Math 396. Determinant bundles

1. Preliminaries

If V is a finite-dimensional vector space over a field F, say with dimension $n \geq 0$, the 1-dimensional top exterior power $\wedge^n(V)$ (understood to mean F if $n = 0$) is sometimes called the *determinant* of V, and is denoted $\det(V)$. If $T : V' \to V$ is a linear map between two n-dimensional vector spaces, there is a naturally associated map $\wedge^n(T) : \det(V') \to \det(V)$ (the identity map on F if $n = 0$); in the special case $V' = V$ with $n > 0$, this is scalar multiplication by the old determinant $\det(T) \in F$.

In the special case $F = \mathbf{R}$, it was explained in an earlier handout on orientations how the specification of an orientation on the vector space V amounts to a choice of connected component in $\det(V) - \{0\}$. In particular, in Examples 1.7, 1.8, and 1.9 of that handout we found some natural isomorphisms among determinants, such as

$$\det(V') \otimes \det(V/V') \simeq \det(V)$$

for any subspace $V' \subseteq V$, as well as

$$\det(V/W_1) \otimes \cdots \otimes \det(V/W_N) \simeq \det(V/W)$$

for subspaces $\{W_i\}$ in V that are mutually transverse with $W = \cap W_i$, and also

$$\det(V^\vee) \simeq (\det V)^\vee$$

for any V. (Such isomorphisms were described using elementary wedge products, and were built over an arbitrary coefficient field F.) As a consequence, we saw that if V is oriented then specifying an orientation on either a subspace V' or a quotient V/V' determines a preferred orientation on the other, if V and a collection of mutually transverse subspaces $\{W_i\}$ are oriented then there is a preferred orientation on $W = \cap_i W_i$, and if V is oriented then there is a preferred orientation on $V^\vee$. We wish to generalize these procedures to the setting of vector bundles, especially the tangent bundle, as that will lead to the notion of orientation on manifolds, a very important bit of structure in the theory of integration on manifolds (due to the presence of an absolute value and not just a raw determinant in the Change of Variables formula).

Although we will later address the subject of orientations on vector bundles (and, via the special case of the tangent bundle, orientations on premanifolds with corners), in the present handout we wish to take up the issue of promoting the above "determinant isomorphisms" in linear algebra to the case of vector bundles. Roughly speaking, we wish to see how these isomorphisms vary as we let the vector spaces and subspaces and quotients range across the fibers of vector bundles and subbundles and quotient bundles. The starting point is:

Definition 1.1. Let X be a C^p premanifold with corners, $0 \leq p \leq \infty$, and let $E \to X$ be a C^p vector bundle over X. On each connected component X_i of X, the bundle $E_i = E|_{X_i}$ has some constant rank $r_i \geq 0$ over X_i. The *determinant bundle* $\det(E) \to X$ is the line bundle given by $\wedge^{r_i}(E_i)$ over X_i (understood to mean the trivial line bundle $X_i \times \mathbf{R}$ if $r_i = 0$).

For each $x \in X$ we have the x-fiber description $(\det E)(x) = \det(E(x))$. If $\{s_1, \ldots, s_n\}$ is an ordered trivializing frame for E over an open set $U \subseteq X$, so in particular E has constant rank n over U, then $(\det E)|_U = \wedge^n(E|_U)$ is trivialized by the non-vanishing global section $s_1 \wedge \cdots \wedge s_n$.

Note also that if $f : X' \to X$ is a C^p map between C^p premanifolds with corners, then there is a unique bundle isomorphism $\det(f^*E) \simeq f^*(\det E)$ inducing $\det(E(f(x'))) \simeq (\det E)(f(x'))$ on x'-fibers for all $x' \in X'$. This is a special case of the general compatibility of pullback and tensor operations (such as exterior powers).

In the special case $E = TX$, the determinant bundle $\det(TX)$ is often called the *orientation bundle* of X; this line bundle is closely related to the theory of orientation on manifolds, as we shall discuss later. For example, the triviality of the orientation bundle will be a necessary and sufficient criterion for X to be "orientable" (which the Möbius strip is not, roughly because of its "one-sided" nature).

