Math 396. Local and global Frobenius theorems

1. Motivation

Let M be a smooth manifold, and E an integrable subbundle of TM. A particularly interesting example is the following. Let $M = G$ be a Lie group and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$. In this case, we have a bundle trivialization $G \times \mathfrak{g} \simeq TG$ given by the construction of left-invariant vector fields. This is the C^∞ bundle map $(g, v) \mapsto (\mathrm{d}\lambda_g(e))(v)$ that is an isomorphism on fibers (the smoothness was shown in the homework), and in this way $G \times \mathfrak{h}$ inside of $G \times \mathfrak{g}$ goes over to a subbundle $\widetilde{\mathfrak{h}}$ in TG given by propogating elements of $\mathfrak{h}$ by left translation. That is, the C^∞ bundle mapping $G \times \mathfrak{h} \to TG$ defined by $(g, v) \mapsto (\mathrm{d}\lambda_g(e))(v)$ is fiberwise injective (it puts $\mathfrak{h}$ inside of $\mathrm{T}_g(G)$ via left translation by g), hence it is a subbundle of TG, and the crux is this: *because* $\mathfrak{h}$ *is a Lie subalgebra* (rather than an arbitrary linear subspace of $\mathfrak{g}$) this subbundle is integrable. To prove this fact, first observe that by construction if $X \in \mathfrak{h} \subseteq \mathfrak{g}$ then the associated left-invariant vector field $\widetilde{X}$ on G is a C^∞ section of the subbundle $\widetilde{\mathfrak{h}} \subseteq TG$ (why?), and so if $X_1, \dots, X_n$ is a basis of $\mathfrak{h}$ then $\widetilde{X}_1, \dots, \widetilde{X}_n$ is clearly a global trivializing frame for $\widetilde{\mathfrak{h}}$ (why?). In general, to prove integrability of a subbundle of the tangent bundle it suffices (as we have seen in class) to prove that the bracket operation applied to members of a trivializing frame over the constituents of an open covering of the base space yields output that is a section of the subbundle. In our case there is the global trivializing frame $\widetilde{X}_1, \dots, \widetilde{X}_n$ of $\widetilde{\mathfrak{h}}$, so to prove integrability of $\widetilde{\mathfrak{h}} \subseteq TG$ we just have to prove that $[\widetilde{X}_i, \widetilde{X}_j] \in \widetilde{\mathfrak{h}}(G)$ inside of $(TG)(G) = \mathrm{Vec}_G(G)$. But $\widetilde{X}_i$ and $\widetilde{X}_j$ are left-invariant vector fields on G, so by the very definition of the Lie algebra structure on $\mathfrak{g} = \mathrm{T}_e(G)$ in terms of the commutator operation on global left-invariant vector fields we have $[\widetilde{X}_i, \widetilde{X}_j] = [X_i, X_j]^\sim$. That is, the bracket of $\widetilde{X}_i$ and $\widetilde{X}_j$ is equal to the left-invariant vector field associated to the tangent vector $[X_i, X_j] \in \mathfrak{g}$. But $X_i, X_j \in \mathfrak{h}$ and by hypothesis $\mathfrak{h}$ *is a Lie subalgebra of* $\mathfrak{g}$. Hence, $[X_i, X_j] \in \mathfrak{h}$, so by construction when this is propogated to a left-invariant vector field on G the resulting global vector field is a section of the subbundle $\widetilde{\mathfrak{h}}$ in TG (by how this subbundle was defined). This concludes the verification that $\widetilde{\mathfrak{h}}$ is indeed an integrable subbundle of TG.

We shall see later in that handout that a maximal integral submanifold H in G to the integrable subbundle $\widetilde{\mathfrak{h}}$ such that H contains the identity is a connected *Lie subgroup* of G (by which we mean an injective immersion of Lie groups $i : H \to G$ that respects the group structures) and that its associated Lie subalgebra $\mathrm{Lie}(H) \subseteq \mathfrak{g}$ is the initial choice of Lie subalgebra $\mathfrak{h}$.

Example 1.1. Let $G = \mathrm{GL}_n(\mathbf{R})$. The Lie algebra is denoted $\mathfrak{gl}_n(\mathbf{R})$, and as a vector space is naturally identified with the vector space $\mathrm{Mat}_{n\times n}(\mathbf{R})$ (as G is an open submanifold of $\mathrm{Mat}_{n\times n}(\mathbf{R})$). I claim that the Lie algebra structure on $\mathfrak{gl}_n(\mathbf{R})$ is thereby identified with the "usual" bracket on $n \times n$ matrices, namely $[A, B] = AB - BA$.

How can we prove this? There is a clever way to prove this using some general principles from the theory of Lie groups, but in the present setting it can be proved rather concretely. Let $A = (a_{ij})$ be an element of $\mathrm{Mat}_{n\times n}(\mathbf{R})$ viewed as $\mathfrak{gl}_n(\mathbf{R})$, which is to say (by the realization of "matrix entries" x_{ij} as a linear coordinate system on the vector space $\mathrm{Mat}_{n\times n}(\mathbf{R})$ containing G as an open submanifold) that A corresponds to the tangent vector $\vec{A} = \sum a_{ij}\partial_{x_{ij}}|_e$ in $\mathrm{T}_e(G)$. By using the method of solution to Exercise 2(*iv*) in Homework 7 (i.e., the formula for matrix multiplication in terms of matrix entries) each $\partial_{x_{ij}}|_e$ extends to the left-invariant vector field $\sum_k x_{ki}\partial_{x_{kj}}$ on G. Hence, the left-invariant vector field with value $\vec{A}$ at the identity is the corresponding $\mathbf{R}$-linear

combination
$$\widetilde{\vec{A}} = \sum_{i,j} a_{ij} \sum_k x_{ki}\partial_{x_{kj}} = \sum_{k,j}(\sum_i a_{ij}x_{ki}) \cdot \partial_{x_{kj}}.$$
This is really to be viewed as a vector field on the open submanifold $G \subseteq \mathrm{Mat}_{n\times n}(\mathbf{R})$, though it makes perfectly good sense even on $\mathrm{Mat}_{n\times n}(\mathbf{R})$.

