Math 396. Bijectivity vs. isomorphism

1. Motivation

Let $f : X \to Y$ be a C^p map between two C^p-premanifolds with corners, with $1 \leq p \leq \infty$. Assuming f is bijective, we would like a criterion to tell us that f^{-1} is a C^p map as well (so f is a C^p isomorphism and $f(\partial X) = \partial Y$). However, this can fail even if f^{-1} is continuous. For example, the map $f : \mathbf{R} \to \mathbf{R}$ defined by $f(t) = t^3$ is a C^∞ bijection, but its inverse $t \mapsto t^{1/3}$ is continuous but is not differentiable at the origin. One necessary condition is that f should be an immersion.

In pathological situations, the "bijective immersion" condition is still not enough: if we let $Y = \mathbf{R}$ with its usual topology and usual C^∞-structure, and we let $X = \mathbf{R}$ with the discrete topology and the associated C^∞-structure, the identity map $X \to Y$ is a C^∞ bijective immersion but the inverse is not even continuous. This latter example is pathological because $\mathbf{R}$ with the discrete topology is not a second-countable space (as an uncountable discrete set cannot have a countable base of opens, as all points are open and there are uncountably many of them). In a sense, the problem is that the dimensions of the two spaces are not the same.

Let us now restrict ourselves to the second-countable case, as this prevents us from doing silly things like making topologies discrete. But even with second-countability and Hausdorff conditions, we still run into problems. For example, let $Y = \mathbf{R}$ (with its usual C^∞-structure) and let X be the disjoint union of $\mathbf{R} - \{0\}$ (with its usual C^∞-structure) and the point $\{0\}$ that we declare to be open. There is an evident C^∞ bijection $X \to Y$ that is also an immersion, but once again the inverse is not even continuous. Here the problem is not that the pointwise dimensions on X are all too small, but that they "drop" somewhere. The same sort of example could be made much more generally, such as taking Y to be open in $\mathbf{R}^n$ and X to be the disjoint union of a hyperplane slice of Y (considered as an open set in an $(n-1)$-dimensional vector space) and the open complement of this hyperplane slice in Y.

Even when dimensions do not drop, there remains one further complication when our spaces have corners. It is best illustrated with an example: $f : [0, 1) \to S^1$ defined by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$ is a bijective immersion between connected 1-dimensional manifolds with corners, but the inverse is not continuous. Here, the problem is that $f(\partial X)$ (a point) is not contained in ∂Y (the empty set). Such a containment was a condition in the inverse function theorem on open sets in sectors, so we should expect to likewise have to require the hypothesis $f(\partial X) \subseteq \partial Y$ in the global setting. (Note that we do not want to have to make more precise assumptions such as that $f(\partial X) = \partial Y$ or even that f preserves the index at singular points; our hope is that, as with the inverse function theorem on open sets in sectors, the weakest necessary condition – $f(\partial X) \subseteq \partial Y$ – will turn out to be sufficient for the desired strong conclusion on the C^p property for the inverse f^{-1}.)

2. A simple criterion

A very basic and handy criterion for checking when a bijective C^p map is a C^p isomorphism is to check if its tangent maps are isomorphisms:

Theorem 2.1. *If $f : X' \to X$ is a bijective C^p map between C^p premanifolds with corners such that $f(\partial X') \subseteq \partial X$ and the tangent map $\mathrm{d}f(x') : \mathrm{T}_{x'}(X') \to \mathrm{T}_{f(x')}(X)$ is a linear isomorphism for all $x' \in X'$ then f is a C^p isomorphism.*

