

MATH 396. DERIVATIVE MAPS, PARAMETRIC CURVES, AND VELOCITY VECTORS

Let (X', \mathcal{O}') and (X, \mathcal{O}) be two C^p premanifolds with corners, $1 \leq p \leq \infty$, and let $F : X' \rightarrow X$ be a C^p mapping. Let $\xi' \in X'$ be a point and let $\xi = F(\xi')$. In class we saw how to define a linear mapping $dF(\xi') : T_{\xi'}(X') \rightarrow T_{\xi}(X)$; explicitly, if $\vec{v}' \in T_{\xi'}(X')$ is a tangent vector (so it is a point-derivation $\vec{v}' : \mathcal{O}'_{\xi'} \rightarrow \mathbf{R}$) then we define

$$(dF(\xi'))(\vec{v}') = \vec{v}' \circ F^* : \mathcal{O}_{\xi} \rightarrow \mathbf{R}$$

with $F^* : \mathcal{O}_{\xi} \rightarrow \mathcal{O}'_{\xi'}$, the “pullback map” defined on germs via $f \mapsto f \circ F$. More precisely, for a germ $[(W, f)]_{\xi}$ at ξ we define $F^*([(W, f)]_{\xi}) = [(F^{-1}(W), f \circ F)]_{\xi'} \in \mathcal{O}'_{\xi'}$, and one easily checks that this latter definition is well-posed in the sense that it really does not depend on the representative pair (W, f) for the chosen germ at ξ . (This may look complicated but it is not so: all we’re saying is that for a function f near ξ , the germ of the composite $f \circ F$ at ξ' only depends on the germ of f on ξ , which is rather obvious if you unwind what it is saying.) Strictly speaking, the notation F^* is a bit abusive since it also depends on the points ξ' and $\xi = F(\xi')$, but the notation $F^*_{\xi', \xi}$ would be too cumbersome. Context will make clear the intended points at which we are considering germs.

In class it was shown that $\vec{v}' \circ F^* : \mathcal{O}_{\xi} \rightarrow \mathbf{R}$ is indeed a point-derivation at ξ (so it lies in $T_{\xi}(X)$), and that the resulting map of sets $dF(\xi') : T_{\xi'}(X') \rightarrow T_{\xi}(X)$ sending \vec{v}' to $\vec{v}' \circ F^*$ is \mathbf{R} -linear. Our aim in this handout is to record proofs of some properties of this derivative mapping, essentially saying that it is a generalization of the classical theory of derivative mappings associated to C^p maps between opens in sectors in vector spaces. As an important application, we will define the concept of velocity vectors to parameterized C^p curves. This is a notion of fundamental importance in differential geometry, as we shall see later (and as should hardly be a surprise, in view of the prominence of velocity vectors in all geometrical problems in physics).

1. PROPERTIES OF DERIVATIVE MAPPINGS

Let (U', φ') and (U, φ) be respective C^p charts around ξ' and ξ in X' and X , with $\varphi' : U' \simeq \varphi'(U') \subseteq \Sigma' \subseteq \mathbf{R}^{n'}$ and $\varphi : U \simeq \varphi(U) \subseteq \Sigma \subseteq \mathbf{R}^n$ having respective component functions $\varphi' = (x_1, \dots, x_{n'})$ and $\varphi = (y_1, \dots, y_n)$ on the source and target. Thus, $T_{\xi'}(X')$ has the ordered basis $\{\partial_{x_j}|_{\xi'}\}$ and $T_{\xi}(X)$ has the ordered basis $\{\partial_{y_i}|_{\xi}\}$. It is natural to ask for the matrix of the linear map $dF(\xi') : T_{\xi'}(X') \rightarrow T_{\xi}(X)$ with respect to these ordered bases.

