

MATH 396. GLUING TOPOLOGIES, THE HAUSDORFF CONDITION, AND EXAMPLES

There are many important topological spaces (and manifolds) that are constructed by “gluing” pieces of spaces. This typically takes the form of gluing along open sets or passing to quotients by (reasonable) equivalence relations. In this handout we explain some general gluing procedures, and we give some examples. Since the Hausdorff property of a space is not local, it is a non-trivial condition on a gluing construction that the end result be Hausdorff, even when the initial pieces are Hausdorff. (A basic example is the line with a doubled point: it is given by gluing the real line to itself along the complement of the origin.) Hence, we also explore different ways to think about the Hausdorff condition in order that we can formulate a clean criterion for a gluing of Hausdorff spaces to be Hausdorff. We conclude with a representative non-trivial construction that will provide an important family of compact Hausdorff manifolds.

1. GLUING AND THE HAUSDORFF PROPERTY

Let  $X$  be a topological space and let  $\{X_i\}$  be an open covering, so each  $X_i$  gets an induced topology. Note that a subset  $U \subseteq X$  is open if and only if  $U \cap X_i$  is open in  $X_i$  for each  $i$ . (Since  $U$  is the union of the  $U \cap X_i$ 's, the key point is that for an open subset  $Y'$  in a topological space  $Y$ , a subset of  $Y'$  is open for the induced topology if and only if it is open in  $Y$ .) We will use this without comment.

If  $f : X \rightarrow Y$  is a continuous map, then by restriction to  $X_i$  we get continuous maps  $f_i : X_i \rightarrow Y$  such that  $f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}$  for all  $i$  and  $j$ . Conversely, if we are given continuous maps  $f_i : X_i \rightarrow Y$  that agree on overlaps (that is,  $f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}$  for all  $i$  and  $j$ ) then there is a unique set-theoretic map  $f : X \rightarrow Y$  satisfying  $f|_{X_i} = f_i$  for all  $i$  and it is continuous. Indeed, for any open  $V \subseteq Y$  we have that  $f^{-1}(V)$  is open in  $X$  because  $f^{-1}(V) \cap X_i = f_i^{-1}(V)$  is open in  $X_i$  for every  $i$ . Hence, we can view continuous maps  $X \rightarrow Y$  as collections of continuous maps  $X_i \rightarrow Y$  that are compatible on overlaps  $X_i \cap X_j$ . We want to run this procedure in reverse.

**Theorem 1.1.** *Let  $X$  be a set, and let  $\{X_i\}$  be a collection of subsets whose union is  $X$ . Suppose on each  $X_i$  there is given a topology  $\tau_i$ , and that the  $\tau_i$ 's are compatible in the following sense:  $X_i \cap X_j$  is open in each of  $X_i$  and  $X_j$ , and the induced topologies on  $X_i \cap X_j$  from both  $X_i$  and  $X_j$  coincide. There is a unique topology on  $X$  that induces upon each  $X_i$  the topology  $\tau_i$ .*

We say that the topology in this theorem is obtained by *gluing* the given topologies on the  $X_i$ 's. (We may also say that the topological space  $(X, \tau)$  is obtained from *gluing* the topological spaces  $(X_i, \tau_i)$ .) Intuitively, the topology on  $X$  declares that points “near”  $x \in X$  are those that are “near” to it in a fixed  $X_i$  containing  $x$ , and the compatibility condition is what ensures that this notion does not depend on the particular choice of  $X_i$  that contains  $x$ .

It is not at all clear that if the  $\tau_i$ 's are metrizable then  $\tau$  is metrizable, and this is false:  $\tau$  can even fail to be Hausdorff even when all  $\tau_i$ 's are Hausdorff. (The line with a doubled point gives a counterexample, as we will see.) The ability to glue without having to simultaneously confront other structures (such as metrics) is a big advantage of working with topological spaces rather than metric spaces as the basic geometric structures.

*Proof.* We first prove uniqueness. If  $\tau$  is a topology on  $X$  inducing  $\tau_i$  for each  $i$  and making  $X_i$  open in  $X$  for each  $i$ , then a subset  $U \subseteq X$  is open for  $\tau$  if and only if  $U \cap X_i$  is open for the induced topology on  $X_i$  for each  $i$  (as  $X_i$  is  $\tau$ -open for every  $i$ ), and hence (by the assumption that the induced topology on  $X_i$  is  $\tau_i$ ) if and only if  $U \cap X_i$  is  $\tau_i$ -open in  $X_i$  for each  $i$ . This final formulation of the openness condition for  $\tau$  is expressed entirely in terms of the  $\tau_i$ 's and so establishes uniqueness: we have no choice as to what the condition of  $\tau$ -openness is to be, and it

must be the case that that  $\tau$ -open sets in  $X$  are exactly those that meet each  $X_i$  in a  $\tau_i$ -open subset of  $X_i$  for each  $i$ .

We now run the process in reverse to verify existence. We *define*  $\tau$  to be the collection of subsets  $U \subseteq X$  such that  $U \cap X_i$  is  $\tau_i$ -open in  $X_i$  for each  $i$ . Is this a topology on  $X$ , and does it make each  $X_i$  open in  $X$  with  $\tau_i$  as its induced topology? By the definition, clearly  $\emptyset$  and  $X$  are in  $\tau$ . If  $\{S_\alpha\}$  is a collection of subsets of  $X$  in  $\tau$  then we need the union  $S$  of the  $S_\alpha$ 's to be in  $\tau$ . That is, we want  $S \cap X_i$  to be  $\tau_i$ -open for each  $i$ . This overlap is the union of the overlaps  $S_\alpha \cap X_i$  that are  $\tau_i$ -open for all  $\alpha$  (due to the assumption that every  $S_\alpha$  is in  $\tau$ ), and so this union is also  $\tau_i$ -open for each  $i$ . Hence, indeed  $S$  is in  $\tau$ . A similar argument works for finite intersections, and so confirms that  $\tau$  is a topology on  $X$ . Since for each fixed  $i_0$  the overlap  $X_{i_0} \cap X_j$  is  $\tau_j$ -open in  $X_j$  for every  $j$  (by the hypotheses on the  $X_i$ 's at the outset), it follows that  $X_{i_0}$  is  $\tau$ -open in  $X$  for every  $i_0$ .

Finally, we have to check that  $\tau$  induces the topology  $\tau_i$  on the open subset  $X_i$  for each  $i$ . Let  $U \subseteq X_i$  be a subset. We must prove that it is open for  $\tau_i$  if and only if it is open for the induced topology from  $\tau$ . Since  $X_i$  is  $\tau$ -open in  $X$ , a subset of  $X_i$  is open for the induced topology from  $\tau$  if and only if it is itself  $\tau$ -open in  $X$ . Thus, we need to prove that  $U \subseteq X_i$  is  $\tau_i$ -open in  $X_i$  if and only if it is  $\tau$ -open in  $X$ . The condition of  $\tau$ -openness means (by definition of  $\tau$ !) that  $U \cap X_j$  is  $\tau_j$ -open in  $X_j$  for each  $j$ , which for  $j = i$  says that  $U$  is  $\tau_i$ -open in  $X_i$  (as  $U \cap X_i = U$ ), and so we need to prove that a subset  $U \subseteq X_i$  that is  $\tau_i$ -open necessarily has  $\tau_j$ -open overlap  $U \cap X_j$  with  $X_j$  for each  $j$ . By assumption  $X_i \cap X_j$  inherits the same topology from both  $X_i$  and  $X_j$ , and it is open in each, and by its  $\tau_j$ -openness in  $X_j$  we see that the subset  $U \cap X_j \subseteq X_i \cap X_j$  is  $\tau_j$ -open in  $X_j$  if and only if it is open for the topology on  $X_i \cap X_j$  induced from  $X_j$ . However, this latter induced topology on  $X_i \cap X_j$  is also the one induced from  $\tau_i$  (by the compatibility hypothesis on the  $\tau_k$ 's!), and so since  $U \cap X_j = U \cap (X_i \cap X_j)$  with  $U$  a  $\tau_i$ -open subset of  $X_i$  (by assumption!) it follows from the definition of "induced topology" that  $U \cap (X_i \cap X_j)$  is indeed open in  $X_i \cap X_j$  for the topology induced by  $\tau_i$ . ■

