

1. MOTIVATION

In the text (as with most books on differential geometry), all notions that ought to be local (such as tangent vectors, connections, curvature, etc.) are initially defined in terms of objects that are far too “global” such as global functions, global vector fields, and so on. A sneaky device using bump functions enables one to show that such global definitions are actually equivalent to the local ones that we shall use.

The aim of this handout is to explain how this equivalence works in the case of tangent spaces at a point: we gave a manifestly local definition in class, but the text gives the usual “global” definition that is found in virtually all introductory books on differential geometry. There are some useful applications of the ability to formulate local notions in terms of global structures, but it is a bit unnatural to do so in the foundations of the theory. Moreover, such a global point of view is not available in the foundations of the theory of real-analytic (or complex-analytic manifolds), due to the lack of interesting analytic bump functions. We will leave it to the interested reader to adapt the method below to verify that other later notions that we define locally are equivalent to the “global” definitions found in the course text or in other books. The local definitions are always adequate for all proofs, so for our purposes it doesn’t really matter than this equivalence holds (and in a sense it is better to not use it early in the development of the theory, in order that one develops good habits of mind that are necessary for the theories of real-analytic and complex-analytic manifolds).

2. ANOTHER PERSPECTIVE ON TANGENT VECTORS

Let  $X$  be a  $C^p$ -premanifold with corners,  $0 \leq p \leq \infty$ , and  $\mathcal{O}_x$  the local ring at  $x \in X$ . Let  $\mathfrak{m}_x$  be the kernel of the evaluation map  $e_x : \mathcal{O}_x \rightarrow \mathbf{R}$  defined by  $f \mapsto f(x)$ . By the very definition of how we add and multiply functions, it is clear that  $e_x$  is a ring homomorphism such that  $e_x(c) = c$  for all  $c \in \mathbf{R}$  (so in particular,  $e_x$  is surjective). To be a bit pedantic, here are the details. Let  $(U, f)$  and  $(V, g)$  be representatives for germs in  $\mathcal{O}_x$ . By definition of the  $\mathbf{R}$ -algebra structure on  $\mathcal{O}_x$ , for any  $a, b \in \mathbf{R}$ ,  $a[(U, f)] + b[(V, g)]$  is represented by  $(U \cap V, a \cdot f|_{U \cap V} + b \cdot g|_{U \cap V})$ . By the definition of the  $\mathbf{R}$ -algebra structure on the set of  $\mathbf{R}$ -valued functions on open sets in  $X$  (such as  $U \cap V$ ), the value of  $a \cdot f|_{U \cap V} + b \cdot g|_{U \cap V}$  at  $x \in X$  is  $a \cdot f(x) + b \cdot g(x) = a \cdot e_x([(U, f)]) + b \cdot e_x([(V, g)])$ . This says exactly that for  $s, s' \in \mathcal{O}_x$  and  $a, a' \in \mathbf{R}$ ,  $e_x(as + a's') = ae_x(s) + a'e_x(s')$ . This shows that  $e_x$  is  $\mathbf{R}$ -linear. The same argument proves  $e_x(c) = c$  for  $c \in \mathbf{R}$  and  $e_x(ss') = e_x(s)e_x(s')$ , so  $e$  is a map of  $\mathbf{R}$ -algebras.

Somewhat more amusing is the fact that such properties uniquely characterize the evaluation map  $e_x$ ; that is, if we focus on the  $\mathbf{R}$ -algebra  $\mathcal{O}_x$  and discard the space  $X$  and point  $x$  that gave rise to it, we can still define the map  $e_x$ . To make sense of this, the key is the following special property of the local ring at  $x$ :

**Lemma 2.1.** *A germ  $f \in \mathcal{O}_x$  has a multiplicative inverse in  $\mathcal{O}_x$  (i.e.,  $fg = 1$  in  $\mathcal{O}_x$  for some  $g \in \mathcal{O}_x$ ) if and only if  $f \notin \mathfrak{m}_x$ . In other words, the kernel  $\mathfrak{m}_x$  of  $e_x$  is precisely the set of  $f \in \mathcal{O}_x$  that do not have a multiplicative inverse.*

We warn the reader that if we choose a representative for a germ, to say that the germ has a multiplicative inverse in the ring of germs does *not* imply that the representative function is *nowhere* zero on its domain; all one can infer is that it has to be non-vanishing *near*  $x$ .