On the open set $U' \cap F^{-1}(U) \subseteq U'$ around ξ' , let $F_j = y_j \circ F \in \mathcal{O}'(U' \cap F^{-1}(U))$. Note that $F_j \in \mathcal{O}'(U' \cap F^{-1}(U))$ (i.e., it is a C^p function on this open set) precisely because F is a C^p mapping! In particular, it makes sense to compute $(\partial F_i / \partial x_j)(\xi')$ for all j . In the classical setup, such evaluated partials of the component functions of the mapping (defined via the target linear coordinates) with respect to the source linear coordinates are the entries in the matrix for the classical derivative mapping. Happily, the same holds in general:

Theorem 1.1. *The matrix of $dF(\xi') : T_{\xi'}(X') \rightarrow T_{\xi}(X)$ with respect to the ordered bases $\{\partial_{x_j}|_{\xi'}\}$ and $\{\partial_{y_i}|_{\xi}\}$ is $((\partial F_i / \partial x_j)(\xi'))$. That is,*

$$dF(\xi') : \partial_{x_j}|_{\xi'} \mapsto \sum_i \frac{\partial F_i}{\partial x_j}(\xi') \partial_{y_i}|_{\xi}.$$

Proof. As can hardly be a surprise, once we strip away the definitions this theorem will ultimately reduce to the classical theorem on Jacobian matrices for derivative mappings (in the special case when the second mapping takes values in \mathbf{R}). By the definition of $dF(\xi')$, the proposed formula

says

$$\partial_{x_j}|_{\xi'} \circ F^* \stackrel{?}{=} \sum_i \frac{\partial F_i}{\partial x_j}(\xi') \partial_{y_i}|_{\xi}$$

in $T_{\xi}(X)$; that is, this is a proposed equality of point-derivations $\mathcal{O}_{\xi} \rightarrow \mathbf{R}$. To verify it, we simply evaluate both sides on an arbitrary germ $f \in \mathcal{O}_{\xi}$. The value of the left side on f is

$$\partial_{x_j}(F^*(f))(\xi') = (\partial_{x_j}(f \circ F))(\xi') = (\partial_j((f \circ F) \circ \varphi'^{-1}))(\varphi'(\xi')) = (\partial_j(f \circ F \circ \varphi'^{-1}))(\varphi'(\xi'))$$

and the value of the right side on f is

$$\sum_i (\partial_{x_j} F_i)(\xi') \cdot (\partial_{y_j} f)(\xi) = \sum_i (\partial_j(F_i \circ \varphi'^{-1}))(\varphi'(\xi')) \cdot (\partial_i(f \circ \varphi'^{-1}))(\varphi(\xi)).$$

Let $H = \varphi \circ F \circ \varphi'^{-1}$ on the open set $W' = \varphi'(U' \cap \varphi^{-1}(U))$ in $\Sigma' \subseteq \mathbf{R}^{n'}$, and let $h = f \circ \varphi^{-1}$ on the open set $W = \varphi(U)$ in $\Sigma \subseteq \mathbf{R}^n$. Hence, $H : W' \rightarrow W$ is a C^p mapping in the classical sense (why?) between opens in sectors in Euclidean spaces and it carries the point $w' = \varphi'(\xi') \in W'$ to the point $w = \varphi(F(\xi')) = \varphi(\xi) \in W$. Also, h is a C^p function on W in the classical sense (why?). The component functions H_i of H are exactly the functions $F_i \circ \varphi'^{-1}$, so the proposed identity becomes

$$\partial_j(h \circ H)(w) \stackrel{?}{=} \sum_i (\partial_j H_i)(w) \cdot (\partial_i h)(H(w)) = \sum_i (\partial_i h)(H(w)) \cdot (\partial_j H_i)(w).$$

The equality of the outer terms for all j is exactly the entrywise equality that encodes the classical Chain Rule $D(h \circ H)(w) = Dh(H(w)) \circ DH(w)$. \blacksquare

In the setting of abstract C^p premanifolds with corners, the *Chain Rule* is:

Theorem 1.2. *Let $G : X'' \rightarrow X'$ and $F : X' \rightarrow X$ be C^p mappings between C^p premanifolds with corners, $1 \leq p \leq \infty$. For any point $\xi'' \in X''$ with $G(\xi'') = \xi' \in X'$ and $F(\xi') = \xi \in X$, the composite linear mapping*

$$(dF)(G(\xi'')) \circ dG(\xi'') : T_{\xi''}(X'') \rightarrow T_{\xi'}(X') \rightarrow T_{\xi}(X)$$

is equal to $d(F \circ G)(\xi'')$.