Proof. By the inverse function theorem (which requires assuming that the map carries boundary points into boundary points!), the hypothesis on tangent spaces implies that f is a local C^p isomorphism. Thus, our problem is to show that a bijective local C^p isomorphism is a C^p isomorphism. That is, we must show that the set-theoretic inverse map f^{-1} is a C^p map. The local isomorphism condition on the bijective f implies that for all $x \in X$ there exist opens $U \subseteq X$ around x and $U' \subseteq X'$ around $x' = f^{-1}(x)$ such that f restricts to a C^p isomorphism between U and U'. In particular, f^{-1} carries U into U' with $f^{-1} : U \to U'$ a C^p map (as it must be the C^p inverse to the C^p isomorphism $U' \simeq U$ induced by f). Since the C^p property for a set-theoretic map between C^p premanifolds with corners is local on the source and target, we conclude (by varying $x \in X$) that f^{-1} is indeed C^p. ■

We now give an application of the theorem, and then explain why it is desired to impose a weaker hypothesis. Let $C \subseteq \mathbf{R}^3$ be the zero locus of $g(x,y,z) = x^2+y^2-1$ (a cylinder centered on the z-axis with radius 1). For all $c = (x_0, y_0, z_0) \in C$ the functional $\mathrm{d}g(c) = 2x_0\mathrm{d}x(c) + 2y_0\mathrm{d}y(c) \in \mathrm{T}_c(\mathbf{R}^3)^\vee$ is nonzero (as $x_0^2 + y_0^2 = 1 \neq 0$), so by the implicit function theorem C is a C^∞ closed hypersurface in $\mathbf{R}^3$ with dimension 2, as is physically obvious. It is geometrically evident that C is a product of S^1 with $\mathbf{R}$, where S^1 parameterizes the angle around the z-axis and $\mathbf{R}$ tracks the z-coordinates. But we want more: C has just been given a C^∞ structure from how it sits in $\mathbf{R}^3$, and so we surely expect that the resulting bijection between C and $S^1 \times \mathbf{R}$ is not merely a set-theoretic (or topological) isomorphism but is even a C^∞ isomorphism (using the product C^∞-structure on $S^1 \times \mathbf{R}$).

Let us now show this rigorously, as an application of the preceding theorem. The map $f : S^1 \times \mathbf{R} \to \mathbf{R}^3$ defined by $f(\theta, t) = (\cos\theta, \sin\theta, t)$ is C^∞ with image inside of the C^∞ embedded submanifold C, so the induced mapping $\overline{f} : S^1 \times \mathbf{R} \to C$ is C^∞. By trigonometry, $\overline{f}$ is bijective. A direct calculation shows that at any point $\xi_0 = (\theta_0, t_0) \in S^1 \times \mathbf{R}$, the tangent map $\mathrm{d}f(\xi_0)$ sends the basis vectors $\partial_\theta|_{\xi_0}$ and $\partial_t|_{\xi_0}$ of $\mathrm{T}_{\xi_0}(S^1 \times \mathbf{R})$ to the respective vectors

$$-\sin(\theta_0)\partial_x|_{f(\xi_0)} + \cos(\theta_0)\partial_y|_{f(\xi_0)}, \partial_z|_{f(\xi_0)} \in \mathrm{T}_{f(\xi_0)}(\mathbf{R}^3)$$

that are clearly linearly independent. Hence, f is an immersion, so $\overline{f}$ is an immersion too (as f factors through $\overline{f}$). But both $S^1 \times \mathbf{R}$ and C have dimension 2 at all points, so for dimension reasons the tangent maps of $\overline{f}$ are linear isomorphisms. Thus, by the theorem it follows that the C^∞ map $\overline{f}$ is a C^∞ isomorphism.

Let us now explain why the isomorphism condition on tangent spaces is not always well-suited to applications, and why we might prefer to get away with a weaker hypothesis. In the preceding example, we had to use the geometric information that the source and target had the same dimension. That is, when proving that the tangent maps for $\overline{f}$ were injections we did not need to know the dimension of C, but to deduce the isomorphism property we did. However, in some abstract situations one is occasionally confronted with the situation of a C^p immersion $f : X \to Z$ between C^p premanifolds such that f is bijective onto a C^p subpremanifold $Z_0 \subseteq Z$ but the dimension of Z_0 may not be easily computed. In such cases, the mapping properties of submanifolds ensure that f factors through a C^p map $\overline{f} : X \to Z_0$ that our hypotheses assure us is bijective, and $\overline{f}$ has to be an immersion because the immersion f factors through it. Thus, $\overline{f}$ is a bijective immersion. It is tempting to hope that this is all we should need to know to infer that $\overline{f}$ is a C^p isomorphism, at least under some mild restrictions on X (but no real restrictions on Z_0). The next section gives an affirmative answer to this dream.

