Math 396. Gluing premanifolds

We have seen earlier how to glue topological spaces along suitable open "overlaps" (subject to a triple overlap condition). We want to enhance this procedure to include gluing of C^p-structures with $1 \leq p \leq \infty$ (the case $p = 0$ offers nothing new), as this is a basic tool in the construction of interesting manifolds. We also illustrate the theory by applying it to a fundamental example: products of premanifolds The Hausdorff and second countability conditions will also be tracked.

1. The basic construction

Let X be a C^p premanifold, with $1 \leq p \leq \infty$, with $\mathscr{O}$ its C^p-structure. For any open subset $U \subseteq X$ (given the induced topology), the opens in U are exactly the open subsets of X that lie in U. In class we have seen that U admits a natural C^p-structure $\mathscr{O}|_U$ that assigns to each open subset $U' \subseteq U$ the $\mathbf{R}$-algebra $\mathscr{O}(U')$ of C^p-functions on U' considered as an open subset of X. Observe that this procedure for putting a C^p-structure on any open subset $U \subseteq X$ is transitive: if $W \subseteq U$ is an open subset, then the C^p-structure on W induced from U is the same as that induced from X. This allows us to work locally on a C^p premanifold, treating all opens as C^p premanifolds in their own right without having to think twice.

Let $\{X_i\}$ be an open covering of X, and give each X_i its natural C^p structure. Let $\phi_i : X_i \to X$ be the natural inclusion map; this is certainly a C^p map, in view of how the C^p-structure on X_i is defined. We likewise endow the open (sometimes empty?) overlaps $X_{ij} = X_i \cap X_j$ with their natural C^p structures. If $f : X \to Y$ is a C^p map to another C^p premanifold, then the restrictions $f_i = f|_{X_i} = f \circ \phi_i : X_i \to Y$ are C^p and they agree on overlaps: $f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}$ for all i, j. We can run this process in reverse, much like the gluing property for open covers of a topological space:

Lemma 1.1. *Let Y be a C^p premanifold, and let $f_i : X_i \to Y$ be set-theoretic maps. These maps are C^p and agree on overlaps if and only if there exists a C^p map $f : X \to Y$ with $f|_{X_i} = f \circ \phi_i$ equal to f_i for all i if and only if f_i and f_j coincide on $X_i \cap X_j$ for all i, j. In this case, such an f is unique.*

Proof. The gluing lemma for open covers of a topological spaces takes care of the topological aspects of the problem (with continuity for the maps), and the C^p property for a continuous mapping is a *local* property on the source in the sense that to check it, we may focus attention on what happens on each of the constituents of an open covering $\{X_i\}$ of X; this is due to the fact that a function on an open U in X is a C^p function if and only if its restriction to the open $U \cap X_i$ in X_i is C^p for all i (as the $U \cap X_i$'s are an open covering of U and C^p-structures are required to be "of local nature" by definition). Hence, the global C^p property for a continuous map $f : X \to Y$ is equivalent to the C^p property for its restrictions $f_i : X_i \to Y$ for all i. ■

Motivated by this lemma, we are led to formulate the notion of a gluing datum and an abstract gluing for C^p premanifolds; this goes exactly as in the topological case that we have already studied, and in fact the following discussion (prior to the proof of Theorem 1.7) is essentially a cut-and-paste from the handout on gluing of topological spaces, except that we have replaced "topological space" with "C^p premanifold" and "continuous map" with "C^p map" everywhere (and a few minor extra comments have been included). The reader who wants to see something that requires a new idea should go to the proof of Theorem 1.7.

Definition 1.2. Let $\{X_i\}$ be a set of C^p premanifolds. A *gluing datum* on the X_i's is the specification of open subsets $X_{ij} \subseteq X_i$ for all i and j and C^p-isomorphisms $\phi_{ji} : X_{ij} \simeq X_{ji}$ such that $X_{ii} = X_i$ and

(1) $\phi_{ji}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$,
(2) "triple overlap" compatibility holds: $\phi_{kj} \circ \phi_{ji} = \phi_{ki}$ as isomorphisms between $X_{ij} \cap X_{ik}$ and $X_{ki} \cap X_{kj}$.

