Math 396. Coordinate-free meaning for d

1. Motivation

Let $(X, \mathscr{O})$ be a smooth premanifold with corners. We have given a general definition of a family operators $\mathrm{d}_U^k : \Omega_X^k(U) \to \Omega_X^{k+1}(U)$ for open $U \subseteq X$ and $k \geq 0$ uniquely characterized by the conditions that the d_U^k's are compatible with shrinking on U (i.e., for $U' \subseteq U$ and $\omega \in \Omega_X^k(U)$, $\mathrm{d}_U^k(\omega)|_{U'} = \mathrm{d}_{U'}^k(\omega|_{U'})$ in $\Omega_X^{k+1}(U')$) and satisfy the two identities $\mathrm{d}_U^{k+1} \circ \mathrm{d}_U^k = 0$ (abbreviated to "$\mathrm{d}^2 = 0$") and the *Leibnitz Rule*

$$\mathrm{d}_U^{k+\ell}(\omega \wedge \eta) = \mathrm{d}_U^k(\omega) \wedge \eta + (-1)^k \omega \wedge \mathrm{d}_U^\ell(\eta)$$

in $\Omega_X^{k+\ell+1}(U)$ for $\omega \in \Omega_X^k(U)$ and $\eta \in \Omega_X^\ell(U)$. The actual construction was given locally in terms of local C^∞ coordinates, and the above properties were used to get independence of local coordinates and agreement on overlaps, whence "globalization" over opens not admitting coordinates.

The above conditions are certainly adequate for all computations. But it is natural to ask for something more, as we now explain. An element $\omega \in \Omega_X^k(U) = (\wedge^k(T^*M))(U) = (\wedge^k(TM))^\vee(U)$ has fiber-value

$$\omega(u) \in (\wedge^k(\mathrm{T}_u(M)))^\vee = \mathrm{Hom}_{\mathbf{R}}(\wedge^k(\mathrm{T}_u(M)), \mathbf{R}) = \mathrm{Alt}^k(\mathrm{T}_u(M), \mathbf{R})$$

at $u \in U$ that is an alternating multilinear functional on $\mathrm{T}_u(M)^{\times k}$. More specifically, if $\vec{v}_1, \dots, \vec{v}_k$ are smooth vector fields on U then we get a function

$$\omega(\vec{v}_1, \dots, \vec{v}_k) : u \mapsto \omega(u)(\vec{v}_1(u), \dots, \vec{v}_k(u)).$$

If there are smooth coordinates $\{x_1, \dots, x_n\}$ on U and $\omega = \sum_{i_1 < \cdots < i_k} a_{i_1,\dots,i_k} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$ and $\vec{v}_q = \sum_p h_{pq} \partial_{x_p}$ then

$$\omega(\vec{v}_1, \dots, \vec{v}_k) = \sum_{i_1 < \cdots < i_k} a_{i_1,\dots,i_k} \det(h_{i_r,s}),$$

so this is a smooth function. Hence, it is natural to ask if we can describe the smooth function $(\mathrm{d}_U^k\omega)(\vec{v}_1, \dots, \vec{v}_{k+1})$ on U in terms of the corresponding smooth functions obtained from evaluating ω on ordered k-tuples of smooth vector fields on U. This would provide an alternative way of describing $\mathrm{d}_U^k\omega$ without requiring the crutch of coordinates. This is not meant to denigrate the explicit formula for d_U^k when U is small enough to admit local coordinates, but nonetheless it would be nice if we can use the viewpoint of multilinear functionals on vector fields to describe $\mathrm{d}_U^k\omega$ without having to use explicit local coordinates (e.g., U so large that it does not admit smooth coordinates).

One point we should emphasize at the outset is that we *cannot* expect to compute the fiber $(\mathrm{d}^k\omega)(u) \in \wedge^{k+1}(\mathrm{T}_u(M)^\vee)$ in terms of $\omega(u) \in \wedge^k(\mathrm{T}_u(M)^\vee)$. Indeed, d_U^k is a kind of derivative and for $k = 0$ we know from calculus that one cannot compute the pointwise value of a partial derivative of a function just from the knowledge of the value of the function at the point. Hence, we have to expect to use data over open sets and not data of values at a single point.