2. Some bundle isomorphisms

The process of making bundle analogues of linear-algebra isomorphisms on top exterior powers is largely a matter of computations with local trivializations. In what follows, we take X to be a C^p premanifold with corners, $0 \le p \le \infty$.

Theorem 2.1. *Let $E \to X$ be a C^p vector bundle, and E' a C^p subbundle of E with quotient E/E'. There is a unique C^p vector bundle isomorphism $\det(E') \otimes \det(E/E') \simeq \det(E)$ that recovers the linear-algebra isomorphism $\det(E'(x)) \otimes \det(E(x)/E'(x)) \simeq \det(E(x))$ on x-fibers.*

Also, there is a unique C^p vector bundle isomorphism $\det(E^\vee) \simeq (\det E)^\vee$ that recovers the linear-algebra isomorphism $\det(E(x)^\vee) \simeq (\det E(x))^\vee$ on x-fibers.

As usual, the meaning of such a theorem is simply that the fibral isomorphisms from linear algebra are given by "universal formulas" in terms of a trivializing frame. A case of much geometric interest is when X is a smooth submanifold of a smooth manifold M, say with inclusion $j : X \to M$, and $E' = TX$ and $E = j^*(TM)$. In this case the quotient bundle $E/E' = j^*(TM)/TX$ is what we have defined to be the *normal bundle* $\mathrm{N}_{X/M}$ in an earlier handout, where we saw (in the presence of a suitably nice family of inner products along the tangent spaces) that $\mathrm{N}_{X/M}$ encodes local information about the geometry of X in M. In language to be introduced later in the course, the first isomorphism of the theorem implies that if M is an oriented manifold then to give an orientation on X is the same as to give an orientation of the normal bundle $\mathrm{N}_{X/M}$.

Proof. We may and do ignore connected components of X over which any of the intervening vector bundles have rank 0 (as the problem is trivial there). Uniqueness of the isomorphism is clear, as we are specifying the maps on fibers. Moreover, since the specified maps on fibers are linear isomorphisms, once we make bundle mappings inducing these on fibers it is automatic that these are isomorphisms. The fibral definitions define bijective maps of sets

$$\det(E') \otimes \det(E/E') \to \det(E), \ \ \det(E^\vee) \simeq (\det E)^\vee$$

respecting projections to X and giving linear maps (even isomorphisms) on fibers over each $x \in X$. Hence, to check that these maps are C^p it suffices to compute in terms of local C^p trivializations. Since a local frame on a subbundle locally extends to a local frame on an ambient vector bundle, we may cover X by open sets U such that $E'|_U$ and $E|_U$ are trivialized by frames $\{s_1, \dots, s_r\}$ and $\{s_1, \dots, s_r, \dots, s_n\}$ respectively, so $\{s_{r+1}, \dots, s_n\}$ in $E(U)$ lifts a trivialization of $(E/E')|_U$. For $r+1 \le j \le n$, let $\overline{s}_j \in (E/E')(U)$ be the image of $s_j \in E(U)$ under the bundle surjection $E \to E/E'$ over X.

The bundles $\det(E')|_U = \det(E'|_U)$ and $\det(E/E')|_U = \det((E/E')|_U)$ are trivialized by the U-sections $s_1 \wedge \dots \wedge s_r$ and $\overline{s}_{r+1} \wedge \dots \wedge \overline{s}_n$ respectively. Also, $\det(E)|_U = \det(E|_U)$ is trivialized by the U-section $s_1 \wedge \dots \wedge s_n$. By *definition*, since $s_j(x) \in E(x)$ lifts $\overline{s}_j(x) \in (E/E')(x) = E(x)/E'(x)$ for $r+1 \le j \le n$, the fibral isomorphism $\det(E'(x)) \otimes \det(E(x)/E'(x)) \simeq \det(E(x))$ from linear algebra satisfies

$$(s_1(x) \wedge \dots \wedge s_r(x)) \otimes (\overline{s}_{r+1}(x) \wedge \dots \wedge \overline{s}_n(x)) \mapsto s_1(x) \wedge \dots s_n(x)$$

for all $x \in X$. Hence, the set-theoretic mapping $\det(E') \otimes \det(E/E') \to \det(E)$ satisfies

$$(s_1 \wedge \cdots \wedge s_r) \otimes (\overline{s}_{r+1} \wedge \cdots \wedge \overline{s}_n) \mapsto s_1 \wedge \cdots \wedge s_n$$

as we can check on x-fibers for all $x \in X$. Hence, this fiberwise-linear map takes a trivializing frame to a trivializing frame, so it is a C^p mapping (and even an isomorphism).