Taking $i = j = k$ in the triple overlap compatibility gives $\phi_{ii} \circ \phi_{ii} = \phi_{ii}$ on $X_{ii} = X_i$ with $\phi_{ii} : X_i \simeq X_i$ an isomorphism, so composing this compatibility condition with ϕ_{ii}^{-1} gives $\phi_{ii} = \mathrm{id}_{X_i}$ automatically. In particular, taking $i = k$, $\phi_{ij} \circ \phi_{ji}$ as a self-map of $X_{ij} \cap X_{ii} = X_{ij}$ is the identity, and likewise for $\phi_{ji} \circ \phi_{ij}$ as a self-map of X_{ji}, so ϕ_{ij} and ϕ_{ji} are automatically inverse to each other.

In terms of these axiomatics, which do not mention any ambient C^p premanifold X at all, we now formulate the notion of how to "glue" X_i to X_j along the identification $\phi_{ji} : X_{ij} \simeq X_{ji}$. The goal is to realize X_i and X_j as opens in a common C^p premanifold such that their overlap meets X_i in X_{ij} and meets X_j in X_{ji}, and such that the composite isomorphism between X_{ij} and X_{ji} through their identification with the overlap of X_i and X_j in the ambient C^p premanifold is exactly ϕ_{ji}. More precisely:

Definition 1.3. Let $(\{X_{ij}\}, \{\phi_{ij}\})$ be a gluing datum on a set of C^p premanifolds $\{X_i\}$. A *gluing* of the X_i's with respect to this gluing datum is a C^p premanifold X equipped with maps $\phi_i : X_i \to X$ such that $\phi_j \circ \phi_{ji} = \phi_i$ on X_{ij} and such that the following universal property holds: for any C^p maps $f_i : X_i \to Y$ to a C^p premanifold Y such that $f_j \circ \phi_{ji} = f_i$ on X_{ij} for all i and j (i.e., f_i and f_j "agree on overlaps"), there exists a unique C^p map $f : X \to Y$ such that $f \circ \phi_i = f_i$ for all i.

The condition $\phi_j \circ \phi_{ji} = \phi_i$ is an axiomatization of the fact that in the motivating example of a C^p premanifold X covered by opens $\{X_i\}$ and inclusions $\phi_i : X_i \to X$, the inclusions of $X_i \cap X_j$ into each of X_i and X_j have respective composites with ϕ_i and X_j that are *equal* (as maps from $X_i \cap X_j$ to X).

Example 1.4. If X is a C^p premanifold covered by open subsets X_i, and we take $X_{ij} = X_i \cap X_j$, $\phi_i : X_i \to X$ the inclusion and $\phi_{ji} : X_{ij} \cap X_{ik} \simeq X_{jk} \cap X_{ji}$ the equality through identifying both sides with $X_i \cap X_j \cap X_k$ in X, then we have seen that this gives a gluing datum such that the structure $(X, \{\phi_i\})$ is a gluing.

In this example, the ϕ_i's are C^p isomorphisms onto open subsets of X, with $\phi_i(X_i) \cap \phi_j(X_j) = \phi_i(X_{ij})$ for all i and j. Although these extra conditions are not part of the requirement in the general definition of a gluing, we will see that they always hold. In particular, the notion of gluing of a gluing datum will be proved to always exist in an essentially unique way, and it is always turns out to look like the basic example arising from an open covering of a C^p premanifold. The true significance of these axiomatics is because in practice we will build *new* C^p premanifolds from old ones by means of gluing.