3. The isomorphism criterion

We now require two properties (in terms of our initial discussion above, with a bijective C^p map $f : X \to Y$ between C^p premanifolds with corners such that $f(\partial X) \subseteq \partial Y$): X should have the same dimension at all points (so no "jumping" occurs) and it should be second-countable (so it is not an artificial "discrete topology" version of Y).

Theorem 3.1. *Let $f : X \to Y$ be a C^p map between non-empty C^p premanifolds with corners with $1 \le p \le \infty$. Assume that $f(\partial X) \subseteq \partial Y$, f is a bijective immersion, and X and Y are second-countable (e.g., manifolds with corners). Also assume that X has the same dimension at all points. The map f is then a C^p isomorphism; in particular, f^{-1} is continuous and $f(\partial X) = \partial Y$.*

Note that in this theorem it is not assumed *a priori* that f preserves the index at boundary points nor that Y has constant dimension at all points or that the dimension of each connected component of X is equal to that of the connected component of Y into which it lands. The theorem is one of the reasons that we require manifolds to be second-countable, and why premanifolds that are paracompact and Hausdorff but have uncountably many connected components are considered to be pathological. Such examples never come up in practice anyway (as one has to do something artificial to make premanifolds with uncountably many connected components).

Proof. If $\{Y_i\}$ is the set of connected components of Y then these are pairwise disjoint opens covering Y and similarly the $f^{-1}(Y_i)$'s are pairwise disjoint opens covering X. It suffices to treat each of the maps $f^{-1}(Y_i) \to Y_i$ separately, and since each of these also satsifies the initial hypotheses on f we may now assume that Y is connected. In particular, Y has the same dimension $d \ge 0$ at all points. The immersion property implies that each tangent space on X has dimension at most d, so the common pointwise dimension at all points of X is at most d. Since X is second-countable, it has only countably many connected components, say $\{X_n\}$ (with n running through a subset of the positive integers). We shall first prove that the common dimension of the X_n's is equal to that of Y, and then we prove that f^{-1} is a C^∞ map (and so *a posteriori* X has to be connected).

First assume that the common dimension of the X_n's is some $d' < d$. In this case, we claim f cannot be surjective, contrary to hypothesis. This requires a preliminary discussion of measure-zero sets in C^p premanifolds with corners. Let us say that a subset S in a C^p premanifold with corners Y has *measure zero* if for each C^p-chart (ϕ, U) on Y, with $\phi : U \to V$ the coordinate map to a finite-dimensional vector space, the subset $U \cap S$ has measure zero with respect to the ϕ-coordinates. That is, $\phi(U \cap S) \subseteq V$ has measure zero in the usual sense. (Recall that the notion of "measure zero" for subsets of $\mathbf{R}^n$ is invariant under linear change of coordinates, and so it makes sense in abstract finite-dimensional $\mathbf{R}$-vector spaces.) This is a good notion in the sense that it can be checked with a single atlas:

Lemma 3.2. *Let S be a subset of a C^p premanifold with corners Y with $1 \le p \le \infty$. Let $\{(\phi_i, U_i)\}$ be a C^p-atlas on Y, with $\phi_i : U_i \to \Sigma_i \subseteq V_i$ a C^p isomorphism onto an open set in a sector Σ_i in a finite-dimensional $\mathbf{R}$-vector space V_i. Assume $\phi_i(U_i \cap S)$ has measure zero in V_i for all i. For any C^p-chart (ϕ, U) on Y with $\phi : U \to V$ the coordinate map, the subset $\phi(S \cap U) \subseteq V$ has measure zero.*

