \leq \infty$. We have developed the notion of a C^p vector bundle over X as a certain kind of C^p mapping $\pi : E \to X$ that is (roughly speaking) a C^p-varying family of finite-dimensional $\mathbf{R}$-vector spaces $E(x)$ parameterized by the points $x \in X$. In order to do interesting things with vector bundles, we wish to apply to them many of the operations of linear algebra. For example, we want to form duals, tensor products, and extensior/symmetric powers of vector bundles. On fibers these operations should recover the familiar ones from linear algebra.

The case of direct sums was worked out in an earlier handout, and there we could carry out the construction in terms of $\mathscr{O}$-modules and pass back to vector bundles with the $V_{\mathscr{M}}$-construction. Due to our wish to not make long digressions into advanced topics in abstract algebra, for more general operations with vector bundles we need a more "direct" approach (not working exclusively with $\mathscr{O}$-modules) to make some constructions. The construction of the tensor product $E_1 \otimes \cdots \otimes E_n$ was given in class. The method use for this case is the template for what we shall do in general.

2. Tensorial operations

We want to review how the tensor product of several vector bundles is built, simultaneously showing how the same technique applies to define good notions of symmetric and exterior powers of a single vector bundle (such that on fibers we recover the usual tensorial constructions).

Let $E_1, \dots, E_n$ and E be C^p vector bundles over X. As sets, we *define*

$$E_1 \otimes \cdots \otimes E_n = \coprod_{x \in X} (E_1(x) \otimes \cdots \otimes E_n(x)),\ \ \mathrm{Sym}^n(E) = \coprod_{x \in X} \mathrm{Sym}^n(E(x)),\ \ \wedge^n(E) = \coprod_{x \in X} \wedge^n(E(x));$$

we map each set to X by sending the subset indexed by $x \in X$ to the point $x \in X$. Thus, each of these sets is equipped with a map π to X such that each fiber $\pi^{-1}(x)$ is the usual linear-algebra tensorial operation applied to the fiber of the given bundles. The real work is to put reasonable topologies and C^p-structures on these sets to make them C^p vector bundles over X.

Remark 2.1. One can and should ask for more: universal mapping properties analogous to what we have in linear algebra. In §4 such properties will be given, and we note here that it is the general viewpoint of $\mathscr{O}$-modules and *not* vector bundles that will be most convenient to use for the formulation and application of such universal properties.

Remark 2.2. The reader can check that the method used below applies equally well to reconstruct the direct sum of vector bundles. In fact, it is essentially the *same* as the earlier construction of the direct sum because the gluing technique used below is the same as that employed in the $V_{\mathscr{M}}$-construction (which was applied in the earlier construction of direct sums).

We first address the topological aspect of the construction problem. We may cover X by open subsets U over which the *finitely many* bundles under consideration become trivial (indeed, any $x \in X$ admits an open neighborhood on which the finitely many given bundles have trivial restriction). For a choice of such a U, fix trivializations of the given bundles. In the tensor-product case we fix isomorphisms $\phi_i : E_i|_U \simeq U \times V_i$ as C^p vector bundles over U (with V_i a finite-dimensional $\mathbf{R}$-vector space), and in the other cases we fix an isomorphism $\phi : E|_U \simeq U \times V$ as C^p vector bundles over U (with V a finite-dimensional $\mathbf{R}$-vector space). Let $\tau = (U; \phi_1, \dots, \phi_n)$ in the tensor product case and let $\tau = (U, \phi)$ in the other cases; this is the "trivialization data" over U for the given bundle(s).

For each $u \in U$, the data in τ provides linear isomorphisms of fibers $\phi_i(u) : E_i(u) \simeq V_i$ in the tensor-product case, and linear isomorphisms $\phi(u) : E(u) \simeq V$ in the other cases. Hence, for the case of tensor products we have bijections

$$\xi_{U,\tau} : \pi^{-1}(U) \simeq U \times (V_1 \otimes \cdots \otimes V_n)$$

given on u-fibers by the linear tensor-product isomorphism

$$\phi_1(u) \otimes \cdots \otimes \phi_n(u) : E_1(u) \otimes \cdots \otimes E_n(u) \simeq V_1 \otimes \cdots \otimes V_n,$$

and in the symmetric and exterior power cases we have bijections

$$\xi_{U,\tau} : \pi^{-1}(U) \to U \times \mathrm{Sym}^n(V), \ \ \xi_{U,\tau} : \pi^{-1}(U) \to U \times \wedge^n(V)$$

given on u-fibers by the linear isomorphisms of symmetric and exterior powers

$$\mathrm{Sym}^n(\phi(u)) : \mathrm{Sym}^n(E(u)) \simeq \mathrm{Sym}^n(V), \ \ \wedge^n(\phi(u)) : \wedge^n(E(u)) \simeq \wedge^n(V).$$

We "force" the bijection $\xi_{U,\tau}$ to be a homeomorphism in each case: let $S_{U,\tau}$ denote the set $\pi^{-1}(U)$ with the topology induced via $\xi_{U,\tau}$ from the topology on its target (using product topology: the product of U and a finite-dimensional vector space). The first main problem is to show that these topologies glue to define topologies on the sets $E_1 \otimes \cdots \otimes E_n$, $\mathrm{Sym}^n(E)$, and $\wedge^n(E)$.

The method of gluing topologies reduces our task to checking two things: for any opens $U, U' \subseteq X$ and trivialization data τ, τ' over these respective opens (for which there are many choices once U and U' have been chosen), we need

(1) the overlap $S_{U,\tau} \cap S_{U',\tau'}$ is an open subset in each of the topological spaces $S_{U,\tau}$ and $S_{U',\tau'}$,
(2) the subspace topologies on this overlap via its inclusion into each of $S_{U,\tau}$ and $S_{U',\tau'}$ are the same topology.

The overlap is the subset $\pi^{-1}(U) \cap \pi^{-1}(U') = \pi^{-1}(U \cap U')$, so the first item follows from the fact that for any topological space Z (such as a finite-dimensional $\mathbf{R}$-vector space) the subset $(U \cap U') \times Z$ in $U \times Z$ and in $U' \times Z$ is open in each. As for the second item, this amounts to proving that the bijective "transition mapping"

$$\xi_{U',\tau'} \circ \xi_{U,\tau}^{-1} : \xi_{U,\tau}(\pi^{-1}(U \cap U')) \to \xi_{U',\tau'}(\pi^{-1}(U \cap U'))$$

is a homeomorphism.

In the case of tensor products, the transition mapping is the self-map of $(U \cap U') \times (V_1 \otimes \cdots \otimes V_n)$ given by

$$(u, t) \mapsto (u, ((\phi_1'(u) \circ \phi_1(u)^{-1}) \otimes \cdots \otimes (\phi_n'(u) \circ \phi_n(u)^{-1}))(t))$$

for $u \in U \cap U'$ and $t \in V_1 \otimes \cdots \otimes V_n$. In the case of symmetric powers, the transition mapping is the self-map of $(U \cap U') \times \mathrm{Sym}^n(V)$ given by

$$(u, t) \mapsto (u, (\mathrm{Sym}^n(\phi'(u) \circ \phi(u)^{-1}))(t))$$

for $u \in U \cap U'$ and $t \in \mathrm{Sym}^n(V)$. In the case of exterior powers, the transition mapping is the self-map of $(U \cap U') \times \wedge^n(V)$ given by

$$(u, t) \mapsto (u, (\wedge^n(\phi'(u) \circ \phi(u)^{-1}))(t))$$

for $u \in U \cap U'$ and $t \in \wedge^n(V)$.

These self-maps of a product of $U \cap U'$ against a finite-dimensional vector space are better than homeomorphisms: they are C^p isomorphisms! We can ignore the inverse map and just check the C^p property because the argument may be applied to the inverse mapping by swapping the roles of (U, τ) and (U', τ'). Upon picking bases of the vector spaces V_i and V we get bases for the tensor products and symmetric/exterior powers in the usual manner, and so the problem is to prove that the linear mappings

$$(\phi_1'(u) \circ \phi_1(u)^{-1}) \otimes \cdots \otimes (\phi_n'(u) \circ \phi_n(u)^{-1}),\ \ \mathrm{Sym}^n(\phi'(u) \circ \phi(u)^{-1}),\ \ \wedge^n(\phi'(u) \circ \phi(u)^{-1})$$

depending on u are given (in these bases) by matrices whose matrix entries have C^p dependence on $u \in U \cap U'$. The linear mappings $\phi_i(u), \phi'(u), \phi(u), \phi'(u)$ are matrix-valued functions on $U \cap U'$ with matrix entries that are C^p functions on $U \cap U'$. Hence, by the "universal" algebraic (polynomial) formulas for the inverse of a matrix and the matrix of the tensor product and of the symmetric/exterior powers of linear mappings, we get the desired C^p property for the transition mappings. In particular, the homeomorphism property is proved.

Having taken care of the definition of the global topology, note that the topology was rigged so that the projection map π to X is continuous (since it is so over each U as above, with $\pi^{-1}(U)$ an open set in the global topology we have constructed). Also, the topology was rigged to force the set-theoretic trivialization $\xi_{U,\tau}$ over U (respecting linear structure on the fibers) to be a topological trivialization. These U's cover X, so each of our constructions is a topological vector bundle over X. Having put a topology on the total spaces, we now go through the construction a second time and use the $\xi_{U,\tau}$'s to put a C^p-structure on each subset $S_{U,\tau}$ (this is $\pi^{-1}(U)$) using the C^p-structure on the target of $\xi_{U,\tau}$ (this target is the product of U against a finite-dimensional vector space).