The calculation of the commutator of global vector fields $\widetilde{\vec{A}}$ and $\widetilde{\vec{B}}$ on G is now a matter of algebra:
$$\begin{aligned}\sum_{k,j,k',j'}\sum_{i,i'} a_{ij}b_{i'j'}[x_{ki}\partial_{x_{kj}}, x_{k'i'}\partial_{x_{k'j'}}] &= \sum_{k,j,k',j'}\sum_{i,i'} a_{ij}b_{i'j'}(\delta_{(k,j),(k',i')}x_{ki}\partial_{x_{k'j'}} - \delta_{(k,i),(k',j')}x_{k'i'}\partial_{x_{kj}}) \\ &= \sum_{i,j,j',k} a_{ij}b_{jj'}x_{ki}\partial_{x_{kj'}} - \sum_{i,j,i',k} a_{ij}b_{i'i}x_{ki'}\partial_{x_{kj}}.\end{aligned}$$
Evaluating at the identity point $(x_{rs}) = (\delta_{rs})$, this collapses to
$$\sum_{i,j,j'} a_{ij}b_{jj'}\partial_{x_{ij'}}|_e - \sum_{i,j,k} a_{ij}b_{ki}\partial_{x_{kj}}|_e = \sum_{r,s}(\sum_m a_{rm}b_{ms} - b_{rm}a_{ms})\partial_{x_{rs}}|_e.$$
The rs-coefficient is the rs-entry of the matrix commutator $AB - BA$, so passing from this vector in $\mathrm{T}_e(G)$ back to the language of $\mathrm{Mat}_{n\times n}(\mathbf{R})$ we obtain the desired description of the Lie algebra structure on $\mathfrak{gl}_n(\mathbf{R})$ as the commutator of $n \times n$ matrices.

Since we have described the Lie algebra structure on $\mathfrak{gl}_n(\mathbf{R}) = \mathrm{Mat}_{n\times n}(\mathbf{R})$ as just the ordinary commutator $AB - BA$ of matrices, one Lie subalgebra jumps out at us: the subspace of trace 0 matrices. This is a hyperplane that is stable under the bracket because *every* bracket in the Lie algebra lies in here (clearly $AB - BA$ has trace 0 for any A and B). This turns out to be the tangent space to the connected (!) closed Lie subgroup $\mathrm{SL}_n(\mathbf{R})$ of matrices with determinant 1. There are lots of other Lie subalgebras of more interesting nature. For example, since $[A, B]^t = [B^t, A^t] = -[A^t, B^t]$ for matrices A and B, if A and B are skew-symmetric then so is $[A, B]$. Hence, the subspace $\mathfrak{so}_n(\mathbf{R}) \subseteq \mathfrak{gl}_n(\mathbf{R})$ of skew-symmetric matrices is a Lie subalgebra. This turns out to be the tangent space to the connected (!) closed Lie subgroup $\mathrm{SO}_n(\mathbf{R})$ of orthogonal matrices with determinant 1.

In this handout, we wish to give a general statement of the local and global Frobenius theorems, some discussions concerning the proofs, and work out the general application to the proof of existence and uniqueness of a *connected* Lie subgroup H of a Lie group G such that $\mathrm{Lie}(H) \subseteq \mathrm{Lie}(G)$ coincides with a given Lie subalgebra $\mathfrak{h} \subseteq \mathrm{Lie}(G)$.

2. Statement of main results

Here is the local theorem:

Theorem 2.1 (Frobenius)**.** *Let E be an fiberwise nonzero integrable subbundle of TM, for M a smooth manifold. There exists a covering of M by C^∞ charts (U, φ) with $\varphi = \{x_1, \dots, x_n\}$ a C^∞ coordinate system with $\varphi(U) = \prod(a_i, b_i) \subseteq \mathbf{R}^n$ a product of open intervals such that for $r = \mathrm{rank}(E|_U)$ the embedded r-dimensional slice submanifolds $\{x_i = c_i\}_{i>r}$ for $(c_{r+1}, \dots, c_n) \in \prod_{i>r}(a_i, b_i)$ are integral manifolds for E. Moreover, all (connected!) integral manifolds for E in U lie in a unique such slice set-theoretically, and hence as C^∞ submanifolds of U due to embeddedness of the slices in U.*

Geometrically, the local coordinates in the theorem have the property that E is the subbundle spanned by the vector fields $\partial_{x_1}, \ldots, \partial_{x_r}$. The *proof* of this local theorem proceeds by induction on the rank of E (which we may take to be constant by passing to connected components of M), and to get the induction started in the case $r = 1$ it is necessary to prove a local theorem concerning a non-vanishing vector field (chosen to locally trivialize the line subbundle E in TM):

Theorem 2.2. *For any non-vanishing smooth vector field on an open subset of a smooth manifold, there are local coordinate systems in which the vector field is ∂_{x_1}.*

We give a proof of this theorem in §3, using the technique of vector flow from the theory of integral curves. In particular, this base case for the inductive proof of the local Frobenius theorem uses the entire force of the theory of ODE's, especially smooth dependence of solutions on varying initial conditions. Given such a local coordinate system as in Theorem 2.2, it is clear from the results we have proved in the case of integral curves for vector fields that the x_1-coordinate lines in a coordinate box (all other coordinates held fixed) *do* satisfy the requirements of the local Frobenius integrability theorem in the case of rank 1. That is, Theorem 2.2 does settle the rank 1 case of the local Frobenius theorem. The general geometric inductive proof of the local Frobenius theorem, building on the special case for rank 1, is given in section 1.60 in the handout from Warner's book.