Proof. The open sets $U_i \cap U$ cover U, but U is homeomorphic to $\phi(U) \subseteq V$, so U is second countable. In particular, every open covering of U has a countable subcover, so countably many of the $U_i \cap U$'s cover U. We may replace Y with the open subset that is the union of the corresponding countably many U_i's (as this open subset of Y still contains U), and so we can assume $\{U_i\}$ is a countable collection. Let $S_i = S \cap U_i \cap U$, so $\phi_i(S_i) \subseteq \phi_i(S \cap U_i)$ inside of V_i. In particular, $\phi_i(S_i)$ has measure zero in V_i for each i.

The overlap $S \cap U$ is the union of its countably many subsets S_i. Hence, $\phi(S \cap U)$ is the union of the countably many subsets $\phi(S_i)$. Since a countable union of measure-zero sets in V has measure zero, if each $\phi(S_i)$ has measure zero then so does $\phi(S \cap U)$. Thus, since $\phi_i(S_i)$ has measure zero in V_i, we may focus our attention on each S_i separately. That is, we fix i and rename S_i as S to get to the following special case: we have a subset $S \subseteq U \cap U'$ for C^p-charts (ϕ, U) and (ϕ', U')

such that $\phi'(S) \subseteq V'$ has measure zero (where V' is the target vector space for the C^p coordinate system ϕ' on U'). We want to prove that $\phi(S) \subseteq V$ also has measure zero. We may assume $U \cap U'$ is non-empty, as otherwise S is empty and the problem is trivial. Hence, $\dim V = \dim V'$, say this common dimension is n.

By the compatibility condition on all C^p-charts for a common C^p-structure, the transition map $\phi \circ \phi'^{-1}$ is a C^p-isomorphism from the open subset $\phi'(U \cap U') \subseteq \Sigma' \subseteq V'$ onto the open subset $\phi(U \cap U') \subseteq \Sigma \subseteq V$ (with Σ and Σ' sectors in V and V'). This map carries $\phi'(S)$ over to $\phi(S)$, so we come down to the following rather concrete question. Consider n-dimensional vector spaces V and V' over $\mathbf{R}$ and a C^p isomorphism $f : A' \simeq A$ between open subsets $A \subseteq \Sigma$ and $A' \subseteq \Sigma'$ where $\Sigma \subseteq V$ and $\Sigma' \subseteq V'$ are sectors. For a subset $S \subseteq A'$, we claim that S has measure zero in V' if and only if $f(S)$ has measure zero in V. It suffices to prove that $f(S)$ has measure zero whenever S has measure zero, as such a general assertion may then be applied to the C^p isomorphism f^{-1} in the other direction to deduce the converse. We may pick linear coordinates on V and V', so our problem becomes: if f is a C^p isomorphism between open sets in sectors in $\mathbf{R}^n$, with $1 \leq p \leq \infty$, then f carries measure-zero sets to measure-zero sets. This follows from the lemma below. ■

Lemma 3.3. *If $\Sigma \subseteq \mathbf{R}^n$ is a sector and $f : U \to \mathbf{R}^n$ is a C^1 mapping on an open set $U \subseteq \Sigma$ then for any measure zero set $A \subseteq U$ the image $f(A) \subseteq \mathbf{R}^n$ has measure zero.*

Proof. We may make a linear change of coordinates on the source so that Σ is a "standard" sector (namely, $\mathbf{R}^n$ or $[0, \infty)^r \times \mathbf{R}^{n-r}$ for some $1 \leq r \leq n$). In this setting, the old proof of the result (in the case $\Sigma = \mathbf{R}^n$) carries over *verbatim*. One can also use the local version of Whitney's extension theorem in conjunction with a countability argument to reduce the problem on sectors to the case of open sets in $\mathbf{R}^n$, and we leave it to the interested reader to work out such an alternative argument (this it is overkill to do this, since the old argument really works on sectors). ■

We now make a definition:

Definition 3.4. For $1 \leq p \leq \infty$, a subset S in a C^p premanifold with corners Y has *measure zero* if for some C^p-atlas $\{(\phi_i, U_i)\}$ covering Y, each $\phi_i(S \cap U_i)$ has measure zero in the target vector space for ϕ_i.