In order to "glue" these C^p structures to a global one, the only problem is to check consistency on overlaps: is $\xi_{U',\tau'} \circ \xi_{U,\tau}^{-1}$ a C^p isomorphism (over $U \cap U'$)? It was *exactly* this stronger property that we verified in the considerations with the topological aspects of the problem. Not only does this provide us with a global C^p structure, but it enhances each $\xi_{U,\tau}$ to a C^p isomorphism that is linear on fibers, and so the local triviality criterion to be a C^p vector bundle is satisfied. This completes the construction of C^p vector bundles

$$E_1 \otimes \cdots \otimes E_n,\ \ \mathrm{Sym}^n(E),\ \ \wedge^n(E)$$

over X with the "desired" fibers (as vector spaces).

Example 2.3. Let us make the preceding constructions very concrete in the language of local frames. First consider tensor products. Let $U \subseteq X$ be an open over which each $E_j|_U$ admits a trivialization via sections $s_{i,j} \in E_j(U)$ (for $1 \le i \le r_j$). Let τ be the "trivialization data" arising from these frames. The set-theoretic sections

$$s_{i_1,1} \otimes \ldots s_{i_n,n} : u \mapsto s_{i_1,1}(u) \otimes \ldots s_{i_n,n}(u) \in E_1(u) \otimes \cdots \otimes E_n(u) \simeq (E_1 \otimes \cdots \otimes E_n)(u)$$

are C^p and moreover give a trivializing frame for $(E_1 \otimes \cdots \otimes E_n)|_U$. Indeed, this is the trivialization $\xi_{U,\tau}$ in the *definition* of the C^p vector bundle structure on $E_1 \otimes \cdots \otimes E_n$.

Next, consider $\mathrm{Sym}^n(E)$ and $\wedge^n(E)$. Let $U \subseteq X$ be an open set such that $E|_U$ has a trivializing frame $\{s_1, \ldots, s_m\}$. (That is, $s_j \in E(U)$ and the $s_j(u)$'s are a basis of $E(u)$ for all $u \in U$.) We have set-theoretic sections

$$s_{i_1} \cdots s_{i_n} : u \mapsto s_{i_1}(u) \cdots s_{i_n}(u) \in \mathrm{Sym}^n(E(u)),\ \ s_{i_1} \wedge \cdots \wedge s_{i_n} : u \mapsto s_{i_1}(u) \wedge \cdots \wedge s_{i_n}(u) \in \wedge^n(E(u))$$

for $1 \le i_1 \le \cdots \le i_n \le m$ in the first case and for $1 \le i_1 < \cdots < i_n \le m$ in the second case. The same method as above shows that these are C^p sections that moreover give trivializations for $\mathrm{Sym}^n(E)$ and $\wedge^n(E)$ over U.

A very useful refinement of the preceding example is that we can drop the "local frame" condition:

Theorem 2.4. *For any open $U \subseteq X$ and arbitrary C^p sections $v_1 \in E_1(U), \ldots, v_n \in E_n(U)$, the set-theoretic section*

$$v_1 \otimes \cdots \otimes v_n : u \mapsto v_1(u) \otimes \cdots \otimes v_n(u) \in E_1(u) \otimes \cdots \otimes E_n(u) = (E_1 \otimes \cdots \otimes E_n)(u)$$

of $E_1 \otimes \cdots \otimes E_n$ *over* U *is a* C^p *section.*

Likewise, for C^p *sections* $v_1, \dots, v_n \in E(U)$ *the set-theoretic sections*

$$v_1 \cdots v_n : u \mapsto v_1(u) \cdots v_n(u) \in \mathrm{Sym}^n(E(u)), \ \ v_1 \wedge \cdots \wedge v_n : u \mapsto v_1(u) \wedge \cdots \wedge v_n(u) \in \wedge^n(E(u))$$

of $\mathrm{Sym}^n(E)$ *and* $\wedge^n(E)$ *over* U *are* C^p *sections.*

Remark 2.5. In contrast with the case of tensor products of vector spaces, it is not *a priori* evident whether or not every element of $(E_1 \otimes \cdots \otimes E_n)(U)$ is a finite $\mathscr{O}(U)$-linear combination of "elementary tensors"

$$v_1 \otimes \cdots \otimes v_n \in (E_1 \otimes \cdots \otimes E_n)(U)$$

for $v_j \in E_j(U)$. At least for $p > 0$ and X a manifold it can be proved that this is true (though we will not use it). The proof requires some serious input (Whitney embedding theorem and Riemannian metrics on bundles), and the complex-analytic analogue is *false*.

Proof. We first explain the case of tensor products. The problem is local over X, and so by shrinking X we can assume that the E_j's are all trivial. Let $\{s_{ij}\}_{1 \le i \le r_j}$ be a trivializing frame for E_j in $E_j(X)$. Hence, $v_j = \sum_i a_{ij} s_{ij}$ in $E_j(X)$ for some C^p functions a_{ij} on X $(1 \le i \le r_j)$ since the v_j's are C^p sections by hypothesis. By the multilinearity rules for tensor products,

$$v_1(x) \otimes \cdots \otimes v_n(x) = \sum_{i_1, \dots, i_n} (\prod_{m=1}^{n} a_{i_m,m}(x)) s_{i_1,1}(x) \otimes \cdots \otimes s_{i_m,m}(x)$$

in $E_1(x) \otimes \cdots \otimes E_n(x)$ for all $x \in X$. That is, when the set-theoretic section $v_1 \otimes \cdots \otimes v_n$ is written as a linear combination (with set-theoretic function coefficients) of the trivializing frame of C^p sections $s_{i_1,1} \otimes \cdots \otimes s_{i_m,m}$ of the tensor product bundle, the coefficient functions are the products $\prod_{m=1}^{n} a_{i_m,m}$ (for $1 \le i_m \le r_m$), and these are all C^p functions on X since the a_{ij}'s are C^p on X. Hence, the set-theoretic section $v_1 \otimes \cdots \otimes v_n$ is a C^p section.

The case of symmetric and exterior powers goes exactly the same way, the only difference being that the formulas for the coefficient functions will be sums of products with conditions on the i_m's and (for exterior powers) some signs, in accordance with the rules for expanding out symmetric and wedge products. ∎

We can also use the above method of chasing "coefficient formulas" to prove:

Theorem 2.6. *Let* $T_j : E'_j \to E_j$ *be* C^p *bundle maps over* X*. There is a unique* C^p *bundle map*

$$T_1 \otimes \cdots \otimes T_n : E'_1 \otimes \cdots \otimes E'_n \to E_1 \otimes \cdots \otimes E_n \tag{1}$$

over X *such that on fibers over* $x \in X$ *it is the tensor product mapping*

$$T_1|_x \otimes \cdots \otimes T_n|_x : E'_1(x) \otimes \cdots \otimes E'_n(x) \to E_1(x) \otimes \cdots \otimes E_n(x).$$

Likewise, if $T : E' \to E$ *is a* C^p *bundle map over* X *then there are unique* C^p *bundle maps*

$$\mathrm{Sym}^n(T) : \mathrm{Sym}^n(E') \to \mathrm{Sym}^n(E), \ \ \wedge^n(T) : \wedge^n(E') \to \wedge^n(E)$$

given on fibers over $x \in X$ *by*

$$\mathrm{Sym}^n(T|_x) : \mathrm{Sym}^n(E'(x)) \to \mathrm{Sym}^n(E(x)), \ \ \wedge^n(T|_x) : \wedge^n(E'(x)) \to \wedge^n(E(x)).$$

Proof. We treat the case of tensor products, and the case of symmetric and exterior powers goes in exactly the same manner (much like in the preceding proof). The problem is one of verifying that a given set-theoretic mapping of bundles (linear on fibers, and respecting projections to X) is a C^p mapping. This problem is local over X, so by working locally over X we may reduce to the case when each E_j and E'_j is trivial. Choose trivializing frames for all E_j and E'_j, so the C^p bundle mappings T_j are described by matrices $[T_j]$ whose entries are C^p functions on X. Using the tensor products of these trivializing frames to make trivializing frames of $E_1 \otimes \cdots \otimes E_n$ and $E'_1 \otimes \cdots \otimes E'_n$ (as in Example 2.3), the mapping under consideration between these two bundles is thereby described in such frames via a matrix whose entries are built up by a "universal formula" as sums of products of entries in the $[T_j]$'s. (This is just the classical universal recipe for computing the matrix for a tensor product of linear mappings between Euclidean spaces of specified dimensions.) Hence, these entries are C^p functions on X, so the set-theoretic mapping of bundles (1) is indeed C^p. ∎

In the preceding development of tensor products and symmetric/exterior powers for bundles, we have checked several aspects: the "right" fibers, the "right" local frames for the output of the construction when given local frames for the input bundles (as in Example 2.3), and the "right" behavior (say, on fibers) for maps between bundles (as in Theorem 2.6). There is one further property we wish to consider: behavior with respect to bundle pullback. We first require a lemma:

Lemma 2.7. *Let $E, E' \rightrightarrows X$ be C^p vector bundles and assume that E' is trivial with a trivializing frame $\{s'_1, \dots, s'_n\}$ for $s'_j \in E'(X)$. For any $s_1, \dots, s_n \in E(X)$, there is a unique C^p bundle map $f : E' \to E$ such that on X-sections it carries s'_j to s_j for all j. If the s_j's are a trivializing frame for E, then f is an isomorphism.*

Proof. We seek to build a unique C^p vector bundle mapping $f : E' \to E$ such that the induced map $E'(X) \to E(X)$ carries s'_j to s_j. Once this is done, then in case the s_j's are a trivializing frame for E we conclude that $f|_x : E'(x) \to E(x)$ is a linear isomorphism for all $x \in X$, so f is an isomorphism.