Proof. We choose a tangent vector $\vec{v}'' \in T_{\xi''}(X'')$, so we want to prove

$$(d(F \circ G)(\xi''))(\vec{v}'') = (dF(G(\xi'')))((dG(\xi''))(\vec{v}''))$$

in $T_{\xi}(X)$. This is an equality of point derivations $\mathcal{O}_{\xi} \rightarrow \mathbf{R}$, and by the definitions of the derivative mappings the left side is $\vec{v}'' \circ (F \circ G)^*$ and the right side is $(\vec{v}'' \circ G^*) \circ F^* = \vec{v}'' \circ (G^* \circ F^*)$. Hence, it suffices to show that the composite of the mappings $F^* : \mathcal{O}_{\xi} \rightarrow \mathcal{O}'_{\xi'}$ and $G^* : \mathcal{O}'_{\xi'} \rightarrow \mathcal{O}''_{\xi''}$ is equal to $(F \circ G)^*$; that is, $(F \circ G)^* = G^* \circ F^*$ (note the order of F and G on the two sides!). These are mappings from \mathcal{O}_{ξ} to \mathbf{R} , so we evaluate on an arbitrary germ $f \in \mathcal{O}_{\xi}$:

$$(G^* \circ F^*)(f) = G^*(F^*(f)) = G^*(f \circ F) = (f \circ F) \circ G = f \circ (F \circ G) = (F \circ G)^*(f).$$

(Observe the similarity with the proof of the identity $(T \circ T')^{\vee} = T'^{\vee} \circ T^{\vee}$ for composites of dual linear mappings.) \blacksquare

We give two interesting examples of derivative mappings, illustrating how the general theory interacts with the classical one on opens in (sectors in) vector spaces when these are viewed as manifolds (with corners).

Example 1.3. Let V be a finite-dimensional vector space and let $L : V \rightarrow V$ be an \mathbf{R} -linear map and $v_0 \in V$ a fixed element. Let $M : V \rightarrow V$ be the map $v \mapsto L(v) + v_0$; this is “affine linear” (composite of a linear map and a translation). Let $X = V$ as a C^p manifold in the usual manner ($1 \leq p \leq \infty$), and let $\xi \in X$ be a point. As we saw in class, there is a natural linear isomorphism $i_\xi : V \simeq T_\xi(X)$ that sends $v \in V$ to the directional derivative operator $D_{v,\xi}$ on germs of C^p functions on V at ξ . There is likewise a natural linear isomorphism $i_{M(\xi)} : V \simeq T_{M(\xi)}(X)$. But we also have a derivative mapping $dM(\xi) : T_\xi(X) \rightarrow T_{M(\xi)}(X)$, so it is natural to ask how to explicitly fill in a commutative square

$$\begin{array}{ccc} T_\xi(X) & \xrightarrow{dM(\xi)} & T_{M(\xi)}(X) \\ i_\xi \uparrow \simeq & & \simeq \uparrow i_{M(\xi)} \\ V & \xrightarrow{??} & V \end{array}$$

The mystery map has to be $i_{M(\xi)}^{-1} \circ dM(\xi) \circ i_\xi$, and this is a linear map, so the problem is this: make the linear map explicit. The mystery linear map cannot be M in general, since M is not linear (due to the intervention of v_0 , which may not be zero). It is reasonable to guess that the map ought to be L , and indeed it is. The intuitive meaning of this “absence” of v_0 in the description is that, roughly speaking, the linear isomorphism $i_\xi : V \simeq T_\xi(X)$ for $\xi \in X = V$ is “translation-invariant” in ξ . (The theory of connections on vector bundles later in the course will clarify the true meaning of this fact.)