The preceding lemma ensures that the condition in this definition is satisfied for all C^p-atlases on Y if it is satisfied for one. Obviously a measure-zero set in a C^p premanifold with corners cannot fill up the whole space, since even in the domain of any C^p chart it cannot fill up that domain (as a non-empty open set in a sector in a finite-dimensional $\mathbf{R}$-vector space does not have measure zero). Also, a countable union of measure zero sets in Y has measure zero. Indeed, by definition we may check this in local C^p-charts, and thus the problem is shifted into open sets in sectors in $\mathbf{R}^n$; it is clear that countable unions of measure zero sets in $\mathbf{R}^n$ have measure zero.

As a consequence of these consideration with sets of measure zero, we may now return to our original problem of proving that a bijective C^p-immersion $f : X \to Y$ between second countable C^p premanifolds with corners is a C^p isomorphism when X has the same dimension at all of its points and $f(\partial X) \subseteq \partial Y$. We have already reduced to the case when Y is connected, say with dimension d, and so each of the countably many connected components X_n of X has a common dimension $d' \leq d$. The key is:

Lemma 3.5. *With notation as above, necessarily $d' = d$.*

Proof. Suppose otherwise, so each X_n has dimension $< d$. If we can show that $f(X_n) \subseteq Y$ has measure zero in Y for all n, then the countable union $f(X)$ of the $f(X_n)$'s would have measure zero in Y. But this would force $f(X)$ to be a proper subset of Y, contradicting the assumption that f

is bijective. To show that each $f(X_n)$ has measure zero in Y, we note that since the topological space X_n is a second-countable space, any open cover has a countable subcover. Likewise, Y has the same property. Let $\{(\phi_m, U_m)\}$ be a covering of Y by countably many C^p-charts, so each open $f^{-1}(U_m)$ in X_n is covered by countably many C^p-charts $\{(\psi_{r,m}, U'_{r,m})\}_{r\geq 1}$. Since $f(X_n)$ is the union of the countably many sets $f(U'_{r,m})$, it suffices to show that each of these has measure zero. Clearly $f(U'_{r,m})$ is contained in U_m, so upon renaming U_m as Y and $U'_{r,m}$ as X we may assume that X is an open set in a sector Σ' in $\mathbf{R}^{d'}$ and Y is an open set in a sector Σ in $\mathbf{R}^d$, and that we are given a C^p map from X to Y. In this case, since $d' < d$ the image $f(X)$ has measure zero in $\mathbf{R}^d$. The reason is as follows. The map f may be factored as the composite of the inclusion of X into the open set $U = X \times \mathbf{R}^{d-d'}$ in the sector $\Sigma' \times \mathbf{R}^{d-d'}$ in $\mathbf{R}^d$ via $x \mapsto (x, 0)$ (where its image in $\mathbf{R}^d$ is a measure zero set, as $d' < d$) followed by the C^p map $U \to \mathbf{R}^d$ defined by $(x, y) \mapsto f(x) + y$. Thus, we get the desired result by Lemma 3.3. ■

We have now proved that the constant dimension of X must equal that of Y. Hence, the injective tangent maps $\mathrm{d}f(x) : \mathrm{T}_x(X) \to \mathrm{T}_{f(x)}(Y)$ must be linear isomorphisms for all $x \in X$, so since $f(\partial X) \subseteq \partial Y$ the inverse function theorem "with corners" may be used to conclude that the map f is a local C^p isomorphism. However, f is bijective, so we conclude that its inverse map f^{-1} is a local C^p isomorphism as well. Thus, f^{-1} is a C^p mapping and hence f is a C^p isomorphism. This concludes the proof of the theorem. ■

4. Applications

We can use Theorem 3.1 to prove uniqueness of differentiable structures on various subsets of manifolds.

