Math 396. Derivations and vector fields

The aim of these notes is to work out the precise correspondence between C^∞ vector fields and derivations on C^∞ functions over open sets in a C^∞ premanifold with corners X. *Throughout what follows, all vector fields, functions, and premanifolds are understood to be of class C^∞, which we may abbreviate by the word "smooth".*

1. Main result

Let $(X, \mathscr{O})$ be a smooth premanifold with corners. For any open set $U \subseteq X$, we let $\mathrm{Vec}_X(U)$ be the set of smooth vector fields $\vec{v}$ on U. Let $\mathrm{Der}_X(U)$ be the set of $\mathbf{R}$-linear derivations D of $\mathscr{O}|_U$, by which we mean a collection $\mathbf{R}$-linear derivations $D_{U'} : \mathscr{O}(U') \to \mathscr{O}(U')$ for all open $U' \subseteq U$ satisfying the compatibility condition with respect to restrictions: if $U'' \subseteq U'$ is an inclusion of opens in U then for all $f \in \mathscr{O}(U')$ we have $D_{U'}(f)|_{U''} = D_{U''}(f|_{U''})$ in $\mathscr{O}(U'')$.

For any $\vec{v} \in \mathrm{Vec}_X(U)$ we define $D_{\vec{v}}$ to be the collection of maps

$$D_{\vec{v},U'} : \mathscr{O}(U') \to \{h : U' \to \mathbf{R}\}$$

given by $D_{\vec{v},U'}(f) : u' \mapsto \vec{v}(u')(f) \in \mathbf{R}$ for open $U' \subseteq U$. Roughly speaking, $D_{\vec{v},U'}$ maps a smooth function on U' to the function whose value at each point $u' \in U'$ is the directional derivative of f in the direction $\vec{v}(u') \in \mathrm{T}_{u'}(U') = \mathrm{T}_{u'}(X)$ at u'. From the definitions it is clear that for any open $U'' \subseteq U'$, the set-theoretic function $D_{\vec{v},U'}(f)$ on U' restricts to $D_{\vec{v},U''}(f|_{U''})$ on U''. It also follows from the $\mathbf{R}$-linearity and the Leibnitz property of $\vec{v}(u) : \mathscr{O}_u \to \mathbf{R}$ for each $u \in U$ that

$$D_{\vec{v},U'}(c_1 f_1 + c_2 f_2)(u') = c_1 \cdot (D_{\vec{v},U'} f_1)(u') + c_2 \cdot (D_{\vec{v},U'} f_2)(u')$$

for all $c_1, c_2 \in \mathbf{R}$ and $f_1, f_2 \in \mathscr{O}(U')$ and

$$(D_{\vec{v},U'}(fg))(u') = f(u')(D_{\vec{v},U'} g)(u') + g(u')(D_{\vec{v},U'} f)(u')$$

for all $f, g \in \mathscr{O}(U')$. Thus, $D_{\vec{v},U'}(c_1 f_1 + c_2 f_2) = c_1 \cdot (D_{\vec{v},U'} f_1) + c_2 \cdot (D_{\vec{v},U'} f_2)$ and

$$D_{\vec{v},U'}(fg) = f D_{\vec{v},U'}(g) + g D_{\vec{v},U'}(f)$$

as $\mathbf{R}$-valued functions on U'. Thus, $f \mapsto D_{\vec{v},U'}(f)$ is an $\mathbf{R}$-linear derivation from $\mathscr{O}(U')$ into the $\mathbf{R}$-algebra of $\mathbf{R}$-valued functions on U'. To say that the $D_{\vec{v},U'}$'s define an element $D_{\vec{v}} \in \mathrm{Der}_X(U)$, it remains to prove a smoothness condition: for all open $U' \subseteq U$, $D_{\vec{v},U'}(f) : U' \to \mathbf{R}$ is a smooth function (and not just a set-theoretic function) for all smooth f on U'. Granting such a property, we would get a map of sets $\mathrm{Vec}_X(U) \to \mathrm{Der}_X(U)$ via $\vec{v} \mapsto D_{\vec{v}}$.