A basic question we must now wish to address is this: if  $X$  is given a topology  $\tau$  obtained from gluing compatible topologies  $\tau_i$  on each  $X_i$ , how can we detect whether or not  $X$  is Hausdorff? Since  $\tau_i$  is the induced topology on  $X_i$  from  $\tau$ , if  $X$  is going to be Hausdorff for  $\tau$  then it is certainly necessary that each  $X_i$  be Hausdorff for  $\tau_i$ . To better understand how the converse can fail, and how we can detect those cases when it holds, we need to recast the Hausdorff condition in a more useful form. The key idea is to look at the diagonal map  $\Delta_X : X \rightarrow X \times X$  defined by  $x \mapsto (x, x)$ . Rather generally, if  $Y$  is any topological space then we have a diagonal map  $\Delta_Y : Y \rightarrow Y \times Y$  given by  $y \mapsto (y, y)$ , and this is not only continuous but even a homeomorphism onto its image. Indeed, if  $U \subseteq Y$  is open then  $(U \times U) \cap \Delta_Y(Y)$  is open in  $\Delta_Y(Y)$  for the subspace topology from  $Y \times Y$ , and its preimage in  $Y$  is  $U$ . It turns out that the Hausdorff property for  $Y$  encodes a global feature of how  $\Delta_Y(Y)$  sits in  $Y \times Y$ :

**Lemma 1.2.** *The Hausdorff property of  $Y$  is equivalent to  $\Delta_Y(Y) \subseteq Y \times Y$  being a closed subset. That is,  $Y$  is Hausdorff if and only if  $\Delta_Y$  is a homeomorphism onto a closed subset of  $Y \times Y$ .*

*Proof.* The closedness of  $\Delta_Y(Y)$  in  $Y \times Y$  is equivalent to the openness of  $(Y \times Y) - \Delta_Y(Y)$  in  $Y \times Y$ . A point in this complement has the form  $(y, y')$  with  $y \neq y'$  in  $Y$ , and by definition of the product topology we see that such openness is exactly the statement that for any pair of distinct points  $y, y' \in Y$  there exist open  $U, U' \subseteq Y$  such that

- $U \times U'$  contains  $(y, y')$  (that is,  $y \in U$  and  $y' \in U'$ ),
- $U \times U' \subseteq (Y \times Y) - \Delta_Y(Y)$ . That is, we require  $(U \times U') \cap \Delta_Y(Y) = \emptyset$ .

The second condition says exactly that  $\Delta_Y^{-1}(U \times U') = \emptyset$ , and this preimage is exactly  $U \cap U'$  due to the definition of  $\Delta_Y$ . Hence, we have arrived at an equivalent formulation of the initial closedness

condition: for distinct  $y, y' \in Y$  there exist disjoint opens  $U, U' \subseteq Y$  such that  $y \in U$  and  $y' \in U'$ . This is precisely the Hausdorff property. ■

Now consider a topological space  $X$  covered by open subsets  $X_i$ . Hence, the opens  $X_i \times X_j$  cover  $X \times X$ . Since closedness is local on the ambient space (by the handout on interior, closure, and boundary), a subset  $A \subseteq X \times X$  is closed if and only if its overlap with each  $X_i \times X_j$  is closed in  $X_i \times X_j$ . By definition of  $\Delta_X(X)$  and the homeomorphism identification of  $X$  with  $\Delta_X(X)$ , clearly  $\Delta_X(X) \cap (X_i \times X_j)$  is identified with  $X_i \cap X_j \subseteq X$ . Hence,  $\Delta_X(X) \subseteq X \times X$  is closed if and only if the map  $X_i \cap X_j \rightarrow X_i \times X_j$  has closed image for each  $i$  and  $j$ . For  $i = j$  this says that  $X_i$  is closed in  $X_i \times X_i$  (via  $\Delta_{X_i}$  for all  $i$ , which is to say that each  $X_i$  is Hausdorff), but there are more conditions! Indeed, we also have the conditions that the maps  $X_i \cap X_j \rightarrow X_i \times X_j$  have closed image for each  $i \neq j$ . Thus, we have established the following Hausdorff criterion:

**Theorem 1.3.** *Let  $(X, \tau)$  be obtained by gluing topological spaces  $\{(X_i, \tau_i)\}$ . The space  $X$  is Hausdorff if and only if each  $X_i$  is Hausdorff and the subset  $X_i \cap X_j$  is closed in the product topological space  $X_i \times X_j$  where the factors are given the respective topologies  $\tau_i$  and  $\tau_j$ .*

*Example 1.4.* Let us return to the example of the line with the doubled origin. In this case  $X_1 = \mathbf{R}$  and  $X_2 = \mathbf{R}$  with  $X_1 \cap X_2$  the complement of the origin in  $X_1$  and  $X_2$  respectively. The compatibility of the topologies on this subset viewed in each of  $X_1$  or  $X_2$  is clear, and  $X_1 \cap X_2$  as a subset of  $X_1 \times X_2 = \mathbf{R} \times \mathbf{R}$  is the complement of the origin in the line  $u = v$ . This is clearly not closed in  $\mathbf{R} \times \mathbf{R}$ !

We will see a more interesting application in the affirmative direction in the next section, but for now we give some additional practical applications of the diagonal description of the Hausdorff property.

**Theorem 1.5.** *Let  $f, g : X \rightrightarrows Y$  be continuous maps to a Hausdorff space  $Y$  such that for a dense subset  $S \subseteq X$  we have  $f|_S = g|_S$ . Then  $f = g$ .*

Recall that a subset of a topological space is *dense* if its closure is the entire space. In the setting of metric spaces, this theorem is easily proved by a limiting argument. In general we cannot use limits in  $X$  when it is non-Hausdorff (and does not have a countable base of opens around all points). Also note that the Hausdorff hypothesis on  $Y$  is crucial: if we let  $Y$  be the line with a doubled origin, there are two natural maps  $\mathbf{R} \rightrightarrows Y$  sending  $0 \in \mathbf{R}$  to each of the two origins; these two maps are distinct, yet they coincide on the dense subset  $\mathbf{R}^\times \subseteq \mathbf{R}$ .

*Proof.* Consider the continuous product map  $(f, g) : X \rightarrow Y \times Y$ . To say  $f = g$  is to say that this map has image contained in  $\Delta_Y(Y)$ , or in other words that the preimage of  $\Delta_Y(Y)$  under  $(f, g)$  is equal to  $X$ . This preimage contains  $S$ , due to the fact that  $f|_S = g|_S$ , and this preimage is *closed* in  $X$  because  $(f, g)$  is continuous and  $\Delta_Y(Y) \subseteq Y \times Y$  is closed (as  $Y$  is Hausdorff). Hence,  $(f, g)^{-1}(\Delta_Y(Y)) \subseteq X$  is a closed set in  $X$  that contains the dense subset  $S$ , and so it contains  $\bar{S} = X$ . ■

An amusing application (which you may skip if you're not interested) is in the setting of topological groups. A *topological group* is a topological space  $G$  endowed with a group structure such that the multiplication law  $G \times G \rightarrow G$  and the inversion map  $G \rightarrow G$  are continuous. Basic interesting examples are a finite-dimensional vector space  $V$  with the operation of addition, and (more interesting)  $G = \text{GL}(V)$ ; in terms of coordinates we identify  $\text{GL}(V)$  with  $\text{GL}_n(\mathbf{R})$ , so the continuity of multiplication and inversion is seen through the habitual formula for matrix multiplication and matrix inversion. In the case of topological groups, there is a very simple criterion for the Hausdorff property to hold:

**Theorem 1.6.** *Let  $G$  be a topological group. It is Hausdorff if and only if the identity point is closed in  $G$ .*

Before we prove the theorem, we make some remarks on the property of closedness for the identity point. For any  $g \in G$  the continuity of multiplication implies that the left multiplication map  $\ell_g : g' \mapsto gg'$  is continuous, and the map  $\ell_{g^{-1}}$  is a continuous inverse. Thus,  $\ell_g : G \rightarrow G$  is a homeomorphism, and it carries the identity point to  $g$ . Thus, by using such maps we see that if one point of  $G$  is closed then all points of  $G$  are closed. Of course, in a Hausdorff space any point is closed (the openness of the complement of a point is immediate from the usual definition of the Hausdorff condition), and so the remarkable aspect of the theorem is that this consequence of the Hausdorff property is also sufficient in the setting of topological groups.

*Proof.* The necessity of the identity point being closed has just been explained, and we seek to prove sufficiency. We want to prove that  $\Delta_G(G)$  is closed in  $G \times G$ . If we let  $i : G \simeq G$  denote the continuous inversion map (which is an isomorphism, as it is its own inverse map), then  $1 \times i : G \times G \rightarrow G \times G$  is an isomorphism of topological spaces, and so  $\Delta_G(G)$  is closed if and only if  $(1 \times i)(\Delta_G(G))$  is closed. This latter set is the set of points of the form  $(g, g^{-1})$ , so it is the preimage of the identity under the continuous multiplication map  $m : G \times G \rightarrow G$ . Hence, closedness of the identity point in  $G$  gives the desired closedness in  $G \times G$ . ■

## 2. GRASSMANNIANS: ALGEBRAIC THEORY

As a fundamental example of a construction of a topology by gluing, we investigate an important example: Grassmannians. We begin with the algebraic development over a general field, and then we specialize to the case  $F = \mathbf{R}$  where we can bring in some topology. (Everything we do over  $\mathbf{R}$  works the same way over  $\mathbf{C}$ , once one sets up the right topological theory over  $\mathbf{C}$ .)