The formula we shall give is also discussed in Theorem 13 in Chapter 7 in the course text. We leave it to the reader to decide if our discussion is preferable to the one in the text.

2. The main formula

We shall now adopt the usual abuse of notation and write $\mathrm{d}\omega$ rather than $\mathrm{d}_U^k\omega$. Here is the main result:

Theorem 2.1. *For $\omega \in \Omega^k_X(U)$ and smooth vector fields $\vec{v}_1, \dots, \vec{v}_{k+1} \in \mathrm{Vec}_X(U)$, $(\mathrm{d}\omega)(\vec{v}_1, \dots, \vec{v}_{k+1})$ is equal to*

$$\sum_{i=1}^{k+1} (-1)^{i+1} \vec{v}_i(\omega(\vec{v}_1, \dots \widehat{\vec{v}_i}, \dots, \vec{v}_{k+1})) + \sum_{i<j} (-1)^{i+j} \omega([\vec{v}_i, \vec{v}_j], \vec{v}_1, \dots, \widehat{\vec{v}_i}, \dots, \widehat{\vec{v}_j}, \dots, \vec{v}_{k+1}), \tag{1}$$

where $\widehat{\cdot}$ means omission of that term and for any $f \in C^\infty(U)$ and $\vec{v} \in \mathrm{Vec}_X(U)$ we write $\vec{v}(f) \in C^\infty(U)$ to denote the smooth function $u \mapsto \vec{v}(u)(f) \in \mathbf{R}$ given in local coordinates by $\sum h_i \partial_{x_i} f$ if $\vec{v} = \sum h_i \partial_{x_i}$.

Remark 2.2. If we evaluate at $m \in U$, then the formula says that for any $v_1, \dots, v_{k+1} \in \mathrm{T}_m(M)$ and any choice of local smooth vector fields $\vec{v}_i$ near m with $\vec{v}_i(m) = v_i$, the value of (1) at m is the number $((\mathrm{d}\omega)(\vec{v}_1, \dots, \vec{v}_{k+1}))(m) = (\mathrm{d}\omega)(m)(v_1, \dots, v_{k+1})$ that is *independent* of the choices of local extensions $\vec{v}_1, \dots, \vec{v}_{k+1}$. Note that $[\vec{v}_i, \vec{v}_j](m)$ depends very much on $\vec{v}_i$ and $\vec{v}_j$ near m and not just on the values $\vec{v}_i(m) = v_i$ and $\vec{v}_j(m) = v_j$ at m, and even the functions in the first sum in (1) depend on the $\vec{v}_i$'s and ω near m (not just at m), whereas the terms in the second sum in (1) only depends on ω through $\omega(m)$. Hence, we see that the formula is consistent with the fact that $(\mathrm{d}\omega)(m)$ depends on ω near m and not merely at m, whereas it is not at all obvious that the value of (1) at m (in contrast with that of each of its parts) only depends on the $\vec{v}_i$'s through the values $\vec{v}_i(m) = v_i$ at m. This is a magical cancellation in the dependences on the extensions $\vec{v}_i$.

Proof. Both sides of the proposed identity are compatible with shrinking on U. We shall next prove that (1) is multilinear over $C^\infty(U)$, and this (together with localizing on X) will reduce the problem to a calculation in a special case. The additivity in each $\vec{v}_i$ (with all others held fixed) is clear for (1), and so for multilinearity it remains to check that if $h \in C^\infty(U)$ then replacing $\vec{v}_{i_0}$ with $h\vec{v}_{i_0}$ in (1) gives output that is h times (1). This is a direct calculation, as follows.