*Proof.* If  $f \in \mathcal{O}_x$  satisfies  $fh = 1$  for some  $h \in \mathcal{O}_x$  then applying  $e_x$  gives  $e_x(f)e_x(h) = e_x(fh) = e_x(1) = 1$  in  $\mathbf{R}$ , so  $e_x(f) \neq 0$ . That is, if  $f \in \mathcal{O}_x$  has a multiplicative inverse then  $f \notin \mathfrak{m}_x$ .

Conversely, if  $f \notin \mathfrak{m}_x$  then we want to show that  $f$  has a multiplicative inverse. In terms of a representative function on an open, it suffices to show that if  $U$  is an open set around  $x$  and  $f \in \mathcal{O}(U)$  satisfies  $f(x) \neq 0$  then on some small open  $U' \subseteq U$  around  $x$  there exists  $h \in \mathcal{O}(U')$  that is a multiplicative inverse to  $f|_{U'}$ . To prove this we may shrink  $U$  around  $x$ , and so by continuity of  $f$  we may assume that  $f$  is non-vanishing on  $U$ . Thus, there is a reciprocal  $1/f$  as an  $\mathbf{R}$ -valued function on  $U$  and we just have to check that it lies in  $\mathcal{O}(U)$ . This is a local problem on  $U$ , and so by working in local  $C^p$ -charts we can shift the problem over to one for  $C^p$ -functions on open domains in sectors in  $\mathbf{R}^n$ : if such a function is non-vanishing then its pointwise reciprocal is also  $C^p$ . This follows from the stability of the  $C^p$  property under composition and the fact that  $x \mapsto 1/x$  is a  $C^\infty$  map from  $\mathbf{R}^\times$  to  $\mathbf{R}^\times$ . (Or one could use Whitney's extension theorem to reduce the problem to the case of non-vanishing  $C^p$  functions on open sets in  $\mathbf{R}^n$ , at least for the non-trivial case  $p > 0$ , but that would be overkill for the present circumstances.) ■

Now we can prove the uniqueness for  $e_x$ :

**Theorem 2.2.** *Let  $e : \mathcal{O}_x \rightarrow \mathbf{R}$  be a ring homomorphism such that  $e(c) = c$  for all  $c \in \mathbf{R}$  (i.e.,  $e$  is a map of  $\mathbf{R}$ -algebras). The map  $e$  must equal  $e_x$ .*

The significance is this: the  $\mathbf{R}$ -algebra map  $e_x$  is intrinsic to the  $\mathbf{R}$ -algebra  $\mathcal{O}_x$  without needing to mention the geometric data such as  $X$  or  $x$  that gave rise to  $\mathcal{O}_x$ .

*Proof.* Since  $e(c) = c$  for all  $c \in \mathbf{R}$ , certainly  $e$  is surjective. We next check that  $e$  kills  $\mathfrak{m}_x$ . If  $f \in \mathfrak{m}_x$  is not killed by  $e$ , so  $e(f) = c \in \mathbf{R}$  is nonzero, then  $e(f - c) = e(f) - e(c) = c - c = 0$ . However, since  $f \in \mathfrak{m}_x$  and  $c \notin \mathfrak{m}_x$  (as  $e_x(c) = c \neq 0$ ), we must have  $f - c \notin \mathfrak{m}_x$  because  $\mathfrak{m}_x = \ker e_x$  is an  $\mathbf{R}$ -linear subspace of  $\mathcal{O}_x$ . Thus, by the lemma it follows that  $f - c \in \mathcal{O}_x$  has a multiplicative inverse, say  $h$ . Applying the ring homomorphism  $e$  to the identity  $(f - c)h = 1$  in  $\mathcal{O}_x$  gives  $e(f - c)e(h) = e(1) = 1$  in  $\mathbf{R}$ , a contradiction since  $e(f - c) = 0$ . Thus,  $e(f)$  must have been zero after all. This shows that  $e$  kills  $\mathfrak{m}_x$ .