The case of dual bundles goes similarly, as follows. The dual sections $s_1^\vee, \dots, s_n^\vee \in E^\vee(U)$ (giving dual basis to $\{s_i(u)\}$ on u-fibers for all $u \in U$) are a trivializing frame for $E^\vee|_U$, so the fiberwise-linear mapping $\det(E^\vee)|_U \to (\det E)^\vee|_U$ carries the trivializing section $s_1^\vee \wedge \cdots \wedge s_n^\vee$ of the line bundle $\det(E^\vee)|_U$ to the trivializing section of $(\det E)^\vee|_U$ dual to the trivializing section $s_1 \wedge \cdots \wedge s_n$ of $(\det E)|_U$. Indeed, this assertion may be checked on fibers over U, where it follows from how the linear algebra isomorphism $\det(E(u)^\vee) \simeq (\det E(u))^\vee$ is *defined*. ■

3. Transversal subbundles

In order to generalize the linear algebra isomorphism

$$\det(V/W_1) \otimes \cdots \otimes \det(V/W_N) \simeq \det(V/W)$$

for mutually transverse subspaces $\{W_i\}$ in V with $W = \cap W_i$, we first need to define the notion of "transversality" for subbundles of a vector bundle. This will be a natural generalization of the notion in linear algebra, but we first consider a motivating example: tangent bundles to mutually transverse submanifolds of a manifold.

Example 3.1. Let $Z_1, \dots, Z_N$ be mutually transverse C^p embedded subpremanifolds in a C^p premanifold M, with $f_i : Z_i \to M$ the C^p embedding. Recall what such mutual transversality means: for all $z \in \cap Z_j$ the tangent spaces $\mathrm{T}_z(Z_j)$ in $\mathrm{T}_z(M)$ are mutually transverse. In such cases, it was proved in the handout on submersions and transverse intersections that $Z = \cap Z_j$ is a C^p embedded subpremanifold in X with $\mathrm{T}_z(Z) = \cap_j \mathrm{T}_z(Z_j)$ for all $z \in Z$.

Let $j : Z \to M$ and $j_i : Z \to Z_i$ be the C^p embeddings, so $f_i \circ j_i = j$ for all i. The embedding f_i identifies TZ_i with a subbundle of $f_i^*(TZ)$, and so by applying j_i^* to the subbundle inclusion $TZ_i \to f_i^*(TZ)$ over Z_i we get a subbundle inclusion $j_i^*(TZ_i) \to j_i^*(f_i^*(TX)) = j^*(TM)$ over Z. Hence, the subbundle TZ in $j^*(TM)$ is a subbundle of each of the subbundles $j_i^*(TZ_i)$ in $j^*(TM)$. (On fibers over $z \in Z$, this mouthful just says that the subspace $\mathrm{T}_z(Z)$ in $\mathrm{T}_z(M)$ is contained in each of the subspaces $\mathrm{T}_z(Z_i)$.)

On Z, consider the C^p vector bundles $E = j^*(TM)$ and $E_i = j_i^*(TZ_i)$ for all i. The subbundle TZ in E is contained in each of the E_i's, and on fibers over each $z \in Z$ we have that inside of the z-fiber $E(z) = \mathrm{T}_z(M)$ the fiber $\mathrm{T}_z(Z)$ of TZ is equal to the intersection $\cap E_i(z)$ of the mutually transverse subspaces $E_i(z)$. In other words, in the case of a mutually transverse collection of submanifolds of a manifold, along the intersection of the submanifolds the tangent spaces to the given submanifolds form a mutually transverse collection of subspaces of the tangent space to the ambient manifold. We have just seen that this aspect of mutual transversality can be expressed in the language of fibers of the subbundles E_i in the vector bundle E over Z: the $E_i(z)$'s are mutually transverse in $E(z)$, and their common intersection is $\mathrm{T}_z(Z)$.