Example 1.5. Let $X_1 = \mathbf{R}$ and $X_2 = \mathbf{R}$, and take $X_{12} = X_1 - \{0\}$ and $X_{21} = X_2 - \{0\}$. Define $\phi_{21} : X_{12} \to X_{21}$ to be the identity. We consider these as C^p premanifolds with their evident C^p-structures. In this case we claim that the gluing as a C^p premanifold exists and its underlying topological space is the line with a doubled origin. Indeed, let X denote this line with doubled origins 0 and $0'$, and for any open $U \subseteq X$ define $\mathscr{O}'(U)$ to be the set of $\mathbf{R}$-valued functions f on U such that the restriction $f|_{U \cap X_i}$ on the open subset $U \cap X_i$ is a C^p function with respect to the standard C^p structure on $X_i = \mathbf{R}$; in other words, a C^p function on U is one that "looks C^p" from the vantage point of both X_1 and X_2 (or rather, the parts of U in each). It is clear that this assignment $\mathscr{O}'(U)$ is of local nature, and thereby gives an $\mathbf{R}$-space structure $\mathscr{O}'$ on X. The resulting $\mathbf{R}$-space structure $\mathscr{O}'|_{X_i}$ on X_i is the *usual* C^p-structure on X_i viewed as the real line because for any open subset $U \subseteq X_1 = \mathbf{R}$ meets X_2 in the complement of the origin in U and a C^p function on an open subset of $\mathbf{R}$ is *automatically* C^p on the complement of the origin in the open set as well (so there is no non-trivial overlap compatibility condition for a member of $\mathscr{O}(U)$ to lie in $\mathscr{O}'(U)$).

Hence, we have a C^p-structure on X, and so X is a C^p premanifold (whose underlying topological space is not Hausdorff!).

The natural maps $X_1 \simeq X - \{0'\}$ and $X_2 \simeq X - \{0\}$ are visibly C^p, the inclusions of $X - \{0'\}$ and $X - \{0\}$ into X are C^p (due to how the C^p-structure on X is *defined*), and so we get C^p composites $\phi_i : X_i \to X$ that are C^p-isomorphisms onto opens in X (with their induced C^p-structure). It is easy to check from the definition of X as a C^p premanifold (by which we mean really the definition of $\mathscr{O}$, as the topological aspects are already understood) that X equipped with ϕ_1 and ϕ_2 is a gluing of X_1 and X_2 along the gluing datum determined by ϕ_{12}. When we prove the existence of gluings in general, that proof applied in the present circumstances does yield X with the above C^p-structure and with these ϕ_i's.

The uniqueness of gluings up to unique C^p isomorphism also goes exactly as in the topological case. We state the theorem, and leave it to the reader to check that the proof in the topological case works *verbatim* once we replace "continuous map" with "C^p map" and "topological space" with "C^p premanifold".

Theorem 1.6. *Let $\{X_i\}$ be a set of C^p premanifolds endowed with a gluing datum $(\{X_{ij}\}, \{\phi_{ij}\})$. Let $\phi_i : X_i \to X$ and $\phi'_i : X_i \to X'$ be two gluings of the X_i's with respect to the given gluing datum. There exists unique C^p maps $\phi : X \to X'$ and $\phi' : X' \to X$ that respect the gluing datum in the sense that $\phi \circ \phi_i = \phi'_i$ and $\phi' \circ \phi'_i = \phi_i$ for all i, and ϕ and ϕ' are inverse to each other. In particular, the two gluings are uniquely C^p-isomorphic to each other in a manner that respects the gluing datum.*

Having proved a very satisfying uniqueness result, we now turn to the existence problem. Given X_i's and a gluing datum $(\{X_{ij}\}, \{\phi_{ij}\})$, does there exist a gluing $(X, \{\phi_i\})$? Moreover, are the maps $\phi_i : X_i \to X$ C^p-isomorphisms onto open subsets of X (with their induced C^p structure), with X_{ij} related to $\phi_i(X_i) \cap \phi_j(X_j)$, exactly as in the test case of an open covering of a C^p premanifold? We shall now provide affirmative answers.