We now turn to the statement of the global Frobenius theorem (see sections 1.62 and 1.64 in the handout from Warner's book). We state it in a slightly stronger form than in Warner's book (but his proof yields the stronger form, as we will explain), and it is certainly also stronger than the version in the course text (which is why we prefer to reference Warner's book for the proof):

Theorem 2.3 (Frobenius)**.** *Let E be an integrable subbundle of TM.*

(1) *For all $m \in M$, there exists a (unique) maximal integral submanifold $i : N \hookrightarrow M$ through m_0.*
(2) *For any C^∞ mapping $M' \to M$ landing in $i(N)$ set-theoretically, the unique factorization $M' \to N$ is continuous and hence smooth.*
(3) *Any* connected *submanifold $i' : N' \hookrightarrow M$ satisfying $\mathrm{T}_{n'}(N') \subseteq E(i'(n'))$ for all $n' \in N'$ lies in a maximal integral submanifold for E.*

Note that in (3), we allow for the possibility that N' might be "low-dimensional" with respect to the rank of E, and so it is a definite strengthening of the property of maximal integral submanifolds for E in M (which are only required to be maximal with respect to other integral submanifolds for E in M, not with respect to connected submanifolds whose tangent spaces are pointwise just contained in – rather than actually equal to – the corresponding fiber of E). Also, in (2) we do not require that the mapping from M' to M be injective. The deduction of smoothness from continuity in (2) follows from an old result in class: the only obstruction to smoothness for a C^∞ map factoring set-theoretically through an injective immersion is topological (i.e., once the first step of the factorization is known to be continuous, the immersion theorem can be used *locally on the source* to infer its smoothness).

In the handout from Warner's book, the above global theorem is proved, except that he omits (3). However, his proof of the "maximal integral submanifold" property in (1) does not use the "maximal dimension" condition on the connected submanifold source, and so it actually proves (3). The method of proof of the global theorem in the course text (pages 194–7) is different from that in Warner's book, and it does not appear to give the result in (3). We will use (3) at one step below.

Before we turn to the task of proving Theorem 2.2, let us explain how to use the global Frobenius theorem to prove a striking result on the existence of connected Lie subgroups realizing a given Lie subalgebra as its Lie algebra. First, a definition:

Definition 2.4. A *Lie subgroup* of a Lie group G is a subgroup $H \subseteq G$ equipped with a C^∞ submanifold structure that makes it a Lie group.

In other words, a Lie subgroup "is" (up to unique isomorphism) an injective immersion $i : H \to G$ of Lie groups with i a group homomorphism. The example of the real line densely wrapped around the torus by the mapping $i : \mathbf{R} \to S^1 \times S^1$ defined by $t \mapsto ((\cos t, \sin t), (\cos(bt), \sin(bt)))$ for b not a rational multiple of π is a Lie subgroup that is *not* an embedded submanifold.

Remark 2.5. The passage between Lie subalgebras and Lie subgroups pervades many arguments in the theory of Lie groups. It is in general hard to tell (in an abstract situation) whether or not the connected Lie subgroup associated to a given Lie subalgebra is actually an embedded submanifold (in which case it turns out to be necessarily a closed submanifold). However, there are some convenient criteria on a Lie subalgebra $\mathfrak{h}$ in $\mathrm{Lie}(G)$ that are sufficient to ensure closedness. For example, if the subspace in $\mathfrak{h}$ spanned by all "brackets" $[x, y]$ with $x, y \in \mathfrak{h}$ is equal to $\mathfrak{h}$ then it turns out that closedness is automatic (this implication is not obvious "by hand"). It may seem that this criterion for closedness is a peculiar one, but it is actually a rather natural one from the perspective of the general structure theory of Lie algebras. Moreover, in practice it is a very mild condition.

As we have seen in class, if $f : H \to G$ is a map of Lie groups (i.e., smooth map of manifolds that is also a group homomorphism) then $(\mathrm{d}f)(e_H) : \mathrm{T}_{e_H}(H) \to \mathrm{T}_{e_G}(G)$ respects the brackets on both sides (i.e., it is a "Lie algebra" map). Hence, in the immersion case we get $\mathrm{Lie}(H)$ as a Lie subalgebra of $\mathrm{Lie}(G)$. It turns out that there is a *bijective* correspondence between *connected* Lie subgroups of G and Lie subalgebras of $\mathfrak{g} = \mathrm{Lie}(G)$:

Theorem 2.6. *Let G be a Lie group, with Lie algebra $\mathfrak{g}$. For every Lie subalgebra $\mathfrak{h}$ there exists a unique connected Lie subgroup H in G with Lie algebra $\mathfrak{h}$ inside of $\mathfrak{g}$. Moreover, if H and H' are connected Lie subgroups then* $\mathrm{Lie}(H) \subseteq \mathrm{Lie}(H')$ *if and only if $H \subseteq H'$ as subsets of G, in which case the inclusion is smooth.*

Before we explain how to prove this theorem using the global Frobenius theorem, we make some remarks. The connectivity is crucial in the theorem. For example, the closed subgroup $\mathrm{O}_n(\mathbf{R})$ of orthogonal matrices in $\mathrm{GL}_n(\mathbf{R})$ for the standard inner product is a Lie subgroup (even a closed submanifold), but it is disconnected with identity component given by the index-2 open subgroup $\mathrm{SO}_n(\mathbf{R})$ of orthogonal matrices with determinant 1. Both $\mathrm{O}_n(\mathbf{R})$ and $\mathrm{SO}_n(\mathbf{R})$ agree near the identity inside of $\mathrm{GL}_n(\mathbf{R})$, so they give the same Lie subalgebra of $\mathfrak{gl}_n(\mathbf{R})$ (consisting of the skew-symmetric matrices in $\mathfrak{gl}_n(\mathbf{R}) = \mathrm{Mat}_{n\times n}(\mathbf{R})$). Beware that there are non-injective immersions of Lie groups, such as $\mathrm{SL}_2(\mathbf{R}) \to \mathrm{SL}_2(\mathbf{R})/\langle -1\rangle$ that induce isomorphisms of Lie algebras. Hence, the passage between the isomorphism problem for connected Lie groups and for Lie algebras is a little subtle and we will not get into it here. The moral of the story is that a good understanding of the structure of $\mathrm{Lie}(G)$ as a Lie algebra does encode a lot of information about the Lie group G. In this way, the structure theory of finite-dimensional Lie algebras over $\mathbf{R}$ (which is a purely algebraic theory that makes sense over any field, though is best behaved in characteristic 0) plays a fundamental role in the theory of Lie groups.