It follows that  $e$  and  $e_x$  agree on  $\mathfrak{m}_x \subseteq \mathcal{O}_x$  (both vanish) and they agree on  $\mathbf{R} \subseteq \mathcal{O}_x$  (both send  $c$  to  $c$  for all  $c \in \mathbf{R}$ ). Hence, to conclude  $e = e_x$  it suffices to show  $\mathbf{R} \oplus \mathfrak{m}_x = \mathcal{O}_x$ . That is, we want that each  $f \in \mathcal{O}_x$  is uniquely expressible as  $f = c + f_0$  with  $c \in \mathbf{R}$  and  $f_0 \in \mathfrak{m}_x$ . The existence follows from the definition of  $\mathfrak{m}_x$  and the identity  $f = f(x) + (f - f(x))$ . As for uniqueness, we just need  $\mathbf{R} \cap \mathfrak{m}_x = 0$ , and this is clear. ■

In the  $C^\infty$  case, we can prove that not only the evaluation map at  $x$  but even the notion of tangent vector at  $x$  is intrinsic to the  $\mathbf{R}$ -algebra  $\mathcal{O}_x$  and does not need to “know” about  $X$  and  $x$ . More specifically, recall that in the definition of tangent vector at  $x$  we had two conditions on the  $\mathbf{R}$ -linear map  $\partial : \mathcal{O}_x \rightarrow \mathbf{R}$  for it to be a point-derivation: it has to satisfy the Leibnitz Rule at  $x$ ,

$$\partial(fg) = f(x)\partial(g) + g(x)\partial(f) = e_x(f)\partial(g) + e_x(g)\partial(f)$$

for all  $f, g \in \mathcal{O}_x$ , and it has to kill all  $f \in \mathcal{O}_x$  vanishing to first order. The Leibnitz Rule condition only uses the data of the  $\mathbf{R}$ -algebra  $\mathcal{O}_x$  and the mapping  $e_x$  that we have proved above is intrinsic to  $\mathcal{O}_x$ . However, the condition on killing elements vanishing to order 1 does not seem to be intrinsic to  $\mathcal{O}_x$  because the notion of “vanishing to order 1” is defined in terms of expressing germs in local  $C^p$  coordinates near  $x$  on  $X$  rather than in terms of  $\mathcal{O}_x$  itself. Remarkably, in the smooth case the concept of “vanishing to order 1” is intrinsic to the ring  $\mathcal{O}_x$ : one can prove that such elements are exactly those elements that are finite sums  $\sum g_i h_i$  with  $g_i, h_i \in \mathfrak{m}_x$ . This is the real content in the proof of the following result which shows that the notion of point-derivation at  $x$  in the smooth case is intrinsic to the  $\mathbf{R}$ -algebra  $\mathcal{O}_x$ :

**Theorem 2.3.** *If  $X$  is smooth, then any  $\mathbf{R}$ -linear map  $\partial : \mathcal{O}_x \rightarrow \mathbf{R}$  satisfying the Leibnitz Rule at  $x$  automatically kills those  $f \in \mathcal{O}_x$  that vanish to first order.*

*Proof.* We shall prove that if  $f$  vanishes to first order at  $x$  and  $\{t_1, \dots, t_n\}$  are local  $C^\infty$  coordinates near  $x$  with  $t_j(x) = 0$  then  $f = \sum t_j h_j$  for  $h_j \in \mathcal{O}_x$  with  $h_j(x) = 0$ . (If  $f$  is  $C^p$  for  $1 \leq p < \infty$  then one only gets  $h_j$  of class  $C^{p-1}$ , so the proof only works in the  $C^\infty$  case.) Using a local  $C^\infty$  chart, we can shift our problem to the origin  $x = 0$  in  $\mathbf{R}^n$  with its standard coordinate functions  $t_1, \dots, t_n$ , and with  $\mathbf{R}^n$  considered as a  $C^\infty$  manifold with its usual  $C^\infty$  structure. Let  $\partial : \mathcal{O}_0 \rightarrow \mathbf{R}$  be an  $\mathbf{R}$ -linear mapping that satisfies  $\partial(fg) = f(0)\partial(g) + g(0)\partial(f)$ . We want to prove that  $\partial(f) = 0$  whenever  $f$  vanishes to first order. We claim that near the origin,  $f = \sum t_j h_j$  for smooth functions  $h_j$  that vanish at the origin. If this is true then  $\partial(f) = \sum \partial(t_j h_j)$  and the Leibnitz Rule at the origin gives  $\partial(t_j h_j) = t_j(0)\partial(h_j) + h_j(0)\partial(t_j) = 0$ , so  $\partial(f) = 0$  as desired.

