Math 396. Handout on compactness criteria

We have seen two ways to think about compactness in metric spaces: in terms of open covers and in terms of sequential convergence. We wish to present two more ways to think about compactness. The first of these will be called the "finite intersection property (FIP)" for closed sets, and turns out to be a (useful!) linguistic reformulation of the open cover criterion. The second point of view we discuss involves a refinement of the concept of boundedness, called "total boundedness". This will rescue the theorem on compactness of closed and bounded sets in $\mathbf{R}^n$ (which is *false* for more general metric spaces) so that we have a version which is a valid compactness criterion for arbitrary metric spaces.

1. FIP

Let X be a topological space.

Definition 1.1. We say that X satisfies the *finite intersection property* (or *FIP*) for closed sets if any collection $\{Z_i\}_{i\in I}$ of closed sets in X with all *finite* intersections

$$Z_{i_1} \cap \cdots \cap Z_{i_n} \neq \emptyset,$$

the intersection $\cap_{i\in I} Z_i$ of all Z_i's is non-empty.

Example 1.2. We give two non-examples to indicate what can go wrong. Let $X = \mathbf{R}$. If we take $Z_n = [n, \infty)$ for $n \in \mathbf{N}$, then all intersections of finitely many Z_i's are visibly non-empty, but these common overlaps "escape to infinity": that is, $\cap_{n\in\mathbf{N}} Z_n = \emptyset$. So $\mathbf{R}$ doesn't satisfy FIP for closed sets. But the fact that $\mathbf{R}$ is "too big" is only half of the problem.

Consider $X = (0, 2)$ and let $Z_n = (0, 1/n]$ for $n \in \mathbf{N}$. Once again, each Z_n is visibly closed *in* X and any finite collection of these has non-empty intersection. But if we intersect all of them, we again get $\emptyset$! Here the problem is that the intersection sort of moves off to the edge which isn't there (in X).

Note that both non-examples are not compact. Quite generally, we have:

Theorem 1.3. *Let X be a topological space. Then X is compact if and only if X satisfies FIP for closed sets.*

Before we prove the theorem, we give an application.

Corollary 1.4. *Choose $a, b \in \mathbf{R}$ with $a \leq b$. Let $C_1, C_2, \ldots$ be a sequence of non-empty closed sets in $[a, b]$ with*

$$C_1 \supseteq C_2 \supseteq \ldots$$

Then $\cap_n C_n \neq \emptyset$.

A special case of this is the classical "bisection method" from calculus (with C_{i+1} a closed half-subinterval of C_i). Let's prove the corollary, and then we'll prove Theorem 1.3.

Proof. Since the sequence of closed sets is decreasing and all are non-empty, if $C_{i_1}, \ldots, C_{i_r}$ is any finite collection of them then for $i = \max(i_1, \ldots, i_r)$ we have

$$C_i = \cap_{j=1}^r C_{i_j},$$

so since $C_i \neq \emptyset$ all such finite intersections are non-empty. Since $[a, b]$ is compact, and thus satisfies FIP for closed sets (by Theorem 1.3), we're done. ■

Now we prove Theorem 1.3:

Proof. This is an exercise in linguistics. For an arbitrary topological space X, to give a collection $\{C_i\}_{i\in I}$ of closed sets in X is exactly the same thing as to give a collection $\{U_i\}_{i\in I}$ of open sets in X, using the rules $U_i = X - C_i$ and $C_i = X - U_i$. We'll use this to translate the FIP condition for closed sets into the open covering criterion for compactness (and vice-versa). Fix a choice of corresponding $\{C_i\}_{i\in I}$ and $\{U_i\}_{i\in I}$. We have

$$\cap_{i\in I} C_i = X - \cup_{i\in I} U_i,$$

so $\cap_i C_i = \emptyset$ if and only if $\{U_i\}$ covers X (i.e., is an open covering of X). Since

$$C_{i_1} \cap \cdots \cap C_{i_r} = X - \cup_{j=1}^{r} U_{i_j},$$

it follows that all finite intersections among the C_i's are non-empty if and only if *no* finite collection among the U_i's covers X. In other words, $\{U_i\}$ is an open cover *without* a finite subcover if and only if $\{C_i\}$ is a counterexample to FIP for closed sets in X. Put another way, X satisfies the open covering criterion for compactness if and only if X satisfies FIP for closed sets. ■

We emphasize that although the proof really was essentially linguistic, Corollary 1.4 shows that FIP is a useful way to think in certain cases (e.g., when doing bisection arguments).

2. Total boundedness and Lebesgue's lemma

Let (X, ρ) be a metric space.

Definition 2.1. We say X is *totally bounded* if, for all $\varepsilon > 0$, X admits a covering by finitely many open ε-balls.

Example 2.2. A totally bounded metric space is bounded, but the converse need not hold. This was studied in Exercise 1, HW 1.