Let X be a C^p premanifold with corners (such as a C^∞ manifold with boundary), with $1 \leq p \leq \infty$, and for $r \geq 0$ let X_r be the locally closed subset of points on X with index r. In §3 in the handout on premanifolds with corners it was shown how to give the subset X_r a structure of C^p premanifold so that it has a universal mapping property: for any C^p premanifold with corners X' and a C^p map $f : X' \to X$ whose image is contained in X_r, the resulting map of sets $f : X' \to X_r$ (which is continuous, as X_r has the subspace topology from X) is a C^p mapping. We can now exploit this property to prove much more. First, the C^p-structure that we have constructed on X_r has an extra property: *by construction*, it makes the inclusion mapping $i_r : X_r \to X$ a C^p mapping that is moreover an immersion: $\mathrm{d}i_r(x) : \mathrm{T}_x(X_r) \to \mathrm{T}_x(X)$ is injective on tangent spaces for all $x \in X_r$. (This basically amounts to the fact that we may take local C^p coordinates $\{y_1, \dots, y_m\}$ on X_r around x to be the restriction to $U \cap X_r$ of the first m members in a system of local C^p coordinates $\{\xi_1, \dots, \xi_M\}$ on an open U around x in X, and $\mathrm{d}i_r(x)$ takes $\partial_{y_j}|_x$ to $\partial_{\xi_i}|_x$.) This provides the input to prove a strong uniqueness theorem that may help to convince the reader that the C^p structure we have put on X_r is not *ad hoc*:

Theorem 4.1. *The C^p-premanifold structure we have on X_r is the unique one with respect to which the inclusion $X_r \to X$ is an immersion.*

We emphasize a special case: if X is a C^p premanifold with boundary, then the C^p structure we have put on the closed subset $\partial X = X_1$ is the unique one with respect to which the inclusion map $\partial X \to X$ is an immersion. This midly generalizes Exercise 12 in Chapter 2 of the course text.

Proof. Let X'_r denote X_r equipped with another such structure, so by hypotheses we have an injective immersion $i'_r : X'_r \to X$. By the mapping property for $i_r : X_r \to X$ using the "usual" C^p structure, we get a unique factorization of i'_r as $i'_r = i_r \circ h$ for a C^p mapping $h : X'_r \to X_r$. The map h must be the identity on underlying topological spaces, and since i'_r and i_r are immersions it follows from the Chain Rule that h is an immersion. Hence, the identity map $X'_r \to X_r$ is an immersion. The equality of C^p structures is exactly the claim that the identity map on underlying spaces is a C^p isomorphism. Hence, our problem is now a special case of the following general claim: if $h : M' \to M$ is a C^p immersion between C^p premanifolds and h is a homeomorphism, prove that it is a C^p isomorphism.

This problem is local over M (i.e., if $\{U_i\}$ is an open covering of M then the restricted maps $h^{-1}(U_i) \to U_i$ have the same properties as h, and these are C^p isomorphisms if and only if h is), so we can replace M with a local chart to ensure that M is second countable and Hausdorff (i.e., a manifold). Since h is a homeomorphism, this forces M' to also be second countable and Hausdorff. We may also work separately over the (open) connected components of M, so we can assume that M is connected and hence (since h is a homeomorphism) M' is also connected. Thus, the pointwise dimensions on M and M' are constant. By Theorem 3.1, we may now conclude that h is a C^p isomorphism. ■

Now let X be a C^p premanifold (no corners!), $1 \le p \le \infty$, and suppose that X is equipped with a properly discontinuous action of a discrete group G. We therefore get a C^p premanifold quotient map $\pi : X \to X/G$ that is a G-invariant local C^p isomorphism and satisfies a universal mapping property for G-invariant C^p maps from X to other C^p premanifolds. We want to give a simple criterion to show that a concretely-given G-invariant surjective C^p map $X \to X'$ "is" the quotient map $X \to X/G$ in disguise.