To define f, recall from class that for a C^p vector bundle $V \to X$, the set $\mathrm{Hom}_X(X \times \mathbf{R}^n, V)$ is in natural bijection with $V(X)^{\times n}$, by chasing images of the constant sections $\underline{e}_j : x \mapsto (x, e_j)$ of $X \times \mathbf{R}^n \to X$ (for $\{e_j\}$ the standard basis of $\mathbf{R}^n$). Thus, there are unique C^p vector bundle morphisms

$$X \times \mathbf{R}^n \xrightarrow{\alpha} E', \;\; X \times \mathbf{R}^n \xrightarrow{\beta} E$$

carrying $\underline{e}_j$ to s'_j and to s_j respectively. The first of these two bundle mappings is an isomorphism on fibers, and hence an isomorphism, so $\beta \circ \alpha^{-1}$ is a C^p vector bundle map

$$E' \simeq X \times \mathbf{R}^n \to E$$

that is as desired on fibers. ∎

Theorem 2.8. *Let $f : X' \to X$ be a C^p mapping between C^p premanifolds with corners. Let $E_1, \dots, E_n, E$ be C^p vector bundles on X. Using the linear fiber isomorphism $(f^*(V))(x') \simeq V(f(x'))$ for all $x' \in X'$ and C^p vector bundles $V \to X$, there are unique isomorphisms*

$$f^*(E_1 \otimes \cdots \otimes E_n) \simeq f^*(E_1) \otimes \cdots \otimes f^*(E_n), \;\; f^*(\mathrm{Sym}^n E) \simeq \mathrm{Sym}^n(f^*E), \;\; f^*(\wedge^n E) \simeq \wedge^n(f^*E)$$

as C^p vectors bundles over X' such that on fibers over $x' \in X'$ these give the usual isomorphisms

$$(E_1 \otimes \cdots \otimes E_n)(f(x')) \simeq E_1(f(x')) \otimes \cdots \otimes E_n(f(x')) \tag{2}$$

and

$$(\mathrm{Sym}^n E)(f(x')) \simeq \mathrm{Sym}^n(E(f(x'))), \;\; (\wedge^n E)(f(x')) \simeq \wedge^n(E(f(x'))).$$

Proof. We work out the case of tensor products and leave it to the reader to check that symmetric and exterior powers carry over the same way. Since we are specifying the linear mapping on fibers, the problem is one of checking that it is C^p. This is local over X (working over an open $U \subseteq X$ allows us to work with the open $f^{-1}(U) \subseteq X'$), so we can assume that the finitely many E_j's are all trivial. Let $\{s_{ij}\}$ be a trivializing frame for E_j (with $1 \le i \le r_j$), so Examples 3.1 and 3.2 in the handout on pullback bundles show that the pullback sections $f^*(s_{ij}) \in (f^*E_j)(X')$ (with value $s_{ij}(f(x'))$ in $(f^*E_j)(x') = E_j(f(x'))$ for all $x' \in X'$) give a trivializing frame for f^*E_j. Hence, by Example 2.3, a trivializing frame for $f^*(E_1) \otimes \cdots \otimes f^*(E_n)$ is given by the "tensor products" $f^*(s_{i_1,1}) \otimes \cdots \otimes f^*(s_{i_n,n})$. Likewise, $f^*(E_1 \otimes \cdots \otimes E_n)$ has a trivializing frame given by the sections

$$f^*(s_{i_1,1} \otimes \cdots \otimes s_{i_n,n})$$

for $1 \leq i_j \leq r_j$. By Lemma 2.7 we may define the C^p isomorphism

$$f^*(E_1 \otimes \cdots \otimes E_n) \to f^*(E_1) \otimes \cdots \otimes f^*(E_n)$$

in terms of such trivializing frames: we require that it satisfies

$$f^*(s_{i_1,1} \otimes \cdots \otimes s_{i_n,n}) \mapsto f^*(s_{i_1,1}) \otimes \cdots \otimes f^*(s_{i_n,n})$$

for all $1 \leq i_j \leq r_j$. On fibers, this gives the desired mapping (2). ■

Example 2.9. Recall that the Möbius strip of infinite height $M_\infty \to S^1$ is a non-trivial line bundle. We claim that its second tensor power $M_\infty^{\otimes 2}$ is a *trivial* line bundle over the circle. Briefly, the reason is that $(-1)^2 = 1$. The precise argument can be carried out in several ways. We will explain it via group actions (recall that M_∞ as a line bundle over the circle is the quotient of the trivial bundle $S^1 \times \mathbf{R}$ by the action $(\theta, t) \mapsto (\theta + \pi, -t)$ covering the 180-degree rotation on the circle). First we explain in general how the tensorial operations interact with group actions on bundles.

Let $E_1, \dots, E_n$ be vector bundles over X and suppose that a group Γ has a right C^p action on X and on the E_j's such that the hypotheses as in Exercise 3 of Homework 5 are satisfied for each E_j. Let $[\gamma]_{j,x} : E_j(x) \to E_j(x.\gamma)$ denote the fibral right action for $x \in X$. We get vector bundles $\overline{E}_j = E_j/\Gamma \to X/\Gamma$ for each j. There is also a unique right C^p action of Γ on $E = E_1 \otimes \cdots \otimes E_n$ over the action on X, given on fibers by the linear isomorphism

$$E(x) \simeq E_1(x) \otimes \cdots \otimes E_n(x) \simeq E_1(x.\gamma) \otimes \cdots \otimes E_n(x.\gamma) \simeq E(x.\gamma) \tag{3}$$

with the middle isomorphism equal to $[\gamma]_{1,x} \otimes \cdots \otimes [\gamma]_{n,x}$. To see that this really is a C^p action on E (over the action on X), one can do an explicit calculation in local frames or argue globally as follows. By the universal property of bundle pullback, the right action of $\gamma \in \Gamma$ on E_j over the action $[\gamma] : x \mapsto x.\gamma$ on X is "the same" as the data of a C^p bundle isomorphism $\phi_{j,\gamma} : E_j \simeq [\gamma]^*(E_j)$ over (the identity on) X. Thus, by Theorem 2.6 and Theorem 2.8 we get C^p bundle isomorphisms

$$\phi_{1,\gamma} \otimes \cdots \otimes \phi_{n,\gamma} : E_1 \otimes \cdots \otimes E_n \simeq [\gamma]^*(E_1) \otimes \cdots \otimes [\gamma]^*(E_n) \simeq [\gamma]^*(E_1 \otimes \cdots \otimes E_n). \tag{4}$$

Let $E = E_1 \otimes \cdots \otimes E_n$. Using the universal property of bundle pullback, (4) gives a C^p bundle mapping $E \to E$ over $[\gamma] : X \to X$, and on fibers over x and $[\gamma](x) = x.\gamma$ it is exactly the map (3). The action of Γ on E is free and properly discontinuous because it covers the action of Γ on X that is free and properly discontinuous.

Now that we have Γ acting freely and properly discontinuously on the right on the E_j's, as well as on their tensor product E, covering the action on X, it is natural to inquire about the relationship between the C^p vector bundle E/Γ over X/Γ and the tensor product bundle $\overline{E}_1 \otimes \cdots \otimes \overline{E}_n$ over X/Γ. Letting $h : X \to X/\Gamma$ be the projection to the quotient, Example 3.5 in the handout on pullback bundles provides natural isomorphisms

$$E \simeq h^*(E/\Gamma), \quad E_j \simeq h^*(\overline{E}_j).$$

Hence, by Theorem 2.8 we get a map

$$E \simeq E_1 \otimes \cdots \otimes E_n \simeq h^*(\overline{E}_1) \otimes \cdots \otimes h^*(\overline{E}_n) \simeq h^*(\overline{E}_1 \otimes \cdots \otimes \overline{E}_n)$$

of C^p vector bundles over X. By the universal property of bundle pullback, this composite map corresponds to a C^p map of bundles

$$\widetilde{h} : E \to \overline{E}_1 \otimes \cdots \otimes \overline{E}_n$$

over the projection $h : X \to X/\Gamma$. By inspecting the construction of $\widetilde{h}$, one sees that it is unaffected by the Γ-action on E, and so by the universal property of quotients by free and properly discontinuous actions the map $\widetilde{h}$ uniquely factors through a C^p mapping

$$E/\Gamma = (E_1 \otimes \cdots \otimes E_n)/\Gamma \to \overline{E}_1 \otimes \cdots \otimes \overline{E}_n \tag{5}$$

over the identity on X/Γ and it is linear on fibers. The map in (5) is a C^p bundle morphism, and in fact it is an *isomorphism* because for $x \in X$ over $\overline{x} \in X/\Gamma$ the $\overline{x}$-fiber map for (5) is identified with the natural isomorphism

$$E(x) \simeq E_1(x) \otimes \cdots \otimes E_n(x).$$

The isomorphism (5) expresses the precise sense in which the formation of Γ-quotients is compatible with tensor products of bundles. (Analogues for symmetric and exterior powers are established similarly.)

Now return to the Möbius strip with infinite height. The line bundle M_∞ over the circle C is the quotient of the trivial bundle $L = S^1 \times \mathbf{R} \to S^1$ by the involution $(\theta, t) \mapsto (\theta + \pi, -t)$ (covering the involution $\theta \mapsto \theta + \pi$ on S^1). This is a right action by the group Γ of order 2. Hence, by the preceding discussion with $X = S^1$, $n = 2$, and $E_1 = E_2 = L$, the bundle $M_\infty^{\otimes 2} \to C$ is the quotient of $L^{\otimes 2} \to S^1$ by the induced "tensor product" Γ-action on $L^{\otimes 2} = S^1 \times \mathbf{R}^{\otimes 2}$. However, in $\mathbf{R} \otimes \mathbf{R}$ the elements $a \otimes b$ and $(-a) \otimes (-b)$ are equal ($(-1)^2 = 1$). Thus, the Γ-action on $L^{\otimes 2} = S^1 \times \mathbf{R}^{\otimes 2}$ is $(\theta, \xi) \mapsto (\theta + \pi, \xi)$ for $\theta \in S^1$ and $\xi \in \mathbf{R}^{\otimes 2}$. The quotient M_∞ is thereby identified with $C \times \mathbf{R}^{\otimes 2}$ *as a line bundle over* C, so $M_\infty^{\otimes 2}$ is trivial over C.