Let us insert $h\vec{v}_{i_0}$ in the role of $\vec{v}_{i_0}$ in (1). By the Leibnitz Rule, the first sum becomes

$$\sum_{i \neq i_0} (-1)^{i+1} \vec{v}_i(h \cdot \omega(\vec{v}_1, \dots, \widehat{\vec{v}_i}, \dots, \vec{v}_{k+1})) + (-1)^{i_0+1} h \cdot \vec{v}_{i_0}(\omega(\vec{v}_1, \dots, \widehat{\vec{v}_{i_0}}, \dots, \vec{v}_{k+1})),$$

and the second sum is

$$\begin{aligned} h \cdot \sum_{i<j; i,j \neq i_0} & (-1)^{i+j} \omega([\vec{v}_i, \vec{v}_j], \vec{v}_1, \dots, \widehat{\vec{v}_i}, \dots, \widehat{\vec{v}_j}, \dots, \vec{v}_{k+1}) \\ & + \sum_{i<i_0} (-1)^{i+i_0} \omega([\vec{v}_i, h\vec{v}_{i_0}], \vec{v}_1, \dots, \widehat{\vec{v}_i}, \dots, \widehat{\vec{v}_{i_0}}, \dots, \vec{v}_{k+1}) \\ & + \sum_{i>i_0} (-1)^{i+i_0} \omega([h\vec{v}_{i_0}, \vec{v}_i], \vec{v}_1, \dots, \widehat{\vec{v}_{i_0}}, \dots, \widehat{\vec{v}_i}, \dots, \vec{v}_{k+1}) \end{aligned}$$

By the Leibnitz Rule, $[\vec{v}_i, h\vec{v}_{i_0}] = h[\vec{v}_i, \vec{v}_{i_0}] + \vec{v}_i(h) \cdot \vec{v}_{i_0}$ and $[h\vec{v}_{i_0}, \vec{v}_i] = h[\vec{v}_{i_0}, \vec{v}_i] - \vec{v}_i(h) \cdot \vec{v}_{i_0}$.

Letting $\varepsilon_{i,i_0} = 1$ for $i < i_0$ and $\varepsilon_{i,i_0} = -1$ for $i > i_0$, we add both parts to get the following formula for (1) when $h\vec{v}_{i_0}$ is inserted in the role of $\vec{v}_{i_0}$:

$$\begin{aligned} h \cdot \sum_i (-1)^{i+1} \vec{v}_i(\omega(\vec{v}_1, \dots, \widehat{\vec{v}_i}, \dots, \vec{v}_{k+1})) \\ + \sum_{i \neq i_0} (-1)^{i_0+1} \vec{v}_i(h) \cdot \omega(\vec{v}_1, \dots, \widehat{\vec{v}_i}, \dots, \vec{v}_{k+1}) \\ + h \cdot \sum_{i<j} (-1)^{i+j} \omega([\vec{v}_i, \vec{v}_j], \vec{v}_1, \dots, \widehat{\vec{v}_i}, \dots, \widehat{\vec{v}_j}, \dots, \vec{v}_{k+1}) \\ + \sum_{i \neq i_0} \varepsilon_{i,i_0} (-1)^{i+i_0} (\vec{v}_i(h)) \omega(\vec{v}_{i_0}, \vec{v}_1, \dots, \widehat{\vec{v}_i}, \dots, \widehat{\vec{v}_{i_0}}, \dots, \vec{v}_{k+1}). \end{aligned}$$

We have abused notation in the indication of the deleted terms: if $i > i_0$ then the notations $\widehat{\vec{v}_i}$ and $\widehat{\vec{v}_{i_0}}$ in the final sum should be swapped.

In this final big sum of four sums, the first and third sums add to give h times (1). Thus, we want the second and fourth terms to add to 0. In fact, we fix $i \neq i_0$ and show that the ith terms in the two sums add to 0. Factoring out $(-1)^{i+1}\vec{v}_i(h)$ leaves us with

$$\omega(\vec{v}_1, \dots, \widehat{\vec{v}_i}, \dots, \vec{v}_{k+1}) - \varepsilon_{i,i_0}(-1)^{i_0} \omega(\vec{v}_{i_0}, \vec{v}_1, \dots, \widehat{\vec{v}_i}, \dots, \widehat{\vec{v}_{i_0}}, \dots, \vec{v}_{k+1}), \tag{2}$$