Going in the other direction, for any $D \in \mathrm{Der}_X(U)$, we can make a set-theoretic vector field $\vec{v}_D$ on U as follows: for each $u \in U$ we define $\vec{v}_D(u) : \mathscr{O}_u \to \mathbf{R}$ to send the germ $s = [(U', f)]$ for open $U' \subseteq U$ around u to $(D_{U'} f)(u) \in \mathbf{R}$. Once this value at $u \in U$ is proved to depend only on the germ $s \in \mathscr{O}_u$ and not on the particular representive (U', f), it will have to be proved that each $\vec{v}_D(u) : \mathscr{O}_u \to \mathbf{R}$ thereby defined is an $\mathbf{R}$-linear point derivation at u and and that the assignment $u \mapsto \vec{v}_D(u) \in \mathrm{T}_u(U) = \mathrm{T}_u(X)$ is a smooth vector field over the open subset $U \subseteq X$. With such a result in hand, $D \mapsto \vec{v}_D$ would then give a map of sets $\mathrm{Der}_X(U) \to \mathrm{Vec}_X(U)$ from the set of $\mathbf{R}$-linear derivations of $\mathscr{O}|_U$ to the set of smooth vector fields on U.

Our aim is to show that the above two procedures do indeed make sense, that they are inverse to each other, and that these bijective maps behave well with respect to $\mathscr{O}(U)$-linear structures. In this precise sense, smooth vector fields on an open set U in a smooth premanifold with corners X are "the same" as $\mathbf{R}$-linear derivations of $\mathscr{O}|_U$, and so we may (and will!) use these two points of view interchangably when working on smooth premanifolds with corners.

Theorem 1.1. *For all open $U \subseteq X$, $D_{\vec{v}}$ is an $\mathbf{R}$-linear derivation of $\mathscr{O}|_U$ for all smooth vector fields $\vec{v}$ on U, and conversely for all $D \in \mathrm{Der}_X(U)$ the assignment $u \mapsto \vec{v}_D(u)$ is a smooth vector field on U. These maps between $\mathrm{Vec}_X(U)$ and $\mathrm{Der}_X(U)$ are inverse $\mathscr{O}(U)$-linear bijections. Moreover, these bijections are of local nature in the following sense: for any open subset $U_0 \subseteq U$, the natural restriction maps $\mathrm{Vec}_X(U) \to \mathrm{Vec}_X(U_0)$ and $\mathrm{Der}_X(U) \to \mathrm{Der}_X(U_0)$ are compatible with the bijections between smooth vector fields and $\mathbf{R}$-linear derivations over U and over U_0.*

Before we prove the theorem, we record a very important corollary that is a consequence of the asserted $\mathscr{O}(U)$-linearity and the knowledge of local descriptions of smooth vector fields in terms of ∂_{x_j}'s with respect to local C^∞ coordinates. Since $\vec{v} = \partial_{x_j} \in \mathrm{Vec}_X(U)$ goes over to $\partial/\partial x_j$ in $\mathrm{Der}_X(U)$, we have:

Corollary 1.2. *If (φ, U) is a C^∞-chart on the open $U \subseteq X$ with $\varphi : U \to \mathbf{R}^n$ a C^∞ isomorphism onto an open subset of a standard sector $\Sigma = \{t_1 \geq c_1, \dots, t_r \geq c_r\}$ in $\mathbf{R}^n$, then*

$$\mathrm{Der}_X(U) = \left\{ \sum_{j=1}^n a_j \frac{\partial}{\partial x_j} \mid a_j \in \mathscr{O}(U) \right\}$$

(with $x_j = t_j \circ \phi : U \to \mathbf{R}$ the component functions of ϕ). That is, the only *compatible systems $D = \{D_{U'}\}_{U' \subseteq U}$ of $\mathbf{R}$-linear derivations $D_{U'} : \mathscr{O}(U') \to \mathscr{O}(U')$ for all open subsets $U' \subseteq U$ are those arising from $\sum a_j \cdot \partial/\partial x_j$ for smooth functions a_j on U, in which case such a_j's are uniquely determined.*

The reader may wish to contemplate giving a direct proof of this corollary entirely in the language of derivations of $\mathscr{O}|_U$ without passing through the theory of smooth vector fields. Such a proof can be given by judicious use of bump functions, but such a method does not carry over to the real-analytic or complex-analytic cases (where the corollary remains valid, by essentially the exact same proof we are about to give via the theory of vector fields).