Let  $F$  be a field and  $V$  a vector space over  $F$  of dimension  $n + 1$  ( $n \geq 1$ ). Let  $\mathbf{G}_d(V)$  denote the set of codimension- $d$  subspaces of  $V$  (with  $1 \leq d \leq n$ ). When  $V = F^{n+1}$  this set is called the *Grassmannian of codimension- $d$  subspaces in  $(n + 1)$ -space* (over  $F$ ) and is denoted  $\mathbf{G}(d, n)(F)$  (or just  $\mathbf{G}(d, n)$  if  $F$  is understood). For  $d = 1$  it is called *projective space* and is denoted  $\mathbf{P}(V)$ . By the dual relationship between subspaces of  $V$  and of  $V^\vee$  (whereby a subspace  $W \subseteq V$  “corresponds” to the subspace  $(V/W)^\vee$  in  $V^\vee$ ),  $\mathbf{G}_d(V)$  can also be viewed as  $\mathbf{G}_{n+1-d}(V^\vee)$ . We want to enhance  $\mathbf{G}_d(V)$  from a set to a topological space when  $F = \mathbf{R}$ . We shall first cover the Grassmannian  $\mathbf{G}_d(V)$  by nice subsets that can be identified with Euclidean spaces (this essentially amounts to finding “standardized” equations for codimension- $d$  subspaces of  $V$ , subject to some geometric constraints on the subspace). In the special case  $F = \mathbf{R}$  there will result a topology by gluing. *We urge the reader to think about the cases  $d = 1, 2$  when formulas below look too complicated.*

You may be wondering at the outset: what is the point of studying these spaces? There are many situations in geometry where one needs to study moving families of linear subspaces of a vector space, such as when one studies a surface  $S$  in  $\mathbf{R}^3$  by slicing it with a family of parallel planes (such as  $x = c$  for  $c \in \mathbf{R}$ ) and examining the topology of the slices as the planes move. To this end, Grassmannians are examples of “parameter spaces”: their points parameterize (at least for now in a set-theoretic sense) some interesting geometric structure (the set of all subspaces of a fixed codimension), and so any investigation using variation of such structures (e.g., varying subspaces of  $V$  with a fixed codimension) is naturally informed by the geometry of the “space” of all such structures (once this “space” is given a reasonable topology which we can in turn interpret in terms of the geometric structure being parameterized!). For example, as we shall see, the Grassmannians for  $F = \mathbf{R}$  are naturally *compact* and *Hausdorff* spaces. This ensures that it is often possible to “pass to the limit” on constructions with moving linear subspaces in  $V$  with a fixed codimension,

at least in the sense that any sequence of points in a compact space (such as sequence of points in a Grassmannian, which is to say a sequence of codimension- $d$  subspaces of  $V$ ) has a convergent subsequence.

You should consider Grassmannians (for the time being) as a concrete collection of parameter spaces whose points classify interesting objects. There is no God-given coordinate system on these spaces, but we will see that Grassmannians are covered by natural subsets that do admit reasonable coordinatizations, and Grassmannians do not *a priori* live inside of any “ambient” Euclidean space. In this sense, they will convince us that it is worthwhile to develop the theory of differential geometry for “abstract” spaces that (like the curved spacetime of General Relativity) do not naturally sit inside of an ambient  $\mathbf{R}^N$ .

Fix an ordered basis  $e_0, \dots, e_n$  of  $V$ .

**Lemma 2.1.** *Let  $I = \{i_1, \dots, i_d\}$  be a set of  $d$  distinct indices  $0 \leq i_1 < \dots < i_d \leq n$ . Let  $U_I \subseteq \mathbf{G}_d(V)$  denote the subset of codimension- $d$  linear subspaces  $W \subseteq V$  for which the  $e_i$ 's with  $i \in I$  project to a basis of the  $d$ -dimensional  $V/W$ . The  $U_I$ 's cover  $\mathbf{G}_d(V)$  as a set.*

*Proof.* Since the  $e_j$ 's span  $V$ , their images span the quotient  $V/W$ . Thus, some subset is a basis for this quotient. Since this quotient has dimension  $d$ , there is some  $d$ -tuple  $I$  of distinct indices such that  $\{e_j \bmod W\}_{j \in I}$  gives a basis of  $V/W$ . Thus,  $W \in \mathbf{G}_d(V)$  lies in  $U_I$ . ■

*Remark 2.2.* We really should write  $U_{I,\mathbf{e}}$ , as it depends on the  $d$ -tuple  $I \subseteq \{0, \dots, n\}$  and  $\mathbf{e}$ , but since  $\mathbf{e}$  is fixed (for now!) we will write  $U_I$  to keep the notation uncluttered.

With notation as above, suppose  $W \in U_I$ , so for each  $0 \leq j \leq n$  we can uniquely write  $e_j \equiv \sum_{i \in I} a_{ij} e_i \bmod W$  with  $a_{ij} \in F$  (by the definition of  $U_I$ ); of course, when  $j \in I$  then  $a_{ij} = 0$  for  $j \neq i$  and  $a_{ii} = 1$ . We may also write  $a_{ij}(W)$  to emphasize the dependence on  $W$  (and that these are “coordinates” that describe  $W$  considered as a “point” in  $U_I$ ). In terms of these coefficients, can we describe those points of  $U_I$  that lie in  $U_I \cap U_{I'}$ ? Indeed we can:

**Lemma 2.3.** *The  $W$ 's in  $U_I$  lying in  $U_I \cap U_{I'}$  are those for which the  $d \times d$  determinant*

$$\text{Det}(a_{ii'}(W))_{i \in I, i' \in I'}$$

*is non-zero.*

*Proof.* Let  $\bar{v} = v \bmod W \in V/W$  for  $v \in V$ . To say that  $W$  lies in  $U_I \cap U_{I'}$  is to say that the  $d$  vectors  $\bar{e}_i$  for  $i \in I$  as well as the  $d$  vectors  $\bar{e}_{i'}$  for  $i' \in I'$  form a basis. The matrix  $(a_{ii'}(W))$  describes (relative to the basis of  $\bar{e}_i$ 's for  $i \in I$ ) the unique linear self-map  $T_{I'}$  of  $V/W$  which sends the ordered set  $\{\bar{e}_{i_1}, \dots, \bar{e}_{i_d}\}$  (with  $i_1 < \dots < i_d$  the elements of  $I$ ) to the ordered set  $\{\bar{e}_{i'_1}, \dots, \bar{e}_{i'_d}\}$  (with  $i'_1 < \dots < i'_d$  the elements of  $I'$ ). The vectors  $\bar{e}_{i'_j}$  span  $V/W$  if and only if they are a basis (as there are  $d$  of them), so given that  $W \in U_I$  it follows that  $W \in U_{I'}$  if and only if  $T_{I'}$  is an isomorphism, or equivalently if and only if its matrix  $(a_{ii'}(W))$  has non-zero determinant. ■

We define a map of sets

$$\phi_I : U_I \rightarrow F^{I \times (\{0, \dots, n\} - I)}$$

by  $\phi_I(W) = (a_{ij}(W))_{i \in I, j \notin I}$ , where  $e_j \equiv \sum_{i \in I} a_{ij}(W) e_i \bmod W$ . We have noted above that such coefficients  $a_{ij}(W)$  are uniquely determined by  $W$  (as we've *fixed* our ordered basis of  $V$ ). Thus,  $\phi_I$  is well-defined. (As with  $U_I$ , we really should write  $\phi_{I,\mathbf{e}}$  since this map depends on the choice of ordered basis  $\mathbf{e}$ , but we omit this from the notation since  $\mathbf{e}$  is fixed for now.) We claim more:

**Lemma 2.4.** *The map  $\phi_I$  is bijective.*

*Proof.* For injectivity, we need to prove that  $\phi_I(W)$  determines  $W$ . In fact, we will explicitly reconstruct  $W$  from the standard coordinates of  $\phi_I(W)$ , which is to say from the  $a_{ij}(W)$ 's for  $i \in I$  and  $j \notin I$ . By the definition of these  $a_{ij}(W)$ 's, for  $j \notin I$  the vector

$$v_{j,W} = e_j - \sum_{i \in I} a_{ij}(W)e_i$$

lies in  $W$ , and as  $j$  runs over all  $n + 1 - d$  indices not in  $I$  these vectors  $v_{j,W}$  are clearly linearly independent of each other (as a given  $v_{j_0,W}$  is the only one of the  $v_{j,W}$ 's ( $j \notin I$ ) with a non-zero coefficient for  $e_j$ ). Thus, their span has dimension  $n + 1 - d$ . But this span lies inside of  $W$  and  $\dim W = n + 1 - d$ . Hence, the  $v_{j,W}$ 's for  $j \notin I$  span  $W$ , so we have universally reconstructed  $W$  from  $\phi_I(W)$ . In particular,  $\phi_I$  is injective.