3. Dual and Hom bundles

We now wish to use similar methods to define a dual vector bundle $E^\vee$ and a Hom-bundle $\mathrm{Hom}(E', E)$ over X (**Warning.** The notation $\mathrm{Hom}(E', E)$ for a certain vector bundle is *not* to be confused with the set $\mathrm{Hom}_X(E', E)$ of bundle mappings over X; in fact, this latter set will turn out to be the set of X-sections of the bundle $\mathrm{Hom}(E', E)$; see Example 3.1). Roughly speaking, $E^\vee$ should be "the" bundle over X whose x-fiber is $E(x)^\vee$ for every $x \in X$, and $\mathrm{Hom}(E', E)$ should be "the" bundle over X whose x-fiber is $\mathrm{Hom}(E'(x), E(x))$ for every $x \in X$.

There is also more: just as we have various isomorphisms in linear algebra such as $V^{\vee\vee} \simeq V$, $V \otimes V'^\vee \simeq \mathrm{Hom}(V', V)$, $(\wedge^n V)^\vee \simeq \wedge^n (V^\vee)$, and so on, we wish to have analogues for dual, Hom, and tensorial operations of bundles (inducing the classical isomorphisms on fibers). As in the discussion of tensor products and symmetric/exterior powers, the basic principle is four-fold: we want the "right" fibers, the "right" local frames for the output of the construction when given local frames for the input bundles (as in Example 2.3), the "right" behavior (say, on fibers) for maps between bundles (as in Theorem 2.6), and good behavior with respect to pullback along C^p mappings $f : X' \to X$.

Let E and E' be C^p vector bundles over X. As sets, define

$$\mathrm{Hom}(E', E) = \coprod_{x \in X} \mathrm{Hom}(E'(x), E(x)).$$

Define the projection $\pi : \mathrm{Hom}(E', E) \to X$ carrying the subset indexed by x onto x. Consider open subsets $U \subseteq X$ over which $E|_U$ and $E'|_U$ are trivial. Let $\tau = (U, \phi, \phi')$ with C^p bundle isomorphisms $\phi : E|_U \simeq U \times V$ and $\phi' : E'|_U \simeq U \times V'$ for finite-dimensional vector spaces V and

V'. For each $u \in U$, the data in τ provides linear isomorphisms of fibers $\phi(u) : E(u) \simeq V$ and $\phi'(u) : E'(u) \simeq V'$. Hence, we have bijections

$$\xi_{U,\tau} : \pi^{-1}(U) \simeq U \times \mathrm{Hom}(V', V)$$

given on u-fibers by the linear isomorphism

$$\phi'(u) \circ (\cdot) \circ \phi(u)^{-1} : \mathrm{Hom}(E'(u), E(u)) \simeq \mathrm{Hom}(V', V)$$

that carries $T \in \mathrm{Hom}(E'(u), E(u))$ to $\phi'(u) \circ T \circ \phi(u)^{-1} \in \mathrm{Hom}(V', V)$. We "force" $\xi_{U,\tau}$ to be a homeomorphism: let $S_{U,\tau}$ denote the set $\pi^{-1}(U)$ with the topology induced via $\xi_{U,\tau}$ from the topology on its target (using product topology: the product of U and a finite-dimensional vector space).

The first main problem is to show that these topologies glue to define a topology on the set $\mathrm{Hom}(E', E)$ as defined "fiberwise" above. The method of gluing topologies reduces our task to checking two things: for any opens $U, U' \subseteq X$ and trivialization data $\tau = (U, \phi, \phi'), \tau' = (U', \psi, \psi')$ over these respective opens (for which there are many choices once U and U' have been chosen), we need

(1) the overlap $S_{U,\tau} \cap S_{U',\tau'}$ is an open subset in each of the topological spaces $S_{U,\tau}$ and $S_{U',\tau'}$,
(2) the subspace topologies on this overlap via its inclusion into each of $S_{U,\tau}$ and $S_{U',\tau'}$ are the same topology.

The overlap is the subset $\pi^{-1}(U) \cap \pi^{-1}(U') = \pi^{-1}(U \cap U')$, so the first item follows from the fact that for any topological space Z (such as a finite-dimensional $\mathbf{R}$-vector space) the subset $(U \cap U') \times Z$ in $U \times Z$ and in $U' \times Z$ is open in each. As for the second item, this amounts to proving that the bijective "transition mapping"

$$\xi_{U',\tau'} \circ \xi_{U,\tau}^{-1} : \xi_{U,\tau}(\pi^{-1}(U \cap U')) \to \xi_{U',\tau'}(\pi^{-1}(U \cap U'))$$

is a homeomorphism.

The transition mapping is the self-map of $(U \cap U') \times \mathrm{Hom}(V', V)$ given by

$$(u, f) \mapsto (u, (\psi(u) \circ \phi(u)) \circ f \circ (\phi'(u)^{-1} \circ \psi'(u)^{-1}))$$

for $u \in U \cap U'$ and $f \in \mathrm{Hom}(V', V)$. This self-map of a product of $U \cap U'$ against a finite-dimensional vector space is better than a homeomorphism: it is a C^p isomorphism! It suffices to verify the C^p property for the above map (ignoring its inverse), as the same argument can be applied to the inverse mapping by swapping the roles of (U, τ) and (U', τ'). Upon picking bases of the vector spaces V and V' we identify $\mathrm{Hom}(V', V)$ with a space of matrices in the usual manner, and so the problem is to prove that the linear mappings

$$(\psi(u) \circ \phi(u)) \circ (\cdot) \circ (\psi'(u) \circ \phi'(u))^{-1}$$

depending on u are given (in these bases) by matrices whose matrix entries have C^p dependence on $u \in U \cap U'$. The linear mappings $\phi(u), \phi'(u), \psi(u), \psi'(u)$ are matrix-valued functions on $U \cap U'$ with matrix entries that are C^p functions on $U \cap U'$. Hence, by the "universal" algebraic (polynomial) formulas for matrix inverse and matrix multiplication in terms of the matrix entries, we get the desired C^p property for the transition mappings. In particular, the homeomorphism property is proved.

Having taken care of the definition of the global topology, note that the topology was rigged so that the projection map π to X is continuous (since it is so over each U as above, with $\pi^{-1}(U)$ an open set in the global topology we have constructed). Also, the topology was rigged to force the set-theoretic trivialization $\xi_{U,\tau}$ over U (respecting linear structure on the fibers) to be a topological trivialization. These U's cover X, so each of our constructions is a topological vector bundle over

X. Having put a topology on the total space, we now go through the construction a second time and use the $\xi_{U,\tau}$'s to put a C^p-structure on each subset $S_{U,\tau}$ (this is $\pi^{-1}(U)$) using the C^p-structure on the target of $\xi_{U,\tau}$ (this target is the product of U against a finite-dimensional vector space).

In order to "glue" these C^p structures to a global one, the only problem is to check consistency on overlaps: is $\xi_{U',\tau'} \circ \xi_{U,\tau}^{-1}$ a C^p isomorphism (over $U \cap U'$)? It was *exactly* this stronger property that we verified in the considerations with the topological aspects of the problem. Not only does this provide us with a global C^p structure, but it enhances each $\xi_{U,\tau}$ to a C^p isomorphism that is linear on fibers, and so the local triviality criterion to be a C^p vector bundle is satisfied. This completes the construction of the C^p vector bundle $\mathrm{Hom}(E', E)$ with the "desired" fibers (as vector spaces). This is called a *Hom-bundle*.

Example 3.1. Let us make the preceding construction very concrete in the language of local frames. Let $U \subseteq X$ be an open over which $E|_U$ and $E'|_U$ admit trivializations via sections $s_i \in E(U)$ and $s'_j \in E'(U)$ (with $1 \le i \le n$, $1 \le j \le n'$), and let τ be the "trivialization data" arising from these frames. The set-theoretic "elementary matrix" sections

$$\varepsilon_{ij} : u \mapsto s_i(u) \otimes s'_j(u)^* \in E(u) \otimes E'(u)^\vee \simeq \mathrm{Hom}(E'(u), E(u)) \simeq (\mathrm{Hom}(E', E))(u) \tag{6}$$

are C^p and moreover give exactly the trivializing frame for $\mathrm{Hom}(E', E)|_U$ corresponding to the C^p trivialization $\xi_{U,\tau}$ in the *definition* of the C^p vector bundle structure on $\mathrm{Hom}(E', E)$.

In particular, every set-theoretic section of $\mathrm{Hom}(E', E)$ over U has the form $\sum a_{ij}\varepsilon_{ij}$ for unique functions $a_{ij} : U \to \mathbf{R}$, and this section of $\mathrm{Hom}(E', E)$ is C^p if and only if the a_{ij}'s are C^p functions on U for all i, j.

Example 3.2. The *dual bundle* $E^\vee$ is $\mathrm{Hom}(E, X \times \mathbf{R})$. This has fibers naturally identified with $\mathrm{Hom}(E(x), \mathbf{R}) = E(x)^\vee$ for all $x \in X$. If $U \subseteq X$ is an open subset and elements $s_i \in E(U)$ give a trivializing frame of $E|_U$, then the preceding example provides C^p sections $s_i^* \in E^\vee(U)$ inducing the dual basis functionals $s_i(u)^* \in E(u)^\vee$ on u-fibers for all $u \in U$. In particular, these give a trivializing frame for $E^\vee|_U$; it is called the *dual frame* to $\{s_i\}$ for $E|_U$.

The behavior of $\mathrm{Hom}(E', E)$ with respect to bundle morphisms and pullback works out nicely:

Theorem 3.3. *Let $T : E_1 \to E_2$ and $T' : E'_2 \to E'_1$ be C^p vector bundle mappings. There is a unique map of C^p vector bundles*

$$\mathrm{Hom}(E'_1, E_1) \to \mathrm{Hom}(E'_2, E_2)$$

that on x-fibers is the map $\mathrm{Hom}(E'_1(x), E_1(x)) \to \mathrm{Hom}(E'_2(x), E_2(x))$ defined by $L \mapsto T|_x \circ L \circ T'|_x^{-1}$.