Proof. Let $i : H \to G$ be an arbitrary connected Lie subgroup. Since the inclusion $i : H \to G$ is a group homomorphism and hence is compatible with left translations by elements of H, it follows that for $h \in H$ the mapping $(\mathrm{d}\lambda_{i(h)})(e_G)$ carries $\mathrm{T}_{e_H}(H) \subseteq \mathrm{T}_{e_G}(G)$ (inclusion via $(\mathrm{d}i)(e_H)$) over to the subspace $\mathrm{T}_h(H) \subseteq \mathrm{T}_{i(h)}(G)$ (inclusion via $(\mathrm{d}i)(h)$). In other words (since $\dim(\mathrm{Lie}(H)) = \dim H$), the connected submanifold H is an integral manifold for the integrable subbundle of $TG \simeq G \times \mathfrak{g}$

given by $G \times \mathrm{Lie}(H)$. It therefore follows that the integral manifold H must factor smoothly through the maximal integral submanifold through e_G for the subbundle $G \times \mathrm{Lie}(H)$ in TG. In particular, once we know that H agrees with this maximal integral submanifold we will get the assertion that one Lie subgroup factors smoothly through another if and only if there is a corresponding inclusion of their Lie algebras inside of $\mathfrak{g}$ (as an inclusion of such Lie subalgebras forces a corresponding inclusion of subbundles of TG, and hence a smooth inclusion of maximal integral submanifolds through e_G by the *third part* of the global Frobenius theorem). This gives the uniqueness (including the manifold structure!) for a connected Lie subgroup of G with a specified Lie algebra inside of $\mathfrak{g}$.

Our problem is now reduced to: given a Lie subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ we seek to prove that the maximal integral submanifold H for the integrable subbundle $G \times \mathfrak{h}$ in $G \times \mathfrak{g} \simeq TG$ is the unique connected Lie subgroup of G with $\mathfrak{h}$ as Lie algebra. First, we prove that this maximal integral submanifold H is in fact a Lie subgroup. That is, we must prove that H is algebraically a subgroup of G and then that the induced group law and inversion mappings are smooth for the manifold structure on H (and on $H \times H$). The stability of H under the group law and inversion will use the maximality, and the uniqueness will use a trick for connected groups.

Pick $h \in H$. We want $hH \subseteq H$. In other words, if $i : H \to G$ is the inclusion for H as a submanifold of G, we want the composite injective immersive mapping $\lambda_{i(h)} \circ i : H \to G$ to factor through $i : H \to G$ set-theoretically (but we'll even get such a factorization smoothly). To make the picture a little clearer, instead of considering the maps $\lambda_{i(h)}$ that are C^∞ automorphisms of the manifold G, let us consider a general smooth automorphism φ of a general manifold M and a general integrable subbundle $E \subseteq TM$. The mapping $\mathrm{d}\varphi$ is an automorphism of TM over φ, so $(\mathrm{d}\varphi)(E)$ is a subbundle of TM, and if N is an integral manifold in M for E then the submanifold $\varphi(N)$ is clearly an integral manifold for $(\mathrm{d}\varphi)(E)$ in M. If N is a maximal integral manifold for E then the integral manifold $\varphi(N)$ must be *maximal* for the subbundle $(\mathrm{d}\varphi)(E)$. Indeed, if it is not maximal then (by the global Frobenius theorem!) $\varphi(N) \to M$ factors smoothly through a strictly larger integral submanifold $N' \to M$ for the subbundle $(\mathrm{d}\varphi)(E)$, and so applying φ^{-1} then gives $\varphi^{-1}(N')$ as an integral submanifold for E in M that strictly contains N, contradicting the assumed maximality of N. (Here we have used that $\mathrm{d}\varphi$ and $\mathrm{d}\varphi^{-1}$ are inverse maps on TM, as follows from the Chain Rule.)

Now in our special situation, the integrable subbundle $E \stackrel{\mathrm{def}}{=} G \times \mathfrak{h}$ in $G \times \mathfrak{g} \simeq TG$ satisfies $(\mathrm{d}\lambda_g)(E) = E$ for all $g \in G$, which is to say $(\mathrm{d}\lambda_g)(g')$ carries $E(g')$ to $E(gg')$ inside of $\mathrm{T}_{gg'}(G)$. This holds because the fibers of the bundle E in TG were *constructed* using the left translation maps on tangent spaces (ultimately due to how the trivialization isomorphism $G \times \mathfrak{g} \simeq TG$ is *defined*). Hence, the preceding generalities imply that λ_g carries maximal integral manifolds for $E = G \times \mathfrak{h} \subseteq G \times \mathfrak{g} \simeq TG$ to maximal integral manifolds for E. In particular, for the maximal integral manifold $i : H \to G$, we conclude that $\lambda_{i(h)} \circ i : H \to G$ is also a maximal integral manifold for E. But the image of the latter contains the point $he = h \in i(H)$, so these two integral submanifold touch! Hence, they must coincide as submanifolds, which is to say that left multiplication by h on G carries H smoothly isomorphically back to itself (as a smooth submanifold of G).

This not only proves that H is algebraically a subgroup, but also that for all $h \in H$ the left multiplication mapping on G restricts to a bijective self-map (even smooth automorphism) of H. Since the identity e lies in H, it follows that $hh' = e$ for some $h' \in H$, which is to say that the unique inverse $h^{-1} \in G$ lies in H. That is, H is stable under inversion, and so it is algebraically a subgroup of G. If we let $\mathrm{inv} : G \simeq G$ be the smooth inversion mapping, then this says that the composite of inv with the smooth inclusion of H into G lands in the subset $H \subseteq G$ set-theoretically. Hence, by (2) in the global Frobenius theorem (applied to the maximal integral manifold H for E

in G) we conclude that $\mathrm{inv}|_H : H \to G$ factors *smoothly* through the inclusion of H into G, which is to say that inversion on the subgroup H of G is a *smooth* self-map of the manifold H.

To conclude that H is a Lie subgroup, it remains to check smoothness for the group law. That is, we want the composite smooth mapping

$$H \times H \stackrel{i \times i}{\to} G \times G \to G$$

(the second step being the smooth group law of G) to factor smoothly through the inclusion i of H into G. But it does factor through this inclusion set-theoretically because H is a subgroup of G, and so again by (2) in the Frobenius theorem we get the desired smooth factorization. Hence, H is a Lie subgroup of G.