Theorem 1.7. *Let $\{X_i\}$ be a set of C^p premanifold, and $(\{X_{ij}\}, \{\phi_{ij}\})$ a gluing datum. There exists a gluing $(X, \{\phi_i\})$, and the maps $\phi_i : X_i \to X$ are C^p-isomorphisms onto open subsets of X that cover X (with these open subsets given their induced C^p-structure). Moreover, $\phi_i(X_i) \cap \phi_j(X_j) = \phi_i(X_{ij}) = \phi_j(X_{ji})$.*

Note in particular from the description that this theorem says that the underlying topological space of an abstract gluing for a gluing datum of C^p premanifolds is in fact the gluing for the gluing datum on underlying topological spaces. In this sense, gluing for C^p premanifolds may (and should!) be viewed as an enhancement of topological gluing that keeps track of the C^p-structures.

Before we prove Theorem 1.7, we make some remarks. Note that once the theorem is proved, the composite C^p-isomorphism $X_{ij} \simeq \phi_i(X_{ij}) = (\phi_j \circ \phi_{ji})(X_{ij}) = \phi_j(X_{ji}) \simeq X_{ji}$ is exactly the C^p-isomorphism ϕ_{ji} because $\phi_i = \phi_j \circ \phi_{ji}$ on X_{ij} with ϕ_i and ϕ_j isomorphisms onto their open images, so when we view X as a gluing of its covering by open subsets $\phi_i(X_i)$ in the usual manner then we see that the initial "abstract" data of the X_i's, X_{ij}'s, and ϕ_{ij}'s has been translated into the concrete data of a covering of a C^p premanifold by open subsets (with their induced C^p structure). In this sense, an abstract gluing may always be identified with the more familiar setup of an open covering of a C^p premanifold. The importance of the abstract axiomatics is to simply provide a mechanism by which we can begin with the X_i's and form X (subject to giving ourselves a gluing datum). Many interesting C^p premanifolds will be constructed in this manner.

Proof. Our problem is one of existence, so it suffices to first make the topological gluing and to then build the right C^p-atlas on this glued topological space. We let X be the topological gluing of the X_i's along the ϕ_{ij}'s (viewed as just homeomorphisms), and on each open subset $\phi_i(X_i) \subseteq X$ we define a C^p-structure by using the given C^p-structure $\mathscr{O}_{X_i}$ on X_i. That is, we make a C^p-atlas on $\phi_i(X_i)$ by using the local charts on X_i (transferred to $\phi_i(X_i)$ by means of the homeomorphism ϕ_i). The data of the ϕ_{ij}'s as C^p-isomorphisms satisfying $\phi_i = \phi_j \circ \phi_{ji}$ on X_{ij} is *exactly* the following statement: for any non-empty open set $U \subseteq \phi_i(X_i) \cap \phi_j(X_j)$, and a pair of C^p-charts $\psi_i : U \to V_i$ and $\psi_j : U \to V_j$ arising from the C^p-structures put on $\phi_i(X_i)$ and $\phi_j(X_j)$ respectively, the "transition map" $\psi_j \circ \psi_i^{-1} : \psi_i(U) \simeq \psi_j(U)$ between open domains in finite-dimensional $\mathbf{R}$-vector spaces is a C^p-isomorphism.

We may now rename $\phi_i(X_i)$ as X_i and restate our problem as follows: we are given a topological space X, an open covering $\{X_i\}$, and a maximal C^p-atlas $\mathscr{A}_i$ on X_i for all i such that for open $U \subseteq X_i \cap X_j$, the C^p-atlases $\mathscr{A}_i|_U$ and $\mathscr{A}_j|_U$ are compatible with each other. Then there exists a unique C^p-structure on X restricting to the one on X_i arising from $\mathscr{A}_i$ for each i. Indeed, once we may such a construction then the property of a continuous map $X \to Y$ being C^p may be checked using $\mathscr{A}_i$ on X_i for every i, due to the ability to check the C^p condition using just a covering of local charts compatible with the C^p-structure on X. In view of the C^p-compatibility hyothesis on the maximal atlases $\mathscr{A}_i|_U$ and $\mathscr{A}_j|_U$ for all i, j, the collection $\mathscr{A}$ of all local C^p-charts from all $\mathscr{A}_i$'s on opens in the X_i's is itself a C^p-compatible system of charts and hence is a C^p-atlas on X that moreover restricts on X_i to be C^p-compatible with the maximal $\mathscr{A}_i$. In view of the bijection between C^p-structures and maximal C^p-atlases, the C^p-structure arising from $\mathscr{A}$ must restrict to the C^p-structure on X_i corresponding to $\mathscr{A}_i$ for each i, so we have made what we need. ■