To see that  $\phi_I$  is surjective, we just run the above argument in reverse: given  $(a_{ij})_{i \in I, j \notin I}$  we define  $W \subseteq V$  to be the span of the  $n + 1 - d$  vectors

$$v_j \stackrel{\text{def}}{=} e_j - \sum_{i \in I} a_{ij}e_i$$

for  $j \notin I$ . These are linearly independent (for the same reason as used in the preceding paragraph with the  $v_{j,W}$ 's), so they span a subspace  $W$  in  $V$  with codimension  $d$ . By construction we see that the  $d$ -dimensional  $V/W$  is spanned by (and hence has as basis) the  $d$  vectors  $\bar{e}_i$  for  $i \in I$ , so  $W \in U_I$  and then clearly  $a_{ij}(W) = a_{ij}$ , so  $\phi_I(W) = (a_{ij})$ . This establishes surjectivity of  $\phi_I$ . ■

Using  $\phi_I$ , points in  $U_I \cap U_{I'}$  can be put in  $F^{I \times (\{0, \dots, n\} - I)}$  (whose coordinates are indexed by ordered pairs  $(i, i')$  with  $i \in I$  and  $i' \notin I$ ), and similarly  $\phi_{I'}$  puts  $U_I \cap U_{I'}$  into  $F^{I' \times (\{0, \dots, n\} - I')}$ . If we're given  $W \in U_I$ , then in terms of the standard coordinates of  $\phi_I(W)$  how do we detect if also  $W \in U_{I'}$ ? That is, we want to describe  $\phi_I(U_I) \cap \phi_I(U_{I'}) = \phi_I(U_I \cap U_{I'})$ . By what we have shown above, the subset  $\phi_I(U_I \cap U_{I'})$  in  $F^{I \times (\{0, \dots, n\} - I)}$  consists of those elements  $(a_{ij})_{i \in I, j \notin I}$  with  $\text{Det}(a_{ii'})_{i \in I, i' \in I'} \neq 0$ , where for  $i' \in I' \cap I$  we define  $a_{ii'} = 0$  when  $i' \neq i$  and  $a_{ii'} = 1$  when  $i' = i$ . A more succinct description is

$$(1) \quad \phi_I(U_I \cap U_{I'}) = \{(a_{ij}) \in F^{I \times (\{0, \dots, n\} - I)} \mid \text{Det}(a_{ii'})_{i \in I - I' \cap I, i' \in I' - I \cap I} \neq 0\}.$$

We have now covered  $\mathbf{G}_d(V)$  by subsets  $U_I$  endowed with bijective maps  $\phi_I$  onto the Euclidean space  $F^{I \times (\{0, \dots, n\} - I)}$ , but we can do better: we can describe the “transition map”

$$\phi_{II'} = \phi_{I'} \circ \phi_I^{-1} : \phi_I(U_I \cap U_{I'}) \simeq \phi_{I'}(U_I \cap U_{I'})$$

between the systems of linear coordinates arising from  $\phi_I$  and  $\phi_{I'}$  on  $U_I \cap U_{I'}$ . From the explicit descriptions in terms of non-vanishing of determinants, the description of  $\phi_{I'} \circ \phi_I^{-1}$  comes down to the following: given  $\mathbf{a}_I = (a_{ij})_{i \in I, j \notin I}$  with  $\text{Det}(a_{ii'})_{i \in I, i' \in I'} \neq 0$  (again using the convention that when  $i' \in I \cap I'$  then  $a_{ii'} = 0$  for  $i' \neq i$  and  $a_{ii'} = 1$  for  $i' = i$ ), with this point corresponding to some  $\phi_I(W)$  for  $W \in U_I \cap U_{I'}$ , we want to describe the “coordinates”  $\mathbf{a}_{I'} = (a_{i'j'})_{i' \in I', j' \notin I'}$  of  $\phi_{I'}(W)$ . Let us introduce the notation

$$M_{\mathbf{a}_I} = (a_{ii'})_{i \in I, i' \in I'}, \quad N_{\mathbf{a}_I} = (a_{ij'})_{i \in I, j' \notin I'}$$

where once again  $a_{ir} = \delta_{ir}$  for  $r \in I$  (i.e., 0 for  $r \neq i$  and 1 for  $r = i$ ). Thus,  $M_{\mathbf{a}_I}$  is an invertible  $d \times d$  matrix and  $N_{\mathbf{a}_I}$  is a  $d \times (n + 1 - d)$  matrix, with  $M_{\mathbf{a}_I}$  having columns giving the  $\bar{e}_i$ -coefficients of the vectors  $\bar{e}_{i'}$  for  $i' \in I'$  and with the columns of  $N_{\mathbf{a}_I}$  giving the  $\bar{e}_i$ -coefficients of the vectors  $\bar{e}_{j'}$  for  $j' \notin I'$ .

Thus,  $M_{\mathbf{a}_I}$  is the “change of basis” matrix from  $\{\bar{e}_{i'}\}$  coordinates to  $\{\bar{e}_i\}$  coordinates, whence  $M_{\mathbf{a}_I}^{-1}N_{\mathbf{a}_I}$  is the  $d \times (n+1-d)$  matrix whose columns are the  $\bar{e}_{i'}$  coordinates of the  $\bar{e}_i$ 's for  $j' \notin I'$ . Thus, we have

$$(2) \quad \phi_{II'} = \phi_{I'} \circ \phi_I^{-1} : \mathbf{a}_I \mapsto M_{\mathbf{a}_I}^{-1}N_{\mathbf{a}_I}.$$

This is some enormous messy expression in general, but at least we know (via Cramer) that it's given by some “universal” polynomial expressions with big determinant denominator.

Before we specialize to the situation  $F = \mathbf{R}$  and make a topology, let us work out these formulas in two representative non-trivial cases.

*Example 2.5.* Suppose  $d = 1$  but  $n \geq 1$  is arbitrary (i.e., hyperplanes in  $F^{n+1}$ , so  $\mathbf{P}(F^{n+1})$ ). In this case  $U_i$  is an  $n$ -dimensional Euclidean space on coordinates  $(a_{ij})_{j \neq i}$ , the correspondence being that such an ordered  $n$ -tuple goes over to the hyperplane whose equation is  $\sum_{j \neq i} a_{ij}x_j + x_i = 0$ . These are exactly the hyperplanes  $H$  whose defining equation (unique up to nonzero scalar multiple) has a nonzero coefficient for  $x_i$ , which is to say that  $e_i \bmod H$  is a basis of the line  $F^{n+1}/H$ .

What is the map  $\phi_{i'} \circ \phi_i^{-1} : \phi_i(U_i \cap U_{i'}) \simeq \phi_{i'}(U_i \cap U_{i'})$ ? Consider a hyperplane  $H$  om  $F^{n+1}$  whose defining linear equation involves both  $x_i$  and  $x_j$ , and write the equation uniquely in the form  $\sum_{j \neq i} a_{ij}x_j + x_i = 0$  with  $a_{ii'} \neq 0$ . Scaling by  $a_{ii'}^{-1}$  makes the  $x_{i'}$  coefficient into 1 and yields the linear form  $\sum_{j \neq i, i'} (a_{ij}/a_{ii'})x_j + (1/a_{ii'})x_i + x_{i'}$  whose zero locus is also  $H$ , and so

$$\phi_{i'} \circ \phi_i^{-1} : (a_{ij})_{j \neq i} \mapsto (a'_{i'j})_{j \neq i'}$$

with  $a'_{i'j} = a_{ij}/a_{ii'}$  for  $j \neq i', i$  and  $a'_{i'i} = 1/a_{ii'}$ . Thus, for example, if  $n = 1$  then we have  $\mathbf{P}(F^2)$  covered by two charts  $U_0$  and  $U_1$  with respective bijections  $\phi_0 : U_0 \simeq F$  and  $\phi_1 : U_1 \simeq F$  that read off the coefficient of  $x_1$  (resp.  $x_0$ ) in the linear equation  $x_0 + \phi_0(L)x_1 = 0$  (resp.  $\phi_1(L)x_0 + x_1 = 0$ ) for a line  $L$  in  $F^2$  distinct from the  $x_1$ -axis (resp. the  $x_0$ -axis). The overlap  $U_0 \cap U_1$  is the locus  $F - \{0\}$  in the coordinatizations of both  $U_0$  and  $U_1$  (i.e.,  $\phi_0(L) \neq 0$  and  $\phi_1(L) \neq 0$  respectively), with the transition relation between  $\phi_0(L)$  and  $\phi_1(L)$  given by  $\phi_0(L) = 1/\phi_1(L)$ . To summarize, the set  $\mathbf{P}(F^2)$  of lines through the origin in the plane  $F^2$  has the set  $U_0$  of lines distinct from the  $x_1$ -axis parameterized by the negative of the  $x_0$ -coordinate of where such lines meet the line  $x_1 = 1$ , and similarly for  $U_1$  with the roles of the  $x_j$ 's reversed, and these respective “coordinates” for a common line  $L$  distinct from the  $x_0$ -axis and  $x_1$ -axis are reciprocal to each other because  $L$  admits unique equations of the form  $x_0 + a'x_1 = 0$  and  $ax_0 + x_1 = 0$  (whence  $-a' = (-a)^{-1}$ ).