2. From vector fields to derivations

Choose $\vec{v} \in \mathrm{Vec}_X(U)$ a smooth vector field over U. We first have to prove that $D_{\vec{v},U'}(f) : U' \to \mathbf{R}$ is smooth for any open $U' \subseteq U$ and $f \in \mathscr{O}(U')$. The definition of $D_{\vec{v},U'}$ only depends on the vector field $\vec{v}|_{U'} \in \mathrm{Vec}_X(U')$, so for the well-posedness of the passage from vector fields on U to $\mathbf{R}$-linear derivations of $\mathscr{O}|_U$ we may rename U' as U and $\vec{v}|_{U'}$ as $\vec{v}$ to reduce to the case $U' = U$. More specifically, for any open set $U'' \subseteq U$ we have $D_{\vec{v},U}(f)|_{U''} = D_{\vec{v}|_{U''},U''}(f|_{U''})$ as $\mathbf{R}$-valued functions on U'', and so since smoothness of an $\mathbf{R}$-valued function on an open set in X is a local property on X, to prove smoothness of $D_{\vec{v}}(f)$ on U we may shrink U around arbitrary points to arrange that there exists a C^∞ coordinate system $\{x_1, \dots, x_n\}$ on U. By smoothness of $\vec{v}$ we have $\vec{v} = \sum a_j \partial_{x_j}$ with $a_j \in \mathscr{O}(U)$. Thus, by definition, the point derivation $\vec{v}(u) : \mathscr{O}_u \to \mathbf{R}$ at u sends the germ of a smooth function f near u to $\sum a_j(u)(\partial_{x_j}|_u)(f) = (\sum a_j \partial_{x_j} f)(u)$. By the definition of $D_{\vec{v},U}$, we then get

$$D_{\vec{v},U}(f) = \sum_{j=1}^n a_j \frac{\partial f}{\partial x_j},$$

and this lies in $\mathscr{O}(U)$ since f and the a_j's are all smooth functions on U.

By the definitions, it is easy to check that the map $\mathrm{Vec}_X(U) \to \mathrm{Der}_X(U)$ is $\mathscr{O}(U)$-linear and (via natural restriction maps) compatible with shrinking U to smaller opens in X. This completes the analysis of one direction of the desired correspondence.

3. From derivations to vector fields

Now choose $D = \{D_{U'}\}_{U' \subseteq U} \in \mathrm{Der}_X(U)$. Our next order of business is to prove that for all $u \in U$ and germs $s = [(U', f)] \in \mathscr{O}_u$, the number $(D_{U'}f)(u)$ only depends on s (and not on the representative (U', f) with $U' \subseteq U$ around u), and that the resulting well-defined map of sets $\vec{v}_D(u) : s \mapsto (D_{U'}f)(u')$ from $\mathscr{O}_{u'}$ to $\mathbf{R}$ is an $\mathbf{R}$-linear point derivation. We will then have to show that $u \mapsto \vec{v}_D(u) \in \mathrm{T}_u(X)$ is a smooth vector field on U, and that the resulting map $\mathrm{Der}_X(U) \to \mathrm{Vec}_X(U)$ gives an inverse to the map constructed above in the other direction.

For the well-definedness on germs, observe that for any open $U'' \subseteq U'$ around u, the definition of $\mathrm{Der}_X(U)$ provides a compatibility condition on the $D_{U'}$'s as we vary U', from which we obtain $(D_{U'}f)|_{U''} = D_{U''}(f|_{U''})$ as $\mathbf{R}$-valued functions on U''. Thus, the germ of the function $D_{U'}f$ around u only depends on the germ of f around u. Hence, we do indeed get a well-defined map $\vec{v}_D(u) : \mathscr{O}_u \to \mathbf{R}$ via $[(U', f)] \mapsto (D_{U'}f)(u) \in \mathbf{R}$, Since any two germs in $\mathscr{O}_u$ can be represented by smooth functions on a common open U' around u, the $\mathbf{R}$-linearity of $D_{U'}$ and the derivation property $D_{U'}(fg) = fD_{U'}(g) + gD_{U'}(f)$ for $D_{U'}$ as a self-map of $\mathscr{O}(U')$ for all open $U' \subseteq U$ implies that evaluation at u makes $\vec{v}_D(u) : \mathscr{O}_u \to \mathbf{R}$ an $\mathbf{R}$-linear point derivation at u

To check that $u \mapsto \vec{v}_D(u) \in \mathrm{T}_u(X)$ is a smooth vector field on U, we can work *locally* on U (as $\vec{v}_D|_{U'}$ only depends on the $\mathbf{R}$-linear derivation $D|_{U'} \in \mathrm{Der}_X(U')$ on $\mathscr{O}|_{U'}$ for any open $U' \subseteq U$ around u). Thus, we may assume that there exists a C^∞ coordinate system $\{x_1, \dots, x_n\}$ on U. We may therefore uniquely write $\vec{v}_D = \sum a_j \partial_{x_j}$ for $\mathbf{R}$-valued functions $a_j : U \to \mathbf{R}$, and the smoothness property of $\vec{v}_D$ on U that we seek to prove is precisely the claim that each a_j is a smooth function on U. For any $u \in U$ we have