Since the Hausdorff property of a C^p premanifold only depends on the underlying topological space, and likewise for the property of being second countable, and we have seen that the underlying topological space of a gluing of C^p premanifolds coincides with the gluing on underlying topological spaces, we immediately obtain the first part of:

Corollary 1.8. *Let X be a C^p premanifold obtained by gluing C^p premanifolds X_i along C^p isomorphisms $\phi_{ji} : X_{ij} \simeq X_{ji}$ forming a gluing datum. The premanifold X is Hausdorff if and only if the X_i's are Hausdorff and the graphs $\Gamma_{\phi_{ji}} \subseteq X_{ij} \times X_{ji}$ are closed as subsets of $X_i \times X_j$ for all $i \neq j$.*

Moreover, X is a C^p manifold if and only if the X_i's are C^p manifolds and X is covered by countably many of the X_i's.

Proof. For the manifold condition we have to track the second countability property. Our problem is topological: if X is a topological space covered by opens X_i, then X is second countable if and only if the X_i's are second countable and countably many of them cover X. This was settled in the homework. ■

2. Applications

Having slogged through the droll mechanics of the existence and uniqueness of C^p gluings, which really involved nothing beyond careful bookkeeping with definitions, we now turn to something a bit more interesting: putting C^p-structures on products. Motivated by the universal property of products in the topological case, we make the:

Definition 2.1. Let X_1 and X_2 be two C^p premanifolds. A C^p *premanifold product* is a triple (P, p_1, p_2) consisting of a C^p premanifold P equipped with C^p-maps $p_i : P \to X_i$ such that for any pair of C^p maps $f_1 : Z \to X_1$ and $f_2 : Z \to X_2$ from a C^p premanifold Z, there exists a unique C^p map $f : Z \to P$ such that $p_i \circ f = f_i$.

The exact same argument as in the topological case shows that if there exist two products (P, p_1, p_2) and (P', p'_1, p'_2) of X_1 and X_2 as C^p premanifolds then there is a unique C^p-isomorphism

$\xi : P \simeq P'$ compatible with the p_i's and p_i''s (i.e., $p_i' \circ \xi = p_i$ for $i = 1, 2$). The existence aspect provides no surprises: it is a natural C^p-structure on the topological product. The construction is simpler to describe in the language of atlases, as follows. Let $P = X_1 \times X_2$ as a topological space, and let $p_i : P \to X_i$ be the usual continuous projections. For local C^p-charts (ϕ_1, U_1) on X_1 and (ϕ_2, U_2) on X_2 with $\phi_i : U_i \to V_i$ a homeomorphism onto an open set $\phi_i(U_i) \subseteq V_i$ for finite-dimensional $\mathbf{R}$-vector spaces V_i, we get a map

$$\phi_1 \times \phi_2 : U_1 \times U_2 \to V_1 \oplus V_2$$

that is a homeomorphism onto the open subset $\phi_1(U_1) \times \phi_2(U_2)$. We can vary U_1 and U_2 to cover X_1 and X_2 respectively, so the opens $U_1 \times U_2$ cover $X_1 \times X_2$. The content is:

Lemma 2.2. *The data $(\phi_1 \times \phi_2, U_1 \times U_2)$ form a C^p-atlas on $X_1 \times X_2$. With respect to the associated C^p premanifold structure, the projections $p_i : X_1 \times X_2 \to X_i$ are C^p and satisfy the universal property to be a product.*

In concrete terms, the local charts on the product are just conjunctions of local coordinates on each of the factors (on products of opens from the factors). Note also that when X_1 and X_2 are manifolds, then so is $X_1 \times X_2$ (as the Hausdorff and second-countability conditions are topological). Whenever we speak of a product of premanifolds, it is always understood that we use the structure in this lemma.