*Example 2.6.* We work out the formulas for  $n = 3$ ,  $d = 2$  (i.e., 2-dimensional subspaces of  $F^4$ ) and  $I = \{0, 1\}$ ,  $I' = \{1, 2\}$ . We have  $U_I = F^{\{0,1\} \times \{2,3\}}$  and  $U_{I'} = F^{\{1,2\} \times \{0,3\}}$ , with coordinates denoted  $(a_{02}, a_{03}, a_{12}, a_{13})$  on  $U_I$  and  $(a_{10}, a_{13}, a_{20}, a_{23})$  on  $U_{I'}$ . Moreover,  $U_I \cap U_{I'}$  corresponds to the non-vanishing locus of the determinant  $a_{01}a_{12} - a_{02}a_{11} = -a_{02}$  in  $U_I$  (the vanishing of this coefficient is the only thing which prevents  $\{\bar{e}_1, \bar{e}_2\}$  from being a basis of  $F^4/W$ , given that  $\{\bar{e}_0, \bar{e}_1\}$  is a basis of  $F^4/W$ , where  $a_{ij} = a_{ij}(W)$  for  $W \in U_I$ ), and likewise the non-vanishing locus of  $a_{10}a_{21} - a_{11}a_{20} = -a_{20}$  in  $U_{I'}$ . Here we have again used the convention for how to define  $a_{ij}$  with  $i, j \in I$ , and likewise for  $a_{i'j'}$  with  $i', j' \in I'$ .

The map  $\phi_{II'}$  is given by considering the columns of the matrix

$$M^{-1} = \begin{pmatrix} a_{01} & a_{02} \\ a_{11} & a_{12} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & a_{02} \\ 1 & a_{12} \end{pmatrix}^{-1} = \begin{pmatrix} -a_{12}/a_{02} & 1 \\ 1/a_{02} & 0 \end{pmatrix}$$

and the columns of the matrix

$$M^{-1} \begin{pmatrix} a_{00} & a_{03} \\ a_{10} & a_{13} \end{pmatrix} = M^{-1} \begin{pmatrix} 1 & a_{03} \\ 0 & a_{13} \end{pmatrix} = \begin{pmatrix} -a_{12}/a_{02} & a_{13} - a_{12}a_{03}/a_{02} \\ 1/a_{02} & a_{03}/a_{02} \end{pmatrix}.$$

These columns give the  $\{\bar{e}_1, \bar{e}_2\}$  coordinates of  $\bar{e}_0$  and  $\bar{e}_3$  respectively (in the quotient  $F^4/W$ ). In other words,

$$\phi_{I'}(a_{02}, a_{03}, a_{12}, a_{13}) = (-a_{12}/a_{02}, 1/a_{02}, a_{13} - a_{12}a_{03}/a_{02}, a_{03}/a_{02}).$$

### 3. GRASSMANNIANS: TOPOLOGICAL THEORY

Finally, we specialize to the case  $F = \mathbf{R}$ , where we can make a topology on the Grassmannians by using Euclidean topologies on the  $U_I$ 's.

**Lemma 3.1.** *Let  $F = \mathbf{R}$ . Fix an ordered basis  $\{e_0, \dots, e_n\}$  of  $V$ . For each strictly increasing  $d$ -tuple  $I = \{0 \leq i_1 < \dots < i_d \leq n\}$  as above, give  $U_I$  the topology induced by the bijection  $\phi_I$ ; that is, declare a subset of  $U_I$  to be open when its image under  $\phi_I$  is open in the finite-dimensional vector space  $\mathbf{R}^{I \times (\{0, \dots, n\} - I)}$ . These topologies are compatible, and so define a topology on  $\mathbf{G}_d(V)$  by gluing.*

*Proof.* The formula (1) shows that  $\phi_I(U_I \cap U_{I'})$  is an open subset of the vector space  $\phi_I(U_I)$ , as it is given by the non-vanishing locus of a determinant polynomial in some of the linear coordinates. Hence, when  $U_I$  is topologized via  $\phi_I$ , the subset  $U_I \cap U_{I'}$  is made into an open subset of  $U_I$ . The compatibility aspect of the topologies on the  $U_I$ 's requires one further condition: the topologies induced on  $U_I \cap U_{I'}$  by both  $U_I$  and  $U_{I'}$  must be proved to coincide. Equivalently, this says that on the open subsets  $\phi_I(U_I \cap U_{I'})$  and  $\phi_{I'}(U_I \cap U_{I'})$  in the Euclidean spaces  $\phi_I(U_I)$  and  $\phi_{I'}(U_{I'})$ , the “transition map”  $\phi_{II'} = \phi_{I'} \circ \phi_I^{-1}$  relating them is a homeomorphism. But we have seen in (2) that  $\phi_{II'}$  is given by rational functions in the standard linear coordinates of these respective Euclidean spaces (with denominator that is non-vanishing on  $\phi_I(U_I \cap U_{I'})$ ). Hence, the topologies are indeed compatible, so we may glue.  $\blacksquare$

There now arises a natural question, in a sense pushing the compatibility problem one step further: since the definition of the set  $\mathbf{G}_d(V)$  does not require a choice of ordered basis on  $V$ , is the topology we have just put on this set independent of the ordered basis  $\mathbf{e} = \{e_i\}$  used above? Since  $U_I = U_{I, \mathbf{e}}$  and  $\phi_I = \phi_{I, \mathbf{e}}$  certainly depend on  $\mathbf{e}$ , there really is something to check! In other words, if  $\mathbf{e}'$  is a second ordered basis of  $V$ , does the compatible system of Euclidean topologies on the subsets  $U_{I, \mathbf{e}'}$  (for varying  $d$ -tuples  $I$  of strictly increasing integers between 0 and  $n$ ) put the same topology on  $\mathbf{G}_d(V)$  that we get from the  $U_{I, \mathbf{e}}$ 's? This will require some further concrete formulas (though it can also be deduced from Theorem 4.1 below and the known independent result in the special case of projective spaces).

Let  $\mathbf{e}' = \{e'_0, \dots, e'_n\}$  be a second ordered basis of  $V$ , so for each ordered  $d$ -tuple  $I$  consisting of a strictly increasing sequence  $1 \leq i_1 < \dots < i_d \leq n$ , we obtain subsets  $U_{I, \mathbf{e}'} \subseteq \mathbf{G}_d(V)$  and bijections  $\phi_{I, \mathbf{e}'}$  of  $U_{I, \mathbf{e}'}$  onto  $\mathbf{R}^{I \times (\{0, \dots, n\} - I)}$ , and the topology obtained from  $\mathbf{e}'$  makes  $U_{I, \mathbf{e}'}$  open with the Euclidean topology. Since a subset of a topological space is open if and only if its overlap with each of the constituents of an open covering is open in each such piece, the problem of agreement of topologies amounts to checking that  $U_{I, \mathbf{e}'}$  is open with its Euclidean topology (obtained via  $\phi_{I, \mathbf{e}'}$ ) when it is considered as a subset of the topological space  $\mathbf{G}_d(V)$  that is topologized using  $\mathbf{e}$ .