To verify the claim, we have to check $dM(\xi) \circ i_\xi = i_{M(\xi)} \circ L$ since $i_{M(\xi)}$ is a linear isomorphism. We pick $v \in V$, and we want to show that $(dM(\xi))(i_\xi(v))$ and $i_{M(\xi)}(Lv)$ in $T_{M(\xi)}(X)$ agree, which is to say that as point derivations on $\mathcal{O}_{M(\xi)}$ they agree. We just have to evaluate each on an arbitrary germ $f \in \mathcal{O}_{M(\xi)}$ and check that we get the same values. By definition,

$$((dM(\xi))(i_\xi(v)))(f) = (i_\xi(v))(f \circ M) = D_{v,\xi}(f \circ M) = (D(f \circ M)(\xi))(v)$$

and

$$(i_{M(\xi)}(Lv))(f) = D_{Lv,M(\xi)}(f) = (Df(M(\xi)))(Lv) = ((Df(M(\xi))) \circ L)(v),$$

so we want $D(f \circ M)(\xi) = (Df(M(\xi))) \circ L$ as linear maps from V to V . By the old Chain Rule, the left side is $(Df(M(\xi))) \circ dM(\xi)$, so we want $dM(\xi) = L$ for all $\xi \in V$. Since $M(x) = L(x) + v_0$ for all $x \in V$, by the definition of the classical derivative mapping we have

$$M(\xi + h) = L(\xi + h) + v_0 = (L(\xi) + v_0) + Lh = M(\xi) + Lh$$

with vanishing error term, so certainly $dM(\xi) = L$.

Example 1.4. We now push the preceding analysis a bit further by making the precise link between our new theory of derivative mappings on tangent spaces and the old theory of derivative mappings on ambient vector spaces for manifolds (with corners) that are open (in sectors) in vector spaces. Let V and V' be finite-dimensional vector spaces and let $\Sigma \subseteq V$ and $\Sigma' \subseteq V'$ be sectors. Let $X \subseteq \Sigma$ and $X' \subseteq \Sigma'$ be open subsets endowed with their natural C^p -structures. Let $F : X' \rightarrow X$ be a C^p map, and let $\xi' \in X'$ be a point with image $\xi = F(\xi') \in X$. We have the new abstract linear map $dF(\xi') : T_{\xi'}(X') \rightarrow T_\xi(X)$ via the theory of premanifolds with corners and the old linear map $DF(\xi') : V' \rightarrow V$ via the theory of calculus on opens in sectors in vector spaces. If the old and new theories are to be related to each other, these maps had better be “the same”. Since they are maps between rather different-looking vector spaces (in the first case depending on ξ' and ξ , and in the second case not at all), to make sense of such “sameness” we have to first linearly identify

the source vector spaces and target vector spaces. But we have already seen how to do this: the “directional derivative” procedure from class provides us with natural linear isomorphisms

$$i_{\xi'} : V' \simeq T_{\xi'}(X'), \quad i_{\xi} : V \simeq T_{\xi}(X),$$

and so the desired agreement of theories of derivative mappings is that the diagram

$$\begin{array}{ccc} T_{\xi'}(X') & \xrightarrow{dF(\xi')} & T_{\xi}(X) \\ i_{\xi'} \uparrow \simeq & & \simeq \uparrow i_{\xi} \\ V' & \xrightarrow{DF(\xi')} & V \end{array}$$

commutes.

To verify such commutativity, we pick $v' \in V'$ and chase it around the square. That is, we want

$$(dF(\xi'))(i_{\xi'}(v')) = i_{\xi}((DF(\xi'))(v'))$$

in $T_{\xi}(X)$, which is an equality of point derivations at ξ . Thus, we pick an arbitrary germ $f \in \mathcal{O}_{\xi}$ and we want to show both sides have the same value on f . The left side has value

$$(i_{\xi'}(v') \circ F^*)(f) = (i_{\xi'}(v'))(f \circ F) = D_{v', \xi'}(f \circ F) = (D(f \circ F)(\xi'))(v')$$

and the right side has value

$$D_{(DF(\xi'))(v'), \xi}(f) = (Df(\xi))((DF(\xi'))(v')),$$

so it suffices to prove $D(f \circ F)(\xi') : V' \rightarrow \mathbf{R}$ is equal to $(Df(\xi)) \circ (DF(\xi'))$, and since $\xi = F(\xi')$ this is just the old Chain Rule on opens in sectors in vector spaces.