Finally, we have to prove the uniqueness aspect: if H' is a connected Lie subgroup of H with Lie algebra equal to the Lie algebra $\mathfrak{h}$ of H inside of $\mathfrak{g}$, then we want $H' = H$ as Lie subgroups of G. The discussion at the beginning of the proof shows that H' must at least smoothly factor through the maximal integral submanifold through the identity for the integrable subbundle $E = G \times \mathfrak{h} \subseteq G \times \mathfrak{g} \simeq TG$, which is to say that H' factors smoothly through H. Hence, we have a smooth injective immersion $H' \hookrightarrow H$ (as submanifolds of G) and we just need this to be an isomorphism. Any Lie group has the same dimension at all points (due to left translation automorphisms that identify all tangent spaces with the one at the identity), so H' and H have the same dimension at all points (as their tangent spaces at the identity coincide inside of $\mathfrak{g}$). Thus, the injective tangent mappings for the immersion $H' \to H$ are forced to be isomorphisms for dimension reasons, so the injective map $H' \to H$ is a local C^∞ isomorphism by the inverse function theorem! As such, it has *open* image onto which it is bijective, so H' is an open submanifold of H and thus is an open Lie subgroup of H.

Now comes the magical trick (which is actually a powerful method for proving global properties of a connected group): I claim that a *connected* topological group (such as H) has no proper open subgroups. This will certainly force the open immersion $H' \to H$ to be surjective and thus $H' = H$ as Lie subgroups of G. Rather more generally, I claim that an open subgroup of a topological group is always *closed* (giving what we need in the connected case). To see closedness, it is equivalent to prove openness of the complement, and by group theory we know that the complement of a subgroup of a group is a disjoint union of left cosets. Since any coset for an open subgroup is open (as it is an image of the open subgroup under a left-translation map that is necessarily a homeomorphism), any union of such cosets is open. ■

We conclude our tour through the dictionary between Lie groups and Lie algebras by posing a natural question: if G and G' are *connected* Lie groups, does any map of the associated Lie algebras $T : \mathfrak{g}' \to \mathfrak{g}$ ($\mathbf{R}$-linear respecting the brackets) necessarily arise from a map $f : G' \to G$ of Lie groups (smooth map of manifolds and group homomorphism)? If such an f exists then it is unique, but there is a topological obstruction to existence. To analyze this, we use the powerful method of graphs.

To see the uniqueness, we note that $f : G' \to G$ gives rise to a smooth graph mapping $\Gamma_f : G' \to G' \times G$ (via $g' \mapsto (g', f(g'))$ that is a connected Lie subgroup of $G' \times G$ when the latter is made into a group with product operations, and via the method of left-translations we also have that the natural identification $\mathrm{T}_{(e',e)}(G' \times G) \simeq \mathrm{T}_{e'}(G') \oplus \mathrm{T}_e(G)$ carries the Lie bracket on the left over to the direct sum of the Lie brackets on the right. That is, $\mathrm{Lie}(G' \times G) \simeq \mathrm{Lie}(G') \oplus \mathrm{Lie}(G)$. In this way, the mapping

$$\mathrm{Lie}(\Gamma_f) : \mathrm{Lie}(G') \to \mathrm{Lie}(G' \times G) \simeq \mathrm{Lie}(G') \oplus \mathrm{Lie}(G)$$

is identified with the linear-algebra "graph" of the map $\mathrm{Lie}(f) = (\mathrm{d}f)(e') : \mathrm{Lie}(G') \to \mathrm{Lie}(G)$ that is assumed to be T. Hence, Γ_f corresponds to a connected Lie subgroup of $G' \times G$ whose associated Lie subalgebra in $\mathfrak{g}' \oplus \mathfrak{g}$ is the graph of the linear map T. By the uniqueness aspect of the passage from connected Lie subgroups to Lie subalgebras, it follows that the mapping $\Gamma_f : G' \to G' \times G$ is uniquely determined (if f is to exist), and so composing it with the projection $G' \times G \to G$ recovers f. This verifies the uniqueness of f.

How about existence? To this end, we try to reverse the above procedure: we use the injective graph mapping $\Gamma_T : \mathfrak{g}' \to \mathfrak{g}' \oplus \mathfrak{g}$ that is a mapping of Lie algebras precisely because T is a map of Lie algebras (and because the direct sum is given the "componentwise" bracket). By the general existence/uniqueness theorem, there is a unique connected Lie subgroup $H \subseteq G' \times G$ whose associated Lie subalgebra is the image of Γ_T. In particular, the first projection $H \to G'$ induces an isomorphism on Lie algebras, and if this mapping of connected Lie groups were an isomorphism then we could compose its inverse with the other projection $H \to G$ to get the desired mapping. (Conversely, it is clear that if the existence problem is to have an affirmative answer, then the first projection $H \to G'$ *must* be an isomorphism.) Hence, the problem is reduced to this: can a mapping $\pi : H \to G'$ between *connected* Lie groups induce an isomorphism on Lie algebras without being an isomorphism?

Such a mapping must be a local isomorphism near the identities (by the inverse function theorem), and so the image subgroup is open (as it contains an open in G' around the identity, and hence around all of its points via left translation in the image subgroup). But we have seen above that *connected* topological groups have no proper open subgroups, so the mapping π must be surjective. Also, $\ker(\pi)$ is a closed subgroup that meets a neighborhood of the identity in H in exactly the identity point (as π is a local isomorphism near identity elements), so the identity is an open point in $\ker(\pi)$. It follows by translations that the closed topological normal subgroup $\ker(\pi)$ must have the *discrete* topology. But if Γ is a discrete closed subgroup of a connected Lie group H then the right multiplication action by Γ on H is certainly free and properly discontinuous, so we know how to make the quotient H/Γ as a C^∞ manifold. Moreover, by mapping properties for quotients by such group actions it is easy to check that if Γ is also normal in H then the multiplication mapping $H \times H \to H \to H/\Gamma$ factors smoothly through the projection $H \times H \to (H/\Gamma) \times (H/\Gamma)$, and likewise for $H \overset{h \mapsto h^{-1}}{\to} H \to H/\Gamma$ factoring through projection from H to H/Γ, so it follows that the natural group law and C^∞ manifold structure on H/Γ make it a Lie group. It is also clear (why?) that if $\pi : H \to G$ is a surjective Lie group map that is an isomorphism on Lie algebras then for the discrete subgroup Γ in H the induced C^∞ map $H/\Gamma \to G$ is a bijective Lie group map that is an isomorphism on Lie algebras and so (via translations!) is an isomorphism between tangent spaces at all points, whence by the inverse function theorem it is a C^∞ isomorphism (whence is an isomorphism of Lie groups).