To find the expression  $f = \sum t_j h_j$  with smooth  $h_j$  vanishing at the origin, we use the first-order Taylor formula: this says that for  $x$  near the origin

$$f(t) = f(0) + \int_0^1 Df(ut)(t)du = \int_0^1 \sum_j (\partial_j f)(ut)t_j du = \sum t_j h_j$$

with  $h_j(t) = \int_0^1 (\partial_j f)(ut)du$ . To see that  $h_j$  is smooth we use the theorem on differentiation through the integral and the fact that  $\partial_j f$  is smooth (as  $f$  is smooth). Clearly  $h_j(0) = 0$  because  $(\partial_j f)(0)$  vanishes due to the assumption that  $f$  vanishes to first order. ■

Note that if we were to try to push through the above proof in the  $C^p$  case with finite  $p \geq 1$ , we would run into a serious problem:  $\partial_j f$  would only be  $C^{p-1}$ , and so  $h_j$  would only be  $C^{p-1}$ . This suggests that in the local ring  $\mathcal{O}_0$  it may not be generally possible to express a  $C^p$  function  $f$  vanishes to first order as a sum  $f = \sum t_j h_j$  with  $h_j \in \mathcal{O}_0$ . Indeed, this *is* usually impossible. A simple counterexample on the real line is  $f(t) = t^{3/2}$ . This is  $C^1$  and vanishes to first order at the origin, but we cannot write  $f = th$  near the origin with  $h$  a  $C^1$  (or even differentiable!) function, as we must take  $h(t) = \sqrt{t}$  and this is not differentiable at the origin. Thus, we see that the smoothness hypothesis is crucial for the truth of the result.

### 3. THE GLOBAL NOTION

Inspired by Theorem 2.3, for smooth manifolds  $X$  we are led to consider the notion of a *global point derivation at  $x \in X$*  to be an  $\mathbf{R}$ -linear map  $D : \mathcal{O}(X) \rightarrow \mathbf{R}$  satisfying the “Leibnitz Rule at  $x$ ”:  $D(fg) = f(x)D(g) + g(x)D(f)$  for all  $f, g \in \mathcal{O}(X)$ . Such  $D$ ’s form an  $\mathbf{R}$ -vector space in the evident manner. In the course book (as in many introductory books on differential geometry), this is often given as the *definition* of a tangent vector at  $x$  (for smooth  $X$ ). We shall prove that this apparently global definition agrees with our local definition of tangent vectors in a precise sense. The key is that, despite the apparently global nature of  $D$ , the value  $D(f)$  only depends on  $f$  near  $x$ :

**Lemma 3.1.** *Assume  $X$  is a smooth Hausdorff premanifold. If  $f, g \in \mathcal{O}(X)$  agree on an open set containing  $x$ , then  $D(f) = D(g)$ . In particular, if the germ  $[(X, f)] \in \mathcal{O}_x$  vanishes then  $D(f) = 0$ .*

*Proof.* Since  $D(f) - D(g) = D(f - g)$ , we may assume  $f$  vanishes on an open  $U$  around  $x$  and we want to conclude  $D(f) = 0$ . Let  $(\phi', U')$  be a  $C^\infty$ -chart with  $U' \subseteq U$ . By the theory of bump functions on open sets in a vector space (which we may restrict to sectors in the vector space), on the open set  $\phi(U')$  in a sectors in a vector space there is a  $C^\infty$  function  $\varphi$  on  $\phi(U')$  with compact support such that  $\varphi$  equals 1 near  $\phi(x)$ . Thus,  $\varphi \circ \phi'^{-1} : U' \rightarrow \mathbf{R}$  is a  $C^\infty$  function that equals 1

near  $x$  and has compact support  $K$ . But  $X$  is Hausdorff, so  $K$  is closed in  $X$  (and not merely in  $U'$ ). It follows from openness of  $X - K$  in  $X$  that  $\varphi \circ \phi'^{-1} \in \mathcal{O}(U')$  and  $0 \in \mathcal{O}(X - K)$  agree on overlaps and hence uniquely glue to a global  $C^\infty$  function.

Thus, we get  $g \in \mathcal{O}(X)$  that equals 1 near  $x$  and vanishes outside of the set  $K \subseteq U$ . But  $f$  vanishes on  $U$ , so  $gf = 0$ . Applying  $D$ ,

$$0 = D(0) = D(fg) = f(x)D(g) + g(x)D(f) = 0 + D(f) = D(f).$$

■

Observe that the preceding proof rested crucially on the existence of bump functions and the Hausdorff property of  $X$ .