We have to show that  $U_{I, \mathbf{e}'} \cap U_{J, \mathbf{e}}$  is open as a subset of both  $U_{I, \mathbf{e}'}$  and  $U_{J, \mathbf{e}}$  when each is given its Euclidean topology, and that the topologies on  $U_{I, \mathbf{e}'} \cap U_{J, \mathbf{e}}$  induced by the Euclidean topologies from  $U_{I, \mathbf{e}'}$  and  $U_{J, \mathbf{e}}$  are the same. Concretely, we must show that when  $U_{I, \mathbf{e}'} \cap U_{J, \mathbf{e}}$  is put inside of the two Euclidean spaces

$$V_{I, \mathbf{e}'} = \mathbf{R}^{I \times (\{0, \dots, n\} - I)}, \quad V_{J, \mathbf{e}} = \mathbf{R}^{J \times (\{0, \dots, n\} - J)}$$



then its images in these Euclidean spaces are open and the resulting bijection  $\phi_{I,\mathbf{e}'} \circ \phi_{J,\mathbf{e}}^{-1}$  between these open subsets (through the identification with  $U_{I,\mathbf{e}'} \cap U_{J,\mathbf{e}}$ ) is a homeomorphism. This follows from the following purely algebraic claim that is valid over any field:

**Theorem 3.2.** *Let  $\mathbf{e}$  and  $\mathbf{e}'$  be two ordered bases of  $V$  over  $F$ . Let  $I$  and  $J$  be two strictly increasing sequences of  $d$  integers between 0 and  $n$ . The subsets*

$$\phi_{I,\mathbf{e}'}(U_{I,\mathbf{e}'} \cap U_{J,\mathbf{e}}) \subseteq V_{I,\mathbf{e}'}, \quad \phi_{J,\mathbf{e}}(U_{I,\mathbf{e}'} \cap U_{J,\mathbf{e}}) \subseteq V_{J,\mathbf{e}}$$

*are non-vanishing loci for certain polynomials in the linear coordinates on  $V_{I,\mathbf{e}'}$  and  $V_{J,\mathbf{e}}$  respectively.*

*Moreover, the bijection  $\phi_{I,\mathbf{e}'} \circ \phi_{J,\mathbf{e}}^{-1}$  between these non-vanishing loci is given by rational functions in the standard linear coordinates on  $V_{I,\mathbf{e}'}$  and  $V_{J,\mathbf{e}}$ .*

*Proof.* Let  $W \subseteq V$  be a codimension- $d$  linear subspace. It lies in  $U_{J,\mathbf{e}}$  precisely when the  $e_j$ 's for  $j \in J$  project to a basis of  $V/W$ , and it lies in  $U_{I,\mathbf{e}'}$  precisely when the  $e'_i$ 's for  $i \in I$  project to a basis of  $V/W$ . Assume  $W \in U_{J,\mathbf{e}}$ . Consider the problem of expressing  $e'_i \bmod W$  (for each  $i \in I$ ) in terms of the basis  $\{e_j \bmod W\}_{j \in J}$  of  $V/W$ . In general, any  $v \in V$  admits an expansion in the  $\mathbf{e}$ -basis of  $V$ , and the standard coordinates of the point  $\phi_{J,\mathbf{e}}(W)$  in the Euclidean space  $V_{J,\mathbf{e}}$  allow us to compute the unique system of coefficients for  $v \bmod W$  as a linear combination of the  $e_j \bmod W$ . More specifically, this computation is given by a universal polynomial formula in the  $\mathbf{e}$ -coefficients of  $v$  in  $F$  and in the standard coordinates of  $\phi_{J,\mathbf{e}}(W) \in V_{J,\mathbf{e}}$ . Applying this with  $v = e'_i$  for each  $i \in I$ , we arrive at  $n-d$  ordered  $(n-d)$ -tuples (one for each  $i \in I$ ), and the resulting  $(n-d) \times (n-d)$  matrix has rows that give the linear coordinates for the  $e'_i \bmod W$  in terms of a known basis  $\{e_j \bmod W\}$  for  $V/W$ . Hence, this matrix has *non-vanishing* determinant if and only if the classes  $e'_i \bmod W$  are also a basis of  $V/W$ . This shows that  $\phi_{J,\mathbf{e}}(U_{I,\mathbf{e}'} \cap U_{J,\mathbf{e}})$  in  $V_{J,\mathbf{e}}$  is the complement of the zero locus of a determinantal expression in the Euclidean coordinates on  $V_{J,\mathbf{e}}$ . We get an analogous conclusion for  $\phi_{I,\mathbf{e}'}(U_{I,\mathbf{e}'} \cap U_{J,\mathbf{e}})$  in  $V_{I,\mathbf{e}'}$ . This settles the first part of the theorem.

Next, we turn to the problem of computing the transition bijection  $\phi_{I,\mathbf{e}'} \circ \phi_{J,\mathbf{e}}^{-1}$  between then subsets of Euclidean spaces. For a point  $W \in U_{I,\mathbf{e}'} \cap U_{J,\mathbf{e}}$ , the point  $\phi_{J,\mathbf{e}}(W) \in V_{J,\mathbf{e}}$  has as its standard coordinates the coefficients of all  $e_j \bmod W$  for  $j \notin J$  when expressed in terms of the basis vectors  $e_j \bmod W$  for  $j \in J$ . Meanwhile, the standard coordinates of  $\phi_{I,\mathbf{e}'}(W) \in V_{I,\mathbf{e}'}$  give the coefficients of all  $e'_i \bmod W$  for  $i \notin I$  when expressed in terms of the basis vectors  $e'_i \bmod W$  for  $i \in I$ . The problem is to exhibit rational functions (with denominator non-vanishing on these domains) that compute how to switch between such systems of coordinates. To do this, in view of the *universal* algebraic formulas for switching between two linear coordinate systems with a known “change of basis matrix” (this involves Cramer’s formula for inverting a matrix), what we need to know is that the change of basis matrix on  $V/W$  between the two ordered bases  $\{e'_i \bmod W\}_{i \in I}$  and  $\{e_j \bmod W\}_{j \in J}$  is given with matrix entries that (on the domain of interest) are rational functions in the Euclidean coordinates with non-vanishing denominator.

First express each  $e'_i$  in the  $\mathbf{e}$ -basis of  $V$ . This introduces some constants (depending on  $\mathbf{e}$  and  $\mathbf{e}'$  but not  $W$ !), and thereby expresses  $e'_i \bmod W$  as a constant linear combination of the  $e_j \bmod W$  for  $0 \leq j \leq n$  (by “constant” we mean “independent of  $W$ ”). Next, each  $e_j \bmod W$  with  $j \notin J$  is a linear combination of  $\{e_j \bmod W\}_{j \in J}$  with coefficients given by the standard coordinates of  $\phi_{J,\mathbf{e}}(W)$ . We insert these linear combinations into the initial constant linear combination expressions, and after recollecting terms we obtain the formula for  $e'_i \bmod W$  as a linear combination of the  $e_j \bmod W$  with  $j \in J$ . The coefficients in these expressions are the standard coordinates of  $\phi_{J,\mathbf{e}}(W)$ , and these give the change of basis matrix in one direction, with non-vanishing determinant precisely because  $W \in U_{I,\mathbf{e}'} \cap U_{J,\mathbf{e}}$ . To go in the other direction, we use the standard coordinates

of  $\phi_{I,\mathbf{e}'}(W)$ . In this manner, we have worked out general rational-function formulas for how to switch between the standard coordinates of  $\phi_{I,\mathbf{e}'}(W) \in V_{I,\mathbf{e}'}$  and  $\phi_{J,\mathbf{e}}(W) \in V_{J,\mathbf{e}}$ , with the rational functions having non-vanishing denominators on the two respective domains

$$\phi_{I,\mathbf{e}'}(U_{I,\mathbf{e}'} \cap U_{J,\mathbf{e}}) \subseteq V_{I,\mathbf{e}'}, \quad \phi_{J,\mathbf{e}}(U_{I,\mathbf{e}'} \cap U_{J,\mathbf{e}}) \subseteq V_{J,\mathbf{e}}.$$

■

The crux to everything in the above work is that the procedures in linear algebra for switching coordinate systems are given by *universal* formulas. If, for example, the determinant of a matrix was not given by a universal polynomial expression in the matrix entries, but was instead some bizarre function of the matrix entries admitting no universal algebraic description, we would have no way of knowing that the above transition manipulations are given by nice functions in Euclidean coordinates on suitable domains.