To summarize, we have found the precise topological obstruction to our problem: if H contains non-trivial discrete normal closed subgroups Γ, then the projection $H \to H/\Gamma$ induces an isomorphism on Lie algebras and it is not an isomorphism. Moreover, and more relevant to our initial question, the inverse map on Lie algebras cannot arise from a map of Lie groups $H/\Gamma \to H$ (as the composite of such a map with the projection $H \to H/\Gamma$ would then give a self-map of H with nontrivial kernel inducing the identity on $\mathrm{Lie}(H)$, contradicting that the identity self-map on the connected Lie group H is the only self-map that induces the identity on the Lie algebra of H). Hence, we see that the problem of going from maps of Lie algebras to maps of Lie groups involves a serious issue, namely the possible existence of discrete normal subgroups of H. For example, $\{\pm \mathrm{id}\} \subseteq \mathrm{SL}_2(\mathbf{R})$ is a discrete normal closed subgroup.

3. Vector fields and local coordinates

We now turn to the task of proving Theorem 2.2. First we consider a more general situation. Let M be a smooth manifold and let $\vec{v}_1, \dots, \vec{v}_n$ be pointwise linearly independent smooth vector fields on an open subset $U \subseteq M$ ($n \geq 1$). One simple example of such vector fields is $\partial_{x_1}, \dots, \partial_{x_n}$ on a coordinate domain for local smooth coordinates $\{x_1, \dots, x_N\}$ on an open set U in M. Can all examples be described in this way (locally) for suitable smooth coordinates?

Choose a point $m_0 \in U$. It is very natural (e.g., to simplify local calculations) to ask if there exists a local C^∞ coordinate system $\{x_1, \dots, x_N\}$ on an open subset $U_0 \subseteq U$ around m_0 such that $\vec{v}_i|_{U_0} = \partial_{x_i}$ in $\mathrm{Vec}_M(U_0) = (TM)(U_0)$ for $1 \leq i \leq n$. The crux of the matter is to have such an identity across an entire open neighborhood of m_0. If we only work in the tangent space *at* the point m_0, which is to say we inquire about the identity $\vec{v}_i(m_0) = \partial_{x_i}|_{m_0}$ in $\mathrm{T}_{m_0}(U_0) = \mathrm{T}_{m_0}(M)$ for $1 \leq i \leq n$, then the answer is trivial (and not particularly useful): we choose local C^∞ coordinates $\{y_1, \dots, y_N\}$ near m_0 and write $\vec{v}_j(m_0) = \sum c_{ij} \partial_{y_i}|_{m_0}$, so the $N \times n$ matrix (c_{ij}) has independent columns. We extend this to an invertible $N \times N$ matrix, and then make a *constant* linear change of coordinates on the y_j's via the inverse matrix to get to the case $c_{ij} = \delta_{ij}$ for $i \leq n$ and $c_{ij} = 0$ for $i > n$. Of course, such new coordinates are only adapted to the situation at m_0. If we try to do the same construction by considering the matrix of functions (h_{ij}) with $\vec{v}_j = \sum h_{ij} \partial_{y_i}$ near m_0, the change of coordinates will now typically have to be *non-constant*, thereby leading to a big mess due to the appearance of differentiation in the transformation formulas for ∂_{t_i}'s with respect to change of local coordinates (having "non-constant" coefficients).

There is a very good reason why the problem over an open set (as opposed to at a single point) is complicated: usually no such coordinates exist! Indeed, if $n \geq 2$ then the question generally has a negative answer because there is an obstruction that is often non-trivial: since the commutator vector field $[\partial_{x_i}, \partial_{x_j}]$ vanishes for any i, j, if such coordinates are to exist around m_0 then the commutator vector fields $[\vec{v}_i, \vec{v}_j]$ must vanish near m_0. (Note that the concept of commutator of vector fields is meaningless when working on a single tangent space; it only has meaning when working with vector fields over open sets. This is "why" we had no difficulties when working at a single point m_0.)

For $n \geq 2$, the necessary condition of vanishing of commutators for pointwise independent vector fields usually fails. For example, on an open set $U \subseteq \mathbf{R}^3$ consider a pair of smooth vector fields

$$\vec{v} = \partial_x + f\partial_z, \quad \vec{w} = \partial_y + g\partial_z$$

for smooth functions f and g on U. These are visibly pointwise independent vector fields but

$$[\vec{v}, \vec{w}] = ((\partial_x g + f\partial_z g) - (\partial_y f + g\partial_z f))\partial_z,$$

so a necessary condition to have $\vec{v} = \partial_{x_1}$ and $\vec{w} = \partial_{x_2}$ for local C^∞ coordinates $\{x_1, x_2, x_3\}$ near $m_0 \in U$ is

$$\partial_x g + f\partial_z g = \partial_y f + g\partial_z f$$

near m_0. There is a special case in which the vanishing condition on the commutators $[\vec{v}_i, \vec{v}_j]$ for all i, j is vacuous: $n = 1$. Indeed, since $[\vec{v}, \vec{v}] = 0$ for any smooth vector field, in the case $n = 1$ we see no obvious reason why our question cannot always have an affirmative answer. The technique of vector flow along integral curves will prove such a result.

In the case $n = 1$, pointwise-independence for the singleton $\{\vec{v}_1\}$ amounts to pointwise non-vanishing. Hence, we may restate the goal we have: if $\vec{v}$ is a smooth vector field on an open set $U \subseteq M$ and $\vec{v}(m_0) \neq 0$ for some $m_0 \in U$ (so $\vec{v}(m) \neq 0$ for m near m_0, by continuity of $\vec{v} : U \to TM$), then there exists a local C^∞ coordinate system $\{x_1, \dots, x_N\}$ near m_0 in U such that $\vec{v} = \partial_{x_1}$ near m_0.