Also, for any C^p mapping $f : X' \to X$, there is a unique isomorphism of C^p vector bundles

$$f^*(\mathrm{Hom}(E', E)) \simeq \mathrm{Hom}(f^*E', f^*E)$$

that induces the natural isomorphism $\mathrm{Hom}(E'(f(x')), E(f(x'))) \simeq \mathrm{Hom}((f^*E')(x'), (f^*E)(x))$ on x'-fibers for all $x' \in X$.

Proof. The goes by the same method as in the analogous results for tensorial operations in Theorem 2.8: we work locally so that all bundles are trivial, and we chase local frames. ∎

Example 3.4. There is a pleasant description of the $\mathscr{O}(U)$-module $(\mathrm{Hom}(E', E))(U)$ for open sets $U \subseteq X$: it is the $\mathscr{O}(U)$-module $\mathrm{Hom}_U(E'|_U, E|_U)$ of C^p bundle mappings $E'|_U \to E|_U$ over U (endowed with its natural $\mathscr{O}(U)$-module structure through the evident way of adding such maps and multiplying them by elements of $\mathscr{O}(U)$; such fiberwise definitions over any open in X do preserve the C^p-property, as is seen by computing with matrices for local frames of E' and E).

To establish the proposed description of the U-sections of the bundle $\mathrm{Hom}(E', E)$, note that any set-theoretic mapping $f : E'|_U \to E|_U$ respecting projections to U and linear on u-fibers for all $u \in U$ gives rise to an element $f|_u \in \mathrm{Hom}(E'(u), E(u)) = (\mathrm{Hom}(E', E))(u)$ for all $u \in U$, and hence gives a set-theoretic section $[f]$ of $\mathrm{Hom}(E', E)$ over U. By definition of $\mathrm{Hom}(E', E)$ as a set, it is clear that $f \mapsto [f]$ is a bijection from the set of set-theoretic mappings $E'|_U \to E|_U$ over U inducing linear maps on fibers onto the set of set-theoretic sections of $\mathrm{Hom}(E', E)$ over U. The problem is to show that f is a C^p map if and only if $[f]$ is a C^p section (and the compatibility with $\mathscr{O}(U)$-module structures is seen by fiberwise calculation over U). Such C^p properties are local over U, so we may work locally on U to reduce to the case in which E and E' are trivial. Upon choosing local C^p frames, f is described by a matrix-valued function on U whose matrix entries are C^p on U if and only if f is a C^p mapping. By Example 3.1, it likewise follows that $[f]$ is a C^p section of $\mathrm{Hom}(E', E)$ if and only if this matrix has entries that are C^p functions on U. This completes the verification.

The special case of dual bundles is sufficiently important that we restate the theorem in this special case:

Theorem 3.5. *Let $T : E_1 \to E_2$ be a C^p vector bundle mapping. There is a unique map of C^p vector bundles $T^\vee : E_2^\vee \to E_1^\vee$ that on x-fibers is the dual map $T|_x^\vee : E_2(x)^\vee \to E_1(x)^\vee$.*

Also, for any C^p mapping $f : X' \to X$, there is a unique isomorphism of C^p vector bundles

$$f^*(E^\vee) \simeq (f^*E)^\vee$$

that induces the natural isomorphism $E^\vee(f(x')) \simeq (f^*E')(x')^\vee$ on x'-fibers for all $x' \in X'$.

The map $T^\vee$ is called the *dual map*, and by checking on fibers it is clear that if $T' : E_2 \to E_3$ is a C^p vector bundle map then $(T' \circ T)^\vee = T^\vee \circ T'^\vee$ as it should be. We expect to have natural (uniquely determined) C^p vector bundle isomorphisms

$$\mathrm{Hom}(E', E) \simeq E \otimes E'^\vee, \;\; E^{\vee\vee} \simeq E, \;\; (E_1 \otimes \cdots \otimes E_n)^\vee \simeq E_1^\vee \otimes \cdots \otimes E_n^\vee,$$

and so on such that on fibers we recover the habitual isomorphisms from linear algebra. Such isomorphisms can be built using local considerations with trivializing frames. However, there is a more elegeant method that is moreover necessary if we wish to gain a greater fluency in working with tensors, Homs, and duals of bundles (especially for our later work with Riemannian manifolds and connections). The main ingredients we need are universal properties to characterize these constructions, so in the next section we take up the issue of universal properties and we use such properties to make bundle analogues of many isomorphisms from linear algebra.

4. Universal properties

We wish to characterize tensorial operations on bundles and Hom and dual bundles in terms of universal properties. To "know" a vector bundle is the same as to "know" its associated $\mathscr{O}$-module, and the language of $\mathscr{O}$-modules turns out to be very well-suited for the formulation of universal properties. (The discussion that follows could be recast in the language of vector bundles, or more precisely fiber bundles, but for later purposes it is the $\mathscr{O}$-module perspective that we shall need and so we have opted to use it here.)

Definition 4.1. Let $\mathscr{M}_1, \ldots \mathscr{M}_n, \mathscr{M}$ be $\mathscr{O}$-modules. A *multilinear mapping* from the $\mathscr{M}_j$'s to $\mathscr{M}$, denoted

$$m : \mathscr{M}_1 \times \cdots \times \mathscr{M}_n \to \mathscr{M},$$

is a compatible family of $\mathscr{O}(U)$-multilinear mappings

$$m_U : \mathscr{M}_1(U) \times \cdots \times \mathscr{M}_n(U) \to \mathscr{M}(U);$$

"compatible" means that for opens $U' \subseteq U$ and elements $s_j \in \mathscr{M}_j(U)$ we have

$$m_U(s_1, \dots, s_n)|_{U'} = m_{U'}(s_1|_{U'}, \dots, s_n|_{U'})$$

in $\mathscr{M}(U')$. For $n = 2$, we say *bilinear.*

For $\mathscr{M}_1 = \cdots = \mathscr{M}_n = \mathscr{M}'$, the notions of *symmetric* and *alternating* multilinear mappings $\mathscr{M}'^{\times n} \to \mathscr{M}$ are defined similarly.

Example 4.2. Let E and E' be C^p vector bundles over X. There is a natural bilinear pairing

$$B : \underline{E'} \times \underline{\mathrm{Hom}(E', E)} \to \underline{E}$$

that on U-sections is the $\mathscr{O}(U)$-bilinear pairing

$$B_U : E'(U) \times \mathrm{Hom}_U(E'|_U, E|_U) \to E(U)$$

sending (s', f) to $f \circ s$. (We have used Example 3.4 to describe the U-sections of the bundle $\mathrm{Hom}(E', E)$.) Each B_U is readily checked to be $\mathscr{O}(U)$-bilinear and compatible with shrinking U, so we indeed have a bilinear pairing B of $\mathscr{O}$-modules. By working locally with trivializing frames for E and E', we see that for open U around $x \in X$ the diagram

$$\begin{tikzcd} E'(U) \times \mathrm{Hom}_U(E'|_U, E_U) \arrow[r, "B_U"] \arrow[d] & E(U) \arrow[d] \\ E'(x) \times \mathrm{Hom}(E'(x), E(x)) \arrow[r] & E(x) \end{tikzcd} \tag{7}$$

commutes, where the bottom side is the natural $\mathbf{R}$-bilinear pairing $(v, T) \mapsto T(v)$. As a special case, taking $E = X \times \mathbf{R}$, there is a natural bilinear pairing

$$\underline{E'} \times \underline{E'^{\vee}} \to \underline{X \times \mathbf{R}} = \mathscr{O}$$

that we call the *evaluation pairing.*

Example 4.3. For C^p vector bundles $E_1, \dots, E_n$ on X, there is a natural multilinear mapping

$$\underline{E}_1 \times \cdots \times \underline{E}_n \to \underline{E_1 \otimes \cdots \otimes E_n} \tag{8}$$

that on U-sections is the mapping

$$E_1(U) \times \cdots \times E_n(U) \to (E_1 \otimes \cdots \otimes E_n)(U)$$

provided by Theorem 2.4: given $v_j \in E_j(U)$ for all j, we send $(v_1, \dots, v_n)$ to the C^p section $v_1 \otimes \cdots \otimes v_n$ whose value in the u-fiber is

$$v_1(u) \otimes \cdots \otimes v_n(u) \in E_1(u) \otimes \cdots \otimes E_n(u) = (E_1 \otimes \cdots \otimes E_n)(u).$$

By computing on fibers we see that this is indeed an $\mathscr{O}(U)$-multilinear mapping, and that this procedure is compatible with shrinking U to open subsets.

The same method (again using Theorem 2.4) works for symmetric and exterior powers, giving symmetric and alternating multilinear mappings

$$\underline{E}^{\times n} \to \underline{\mathrm{Sym}^n(E)}, \;\; \underline{E}^{\times n} \to \underline{\wedge^n E}.$$

Example 4.4. For C^p vector bundles E and E' over X, there is a natural bilinear pairing

$$\underline{E} \times \underline{E'^{\vee}} \to \underline{\mathrm{Hom}(E', E)} \tag{9}$$

defined on U-sections as follows: to $s \in E(U)$ and $\ell' \in E'^{\vee}(U) = \mathrm{Hom}_U(E'|_U, U \times \mathbf{R})$ we associate the element in $(\mathrm{Hom}(E', E))(U) = \mathrm{Hom}_U(E'|_U, E|_U) = \mathrm{Hom}_{\mathscr{O}|_U-\mathrm{mod}}(\underline{E'}|_U, \underline{E}|_U)$ given by the compatible collection of $\mathscr{O}(U')$-linear maps $s' \mapsto \ell'_{U'}(s') \cdot s|_{U'}$ for open $U' \subseteq U$. This definition is readily checked to indeed be a bilinear pairing (i.e., $\mathscr{O}(U)$-bilinear in the pair (s, ℓ') and compatible with shrinking U). By working locally with trivializing frames, we see that for open U containing $x \in X$, the diagram

$$\begin{tikzcd} E(U) \times E'^{\vee}(U) \arrow[r] \arrow[d] & \mathrm{Hom}_U(E'|_U, E|_U) \arrow[d] \\ E(x) \times E'(x)^{\vee} \arrow[r] & \mathrm{Hom}(E'(x), E(x)) \end{tikzcd} \tag{10}$$

commutes, where the bottom side is the natural $\mathbf{R}$-bilinear map $(v, \ell') \mapsto (v' \mapsto \ell'(v')v)$.