where it must be understood that if $i > i_0$ then the notations $\widehat{\vec{v}_i}$ and $\widehat{\vec{v}_{i_0}}$ in the final term should be swapped. Let us analyze the term being subtracted in (2), and more specifically the placement of the entry $\vec{v}_{i_0}$. In the case $i > i_0$, we can move this initial entry past the next $i_0 - 1$ entries $\vec{v}_1, \dots, \vec{v}_{i_0-1}$ to arrive at the initial term in (2) at the expense of introducing a sign of $(-1)^{i_0-1}$ due to the alternating nature of ω at every point of U. But $\varepsilon_{i,i_0}(-1)^{i_0}(-1)^{i_0-1} = 1$ for $i > i_0$, so we get the desired cancellation for $i > i_0$. If instead $i < i_0$ then to move $\vec{v}_{i_0}$ "into position" (as in the first term in (2)) we only move it past $i_0 - 2$ terms because the term $\vec{v}_i$ with $i < i_0$ has been deleted! Hence, in the calculation for $i < i_0$ the relevant sign calculation is $\varepsilon_{i,i_0}(-1)^{i_0}(-1)^{i_0-2} = 1$ for $i < i_0$. We again get the required cancellation. This completes the proof that the right side of (1) is $C^\infty(U)$-multilinear in the $\vec{v}_i$'s.

Now we are ready to prove (1) holds. Since (1) is a proposed identity among smooth functions on U, to verify it we may work locally on U. Both sides behave well with respect to shrinking on U (i.e., the restriction to an open subset $U' \subseteq U$ is given by replacing ω and the $\vec{v}_i$'s with their restrictions to U'; check!), so since U is covered by coordinate charts we may suppose U has C^∞ coordinates $\{x_1, \dots, x_n\}$. Thus, each $\vec{v}_j$ is a $C^\infty(U)$-linear combination of the ∂_{x_i}'s. But both sides of (1) are $C^\infty(U)$-multilinear in the $\vec{v}_j$'s! Hence, both expand out the same way when we insert the $C^\infty(U)$-linear expressions for the $\vec{v}_j$'s in terms of the ∂_{x_i}'s, so it suffices to treat the case when each $\vec{v}_j$ is equal to some ∂_{x_i}. This simplifies the nature of the input vector fields.

Next, we simplify the nature of ω. We can write

$$\omega = \sum_I a_I \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$$

for $a_I \in C^\infty(U)$, and both sides of (1) behave the same way with respect to addition in ω. Hence, it suffices to treat the case $\omega = h\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$ for some $h \in C^\infty(U)$ and $i_1 < \cdots < i_k$. We have $\omega = h\omega_0$ with ω_0 a wedge product of some $\mathrm{d}x_i$'s, so to reduce to treating such ω_0's (i.e., the case $h = 1$) we must now check that both sides of (1) have the same behavior upon replacing ω with $h\omega$ (i.e., if the formula holds for some ω, then it holds for $h\omega$ for any $h \in C^\infty(U)$). Since

$\mathrm{d}(h\omega) = \mathrm{d}h \wedge \omega + h\mathrm{d}\omega$ and $\vec{v}_i(hg) = h\vec{v}_i(g) + g\vec{v}_i(h)$, the good behavior of (1) with respect to multiplication by h comes down to checking

$$(\mathrm{d}h \wedge \omega)(\vec{v}_1, \ldots, \vec{v}_{k+1}) \stackrel{?}{=} \sum_i (-1)^{i+1} \vec{v}_i(h) \cdot \omega(\vec{v}_1, \ldots, \widehat{\vec{v}_i}, \ldots, \vec{v}_{k+1})$$

in $C^\infty(U)$. What does such an equality say at each point $u \in U$? Note that $(\vec{v}_i(h))(u) = \vec{v}_i(u)(h) = (\mathrm{d}h(u))(\vec{v}_i(u))$ by the *definition* of the *linear functional* $\mathrm{d}h(u)$ on $\mathrm{T}_u(X)$. (It is the coefficient of $\partial_t|_{h(u)}$ for the tangent mapping $\mathrm{d}h(u) : \mathrm{T}_u(X) \to \mathrm{T}_{h(u)}(\mathbf{R}) = \mathbf{R} \cdot \partial_t|_{h(u)}$.) Letting $V = \mathrm{T}_u(X)$, $v_i = \vec{v}_i(u) \in V$, $\ell = \mathrm{d}h(u) \in \mathrm{T}_u(X)^\vee$, and $\psi = \omega(u) \in \wedge^k(\mathrm{T}_u(X)^\vee) \simeq (\wedge^k \mathrm{T}_u(X))^\vee$ viewed as an alternating k-multilinear functional on $\mathrm{T}_u(X) = V$, the required identity of values at u becomes