Example 3.1. Consider the circular vector field $\vec{v} = -y\partial_x + x\partial_y$ on $M = \mathbf{R}^2$ with constant speed $r \geq 0$ on the circle of radius r centered at the origin. This vector field vanishes at the origin, but for $m_0 \neq (0,0)$ we have $\vec{v}(m_0) \neq 0$. Let $U_0 = \mathbf{R}^2 - L$ for a closed half-line L emanating from the origin and not passing through m_0. For a suitable θ_0, trigonometry provides a C^∞ parameterization $(0,\infty) \times (\theta_0, \theta_0 + 2\pi) \simeq U_0$ given by $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$, and $\partial_\theta = \vec{v}|_{U_0}$. Thus, in this special case we get lucky: we already "know" the right coordinate system to solve the problem. But what if we didn't already know trigonometry? How would we have been able to figure out the answer in this simple special case?

Example 3.2. In order to appreciate the non-trivial nature of the general assertion we are trying to prove, let us try to prove it in general "by hand" (i.e., using just basic definitions, and no substantial theoretical input such as the theory of vector flow along integral curves). We shrink U around m_0 so that there exist local C^∞ coordinates $\{y_1, \dots, y_N\}$ on U. Hence, $\vec{v} = \sum h_j \partial_{y_j}$, and since $\vec{v}(m_0) = \sum h_j(m_0)\partial_{y_j}|_{m_0}$ is nonzero, we have $h_j(m_0) \neq 0$ for some j. By relabelling, we may assume $h_1(m_0) \neq 0$. By shrinking U around m_0, we may assume h_1 is non-vanishing on U (so $\vec{v}$ is non-vanishing on U). We wish to find a C^∞ coordinate system $\{x_1, \dots, x_N\}$ near m_0 inside of U such that $\vec{v} = \partial_{x_1}$ near m_0.

What conditions are imposed on the x_i's in terms of the y_j's? For any smooth coordinate system $\{x_i\}$ near m_0, $\partial_{y_j} = \sum (\partial_{y_j} x_i)\partial_{x_i}$ near m_0, so near m_0 we have

$$\vec{v} = \sum_j h_j \sum_i (\partial_{y_j} x_i)\partial_{x_i} = \sum_i (\sum_j h_j \partial_{y_j}(x_i))\partial_{x_i}.$$

Thus, the necessary and sufficient conditions are two-fold: $x_1, \dots, x_N$ are smooth functions near m_0 such that $\det((\partial_{y_j} x_i)(m_0)) \neq 0$ (this ensures that the x_i's are local smooth coordinates near m_0, by the inverse function theorem) and

$$\sum_j h_j \partial_{y_j}(x_i) = \delta_{ij}$$

for $1 \leq i \leq N$. This is a system of linear first-order PDE's in the N unknown functions $x_i = x_i(y_1, \dots, y_N)$ near m_0. We have already seen that the theory of first-order linear ODE's is quite substantial, and here were are faced with a PDE problem. Hence, our task now looks to be considerably less straightforward than it may have seemed to be at the outset.

The apparent complications are an illusion: it is because we have written out the explicit PDE's in local coordinates that things look complicated. As will be seen in the proof below, when we restate our problem in *geometric* language the idea for how to solve the problem essentially drops into our lap without any pain at all. This is reminiscent of a basic principle we learned in linear algebra: geometric language is very effective at cutting through apparent difficulties in coordinatized problems.

The fundamental theorem is this (a restatement of Theorem 2.2):

Theorem 3.3. *Let M be a smooth manifold and $\vec{v}$ a smooth vector field on an open set $U \subseteq M$. Let $m_0 \in U$ be a point such that $\vec{v}(m_0) \neq 0$. There exists a local C^∞ coordinate system $\{x_1, \dots, x_N\}$ on an open set $U_0 \subseteq U$ containing m_0 such that $\vec{v}|_{U_0} = \partial_{x_1}$.*

This theorem is proved in the course text as Theorem 7 in Chapter 5. You may like the picture there, and perhaps you may also prefer the proof there. (It is the same proof as we give, except we include some more details and geometric explanation.)

Proof. What is the geometric meaning of what we are trying to do? We are trying to find local coordinates $\{x_i\}$ an open open U_0 in U around m_0 so that the integral curves for $\vec{v}|_{U_0}$ are exactly flow along the x_1-direction at unit speed. That is, in this coordinate system for any point ξ near m_0 the integral curve for $\vec{v}$ through ξ is coordinatized as $c_\xi(t) = (t + x_1(\xi), x_2(\xi), \dots, x_N(\xi))$ for t near 0. This suggests that we try to find a local coordinate system around m_0 such that the first coordinate is "time of vector flow". Recall from our study of openness of the domain of definition for vector flow along integral curves in manifolds that for a sufficiently small open $U_0 \subseteq U$ around m_0 there exists $\varepsilon > 0$ such that for all $\xi \in U_0$ the maximal interval of definition for the integral curve c_ξ contains $(-\varepsilon, \varepsilon)$. More specifically, we proved that the vector-flow mapping

$$X_{\vec{v}} : \mathscr{D}(\vec{v}) \to M$$

defined by $(t, \xi) \mapsto c_\xi(t)$ has *open* domain of definition in $\mathbf{R} \times M$ and is a smooth mapping. Thus, for small $\varepsilon > 0$ and small $U_0 \subseteq U$ around m_0, we have that $(-\varepsilon, \varepsilon) \times U_0$ is contained in $\mathscr{D}(\vec{v})$ (as $\{0\} \times M \subseteq \mathscr{D}(\vec{v})$). The mapping $X_{\vec{v}}$, restricted to $(-\varepsilon, \varepsilon) \times U_0$, will be the key to creating a coordinate system on M near m_0 such that the time-of-flow parameter t is the first coordinate.