Definition 4.5. A *tensor product* of $\mathscr{O}$-modules $\mathscr{M}_1, \dots, \mathscr{M}_n$ is a pair $(\mathscr{M}, t)$ consisting of an $\mathscr{O}$-module $\mathscr{M}$ and a multilinear mapping

$$t : \mathscr{M}_1 \times \cdots \times \mathscr{M}_n \to \mathscr{M}$$

with the universal property that for any multilinear mapping

$$\mu : \mathscr{M}_1 \times \cdots \times \mathscr{M}_n \to \mathscr{M}'$$

there is a unique $\mathscr{O}$-linear mapping $T : \mathscr{M} \to \mathscr{M}'$ such that $T_U \circ t_U = \mu_U$ for all opens $U \subseteq X$.

For an $\mathscr{O}$-module $\mathscr{M}$, its nth *symmetric power* and nth *exterior power* are defined similarly in terms of universality for symmetric and alternating multilinear mappings on $\mathscr{M}^{\times n}$ for $n \geq 1$.

It can be proved (with the help of some elementary sheaf theory) that such tensorial objects always exist (without restrictions such as local freeness of finite rank), but this is too general for us; we merely require:

Theorem 4.6. *For C^p vector bundles $E_1, \dots, E_n$ over X, the multilinear mapping in* (8) *is universal: it identifies the $\mathscr{O}$-module associated to $E_1 \otimes \cdots \otimes E_n$ as the tensor product of the $\underline{E}_j$'s.*

Likewise, for any C^p vector bundle E over X the symmetric and alternating mappings

$$\underline{E}^{\times n} \to \underline{\mathrm{Sym}^n(E)}, \;\; \underline{E}^{\times n} \to \underline{\wedge^n E}$$

resting on Theorem 2.4 *are universal: these identify the $\mathscr{O}$-modules associated to* $\mathrm{Sym}^n(E)$ *and* $\wedge^n E$ *as the nth symmetric and exterior powers of $\underline{E}$.*

The proof of this theorem is somewhat lengthy, and is largely a matter of artful reduction to the local case where local triviality of vector bundles can be brought in. It may be reasonable to skip the proof on a first reading.

Proof. We work out the case of tensor products; the symmetric and exterior powers are handled similarly. Let $E = E_1 \otimes \cdots \otimes E_n$. Consider a compatible family of $\mathscr{O}(U)$-multilinear mappings

$$m_U : E_1(U) \times \cdots \times E_n(U) \to \mathscr{M}(U)$$

for an $\mathscr{O}$-module $\mathscr{M}$. We want to prove the existence *and uniqueness* of an $\mathscr{O}$-linear map $T : \underline{E} \to \mathscr{M}$ such that $T_U(s_1 \otimes \cdots \otimes s_n) = m_U(s_1, \dots, s_n)$ in $\mathscr{M}(U)$ for all $s_j \in E_j(U)$ for all open $U \subseteq X$.

Step 1. In order to make the problem a local one, it is convenient to first prove uniqueness. Suppose that $T = \{T_U\}$ and $T' = \{T'_U\}$ are two such $\mathscr{O}$-linear maps. We want $T_U = T'_U$ for all open $U \subseteq X$, so by passing to $T - T' = \{T_U - T'_U\}$ we just have to show that if $T : \underline{E} \to \mathscr{M}$ is an $\mathscr{O}$-linear map satisfying $T_U(s_1 \otimes \cdots \otimes s_n) = 0$ in $\mathscr{M}(U)$ for *all* open $U \subseteq X$ and all $s_j \in E_j(U)$ then $T = 0$ (i.e., $T_U = 0$ for all open U). Fix a choice of open $U \subseteq X$ for which we want to prove $T_U = 0$. Pick $s \in E(U)$, so we want $T_U(s) \in \mathscr{M}(U)$ to vanish. If s were a finite $\mathscr{O}(U)$-linear combination of elementary tensors, then the vanishing would follow from the vanishing hypothesis on T. In general we do not know if s admits such an expression, but we can easily get around this: in general U admits an open covering $\{U_i\}$ such that all $E_j|_{U_i}$ are trivial, and so all elements of $E(U_i)$ are finite $\mathscr{O}(U_i)$-linear combinations of elementary tensors of elements of local frames for the $E_j|_{U_i}$'s ($1 \leq j \leq n$). Hence, $s|_{U_i}$ is a finite $\mathscr{O}(U_i)$-linear combination of elementary tensors, and so $T_{U_i}(s|_{U_i}) = 0$ in $\mathscr{M}(U_i)$ for all i. But since T is an $\mathscr{O}$-linear mapping we have $T_U(s)|_{U_i} = T_{U_i}(s|_{U_i})$ for all i, so the element $T_U(s) \in \mathscr{M}(U)$ restricts to 0 in $\mathscr{M}(U_i)$ for opens U_i that cover U. But the element $0 \in \mathscr{M}(U)$ has such restrictions, and so by the "unique gluing" axiom for $\mathscr{O}$-modules it follows that $T_U(s)$ must vanish in $\mathscr{M}(U)$. This completes the proof of uniqueness.

Step 2. To prove existence, the key point is that the existence problem is local on X because of uniqueness. Indeed, assume that $\{X_i\}$ is an open cover of X such that we can solve the existence problem over X_i. Thus, we have $\mathscr{O}|_{X_i}$-linear maps $T_i : \underline{E}|_{X_i} \to \mathscr{M}|_{X_i}$ such that for all opens $U \subseteq X_i$ and all choices of elements $s_j \in E_j(U)$,

$$(T_i)_U(s_1 \otimes \cdots \otimes s_n) = m_U(s_1, \ldots, s_n).$$

Observe that $T_i|_{X_i \cap X_j}$ and $T_j|_{X_i \cap X_j}$ solve the *same* mapping problem over $X_i \cap X_j$ (for factoring $m|_{X_i \cap X_j}$ through the tensor pairing of the $\underline{E}_k|_{X_i \cap X_j}$'s into $\underline{E}|_{X_i \cap X_j}$), whence by uniqueness (Step 1) they are *equal*. That is, for any open $U \subseteq X_i \cap X_j$ we have $(T_i)_U = (T_j)_U$ as $\mathscr{O}(U)$-linear maps from $E(U)$ to $\mathscr{M}(U)$. Using the equality of $T_i|_{X_i \cap X_j}$ and $T_j|_{X_i \cap X_j}$ for all i and j, we claim that there is a (unique) $\mathscr{O}$-linear map $T : \underline{E} \to \mathscr{M}$ such that $T|_{X_i} = T_i$ for all i (i.e., for opens $U \subseteq X_i$, $T_U = (T_i)_U$). That is, we seek to built a (unique) compatible family of $\mathscr{O}(U)$-linear maps $T_U : E(U) \to \mathscr{M}(U)$ such that $T_U = (T_i)_U$ whenever $U \subseteq X_i$. How are we to do this?

Pick open $U \subseteq X$ and $s \in E(U)$. We need to define $T_U(s) \in \mathscr{M}(U)$. This will be done via the gluing axiom for $\mathscr{M}$. It is necessary that for all i we have

$$T_U(s)|_{U \cap X_i} = T_{U \cap X_i}(s|_{U \cap X_i}) = (T_i)_{U \cap X_i}(s|_{U \cap X_i})$$

in $\mathscr{M}(U \cap X_i)$. Since the opens $U \cap X_i$ cover U, the gluing axiom (and transitivity of restriction) ensures that there is at most one element in $\mathscr{M}(U)$ whose restriction to $U \cap X_i$ is $(T_i)_{U \cap X_i}(s|_{U \cap X_i})$ for all i, and that such an element exists if and only if the elements $(T_i)_{U \cap X_i}(s|_{U \cap X_i}) \in \mathscr{M}(U \cap X_i)$ coincide on the overlap $(U \cap X_i) \cap (U \cap X_j) = U \cap (X_i \cap X_j)$ for all i and j. We now compute

$$\begin{aligned} (T_i)_{U \cap X_i}(s|_{U \cap X_i})|_{U \cap X_i \cap X_j} = (T_i)_{U \cap X_i \cap X_j}(s|_{U \cap X_i \cap X_j}) &= (T_j)_{U \cap X_i \cap X_j}(s|_{U \cap X_i \cap X_j}) \\ &= (T_j)_{U \cap X_j}(s|_{U \cap X_j})|_{U \cap X_j}, \end{aligned}$$

with the second equality due to the identity $(T_i)_{U \cap X_i \cap X_j} = (T_j)_{U \cap X_i \cap X_j}$ (using that $U \cap X_i \cap X_j \subseteq X_i \cap X_j$). Hence, we may indeed uniquely glue to define $T_U(s) \in \mathscr{M}(U)$. By construction, if $U \subseteq X_i$ (so $U \cap X_i = U$) then $T_U(s) = (T_i)_U(s)$. Since the $(T_i)_{U \cap X_i}$'s are $\mathscr{O}(U \cap X_i)$-linear, for any $s_1, s_2 \in E(U)$ and $a_1, a_2 \in \mathscr{O}(U)$ we see by restriction to the $U \cap X_i$'s that $a_1 T_U(s_1) + a_2 T_U(s_2)$ satisfies the restriction conditions (in the $\mathscr{M}(U \cap X_i)$'s) that *uniquely* characterize $T_U(a_1 s_2 + a_2 s_2)$. Hence, $s \mapsto T_U(s)$ is an $\mathscr{O}(U)$-linear map from $E(U)$ to $\mathscr{M}(U)$. Also, if $U' \subseteq U$ is an open subset and $s \in E(U)$ is an element then $T_U(s)|_{U'} \in \mathscr{M}(U')$ satisfies the restriction properties (in the $\mathscr{M}(U' \cap X_i)$'s) that uniquely characterize $T_{U'}(s|_{U'})$. Hence, $T_U(s)|_{U'} = T_{U'}(s|_{U'})$. This verifies that $T = \{T_U\}$ is an $\mathscr{O}$-linear map from $\underline{E}$ to $\mathscr{M}$ such that $T|_{X_i} = T_i$ for all i.