If X is compact as a metric space, then X is complete (as we saw in lecture) and totally bounded (obvious). Remarkably, the converse is true: a complete and totally bounded metric space is compact. Before we prove this, we note that in the case of $\mathbf{R}^n$ this recovers the classification of compacts in $\mathbf{R}^n$ as those subsets which are closed and bounded relative to a norm metric:

Theorem 2.3. *Let V be a finite-dimensional normed vector space over $\mathbf{R}$, given the metric induced by its norm (so V is complete). A subset $Z \subseteq V$ is closed if and only if it is complete, and Z is bounded if and only if it is totally bounded.*

Proof. In lecture we saw that a subset of a complete metric space is closed if and only if it is complete with respect to the induced metric. That settles the first part. It is obvious that a totally bounded set is bounded (this is true in any metric space whatsoever). Conversely, if Z is bounded then we wish to prove that Z is totally bounded (as a metric space in its own right). As a preliminary step, we wish to show that for any metric space X and any subset Z, boundedness and total boundedness of the metric space Z can be reformulated in terms of open balls of X (rather than of Z).

It is clear from the definitions that the metric space Z is bounded if and only if it is contained inside of some large open ball in X. As for total boundedness, if Z is covered by finitely many open ε-balls *of* X for each $\varepsilon > 0$, then we claim that Z is totally bounded as a metric space (the converse is obvious, since each open ball in Z is obtained by intersecting Z with an open ball of X with the same radius and center). The only delicate point is that a covering of Z by finitely many open ε-balls of X might have all centers outside of Z. Hence, we have to fiddle a bit with radii and re-centering.

Suppose Z is covered by $B_\varepsilon(x_{i,\varepsilon})$ for $1 \le i \le n_\varepsilon$ and $x_{i,\varepsilon} \in X$. We can assume each such ball actually meets Z by simply dropping from consideration those that don't (this doesn't affect the property of the collection of such balls covering Z). The triangle inequality then ensures

$$B_\varepsilon(x_{i,\varepsilon}) \subseteq B_{2\varepsilon}(z_{i,\varepsilon})$$

for any $z_{i,\varepsilon} \in Z \cap B_\varepsilon(x_{i,\varepsilon}) \ne \emptyset$ (use that $\rho(x_{i,\varepsilon}, z_{i,\varepsilon}) < \varepsilon$ for any such $z_{i,\varepsilon}$). Thus, the finitely many open balls $B_{2\varepsilon}(z_{i,\varepsilon}) \cap Z$ of Z actually cover Z. That is, the metric space Z is covered by finitely many 2ε-balls for all $\varepsilon > 0$, which yields total boundedness.

Thinking in terms of our ambient metric space V, we need to show that a subset Z of V is totally bounded as a metric space if it is bounded as a metric space. Assuming Z to be bounded, in order to show that Z is totally bounded, it suffices (in view of what we have just argued) to prove that Z is covered by finitely many ε-balls of V (with $\varepsilon > 0$ arbitrary). Note that this statement holds for Z if it holds for a larger set within V. Since the bounded Z in V must be contained inside of some $B_r(0)$, and hence inside of $\overline{B}_r(0)$, it suffices to prove that $\overline{B}_r(0)$ is totally bounded. It is an easy exercise to check that total boundedness is unaffected by passage to an equivalent norm (and likewise for boundedness), so we may assume $V = \mathbf{R}^n$ with the box norm. Then $\overline{B}_r(0)$ is the standard cube of side-length $2r$ centered at the origin, and this is trivially covered by finitely many open cubes of "radius" ε for each $\varepsilon > 0$ (just think about a big grid of open boxes with faces parallel to coordinate hyperplanes, each one slightly thickened-up: you should be able to convert this into a rigorous argument by explicitly defining the cubes as suitable products of open intervals). ■

Now we prove the equivalence of compactness and the conjunction of completeness and total boundedness.

Theorem 2.4. *Let X be a metric space. Then X is compact if and only if X is complete and totally bounded.*

Proof. We have already noted that the "⇒" implication is clear. The interesting part is the converse. Suppose that X is both complete and totally bounded. We wish to prove that X is compact. In order to do this, we need a new idea: the Lebesgue covering number. This is a number arising in the following important definition.

Definition 2.5. Let $\{U_i\}_{i\in I}$ be an open covering of a metric space X. A *Lebesgue number* for this covering is a $\delta > 0$ such that for all $x \in X$ we have $B_\delta(x) \subseteq U_{i(x)}$ for some $i(x) \in I$.

A Lebesgue covering number is a certain "uniformity" across the covering. The mere fact that $\{U_i\}_{i\in I}$ covers X ensures that for all $x \in X$ there is some $i(x) \in I$ and some $\delta(x) > 0$ such that $B_{\delta(x)}(x) \subseteq U_{i(x)}$. The magical property of a Lebesgue number δ is that it can be taken as $\delta(x)$ for *all* $x \in X$. This is somewhat similar in spirit to the idea of uniform continuity. The importance of Lebesgue numbers for our purposes is:

Lemma 2.6. (Lebesgue's covering lemma) *If X is complete and totally bounded, every open covering admits a Lebesgue number.*

We stress that the proof is by contradiction, and is thereby non-constructive.