Here is the construction. We first choose an arbitrary smooth coordinate system $\phi : W \to \mathbf{R}^N$ on an open around m_0 that "solves the problem at m_0". That is, if $\{y_1, \dots, y_N\}$ are the component functions of ϕ, then $\partial_{y_1}|_{m_0} = \vec{v}(m_0)$. This is the trivial pointwise version of the problem that we considered at the beginning of this handout (and it has an affirmative answer precisely because the singleton $\{\vec{v}(m_0)\}$ in $\mathrm{T}_{m_0}(M)$ is an independent set; i.e., $\vec{v}(m_0) \neq 0$). Making a constant translation (for ease of notation), we may assume $y_j(m_0) = 0$ for all j. In general this coordinate system will fail to "work" at any other points, and we use vector flow to fix it. Consider points on the slice $W \cap \{y_1 = 0\}$ in M near m_0. In terms of y-coordinates, these are points $(0, a_2, \dots, a_N)$ with small $|a_j|$'s. By openness of the domain of flow $\mathscr{D}(\vec{v}) \subseteq \mathbf{R} \times M$, there exists $\varepsilon > 0$ such that, after perhaps shrinking W around m_0, $(-\varepsilon, \varepsilon) \times W \subseteq \mathscr{D}(\vec{v})$.

By the definition of the y_i's in terms of ϕ, $\phi(W \cap \{y_1 = 0\})$ is an open subset in $\{0\} \times \mathbf{R}^{N-1} = \mathbf{R}^{N-1}$, and ϕ restricts to a C^∞ isomorphism from the smooth hypersurface $W \cap \{y_1 = 0\}$ onto $\phi(W \cap \{y_1 = 0\})$. Consider the vector-flow mapping

$$\Psi : (-\varepsilon, \varepsilon) \times \phi(W \cap \{y_1 = 0\}) \to M$$

defined by

$$(t, a_2, \dots, a_N) \mapsto X_{\vec{v}}(t, \phi^{-1}(0, a_2, \dots, a_N)) = c_{\phi^{-1}(0, a_2, \dots, a_N)}(t).$$

By the theory of vector flow, this is a *smooth* mapping. (This is the family of solutions to a first-order initial-value problem with varying initial parameters $a_2, \dots, a_N$ near 0. Thus, the smoothness of the map is an instance of smooth dependence on varying initial conditions for solutions to first-order ODE's.) Geometrically, we are trying to parameterize M near m_0 by starting on the hypersurface $H = \{y_1 = 0\}$ in W (with coordinates given by the restrictions $y_2', \dots, y_N'$ of $y_2, \dots, y_N$ to H) and flowing away from H along the vector field $\vec{v}$; the time t of flow provides the first parameter in our attempted parameterization of M near m_0.

Note that $\Psi(0, 0, \dots, 0) = c_{m_0}(0) = m_0$. Is Ψ a parameterization of M near m_0? That is, is Ψ a local C^∞ isomorphism near the origin? If so, then its local inverse near m_0 provides a C^∞ coordinate system $\{x_1, \dots, x_N\}$ with $x_1 = t$ measuring time of flow along integral curves for $\vec{v}$ with their *canonical* parameterization (as integral curves). Thus, it is "physically obvious" that in such a coordinate system we will have $\vec{v} = \partial_{x_1}$ (but we will also derive this by direct calculation below). To check the local isomorphism property for Ψ near the origin, we use the inverse function theorem: we have to check $\mathrm{d}\Psi(0, \dots, 0)$ is invertible. In terms of the local C^∞ coordinates $\{t, y_2', \dots, y_N'\}$ near the origin on the source of Ψ and $\{y_1, \dots, y_N\}$ near $m_0 = \Psi(0, \dots, 0)$ on the target of Ψ, the

$N \times N$ Jacobian matrix for $\mathrm{d}\Psi(0,\ldots,0)$ has lower $(N-1)\times(N-1)$ block given by the identity matrix (i.e., $(\partial_{y'_j} y_i)(0,\ldots,0) = \delta_{ij}$) because $\partial_{y'_j} y_i = \delta_{ij}$ at points on $W \cap \{y_1 = 0\}$ (check! It is *not* true at most other points of W).

What is the left column of the Jacobian matrix at $(0,\ldots,0)$? Rather generally, if ξ is the point with y-coordinates $(t_0, a_2, \ldots, a_N)$ then the t-partials $(\partial_t y_i)(t_0, a_2, \ldots, a_N)$ are the coefficients of the velocity vector $c'_\xi(t_0)$ to the integral curve c_ξ of $\vec{v}$ at time t_0, and such a velocity vector is equal to $\vec{v}(c_\xi(t_0))$ by the *definition* of the concept of integral curve. Hence, setting $t_0 = 0$, $c'_\xi(0) = \vec{v}(c_\xi(0)) = \vec{v}(\xi)$, so taking $\xi = m_0 = \Psi(0,\ldots,0)$ gives that $(\partial_t y_i)(0,\ldots,0)$ is the coefficient of $\partial_{y_i}|_{m_0}$ in $\vec{v}(m_0)$. Aha, but recall that we *chose* $\{y_1,\ldots,y_N\}$ at the outset so that $\vec{v}(m_0) = \partial_{y_1}|_{m_0}$. Hence, the left column of the Jacobian matrix at the origin has $(1,1)$ entry 1 and all other entries equal to 0. Since the lower right $(N-1)\times(N-1)$ block of the Jacobian matrix is the identity, this finishes the verification of invertibility of $\mathrm{d}\Psi(0,\ldots,0)$, so Ψ gives a local C^∞ isomorphism between opens around $(0,\ldots,0)$ and m_0.

Let $\{x_1,\ldots,x_N\}$ be the C^∞ coordinate system near m_0 on M given by the local inverse to Ψ. We claim that $\vec{v} = \partial_{x_1}$ near m_0. By definition of the x-coordinate system, $(a_1,\ldots,a_n)$ is the tuple of x-coordinates of the point $X_{\vec{v}}(a_1, \phi^{-1}(0,a_2,\ldots,a_n)) \in M$. Thus, ∂_{x_1} is the field of velocity vectors along the parameteric curves $X_{\vec{v}}(t, \phi^{-1}(0,a_2,\ldots,a_n)) = c_{\phi^{-1}(0,a_2,\ldots,a_n)}(t)$ that are the integral curves for the smooth vector field $\vec{v}$ with initial positions (time 0) at points $\phi^{-1}(0,a_2,\ldots,a_n) \in W \cap \{y_1 = 0\}$ near m_0. Thus, the velocity vectors along these parametric curves are exactly the vectors from the smooth vector field $\vec{v}$! This shows that the smooth vector fields ∂_{x_1} and $\vec{v}$ coincide near m_0. ■