*Example 3.3.* We claim that the topology on  $\mathbf{P}(V) = \mathbf{G}_1(V)$  is compact and Hausdorff. To verify the Hausdorff condition, we fix an ordered basis  $\mathbf{e} = \{e_0, \dots, e_n\}$  of  $V$  and consider the resulting covering of  $\mathbf{P}(V)$  by open sets  $U_i$  equipped with homeomorphisms  $\phi_i : U_i \simeq \mathbf{R}^n$ . Each of these opens is Hausdorff by Theorem 1.3, so the Hausdorff property for the glued topology is equivalent to  $U_i \cap U_j$  being closed in  $U_i \times U_j$  for all  $i \neq j$ . In terms of the standard coordinates  $(a_{ki})_{k \neq i}$  on  $U_i$  and  $(a_{kj})_{k \neq j}$  on  $U_j$ ,  $U_i \cap U_j$  as a subset of  $U_i \times U_j$  consists of points satisfying  $a_{ji}a_{ij} = 1$  and  $a_{kj}a_{ji} = a_{ki}$  ( $k \neq i, j$ ). These are closed conditions on  $U_i \times U_j$ ! The most concrete case is when  $n = 1$ , in which case  $U_0 \times U_1$  is  $\mathbf{R} \times \mathbf{R}$  and  $U_0 \cap U_1$  is the hyperbola  $a_{01}a_{10} = 1$ .

To see that  $\mathbf{P}(V)$  is compact, we simply note that any hyperplane in  $V$  can be given by a linear equation (relative to  $\mathbf{e}$ -coordinates) with coefficients having absolute value at most 1. Hence, the compact unit cubes  $[-1, 1]^n$  in each of the finitely many  $U_i$ 's cover  $\mathbf{P}(V)$ , and so this space must be compact.

#### 4. TOPOLOGICAL PROPERTIES OF GRASSMANNIANS

To wrap up our topological discussion, we prove a fundamental fact concerning the topology of Grassmannians:

**Theorem 4.1.** *Let  $V$  be a finite-dimensional vector space over  $\mathbf{R}$  with dimension  $n + 1 \geq 2$ . For  $1 \leq d \leq n$ , the topological space  $\mathbf{G}_d(V)$  is naturally homeomorphic to a closed subset of the projective space  $\mathbf{P}(\wedge^d(V))$ . In particular, Grassmannians are compact and Hausdorff.*

The final part follows from the fact that projective spaces are compact and Hausdorff (by Example 3.3). The injection in this theorem is called the *Plücker embedding*.

*Proof.* This theorem turns out to be largely algebraic, and only briefly involves using that the field  $F$  is  $\mathbf{R}$ . We first work in an algebraic setting, and wish to define a natural injective map  $\mathbf{G}_d(V) \rightarrow \mathbf{P}(\wedge^d(V))$ . For each codimension- $d$  linear subspace  $W \subseteq V$ , we have a quotient map  $V \twoheadrightarrow V/W$ , and so we clearly get a surjective map  $\wedge^d(V) \twoheadrightarrow \wedge^d(V/W)$  whose target is *one-dimensional*. The kernel of this is a hyperplane (slightly complicated to describe) in  $\wedge^d(V)$ , and so corresponds to a point in  $\mathbf{P}(\wedge^d(V))$ . Hence, the assignment

$$W \mapsto \ker(\wedge^d(V) \twoheadrightarrow \wedge^d(V/W))$$

defines a map of sets  $\mathbf{G}_d(V) \rightarrow \mathbf{P}(\wedge^d(V))$ . In class it was proved that this map is injective. That is, if  $W, W' \subseteq V$  are linear subspaces of  $V$  with codimension  $d$  and the projections of  $\wedge^d(V)$  onto  $\wedge^d(V/W)$  and  $\wedge^d(V/W')$  have the same kernel, then  $W = W'$  inside of  $V$ .

There is further algebraic work to be done before we turn to the case  $F = \mathbf{R}$  and sort out topological consequences. Let us fix an ordered basis  $\mathbf{e} = \{e_0, \dots, e_n\}$  of  $V$ . For each ordered  $d$ -tuple  $\underline{i} = (i_1, \dots, i_d)$  of strictly increasing indices, define  $e_{\underline{i}} = e_{i_1} \wedge \dots \wedge e_{i_d}$ , so these give a basis of  $\wedge^d(V)$ . The  $e_{\underline{i}}$ 's give a basis of  $\wedge^d(V)$  that we shall denote  $\wedge^d(\mathbf{e})$  (and that we make into an ordered basis with a choice of ordering; the choice does not matter). Let  $U_{\underline{i}} \subseteq \mathbf{P}(\wedge^d(V))$  be the subset of hyperplanes in  $\wedge^d(V)$  whose defining equation (unique up to scaling) has non-vanishing coefficient for  $e_{\underline{i}}$ : that is,  $U_{\underline{i}}$  consists of all hyperplanes  $H$  in  $\wedge^d(V)$  such that the image of  $e_{\underline{i}}$  is *non-zero* and hence a *basis* in  $\wedge^d(V)/H$ . Of course,  $U_{\underline{i}}$  should be denoted  $U_{\underline{i}, \wedge^d(\mathbf{e})}$ , but  $\mathbf{e}$  will not be changing.

**Lemma 4.2.** *The preimage  $\iota_{d,V}^{-1}(U_{\underline{i}}) \subseteq \mathbf{G}_d(V)$  is the subset  $U_I = U_{I,\mathbf{e}}$  as defined earlier, with  $I = \{i_1, \dots, i_d\}$ . Moreover, the resulting map  $\iota_{d,n} : U_I \rightarrow U_{\underline{i}}$  is given in coordinates by*

$$(a_{ij})_{i \in I, j \notin I} \mapsto (\text{Det}(a_{ij})_{i \in I, j \in I'})_{I' \neq I},$$

where  $I'$  runs over all strictly increasing  $d$ -tuples distinct from  $I$  (corresponding to all  $e_{\underline{i}'}$ 's with  $\underline{i}' \neq \underline{i}$ ) and where, as usual, we define  $a_{ij}$  for  $j \in I$  to be 1 when  $j = i$  and to be 0 when  $j \neq i$ .

The significance of this lemma is that  $\iota_{d,n} : U_I \rightarrow U_{\underline{i}}$  is a “polynomial map” in terms of the standard linear coordinates on these domains (using  $\phi_{I,\mathbf{e}}$  and  $\phi_{\underline{i}, \wedge^d(\mathbf{e})}$ ). In particular, for  $F = \mathbf{R}$  this implies (by the local nature of continuity) that  $\iota_{d,V}$  is *continuous*.

*Proof.* If  $W \subseteq V$  is a codimension- $d$  subspace, then the image of  $e_{\underline{i}}$  in  $\wedge^d(V/W)$  is non-zero if and only if the image vectors  $\bar{e}_i \in V/W$  for  $i \in I$  have non-zero wedge product in  $\wedge^d(V/W)$ . But such a wedge product is non-zero if and only if  $\{\bar{e}_i\}_{i \in I}$  is a linearly independent set in  $V/W$  (why?). Since  $\dim(V/W) = d$  and  $I$  has size  $d$ , we arrive at the equivalent statement that the  $\bar{e}_i$ 's for  $i \in I$  form a basis of  $V/W$ . This is exactly the criterion for the point  $W$  in the Grassmannian to lie in  $U_I$ , due to how we *defined*  $U_I$ . This settles the determination of  $\iota_{d,n}^{-1}(U_{\underline{i}})$ .

Given a point in  $U_I$  corresponding to some  $W$ , it is cut out by equations

$$e_j - \sum_{i \in I} a_{ij} e_i = 0$$

for  $j \notin I$ . Recall how we defined coordinates on various pieces  $U_{\underline{i}}$  of  $\mathbf{P}(\wedge^d(V))$  using the preceding general constructions (applied to the projective space viewed as  $\mathbf{G}_1(\wedge^d(V))$ , with the basis  $\wedge^d(\mathbf{e})$  for  $\wedge^d(V)$ ); for  $\underline{i}' \neq \underline{i}$ , the  $\underline{i}'$ -coordinate of the point  $\wedge^d(V/W) \in U_{\underline{i}, \wedge^d(\mathbf{e})}$  is the coefficient that expresses the image of  $e_{\underline{i}'}$  in the 1-dimensional quotient  $\wedge^d(V/W)$  as a multiple of the image of  $e_{\underline{i}}$ . Since  $e_j \equiv \sum_{i \in I} a_{ij} e_i \pmod{W}$  for all  $j$  (using the usual convention of  $a_{ij}$ 's when  $j \in I$ ), when computing in  $\wedge^d(V/W)$  we can replace any appearance of an  $e_j$  with  $\sum_{i \in I} a_{ij} e_i$ . Thus, for any  $\underline{i}' \neq \underline{i}$  we just use the computation

$$e_{\underline{i}'_1} \wedge \dots \wedge e_{\underline{i}'_d} \rightsquigarrow \left( \sum_{i \in I} a_{ii'_1} e_i \right) \wedge \dots \wedge \left( \sum_{i \in I} a_{ii'_d} e_i \right) = \text{Det}(a_{ii'})_{i \in I, i' \in I'} e_{i_1} \wedge \dots \wedge e_{i_d}.$$

■

Before we apply this lemma to complete the proof of the theorem, we briefly digress to work out an example.