**Lemma 3.2.** *Assume  $X$  is smooth and Hausdorff. The natural map of  $\mathbf{R}$ -algebras  $\pi_x : \mathcal{O}(X) \rightarrow \mathcal{O}_x$  given by  $f \mapsto [(X, f)]$  is surjective.*

*Proof.* Pick a representative  $f : U \rightarrow \mathbf{R}$  for an element of  $\mathcal{O}_x$ . We seek an element in  $\mathcal{O}(X)$  whose restriction to  $U$  agrees with  $f$  near  $x$ . As in the preceding proof, we can find  $g \in \mathcal{O}(X)$  with  $g = 1$  near  $x$  and  $g = 0$  outside of a compact set  $K \subseteq U$ . Hence,  $g|_U \cdot f \in \mathcal{O}(U)$  agrees with  $f$  near  $x$  but vanishes outside of  $K$ . Since  $X - K$  is open, we can glue  $g|_U \cdot f \in \mathcal{O}(U)$  and  $0 \in \mathcal{O}(X - K)$  to get  $\tilde{f} \in \mathcal{O}(X)$  that agrees with  $f$  near  $x$ . ■

For the remainder of the discussion, we assume  $X$  to be smooth and Hausdorff. By Lemma 3.2, we view  $\mathcal{O}_x$  as a quotient of  $\mathcal{O}(X)$  (say as  $\mathbf{R}$ -vector spaces). By Lemma 3.1, the  $\mathbf{R}$ -linear map  $D : \mathcal{O}(X) \rightarrow \mathbf{R}$  defined by a global point derivation at  $x$  kills the kernel of the quotient map  $\pi_x : \mathcal{O}(X) \rightarrow \mathcal{O}_x$  and hence uniquely factors as  $D = [D] \circ \pi_x$  for an  $\mathbf{R}$ -linear map  $[D] : \mathcal{O}_x \rightarrow \mathbf{R}$ . This linear map satisfies the Leibnitz Rule at  $x$  (and so it is a tangent vector at  $x$  by Theorem 2.3!). Indeed, for any  $f_x, g_x \in \mathcal{O}_x$  we have  $f_x = \pi_x(f)$  and  $g_x = \pi_x(g)$  for  $f, g \in \mathcal{O}(X)$ , so since  $\pi_x$  carries products to products we get

$$[D](f_x g_x) = [D](\pi_x(fg)) = D(fg) = f(x)D(g) + g(x)D(f) = f_x(x)[D](g_x) + g_x(x)[D](f_x).$$

Conversely, if  $\partial \in T_x(X)$  is a tangent vector in our sense then composing  $\partial : \mathcal{O}_x \rightarrow \mathbf{R}$  with  $\pi_x$  gives  $D = \partial \circ \pi_x$  that is readily checked to be a global point derivation at  $x$ . Hence,  $D \mapsto [D]$  defines a surjective  $\mathbf{R}$ -linear map from the  $\mathbf{R}$ -vector space of global point derivations at  $x$  onto  $T_x(X)$ . Lemma 3.1 and Lemma 3.2 ensure that  $[D] = 0$  only if  $D = 0$ , and hence we really have an isomorphism from the  $\mathbf{R}$ -vector space of global point derivations at  $x$  onto  $T_x(X)$ . Thus, for smooth  $X$  we have obtained an  $\mathbf{R}$ -linear isomorphism between the tangent space at  $x \in X$  as defined in the course text and the local definition that we have used as our foundation.

Note that in the preceding discussion, if we relaxed  $C^\infty$  to  $C^p$  for  $1 \leq p < \infty$  then we would get an isomorphism between the  $\mathbf{R}$ -vector space of global derivations at  $x$  and the  $\mathbf{R}$ -vector space of  $\mathbf{R}$ -linear maps  $\partial : \mathcal{O}_x \rightarrow \mathbf{R}$  satisfying the Leibnitz Rule at  $x$ . But this is useless unless we know that satisfying the Leibnitz Rule at  $x$  forces annihilation of germs vanishing to order 1. Hence, only in the  $C^\infty$  Hausdorff case can we get something interesting, thanks to Theorem 2.3.