Proof. Suppose otherwise, so for all $n \in \mathbf{N}$ there is some $z_n \in X$ with $B_{1/n}(z_n)$ not contained in any U_i. Let $x_{1,m}, \dots, x_{k_m,m} \in X$ be a finite set for which the *finitely many* open balls $B_{1/m}(x_{j,m})$ cover the totally bounded X. Since there are only finitely many balls $B_1(x_{j,1})$ and these *cover* X, and we have points z_n for *infinitely many* n, by the pigeonhole principle there must be an infinite set of n's for which the corresponding points z_n lie in a common one of these balls (for if $z_n \in B_1(x_{j,m})$ for only n in some finite set of positive integers S_m, we encounter a contradiction to the covering property when we look at z_n for n not in any of the finitely many finite sets S_m). That is, for some ball $B_1(x_{j_1,1})$ there is an infinite subsequence $C_1 = \{z_{n_1}, z_{n_2}, \dots\}$ of $\{z_1, z_2, \dots\}$ entirely contained in $B_1(x_{j_1,1})$.

Arguing in the same way with the covering by finitely many balls $B_{1/2}(x_{j,1/2})$ and the infinite sequence C_1 in X (as opposed to the infinite sequence $\{z_n\}$ in X which we used above), there is some $B_{1/2}(x_{j_2,2})$ which meets C_1 in an infinite subsequence C_2. We can continue this process ad infinitum, getting a decreasing chain of infinite subsequences $C_1, C_2, \dots$ and open balls $B_{1/n}(x_{j_n,n})$ meeting each C_n in an infinite subsequence C_{n+1}.

We claim that the sequence of centers $\{x_{j_r,r}\}_{r\in\mathbf{N}}$ is a *Cauchy* sequence. Indeed, if we pick $\varepsilon > 0$ then for $n \geq m > 1/\varepsilon$ we have

$$\rho(x_{j_n,n}, x_{j_m,m}) \leq \rho(x_{j_n,n}, z) + \rho(z, x_{j_m,m}) \leq 1/n + 1/m < 2\varepsilon$$

for any $z \in C_n$ (since C_n is a subsequence of C_m, as $n \geq m$). Note that if we visualize the $x_{n,m}$'s as arranged in a large grid, then $\{x_{j_r,r}\}$ is sort of a zig-zag "digonal"-like path through the grid. This style of argument is the old "diagonal trick" that is so very useful device when dealing with infinite collections of (finite or infinite) sequences.

Since X is *complete*, the sequence $\{x_{j_n,n}\}$ has a limit $x \in X$. The U_i's cover X, so some $B_\varepsilon(x)$ lies inside of some U_{i_0}. But $x_{j_k,k} \to x$, so for large k we have $x_{j_k,k} \in B_{\varepsilon/4}(x)$. Taking $k > 4/\varepsilon$, we have

$$B_{1/k}(x_{j_k,k}) \subseteq B_{\varepsilon/4}(x_{j_k,k}) \subseteq B_{\varepsilon/2}(x),$$

the last inclusion by the triangle inequality (since $\rho(x_{j_k,k}, x) < \varepsilon/4$ and $\varepsilon/4 + \varepsilon/4 = \varepsilon/2$). Fix such a k_0. Recall that for each n, the open ball $B_{1/n}(x_{j_n,n})$ contains z_m for infinitely many m. Thus, for infinitely many m we have

$$z_m \in B_{1/k_0}(x_{j_{k_0},k_0}) \subseteq B_{\varepsilon/2}(x).$$

Choose such m_0 with $m_0 > 2/\varepsilon$. Hence,

$$B_{1/m_0}(z_{m_0}) \subseteq B_{\varepsilon/2}(z_{m_0}) \subseteq B_\varepsilon(x) \subseteq U_{i_0}.$$

Aha, but recall that the z_m's had the property that (for all m) the open ball $B_{1/m}(z_m)$ is *never* contained in a single U_i. Contradiction! It follows that the original hypothesis of non-existence of a Lebesgue number is false, and hence a Lebesgue number must exist. ■

Using Lebesgue covering numbers, it is now a simple matter to complete the proof of Theorem 2.4. For X complete and totally bounded, and $\{U_i\}$ an open cover, we wish to find a finite subcover. By Lebesgue's covering lemma, there exists some δ such that $B_\delta(x) \subseteq U_{i(x)}$ for all $x \in X$. But since X is totally bounded, there exists a *finite* covering of X by open δ-balls, say $B_\delta(x_1), \dots, B_\delta(x_n)$. Then $U_{i(x_1)}, \dots, U_{i(x_n)}$ has union containing all $B_\delta(x_j)$'s, so the $U_{i(x_j)}$'s cover X. That gives us the desired finite subcover. ■

Now that Theorem 2.4 has been proven, we should note that Lebesgue's covering lemma is usually not stated in the way we have given it. Instead, one always formulates it as a theorem about compact metric spaces (rather than complete and totally bounded ones). Just keep in mind the order of the logic: we really prove the covering lemma as a result about complete and totally bounded spaces, and then deduced from this that the class of complete and totally bounded metric spaces coincides with the class of compact metric spaces.