In view of what has gone above, we shall have to assume that X is second-countable. Note that this forces X/G to be second-countable (as follows immediately from the construction of the topology on X/G, with all sufficiently small opens obtained as images of opens in X). This motivates the hypotheses is:

Theorem 4.2. *For X and G as above, in particular with X second countable, suppose that the map $f : X \to X'$ is a surjective C^p map onto a second-countable C^p premanifold X' such that $f(x.g) = f(x)$ for all $x \in X$ and $g \in G$. Assume also that X has constant pointwise dimension and that each fiber $f^{-1}(x')$ is a G-orbit in X. If f is an immersion then the induced C^p map $\overline{f} : X/G \to X'$ is a C^p isomorphism. In particular, $f : X \to X'$ "is" the C^p premanifold quotient of X by the action of G.*

This theorem gives a concrete condition for identifying a C^p surjection $f : X \to X'$ as the premanifold quotient by the G-action: f should be an immersion and the the fibers should be G-orbits. Specific examples will be given after the proof.

Proof. Since $X \to X/G$ induces isomorphisms on tangent spaces, the induced C^p map $\overline{f} : X/G \to X'$ is an immersion and X/G has constant pointwise dimension (the same as that of X). Also, X/G is second countable (since X is), and $\overline{f}$ is surjective because f is. Since it is assumed that the fibers of f are G-orbits, it follows that $\overline{f}$ is also injective and hence bijective. By Theorem 3.1, $\overline{f}$ is therefore a C^p isomorphism. ■

We now apply this theorem to efficiently handle some earlier problems that we had concerning the comparison of multiple C^p-structures on the same topological space.

Example 4.3. The map $\mathbf{R} \to S^1 \subseteq \mathbf{R}^2$ given by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$ is a C^∞ map from $\mathbf{R}$ to $\mathbf{R}^2$, and since S^1 is a closed submanifold of the plane it follows that this map factors through a C^∞ map $\mathbf{R} \to S^1$ that is clearly invariant with respect to the $\mathbf{Z}$-action on $\mathbf{R}$ by additive translation. By trigonmetry, this map is surjective and its fibers are $\mathbf{Z}$-orbits, so the induced C^∞ map $\mathbf{R}/\mathbf{Z} \to S^1$ is bijective and hence (by Theorem 3.1) is a C^∞ isomorphism. This says that the C^∞-structure on S^1 through planar coordinates is the same as through polar coordinates, but it avoids the need to do explicit calculations: we let the *properties* of the trigonometric functions do all of the work. (These properties are hiding in the fact that $\overline{f}$ is bijective and f is a C^∞ map from $\mathbf{R}$ to the plane.)

Example 4.4. Let us revisit the problem of showing that the C^∞-structures on $\mathbf{P}^n(\mathbf{R})$ through its standard atlas of $n+1$ Euclidean spaces and through its presentation as the antipodal quotient of the standard n-sphere $S^n \subseteq \mathbf{R}^{n+1}$ are in fact the same C^∞-structure. This was proved in an earlier handout by explicit computations with charts. We now do it with essentially no explicit computations, letting the geometry of tangent spaces and an intelligent choice of coordinates at the end do all of the work.