Definition 3.2. A collection of C^p subbundles $\{E_i\}$ in a C^p vector bundle E over X is *mutually transverse* if the subspaces $E_i(x)$ in $E(x)$ are mutually transverse (in the sense of linear algebra) for all $x \in X$.

The preceding example shows that along the intersection Z of a collection of mutually transverse embedded submanifolds Z_i of a manifold M, the pullbacks of the tangent bundles of the Z_i's are mutually transverse subbundles of the pullback (to Z) of the tangent bundle of M. Thus, many concepts of interest for transverse intersections of submanifolds of a manifold can be defined more generally in the setting of mutually transverse subbundles of a vector bundle (and then be applied to pullbacks of tangent bundles). One pleasant property of mutually transverse subbundles is that their fiberwise intersections form a subbundle too:

Theorem 3.3. *Let $E_1, \dots, E_N$ be mutually transverse subbundles of a C^p vector bundle $E \to X$. The $\mathscr{O}$-module $U \mapsto E_1(U) \cap \cdots \cap E_N(U) \subseteq E(U)$ is locally free of finite rank, and its associated C^p vector bundle is a subbundle of E whose x-fiber is $\cap E_i(x)$ for all $x \in X$. This is the unique subbundle of E whose x-fiber is $\cap E_i(x)$ for all $x \in X$.*

We call the subbundle in this lemma the (tranverse) *intersection* of the E_i's inside of E, and it is denoted $\cap E_i$. The most important case is $E_i = j_i^*(TZ_i)$ and $E = j^*(TM)$ for mutually transverse embedded submanifolds $Z_i \hookrightarrow M$ in a manifold M, with $Z = \cap Z_i$ and $j_i : Z \to Z_i$ and $j : Z \to M$ the embeddings. In this case, the subbundle TZ in $j^*(TX)$ has z-fiber $\mathrm{T}_z(Z) = \cap \mathrm{T}_z(Z_i) = \cap E_i(z)$ for all $z \in Z$, so it is the intersection of the E_i's inside of E.

Proof. The uniqueness is clear, since we are specifying the fibers of the subbundle inside of E. (See Lemma 2.1 in the handout on subbundles and quotient bundles.) We need to prove that $U \mapsto \cap E_i(U)$ is a locally free $\mathscr{O}$-module of finite rank, and that the associated C^p vector bundle is a subbundle of E with x-fiber $\cap E_i(x)$ for all $x \in X$. This can be done "by hand" (using induction to reduce to the case $N = 2$, and exerting some efforts for this case), but it is cleaner to use our earlier work to avoid reinventing the wheel, as follows. Consider the bundle mapping

$$f : E \to (E/E_1) \oplus \cdots \oplus (E/E_N)$$

over X. On x-fibers this is the mapping $E(x) \to (E(x)/E_1(x)) \oplus \cdots \oplus (E(x)/E_N(x))$ whose kernel is $\cap E_i(x)$, and by transversality it follows that $\mathrm{codim}(\cap E_i(x)) = \sum \mathrm{codim}(E_i(x))$ (all codimensions in $E(x)$), so

$$\dim \ker(f|_x) = \dim(\cap E_i(x)) = \sum \dim E_i(x) - (N-1) \dim E(x)$$

is locally constant in x since the fiberwise rank of a vector bundle (such as the E_i's and E) is locally constant on the base space.

Now comes the key: local constancy of the dimension of the fiberwise kernel allows us to use Theorem 2.6 in the handout on subbundles and quotient bundles to infer that

$$U \mapsto \ker(\underline{f}_U : E(U) \to (\oplus(E/E_i))(U) = \oplus(E/E_i)(U))$$

is a locally free $\mathscr{O}$-module of finite rank whose associated vector bundle is a subbundle of E with x-fiber $\ker(f|_x) = \cap E_i(x)$ for all $x \in X$. It remains to explain why the kernel of $\underline{f}_U$ is $\cap E_i(U)$ for all open $U \subseteq X$. It suffices to show that $E(U) \to (E/E_i)(U)$ has kernel $E_i(U)$ for all open U in X. That is, the kernel subbundle of the bundle surjection $E \to E/E_i$ should be E_i, and this follows from the definition of the quotient bundle E/E_i. ∎

With the "intersection" of mutually transverse subbundles now proved to be a good notion (both on the level of fibers and sections over opens in the base), it is reasonable to contemplate the formation of a line-bundle isomorphism $\det(E/E_1) \otimes \cdots \otimes \det(E/E_N) \simeq \det(E/E')$ for mutually transverse subbundles E_i in E with $E' = \cap E_i$.