2. PARAMETRIC CURVES AND VELOCITY VECTORS

Let X be a C^p premanifold with corners, $0 \leq p \leq \infty$. A *parameterized C^p curve* at $\xi \in X$ is a C^p map $c : I \rightarrow X$ with $I \subseteq \mathbf{R}$ a nontrivial interval (not a point or the empty set), $0 \in I$, and $c(0) \in \xi$. Usually it is also required that 0 be on the interior of I when $\xi \notin \partial X$. We emphasize that a parameterized curve is the data of the mapping c and it may not be injective or (even for $p = \infty$) have “smooth” image (e.g., $c(t) = (t^2, t^3)$ has image equal to the locus $y^2 = x^3$ in the plane that has a “cusp” at the origin). If we work with the interval $\tilde{I} = (1/2)I$ and the mapping $\tilde{c}(t) = c(2t)$ that “moves twice as quickly” then we consider it to be a different parameterized curve even though the image sets $c(I), \tilde{c}(\tilde{I}) \subseteq X$ coincide.

It is not possible to assign a good notion of velocity vector at a point on a “curve” without the data of a parameterization; think about the physical meaning of this statement (the path of motion does not know how quickly the particle is moving, but the velocity vectors at each time sure do!). This leads to use a very important definition:

Definition 2.1. Let $J \subseteq \mathbf{R}$ be a nontrivial interval and let $c : J \rightarrow X$ be a C^p mapping. For each $t_0 \in J$, the *velocity vector* to c at t_0 is

$$c'(t_0) \stackrel{\text{def}}{=} dc(t_0)(\partial_t|_{t_0}) \in T_{c(t_0)}(X)$$

where $\partial_t|_{t_0} \in T_{t_0}(J)$ is the canonical basis vector (sending a C^p germ f at t_0 to the old-fashioned derivative $f'(t_0)$ in the sense of calculus).

Note that $c'(t_0)$ lies in $T_{c(t_0)}(X)$, which is the right place for it to be (think about it!), and if $c(t_1)$ and $c(t_0)$ are equal to a common point $\xi \in X$ for some $t_1 \neq t_0$ then it could happen that $c'(t_1) \neq c'(t_0)$ in $T_\xi(X)$; that is, if the curve has self-intersection in X then as it passes through the same point ξ several times it may do so with different velocity vectors at each time. The notion of velocity vector is best illustrated with a few examples to convince us that it is the “right” definition.

Example 2.2. Let us first consider the most classical case with X an open set in a sector Σ in $V = \mathbf{R}^n$. In this case, the C^p mapping $c : J \rightarrow X$ is exactly a C^p mapping from J to $V = \mathbf{R}^n$ whose image is set-theoretically contained in X , and so c is described by an ordered n -tuple of C^p functions $c(t) = (c_1(t), \dots, c_n(t))$ with $c_i = x_i \circ c : J \rightarrow \mathbf{R}$ a C^p function for each $1 \leq i \leq n$ (here, x_1, \dots, x_n are the standard coordinate functions on \mathbf{R}^n , considered as C^p functions on X via restriction). By Theorem 1.1 applied to $F = c$ and the standard coordinate t on J and the standard coordinates x_1, \dots, x_n on X , for each $t_0 \in J$ we compute

$$c'(t_0) = dc(t_0)(\partial_t|_{t_0}) = \sum_j \frac{\partial c_j}{\partial t}(t_0) \cdot \frac{\partial}{\partial x_j}|_{c(t_0)} = \sum_j c'_j(t_0) \partial_{x_j}|_{c(t_0)} \in T_{c(t_0)}(X).$$