$$(\ell \wedge \psi)(v_1, \ldots, v_{k+1}) \stackrel{?}{=} \sum_i (-1)^{i+1} \ell(v_i)\psi(v_1, \ldots, \widehat{v_i}, \ldots, v_{k+1})$$

in $\mathbf{R}$ for $v_1, \ldots, v_{k+1} \in V$. This is really an identity in linear algebra:

Lemma 2.3. *Let V be a finite-dimensional vector space over a field F with dimension $n \geq 2$, and choose $1 \leq k < \dim V$. For any $v_1, \ldots, v_{k+1} \in V$, $\ell \in V^\vee$, and $\psi \in \wedge^k(V^\vee) \simeq (\wedge^k V)^\vee$, the alternating $(k+1)$-multilinear functional $\ell \wedge \psi \in \wedge^{k+1}(V^\vee) \simeq (\wedge^{k+1} V)^\vee$ on V is given by*

$$(\ell \wedge \psi)(v_1, \ldots, v_{k+1}) = \sum_i (-1)^{i+1} \ell(v_i)\psi(v_1, \ldots, \widehat{v_i}, \ldots, v_{k+1}) \in F.$$

Proof. This will ultimately come down to the formula for computing a determinant by expanding along the first row of a matrix. Since ψ is a finite F-linear combination of elementary wedge products (of elements of $V^\vee$) and both sides of the proposed identity are linear in ψ, it suffices to treat the case when ψ is an elementary wedge product. We next have to recall how the duality between $\wedge^r V$ and $\wedge^r V^\vee$ was *defined* for any $1 \leq r \leq \dim V$: it corresponds to the unique bilinear pairing

$$\langle \cdot, \cdot \rangle : \wedge^r V \times \wedge^r (V^\vee) \to F$$

satisfying

$$\langle v_1 \wedge \cdots \wedge v_r, \ell_1 \wedge \cdots \wedge \ell_r \rangle = \det(\ell_i(v_j)) = \det(\ell_j(v_i)).$$

Hence, if $\psi = \ell_2 \wedge \cdots \wedge \ell_{k+1}$ with $\ell_j \in V^\vee$ and we define $\ell_1 = \ell$ then

$$(\ell \wedge \psi)(v_1, \ldots, v_{k+1}) = \det(\ell_i(v_j)),$$

and expanding along the first row gives

$$\det(\ell_i(v_j)) = \sum_i (-1)^{i+1} \ell_1(v_i) \det(\ell_r(v_s))_{r \neq 1, s \neq i},$$

with the determinant in the ith term of the sum equal to $\psi(v_1, \ldots, \widehat{v_i}, \ldots, v_{k+1})$. ∎

Resuming the proof that (1) computes $\mathrm{d}\omega$, we have reduced ourselves to the following special case: U admits C^∞ coordinates $x_1, \ldots, x_n$, each vector field $\vec{v}_j$ is one of the ∂_{x_i}'s, and ω is a wedge product of k of the $\mathrm{d}x_i$'s. In this special case, we have $\mathrm{d}\omega = 0$ in $\Omega_X^{k+1}(U)$ and so we need to prove that the right side of (1) vanishes on U. All commutators $[\vec{v}_i, \vec{v}_j]$ vanish since the operators ∂_{x_r} and ∂_{x_s} on $C^\infty(U)$ commute for any r and s. Hence, the first sum in (1) has all terms vanishing. All terms in the second sum in (1) also vanish, since for our special ω's and $\vec{v}_i$'s each smooth function $\omega(\vec{v}_1, \ldots, \widehat{\vec{v}_i}, \ldots, \vec{v}_{k+1})$ is constant (either 0 or ± 1) and so is killed by the first-order differential operator $\vec{v}_i$ (equal to some ∂_{x_r}).

∎