With the $\mathscr{O}$-linear "gluing" T of the $\mathscr{O}|_{X_i}$-linear T_i's now at our disposal, for any open $U \subseteq X$ and $s_j \in E_j(U)$ we compute

$$\begin{aligned} T_U(s_1 \otimes \cdots \otimes s_n)|_{U\cap X_i} &= (T_i)_{U\cap X_i}(s_1|_{U\cap X_i} \otimes \cdots \otimes s_n|_{U\cap X_i}) \\ &= m_{U\cap X_i}(s_1|_{U\cap X_i}, \ldots, s_n|_{U\cap X_i}) \\ &= m_U(s_1, \ldots, s_n)|_{U\cap X_i}, \end{aligned}$$

and hence $T_U(s_1 \otimes \cdots \otimes s_n) = m_U(s_1, \ldots, s_n)$ in $\mathscr{M}(U)$ since these elements have the same restriction in each $\mathscr{M}(U \cap X_i)$ with opens $U \cap X_i$ that cover U (as the X_i's cover X). In other words, T solves the existence problem. The construction of T was predicated on the ability to solve the existence problem over opens X_i that cover X, so rather than actually solving the existence problem all we have proved is this: the solvability of the existence problem is local on X.

Step 3. Let $\{X_i\}$ be a covering of X by opens such that the finitely many E_j's are all trivial on each X_i. (Such X_i's exist since finite intersections of open sets are open, and each E_j is trivial on some open around each point of X.) In view of Step 2, it suffices to solve the existence problem over each X_i. Hence, we can assume now that all E_j's are trivial. Let $\{s_{ij}\}$ in $E_j(X)$ for $1 \le i \le r_j$ be a trivializing frame for E_j. Hence, by Example 2.3 the sections

$$s_I := s_{i_1,1} \otimes \cdots \otimes s_{i_n,n} \in (E_1 \otimes \cdots \otimes E_n)(X) = E(X)$$

for all $I = (i_1, \ldots, i_n)$ with $1 \le i_j \le r_j$ are a trivializing frame for E. For any open $U \subseteq X$, the $\mathscr{O}(U)$-module $E(U)$ is free on the basis of $s_I|_U$'s, and so just as we uniquely define linear maps on a vector space by specifying the images of elements of an ordered basis we may uniquely define an $\mathscr{O}(U)$-linear map $T_U : \underline{E}(U) = E(U) \to \mathscr{M}(U)$ by the condition

$$T_U(\sum a_I s_I|_U) = \sum a_I m_U(s_{i_1,1}, \ldots, s_{i_n,n}) \in \mathscr{M}(U).$$

For $U' \subseteq U$ it is clear that $T_U(s)|_{U'} = T_{U'}(s|_{U'})$ for $s \in E(U)$ (express s uniquely as an $\mathscr{O}(U)$-linear combination of the $s_I|_U$'s), so $T = \{T_U\}$ is an $\mathscr{O}$-linear map from $\underline{E}$ to $\mathscr{M}$.

To see that T "works" (and thereby solves the existence problem), we need to prove that for open $U \subseteq X$ and $v_j \in E_j(U)$

$$T_U(v_1 \otimes \cdots \otimes v_n) = m_U(v_1, \ldots, v_n) \tag{11}$$

in $\mathscr{M}(U)$. Note that both sides are $\mathscr{O}(U)$-multilinear in the v_j's! Recall from linear algebra that if two multilinear mappings agree on n-tuples from spanning sets of the factors then by the rules of multilinearity they are equal (just as two linear maps that agree on a spanning set of the source are equal). That argument is not special to linear algebra: it applies *verbatim* to multilinear mappings on modules over any commutative ring. Thus, since the elements $s_{ij} \in E_j(U)$ span $E_j(U)$ as an $\mathscr{O}(U)$-module, it suffices to treat the case $v_j = s_{i_j,j}$ with $1 \le i_j \le r_j$. But T_U was *defined* to make the desired identity (11) hold in such special cases! ■

We now get lots of nice consequences:

Example 4.7. The bilinear pairing (9) gives an $\mathscr{O}$-linear mapping $\underline{E \otimes E'^{\vee}} \to \underline{\mathrm{Hom}(E', E)}$ such that on fibers over $x \in X$ it is the natural map $E(x) \otimes E'(x)^{\vee} \to \mathrm{Hom}(E'(x), E(x))$ that is an isomorphism. (Here we have used commutativity of (10).) Thus, we get a bundle morphism

$$E \otimes E'^{\vee} \to \mathrm{Hom}(E', E)$$

that is the natural isomorphism on fibers, so it is an isomorphism of bundles.

Likewise using Example 4.2 and the commutativity of (7), we get a unique C^p bundle morphism

$$E' \otimes \mathrm{Hom}(E', E) \to E$$

such that on x-fibers it is the natural map

$$E'(x) \otimes \mathrm{Hom}(E'(x), E(x)) \to E(x)$$

from linear algebra. As a special case, taking $E = X \times \mathbf{R}$ gives a natural bundle morphism

$$E' \otimes E'^{\vee} \to X \times \mathbf{R}$$

that induces the evaluation pairing $E'(x) \otimes E'(x)^{\vee} \to \mathbf{R}$ on x-fibers for all $x \in X$.

Example 4.8. To give a bundle map $E \otimes E' \to E''$ is "the same" as to give a bundle map $E \to \mathrm{Hom}(E', E'')$ (just like in linear algebra!). Explicitly, a bundle map $L : E \otimes E' \to E''$ is the same as a bilinear pairing $\underline{E} \times \underline{E}' \to \underline{E}''$, which is to say compatible $\mathscr{O}(U)$-bilinear pairings

$$B_U : E(U) \times E'(U) \to E''(U)$$

for all open $U \subseteq X$. Likewise, a bundle map $E \to \mathrm{Hom}(E', E'')$ is the same as an $\mathscr{O}$-linear map between the associated $\mathscr{O}$-modules, which is to say a compatible collection of $\mathscr{O}(U)$-linear maps $T_U : E(U) \to \mathrm{Hom}_U(E'|_U, E''|_U)$.

Given B_U we define T_U to send $s \in E(U)$ to the bundle mapping $E'_U \to E''|_U$ that "is" the collection of compatible $\mathscr{O}(U_0)$-linear maps $B_{U_0}(s|_{U_0}, \cdot) : E'(U_0) \to E''(U_0)$ for opens $U_0 \subseteq U$. Conversely, given T_U we define $B_U(s, s') = (T_U(s))_U(s') \in E''(U)$. These are readily checked to have the right linearity and bilinearity properties, and to be inverse constructions.

By chasing local trivializing frames (which lift bases on fibers), one sees (check!) that if bundle maps $L : E \otimes E' \to E''$ and $T : E \to \mathrm{Hom}(E', E'')$ "correspond" under the above dictionary then their induced linear fibral maps

$$L|_x : E(x) \otimes E'(x) \to E''(x), \;\; T|_x : E(x) \to \mathrm{Hom}(E'(x), E''(x))$$

likewise correspond under the recipes in linear algebra.

An important special case is $E'' = X \times \mathbf{R}$: to give a bundle map $L : E \times E' \to X \times \mathbf{R}$ (or equivalently, a bilinear pairing $\underline{E} \times \underline{E}' \to \mathscr{O}$) is "the same" as to give a bundle map $T : E \to E'^{\vee}$, with $L|_x : E(x) \otimes E'(x) \to \mathbf{R}$ "corresponding" to $T|_x : E(x) \to E'(x)^{\vee}$ as in linear algebra. An especially interesting case is the evaluation pairing $L : E \otimes E^{\vee} \to X \otimes \mathbf{R}$ (that is the usual evaluation pairing on fibers); this L was built in Example 4.7. For this L we obtain a bundle map $T : E \to E^{\vee\vee}$ that is (!) the usual double duality isomorphism on fibers and hence is an isomorphism. This gives "double duality" for vector bundles.

Observe how the next example deftly argues almost "as if" we were in the setting of linear algebra, not once having to appeal to explicit formulas in local frames (the approach used in the 19th century, and by many today).

Example 4.9. We can build a bundle isomorphism

$$(E_1 \otimes \cdots \otimes E_n) \otimes (E'_1 \otimes \cdots \otimes E'_{n'}) \simeq E_1 \otimes \cdots \otimes E_n \otimes E'_1 \otimes \cdots \otimes E'_{n'}$$

and bundle maps

$$\mathrm{Sym}^n(E) \otimes \mathrm{Sym}^m(E) \to \mathrm{Sym}^{n+m}(E), \;\; \wedge^n(E) \otimes \wedge^m(E) \to \wedge^{n+m}(E)$$

that on fibers are the natural maps from linear algebra (as in the handout on tensors and duality). By Example 4.8, this amounts to constructing bundle maps

$$E_1 \otimes \cdots \otimes E_n \to \mathrm{Hom}(E'_1 \otimes \cdots \otimes E'_{n'}, E_1 \otimes \cdots \otimes E_n \otimes E'_1 \otimes \cdots \otimes E'_{n'})$$

and

$$\mathrm{Sym}^n(E) \to \mathrm{Hom}(\mathrm{Sym}^m(E), \mathrm{Sym}^{n+m}(E)), \;\; \wedge^n E \to \mathrm{Hom}(\wedge^m E, \wedge^{n+m} E)$$

that recover the habitual maps on fibers (via the pairings of tensor products and symmetric/exterior powers).