For any $\xi \in X$ the *canonical* isomorphism

$$\mathbf{R}^n = V \simeq T_\xi(\Sigma) = T_\xi(X)$$

carries the j th standard basis vector e_j to $\partial_{x_j}|_\xi$ (why?), so for $\xi = c(t_0)$ it carries e_j to $\partial_{x_j}|_{c(t_0)}$. Hence, the natural linear isomorphism $T_{c(t_0)}(X) \simeq V = \mathbf{R}^n$ carries $c'(t_0)$ to $\sum_j c'_j(t_0)e_j = (c'_1(t_0), \dots, c'_n(t_0)) \in \mathbf{R}^n$. This thereby recovers the “classical” *ad hoc* method for computing the velocity vector to a parametric curve in (a sector in) Euclidean space via componentwise differentiation, and it helps us to see the fundamental advantage brought in through the manifold perspective: we see that the velocity vector $c'(t_0)$ really belongs to the vector space $T_{c(t_0)}(X)$ that *depends on* $c(t_0)$. This is the modern translation of the classical idea that the velocity vector to a parameterized curve at a point should be “based at” the point. The natural isomorphisms $i_\xi : V \simeq T_\xi(X)$ for $\xi = c(t)$ with varying t provide the dictionary that relates the modern definitions with the classical calculations of all velocity vectors as ordered n -tuples in a *common* Euclidean space $V = \mathbf{R}^n$.

Example 2.3. Let $c : \mathbf{R} = J \rightarrow X = \mathbf{R}^2$ be the map $c(t) = (\cos t, \sin t)$. This is a smooth curve (considering $X = \mathbf{R}^2$ as a smooth manifold in the usual manner), and at any t_0 the velocity vector is

$$c'(t_0) = c'_1(t_0)\partial_x|_{c(t_0)} + c'_2(t_0)\partial_y|_{c(t_0)} = -\sin(t_0)\partial_x|_{c(t_0)} + \cos(t_0)\partial_y|_{c(t_0)}.$$

Since $c(t_0) = (\cos(t_0), \sin(t_0))$ we see that this is $\partial_\theta|_{c(t_0)}$ as it “should” be (a “unit vector” point in the “ θ -direction”)!

In this example, whenever $c(t_0)$ and $c(t_1)$ are equal to a common point $\xi = (a, b)$ on the circle $x^2 + y^2 = 1$ in \mathbf{R}^2 , we have $c'(t_0) = c'(t_1)$ in $T_\xi(\mathbf{R}^2)$ since each is equal to the tangent vector $-b\partial_x|_\xi + a\partial_y|_\xi$ at ξ that is determined by the standard coordinates of ξ . That is, as c retraces itself it always does so with the same velocity vector. However, this is just a quirk of the special parameterization; if we use $c(t) = (\cos(t^3), \sin(t^3))$ then we trace out the same path in the same cyclic pattern but at wildly varying speeds (and when $c(t_0) = c(t_1)$ with $t_0 \neq t_1$ we always have $c'(t_0) \neq c'(t_1)$ in the common tangent space where these velocity vectors live).

Example 2.4. Consider the smooth parameterized curve $c : \mathbf{R} = I \rightarrow X = \mathbf{R}^2$ given by $c(t) = (t^2, t^3)$. This is called a “smooth” parameterized curve because the mapping c is C^∞ , thought note that the image is “bad”: it is the locus $y^2 = x^3$ that has a cuspidal singularity at $(0, 0)$ (and in terminology to be defined later, it is not a smooth submanifold of \mathbf{R}^2 near the point $(0, 0)$). The velocity vector at any t is $c'(t) = 2t\partial_x|_{c(t)} + 3t^2\partial_y|_{c(t)}$. Note that this is always nonzero except for

at precisely the time $t = 0$ which is exactly the time when $c(t)$ is equal to the point $(0, 0)$ that looks problematic on the image of c . It will later be made clear why this is not a coincidence.

Example 2.5. Finally, consider the parameterized smooth curve $c : \mathbf{R} \rightarrow \mathbf{R}^2$ defined by $c(t) = (t^2, t(t^2 - 1))$. This curve lives in the right half-plane $x \geq 0$ and is symmetric about the x -axis. For $t \rightarrow -1^-$ it approaches the point $c(-1) = (1, 0)$ from the lower right, after which it loops around to the origin (at $t = 0$) to then return to the point $c(1) = (1, 0)$ now aiming in the trajectory pointing to the upper right. Explicitly, the velocity vector at any time t is $c'(t) = 2t\partial_x|_{c(t)} + (3t^2 - 1)\partial_y|_{c(t)}$ and a direct calculation shows that $c'(t) \neq 0$ for all t . Note in particular that for $t = \pm 1$ we get different velocity vectors at the common point $c(1) = c(-1) = (1, 0)$:

$$c'(-1) = -2\partial_x|_{(1,0)} + 2\partial_y|_{(1,0)}, \quad c'(1) = 2\partial_x|_{(1,0)} + 2\partial_y|_{(1,0)}$$

in $T_{(1,0)}(\mathbf{R}^2)$. These have the same vertical components but have opposite horizontal components.