Theorem 3.4. *Let $\{E_i\}$ be mutually transverse subbundles of a vector bundle E over X. Let $E' = \cap E_i$ be the intersection subbundle. There is a unique bundle isomorphism*

$$\det(E/E_1) \otimes \cdots \otimes \det(E/E_N) \simeq \det(E/E')$$

over X such that for each $x \in X$ on x-fibers it is the isomorphism

$$\det(E(x)/E_1(x)) \otimes \cdots \otimes \det(E(x)/E_N(x)) \simeq \det(E(x)/E'(x))$$

from linear algebra, with $E'(x) = \cap E_i(x)$ the intersection of the mutually transverse subspaces $E_i(x)$ in $E(x)$.

Proof. The uniqueness is immediate, since we are specifying the map on fibers. As with the other determinant-bundle isomorphisms built in this handout (recovering ones from linear algebra on fibers), the only real issue is to calculate that the set-theoretic map given by the specified recipe on fibers of these line bundles carries a trivializing section to a trivializing section over a collection of opens that cover X.

Consider opens sets U in X over which the vector bundles E/E_i are trivial. Let c_i be the constant rank of E/E_i over U. Let $c = \sum c_i$ denote the constant codimension of $\cap E_i(u)$ in $E(u)$ for all $u \in U$. Shrinking some more, we may suppose that choices of trivializing frames $\{\overline{s}_{i1}, \dots, \overline{s}_{i,c_i}\}$ on each $(E/E_i)|_U$ lift to U-sections $\{s_{i1}, \dots, s_{i,c_i}\}$ of E/E'. We fix a choice of such an open U; these opens cover X. By passing to fibers over each $u \in U$ we infer from the work involved in proving Theorem 2.6 in the handout on tensor algebras that the sections $s_{ij} \in (E/E')(U)$ are a trivializing frame for E/E' over U. By using the definiton of the fibral isomorphism, our set-theoretic map between the line bundles over X satisfies

$$(\overline{s}_{11} \wedge \dots \wedge \overline{s}_{1,c_1}) \otimes \dots \otimes (\overline{s}_{N,1} \wedge \dots \wedge \overline{s}_{N,c_N}) \mapsto s_{11} \wedge \dots \wedge s_{1,c_1} \wedge \dots \wedge s_{N,1} \wedge \dots \wedge s_{N,c_N}$$

on U-sections (as we may check on fibers), so it carries a trivializing U-section to a trivializing U-section. Thus, our set-theoretic map is an isomorphism of C^p line bundles over X. ■

The most important example of Theorem 3.4 arises when $Z_1, \dots, Z_N$ are mutually transverse smooth embedded submanifolds of a smooth manifold M, and $X = \cap Z_i$. Letting E be the pullback of TM to X, and E_i the pullback of TZ_i to X, the E_i's are mutually transverse subbundles of E with intersection $\cap E_i = TX$ inside of E. The quotient bundles in Theorem 3.4 in this case are determinants of normal bundles: $\det \mathrm{N}_{X/M} = E/E'$ and $j_i^*(\det \mathrm{N}_{X/Z_i}) = E/E_i$ for the embedding $j_i : X \to Z_i$. Hence, the theorem gives a natural isomorphism

$$j_1^*(\det \mathrm{N}_{X/Z_1}) \otimes \dots \otimes j_N^*(\det \mathrm{N}_{X/Z_N}) \simeq \det \mathrm{N}_{X/M}$$

over X. In language to be used later, this implies that if M is oriented and the Z_i's are oriented then the transverse intersection $X = \cap Z_i$ inherits a canonical induced orientation.