By Theorem 4.6, it is equivalent to build compatible families of multilinear (resp. symmetric, alternating) $\mathscr{O}(U)$-module mappings

$$E_1(U) \times \cdots \times E_n(U) \to \mathrm{Hom}_U((E'_1 \otimes \cdots \otimes E'_{n'})|_U, (E_1 \otimes \cdots \otimes E_n \otimes E'_1 \otimes \cdots \otimes E'_{n'})|_{U'}),$$

$$E(U)^{\times n} \to \mathrm{Hom}_U(\mathrm{Sym}^m(E)|_U, \mathrm{Sym}^{n+m}(E)|_U), \;\; E(U)^{\times n} \to \mathrm{Hom}_U((\wedge^m E)|_U, (\wedge^{n+m} E)|_U)$$

compatibly with the linear algebra analogues on fibers (via commutative diagrams in the spirit of (10)). We may make such constructions *exactly* as in the case of linear algebra. We illustrate with the case of exterior powers. Pick an ordered n-tuple $s_1, \dots, s_n$ in $E(U)$. We seek to define a bundle mapping

$$\mu_{s_1,\dots,s_n} : (\wedge^m E)|_U \to (\wedge^{n+m} E)|_U$$

that is the habitual map on fibers (using the vectors $s_j(x) \in E(x)$), is compatible with shrinking U, and depends on the s_j's in an alternating $\mathscr{O}(U)$-multilinear manner. The fibral requirement forces the rest because a map of bundles is determined by its effect on fibers. The source and target of $\mu_{s_1,\dots,s_n}$ are identified with $\wedge^m(E|_U)$ and $\wedge^{n+m}(E|_U)$ respectively, and so by the universal property of exterior powers of bundles (now over U) we seek to define a suitable compatible family of alternating $\mathscr{O}(U')$-multilinear mappings

$$\mu_{s_1,\dots,s_n,U'} : E(U')^{\times m} \to (\wedge^{n+m} E)(U')$$

for all opens $U' \subseteq U$. We define

$$\mu_{s_1,\dots,s_n,U'}(s'_{n+1}, \dots, s'_{n+m}) = s_1|_{U'} \wedge \cdots \wedge s_n|_{U'} \wedge s'_{n+1} \wedge \cdots \wedge s'_{n+m}.$$

This has all of the desired properties. (To check that we get the desired mapping on fibers, note that any m-tuple of elements in $E(x)$ can be lifted to an m-tuple in $E(U')$ for a small open U' around x.)

Remark 4.10. Let E be a C^p vector bundle on X. If U lies in an open over which E has a trivialization then elements of $E(U)$ are finite $\mathscr{O}(U)$-linear combinations of elementary wedge products of elements from the trivializing frame. The preceding example gives well-defined $\mathscr{O}(U)$-bilinear pairings

$$(\wedge^n E)(U) \times (\wedge^m E)(U) \xrightarrow{\wedge} (\wedge^{n+m} E)(U)$$

denoted $(\omega, \eta) \mapsto \omega \wedge \eta$ that are compatible with shrinking U. On fibers over $x \in X$ this is the wedge-product pairing

$$\wedge^n(E(x)) \times \wedge^m(E(x)) \to \wedge^{n+m}(E(x)),$$

and hence from the associativity and sign results in the case of linear algebra we conclude that these wedge-product pairings on U-sections in general satisfy associativity and $\omega \wedge \eta = (-1)^{nm} \eta \wedge \omega$. An important example is the case when $p \geq 1$ and $E = T^*X := (TX)^\vee$ is the cotangent bundle (of class C^{p-1}), in which case $\underline{\wedge^n E}$ is denoted Ω^n_X and elements of $\Omega^n_X(U)$ are called *differential n-forms* over U. If U has C^p coordinates $\{x_1, \dots, x_r\}$ then $T^*X|_U$ is trivialized with the sections $\mathrm{d}x_j$, so elements of $\Omega^n_X(U)$ are finite $\mathscr{O}'(U)$-linear combinations of elementary wedge products $\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_n}$ for the C^{p-1}-structure $\mathscr{O}'$ on X.

To give a final application of Theorem 4.8, we discuss the relationship between duality and tensorial operations on bundles.

Theorem 4.11. *For C^p vector bundles $E_1, \dots, E_n$, there is a unique C^p bundle isomorphism*

$$E_1^\vee \otimes \cdots \otimes E_n^\vee \simeq (E_1 \otimes \cdots \otimes E_n)^\vee$$

that induces the linear-algebra isomorphism on fibers. Also, for any C^p vector bundle E on X and any $n \geq 1$ there are unique C^p bundle isomorphisms

$$\mathrm{Sym}^n(E^\vee) \simeq (\mathrm{Sym}^n E)^\vee, \ \wedge^n(E^\vee) \simeq (\wedge^n E)^\vee$$

that induce the linear-algebra isomorphisms on fibers.

The isomorphisms in the theorem give rise to bilinear $\mathscr{O}$-module pairings

$$\underline{E_1 \otimes \cdots \otimes E_n} \times \underline{E_1^\vee \otimes \cdots \otimes E_n^\vee} \to \mathscr{O},$$
$$\underline{\mathrm{Sym}^n(E)} \times \underline{\mathrm{Sym}^n(E^\vee)} \to \mathscr{O}, \ \underline{\wedge^n E} \times \underline{\wedge^n(E^\vee)} \to \mathscr{O}$$

that (as we can check by passage to fibers!) have the expected effect on elementary tensors and elementary symmetric/wedge products over opens in X.

Proof. By two applications for the universal property of tensor products (resp. symmetric/exterior powers) of bundles, we have to show that for open $U \subseteq X$ and $\ell_j \in E_j^\vee(U) = \mathrm{Hom}_U(E_j, U \times \mathbf{R})$ (resp. $\ell_1, \dots, \ell_n \in E^\vee(U) = \mathrm{Hom}_U(E|_U, U \times \mathbf{R})$) and any open $U' \subseteq U$ there are $\mathscr{O}(U')$-multilinear (resp. symmetric, alternating) mappings

$$\mu_{\ell_1, \dots, \ell_n, U'} : E_1(U') \times \cdots \times E_n(U') \to \mathscr{O}(U')$$

(resp. $\mu_{\ell_1, \dots, \ell_n, U'} : E(U')^{\times n} \to \mathscr{O}(U')$) that are compatible with the old linear-algebra mappings on fibers (as then all compatibility requirements for shrinking U' and U will be satisfied). In the case of tensor products we define

$$\mu_{\ell_1, \dots, \ell_n}(s_1', \dots, s_n') = \prod_{i=1}^n (\ell_i|_{U'})(s_i') \in \mathscr{O}(U')$$

and in the cases of symmetric and exterior powers we define $\mu_{\ell_1, \dots, \ell_n}(s_1', \dots, s_n')$ to respectively be

$$\sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n (\ell_i|_{U'})(s_{\sigma(i)}'), \quad \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sign}(\sigma) \prod_{i=1}^n (\ell_i|_{U'})(s_{\sigma(i)}') = \det((\ell_i|_{U'})(s_j'))$$

in $\mathscr{O}(U')$. ∎

Example 4.12. Suppose $p \geq 1$ and let $\mathscr{O}'$ be the C^{p-1} structure on X. A C^{p-1} *tensor field of type* (r, s) over an open set $U \subseteq X$ is a U-section ω of $(TX)^{\otimes r} \otimes (T^*X)^{\otimes s}$ of class C^{p-1} (its "value" at each $u \in U$ is an element in $\mathrm{T}_u(X)^{\otimes r} \otimes (T_u(X)^\vee)^{\otimes s}$ that has C^{p-1}-dependence on u). That is, if $\{x_1, \dots, x_n\}$ are local C^p coordinates on a small open $U_0 \subseteq U$ around u then

$$\omega|_{U_0} = \sum a_{I,J} \partial_{x_{i_1}}|_{U_0} \otimes \dots \partial_{x_{i_r}}|_{U_0} \otimes \mathrm{d}x_{j_1}|_{U_0} \otimes \cdots \otimes \mathrm{d}x_{j_s}|_{U_0}$$

with $a_{I,J} \in \mathscr{O}'(U_0)$ for $I = \{i_1, \dots, i_r\}$ and $J = \{j_1, \dots, j_s\}$.

In Riemannian geometry with $p = \infty$ one essentially puts a smoothly varying inner product on the tangent bundle (a "Riemannian metric"), and this has the effect of providing an identification of the bundles TX and T^*X. Consequently, in classical (as well as some modern!) books on differential geometry one sees smooth tensor fields of type (r, s) written as if they are sections of $(TX)^{\otimes r} \otimes (TX)^{\otimes s} \simeq (TX)^{\otimes (r+s)}$. The way the classical geometers kept track of the opposite behaviors of TX and T^*X with respect to mappings (much like the dichotomy in order of composition when passing to dual maps) was to indicate with sub/superscript indices that certain "tensor variables" were either covariant (really sections of TX) or contravariant (really sections of T^*X), up to the annoyance that the old-timers actually used the words co/contravariant in the opposite places (contrary to current meaning of these words in modern algebra). The indiscriminate identification of TX and T^*X sometimes has the effect of making constructions appear to depend on the Riemannian metric, whereas often the constructions make perfectly good sense without it if one keeps track of which TX's should really be T^*X's.