Velocity vectors are very well-behaved with respect to C^p mappings:

Lemma 2.6. *Let $F : X' \rightarrow X$ be a C^p mapping between C^p premanifolds with corners, $1 \leq p \leq \infty$, and let $c : J \rightarrow X'$ be a parameterized C^p curve, so $F \circ c : J \rightarrow X$ is a parameterized C^p curve as well. Choose $t_0 \in J$ and let $\xi' = c(t_0)$, $\xi = F(\xi') = (F \circ c)(t_0)$. The linear map $dF(\xi') : T_{\xi'}(X') \rightarrow T_{\xi}(X)$ carries the velocity vector $c'(t_0)$ to the velocity vector $(F \circ c)'(t_0)$.*

Proof. This is just the manifold version of the Chain Rule:

$$(dF(\xi'))((dc(t_0))(\partial_t|_{t_0})) = (dF(c(t_0)) \circ dc(t_0))(\partial_t|_{t_0}) = (d(F \circ c)(t_0))(\partial_t|_{t_0}) = (F \circ c)'(t_0). \quad \blacksquare$$

Having seen a variety of phenomena that are exhibited by velocity vectors to curves, we conclude by linking up our “abstract” notion of tangent vectors with the more geometrically appealing notion of velocity vector to a curve. By definition, if $c : I \rightarrow X$ is a parameterized C^p curve at $\xi \in X$ with X a C^p premanifold with corners ($1 \leq p \leq \infty$) then the velocity vector $c'(0) \in T_{c(0)}(X) = T_{\xi}(X)$ lies in the tangent space to X at ξ . We claim that this construction gives all tangent vectors at ξ (i.e., every tangent vector at ξ is the velocity vector to *some* C^p curve at ξ), at least if ξ is not a point with index > 1 (so this is no restriction if X is a C^p premanifold, or more generally if it is a C^p premanifold with boundary). The idea is to use straight lines in coordinates:

Theorem 2.7. *If $\xi \in X$ has index ≤ 1 then every $\vec{v} \in T_{\xi}(X)$ is the velocity vector $c'(0)$ for some C^p curve $c : I \rightarrow X$ at ξ . If $\xi \notin \partial X$, we can take 0 to be an interior point in I .*

Points with index > 1 present obstacles, as is most easily seen by considering the point $\xi = (0, 0)$ in $X = [0, 1] \times [0, 1]$ and the tangent vector $\vec{v} = -\partial_x|_{\xi} + \partial_y|_{\xi}$ that points along the line $x + y = 0$ that meets X only at ξ . (Briefly put, lines that only touch a sector at a boundary point and nowhere else cannot be captured via C^p curves in the sector for $p \geq 1$; this difficulty never happens at points of index ≤ 1 .)

Proof. The problem is local around ξ in X , and by Lemma 2.6 applied to C^p isomorphisms we see that the problem is unaffected by passing to a C^p -isomorphic pair (X', ξ') . Hence, by shrinking X around ξ and picking a local C^p -chart at ξ we may suppose $X = [0, \varepsilon]^r \times (-\varepsilon, \varepsilon)^{n-r}$ in \mathbf{R}^n with $\xi = 0$ and $r \leq 1$. The vectors $\partial_j|_0$ in $T_{\xi}(X)$ form an ordered basis. In the case $r = 0$, for any $v = \sum a_j \partial_j|_0$ we choose the curve $c(t) = (a_1 t, \dots, a_n t)$ with $|t|$ very small so that c has image inside of X . Clearly $c'(0) = v$. In the case $r = 1$ with $a_1 \geq 0$ we take small $t \geq 0$ and use the same mapping c . If $r = 1$ and $a_1 < 0$ then we use $t \leq 0$ with $|t|$ small. \blacksquare