$$a_j(u) = (\vec{v}_D(u))(x_j) = (D_U(x_j))(u)$$

(the final equality by the definition of $\vec{v}_D$), so $a_j = D_U(x_j)$ as $\mathbf{R}$-valued functions on U. However, D_U is a self-map of $\mathscr{O}(U)$, and so $D_U(x_j)$ is smooth on U for all j. Thus, the a_j's are all smooth as U, and hence $\vec{v}_D \in \mathrm{Vec}_X(U)$. We now have the desired set-theoretic map $\mathrm{Der}_X(U) \to \mathrm{Vec}_X(U)$, namely $D \mapsto \vec{v}_D$. It is a straightforward check from the definitions that the map $D \mapsto \vec{v}_D$ from $\mathrm{Der}_X(U)$ to $\mathrm{Vec}_X(U)$ satisfies

$$\vec{v}_{h \cdot D} = h \cdot \vec{v}_D, \ \ \vec{v}_{D_1 + D_2} = \vec{v}_{D_1} + \vec{v}_{D_2}$$

for all $h \in \mathscr{O}(U)$, which is to say that this map is $\mathscr{O}(U)$-linear.

The final step, and the most important of all, is to check that our two maps $\mathrm{Der}_X(U) \to \mathrm{Vec}_X(U)$ and $\mathrm{Vec}_X(U) \to \mathrm{Der}_X(U)$ via $D \mapsto \vec{v}_D$ and $\vec{v} \mapsto D_{\vec{v}}$ are indeed inverse to each other. In other words, for each $\vec{v} \in \mathrm{Vec}_X(U)$ we want to prove that the map $\mathrm{Der}_X(U) \to \mathrm{Vec}_X(U)$ carries $D_{\vec{v}}$ to $\vec{v}$, and likewise that for each $D \in \mathrm{Der}_X(U)$ the map $\mathrm{Vec}_X(U) \to \mathrm{Der}_X(U)$ carries $\vec{v}_D$ to D. Since *both* maps (from the set of derivations of $\mathscr{O}|_U$ to the set of smooth vector fields over U, and *vice-versa*) are compatible with localization on the open set U, the problem of checking these maps are inverse to each other is a local problem: it suffices to check it in a sufficiently small open set around each point in U. In particular, for both directions of the problem we may suppose U is so small that it admits a C^∞ coordinate system $\{x_1, \dots, x_n\}$.

In terms of these coordinates, for any $\vec{v} \in \mathrm{Vec}_X(U)$ we may uniquely write $\vec{v}(u) = \sum a_j(u) \partial_{x_j}|_u$ in $\mathrm{T}_u(X)$ for $a_j(u) \in \mathbf{R}$, and the smoothness of $\vec{v}$ gives that the functions $a_j : U \to \mathbf{R}$ are smooth. The calculations in local coordinates in the preceding section apply to show that $D_{\vec{v}} \in \mathrm{Der}_X(U)$ is the collection of derivations of the $\mathscr{O}(U')$'s for open $U' \subseteq U$ given by $\sum a_j \partial / \partial x_j$ acting on $\mathscr{O}(U')$. Thus, the smooth vector field on U associated to $D_{\vec{v}} \in \mathrm{Der}_X(U)$ has value at u in $\mathrm{T}_u(X)$ given by the point derivation $\mathscr{O}_u \to \mathbf{R}$ satisfying

$$[(U', f)] \mapsto (D_{\vec{v},U'}f)(u) = (\sum_j a_j(\partial f/\partial x_j)(u) = \sum_j a_j(u)(\partial_{x_j}|_u)(f) = (\vec{v}(u))([(U', f)]),$$

where the last equality uses the definition of the a_j's in terms of $\vec{v}$. Hence, the map $\mathrm{Der}_X(U) \to \mathrm{Vec}_X(U)$ indeed carries $D_{\vec{v}}$ back to $\vec{v}$.

Conversely, choose $D \in \mathrm{Der}_X(U)$. We want to prove that the map $\mathrm{Vec}_X(U) \to \mathrm{Der}_X(U)$ carries $\vec{v}_D$ back to D. That is, for open $U' \subseteq U$ and $f \in \mathscr{O}(U')$, we want $u' \mapsto (\vec{v}_D(u'))([(U', f)]) \in \mathbf{R}$ to equal the function $u' \mapsto (D_{U'}f)(u') \in \mathbf{R}$. But this is exactly the definition of $\vec{v}_D$ (modulo the issues of well-definedeness in its definition that we have settled above). This completes the proof of Theorem 1.1.