By Theorem 4.2, it suffices to show that when $\mathbf{P}^n(\mathbf{R})$ is given the C^∞ structure arising from the atlas of Euclidean charts, then the natural topological quotient map $S^n \to \mathbf{P}^n(\mathbf{R})$ (which is surjective with antipodal pairs as fibers) is an immersion. Let $V = \mathbf{R}^{n+1}$ with its standard inner product, so $\mathbf{P}^n(\mathbf{R}) = \mathbf{P}(V)$ and S^n is identified with the unit sphere S in $V^\vee$. The quotient map $S^n \to \mathbf{P}^n(\mathbf{R})$ is then identified with the composite of the C^∞ embedding $\iota : S \to V^\vee - \{0\}$ and the submersive C^∞ map $F : V^\vee - \{0\} \to \mathbf{P}(V)$ that was studied in an earlier homework. (Set-theoretically, F sends a functional $\ell \in V^\vee - \{0\}$ to the hyperplane $\ker \ell$ as a point of $\mathbf{P}(V)$.) This shows that the map $S \to \mathbf{P}(V)$ is C^∞ with respect to the C^∞-structure we have put on $\mathbf{P}(V)$.

To check the immersion criterion in Theorem 4.2, by the Chain Rule we just have to make sure that for each $s \in S$ the hyperplane $\mathrm{T}_s(S)$ inside of $\mathrm{T}_s(V^\vee - \{0\})$ has trivial intersection with the kernel line of the surjective map

$$\mathrm{d}F(s) : \mathrm{T}_s(V^\vee - \{0\}) \to \mathrm{T}_{F(s)}(\mathbf{P}(V)).$$

(Indeed, this implies that $\mathrm{d}(F \circ \iota)(s)$ is injective for all $s \in S$, as desired.) Let $L \subseteq V^\vee$ be the line spanned by the nonzero point s, so $L - \{0\}$ in $V^\vee$ is a punctured line passing through s whose image in $\mathbf{P}(V)$ under F is the single point $F(s)$ (as all nonzero multiples of s are nonzero scalar multiples of a common functional on V and hence all have the same kernel hyperplane in V). Since the map $L - \{0\} \to \mathbf{P}(V)$ factors through a point, its induced map on tangent spaces must be zero because (by the Chain Rule) it factors through the vanishing tangent space of a 1-point manifold. Thus, $\mathrm{T}_s(L - \{0\})$ is a line in the kernel of the surjection $\mathrm{d}F(s)$, yet for dimension reasons this surjective map has 1-dimensional kernel. We conclude that $\ker \mathrm{d}F(s) = \mathrm{T}_s(L - \{0\})$, and so our problem is to check that the subspaces $\mathrm{T}_s(S)$ and $\mathrm{T}_s(L - \{0\})$ in $\mathrm{T}_s(V^\vee - \{0\})$ have trivial intersection. This is a problem that is internal to $V^\vee$: the projective space $\mathbf{P}(V)$ has now dropped out of the picture.

Renaming $V^\vee$ as W, our problem is to show that if W is a finite-dimensional inner product space, s is a point on the unit sphere S in W, and $L \subseteq W$ is the line spanned by $s \neq 0$, the subspaces $\mathrm{T}_s(S)$ and $\mathrm{T}_s(L)$ have vanishing intersection inside of $\mathrm{T}_s(W)$. We may now choose an orthonormal basis $\mathbf{e} = \{e_i\}$ of W with $e_1 = s$, so $W = \mathbf{R}^m$ with the standard inner product, $S = \{\sum x_i^2 = 1\}$, $s = (1, 0, \dots, 0)$, and L is the x_1-axis (given by $x_2 = \cdots = x_n = 0$). In $\mathrm{T}_s(W)$ with its standard basis $\{\partial_{x_i}|_s\}$, the line $\mathrm{T}_s(L)$ is the span of $\partial_{x_1}|_s$. Since S is the zero locus of $f = \sum x_i^2 - 1$, the hyperplane $\mathrm{T}_s(S)$ is the kernel of the functional

$$\mathrm{d}f(s) : \sum a_i \partial_{x_i}|_s \mapsto \sum 2x_i(s) a_i = 2a_1$$

on $\mathrm{T}_s(W)$. Hence, $\mathrm{T}_s(S)$ is spanned by the $\partial_{x_i}|_s$'s for $i > 1$, so it indeed has trivial intersection with the line $\mathrm{T}_s(L)$ that is spanned by $\partial_{x_1}|_s$.