*Example 4.3.* We wish to give an explicit formula for  $\iota_{2,F^4} : U_I \rightarrow U_{\underline{i}}$  for the special case  $V = F^4$ ,  $d = 2$ , and  $\underline{i} = (0, 1)$  (so  $I = \{0, 1\}$ ). To do this, first recall that  $(a_{02}, a_{03}, a_{12}, a_{13})$  are coordinates

on  $U_I$ , denoting the conditions

$$e_2 \equiv a_{02}e_0 + a_{12}e_1 \pmod{W}, \quad e_3 \equiv a_{03}e_0 + a_{13}e_1 \pmod{W}$$

for  $W$  a point in  $U_I$ . We will need to compute the images of lots of wedge products  $e_i \wedge e_j$  in  $\wedge^2(V/W)$ , and more specifically we will need 5 coordinates for  $U_{\underline{i}}$  in this example.

To be more precise, to compute  $\iota_{2,F^4} : U_{\{0,1\}} \rightarrow U_{(0,1)}$  we use the following computations in  $\wedge^2(F^4/W)$  where  $W$  is spanned by  $e_2 - a_{02}e_0 - a_{12}e_1$  and  $e_3 - a_{03}e_0 - a_{13}e_1$ :

$$\begin{aligned} e_0 \wedge e_2 &\rightsquigarrow e_0 \wedge (a_{02}e_0 + a_{12}e_1) = a_{12}e_0 \wedge e_1, & e_0 \wedge e_3 &\rightsquigarrow a_{13}e_0 \wedge e_1, \\ e_1 \wedge e_2 &\rightsquigarrow -a_{02}e_0 \wedge e_1, & e_1 \wedge e_3 &\rightsquigarrow -a_{03}e_0 \wedge e_1, \end{aligned}$$

and

$$e_2 \wedge e_3 \rightsquigarrow (a_{02}e_0 + a_{12}e_1) \wedge (a_{03}e_0 + a_{13}e_1) = (a_{02}a_{13} - a_{03}a_{12})e_0 \wedge e_1$$

so relative to the lexicographical ordering of  $e_i \wedge e_j$ 's for  $i < j$ , we have (in terms of the ‘‘usual’’  $\phi_I$  and  $\phi_{\underline{i}}$  coordinates on the domains  $U_I$  and  $U_{\underline{i}}$  under consideration)

$$\iota_{2,F^4}(a_{02}, a_{03}, a_{12}, a_{13}) = (a_{12}, a_{13}, -a_{02}, -a_{03}, a_{02}a_{13} - a_{03}a_{12}).$$

In terms of homogeneous coordinates  $[c_{(i,j)}]_{i < j}$  on  $\mathbf{P}(\wedge^2(F^4))$  corresponding to the linear functional  $\sum_{i < j} c_{(i,j)} e_i^* \wedge e_j^*$  on  $\wedge^2(F^4)$ , we conclude that the quadric Plücker relation

$$c_{(0,1)}c_{(2,3)} - c_{(0,2)}c_{(1,3)} + c_{(0,3)}c_{(1,2)} = 0$$

on  $\mathbf{P}(\wedge^2(F^4))$  meets the open locus  $U_{(0,1)}$  in precisely its overlap  $\iota_{2,F^4}(U_{\{0,1\}})$  with  $\iota_{2,F^4}(\mathbf{G}_2(F^4))$ . It is also straightforward to check by hand that any permutation of the set  $\{0, 1, 2, 3\}$  carries this equation back to itself up to a sign (it suffices to treat the transpositions (01), (12), and (23) that generate the symmetric group  $S_4$ !). Hence, the global homogeneous Plücker relation on  $\mathbf{P}(\wedge^2(F^4))$  cuts out exactly the image of the Grassmannian  $\mathbf{G}_2(F^4)$  under  $\iota_{2,F^4}$ .

Let us now push this example further, showing in general that every  $\phi_{I,\mathbf{e}}$ -coordinate  $a_{ij}$  (for  $i \in I, j \notin I$ ) on  $U_I$  shows up (up to sign) as one of the component functions of the ‘‘coordinatized’’ map

$$(3) \quad \phi_{\underline{i}, \wedge^d(\mathbf{e})} \circ \iota_{d,V}|_{U_{I,\mathbf{e}}} \phi_{I,\mathbf{e}}^{-1} : \phi_{I,\mathbf{e}}(U_{I,\mathbf{e}}) \rightarrow \phi_{\underline{i}, \wedge^d(\mathbf{e})}(U_{\underline{i}, \wedge^d(\mathbf{e})}).$$

For any  $i \in I$  and  $j \notin I$  the  $d$ -tuple  $I' = (I - \{i\}) \cup \{j\}$  is such that the  $a_{I'}$ -coordinate function of (3) is given up to sign by the coefficient which expresses the image of  $\omega = e_j \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_{i_n}$  (with no  $e_i$  term) in  $\wedge^d(V/W)$  as a multiple of the image of  $e_{i_1} \wedge \cdots \wedge e_{i_d}$ . But replacing  $e_j$  with  $a_{i_1 j} e_{i_1} + \cdots + a_{i_d j} e_{i_d}$  in the wedge product  $\omega$  kills off all terms except for the  $i$ th, leaving us with  $\pm a_{ij} e_{i_1} \wedge \cdots \wedge e_{i_d}$ . Thus, we get the desired  $\pm a_{ij}$  as a coordinate function.

Now we assume  $F = \mathbf{R}$  and we are ready to deduce that the map  $\iota_{d,V}$  (which we have proved to be continuous) is a homeomorphism onto a closed subset of the target projective space.

Due to the proved appearance of source coordinates (up to sign) as component functions of (3), we can find a *continuous* projection map  $p : U_{\underline{i}} \rightarrow U_I$  (with some sign interventions) such that  $p \circ \iota_{d,n} : U_I \rightarrow U_I$  is the identity. Thus, to show that (3) is a *closed embedding* (i.e., a homeomorphism onto a closed set in its target), from which we shall deduce that  $\iota_{d,n}$  is a closed embedding, it suffices to show rather generally that if  $f : X \rightarrow Y$  is any continuous map to a Hausdorff topological space such that there exists a continuous  $p : Y \rightarrow X$  with  $p \circ f = \text{id}_X$ , then  $f$  must be a homeomorphism of  $X$  onto a closed subset of  $Y$ . Indeed,  $f$  is certainly injective, and  $f(U) = f(X) \cap p^{-1}(U)$  is open in  $f(X)$  for any open  $U$  in  $X$ , so  $f$  is a homeomorphism onto its image. To see that  $f(X)$  is closed, we use Lemma 1.2:  $f(X) \subseteq Y$  is the preimage of the closed

set  $\Delta_Y(Y) \subseteq Y \times Y$  under the continuous map  $(f \circ p, 1) : Y \rightarrow Y \times Y$  because if  $y = f(x)$  then  $p(y) = x$  (so  $y = f(p(y))$ ).

We now have a covering of  $\mathbf{P}(\wedge^d(V))$  by opens  $U_{\underline{i}}$  such that  $\iota_{d,V}^{-1}(U_{\underline{i}}) \rightarrow U_{\underline{i}}$  is a closed embedding. Thus, it remains to check quite generally that if  $f : X \rightarrow Y$  is a continuous map of topological spaces and  $\{V_\alpha\}$  is an open covering of  $Y$  for which the maps  $f^{-1}(V_\alpha) \rightarrow V_\alpha$  are closed embeddings for every  $\alpha$ , then necessarily  $f$  is a closed embedding. In more suggestive terms, the property of a continuous map  $f$  being a closed embedding is “local on the target space”. To check it, we first note that  $f$  is definitely injective (for if  $x, x' \in X$  with  $f(x) = f(x') \in V_\alpha$  then  $x, x' \in f^{-1}(V_\alpha)$ , with  $f$  injective on  $f^{-1}(V_\alpha)$  (so  $x = x'$ ). Thus,  $f : X \rightarrow f(X)$  is a continuous bijection which we want to have closed image and continuous inverse. Since

$$f(X) \cap V_\alpha = f(f^{-1}(V_\alpha))$$

with  $f : f^{-1}(V_\alpha) \rightarrow V_\alpha$  a closed embedding, we deduce that  $f(X) \cap V_\alpha$  is closed in  $V_\alpha$  for all  $\alpha$  (so  $f(X)$  is closed in  $Y$ ) and likewise  $f : X \rightarrow f(X)$  is even a homeomorphism since this is true over the open covering of  $f(X)$  by  $f(X) \cap V_\alpha$ 's. ■