Proof. Let (ϕ_i, U_i) amd (ϕ_i', U_i') be two C^p-charts on X_i (with ϕ_i' taking values in a finite-dimensional $\mathbf{R}$-vector space V_i'), and suppose $U_1 \times U_2$ meets $U_1' \times U_2'$ in $X_1 \times X_2$. Since $(U_1 \times U_2) \cap (U_1' \times U_2') = (U_1 \cap U_1') \times (U_2 \times U_2')$ inside of $X_1 \times X_2$, we have to check that the composite homeomorphism

$$(\phi_1' \times \phi_2') \circ (\phi_1 \times \phi_2)^{-1} : (\phi_1 \times \phi_2)((U_1 \cap U_1') \times (U_2 \cap U_2')) \to (\phi_1' \times \phi_2')((U_1 \cap U_1') \times (U_2 \cap U_2'))$$

between respective opens in $V_1 \oplus V_2$ and $V_1' \oplus V_2'$ is a C^p-isomorphism. But this map is the product of the homeomorphisms

$$\phi_1' \circ \phi_1^{-1} : \phi_1(U_1 \cap U_1') \simeq \phi_1'(U_1 \cap U_1'), \ \ \phi_2' \circ \phi_2^{-1} : \phi_2(U_2 \cap U_2') \simeq \phi_2'(U_2 \times U_2')$$

between opens in V_1 and V_1', and in V_2 and V_2'. Both of these are C^p-isomorphisms because we are comparing local charts from a common atlas on each of X_1 and X_2, and hence we just have to note that in the setting of multivariable calculus on vector spaces a product of C^p maps is again C^p (as the C^p property may be checked using component functions with respect to a choice of linear coordinates on the target). This completes the verification that we have a C^p-atlas on $X_1 \times X_2$.

By construction, the projections $X_1 \times X_2 \to X_i$ are C^p, as working with the local charts in the above atlas reduces this to the classical fact that if $U \subseteq V$ and $U' \subseteq V'$ are open domains in finite-dimensional $\mathbf{R}$-vector spaces then upon viewing $U \times U'$ as an open domain in the vector space $V \oplus V'$ the projection $U \times U' \to U$ is C^p; it is even C^∞ since it is the restriction of the linear projection $V \oplus V' \to V$. ■

Of course, the same construction can be iterated for finite products in the evident manner.

Example 2.3. Consider $\mathbf{R}^n$. This is a C^∞ manifold in the usual manner, but as a topological space it is a product of copies of $\mathbf{R}$. Does the product manifold structure (using that on each of the factors $\mathbf{R}$) recover the usual C^∞ manifold structure on $\mathbf{R}^n$? Indeed it does; this follows from inspecting the construction using the identity map on $\mathbf{R}$ as a global chart on each copy of $\mathbf{R}$. The same argument shows that for finite-dimensional vector spaces V and W, the product C^∞-manifold structure on $V \times W$ coincides with the usual one via the vector space structure on the direct sum $V \oplus W$. The same goes through if we replace C^∞ with C^p everywhere for $1 \leq p < \infty$.

The preceding example has the following pleasing consequence:

Corollary 2.4. *Let X be a C^p premanifold, and let $f_i \in \mathscr{O}(X)$ be elements. The map $f : X \to \mathbf{R}^n$ given by $x \mapsto (f_i(x))$ is a C^p mapping. Conversely, any C^p map $X \to \mathbf{R}^n$ arises in this way.*

This is the sort of statement that one wants to use without thinking twice, and fortunately we can. (If we couldn't, there would be something seriously wrong with the definitions.)

Proof. This is largely a matter of unwinding definitions, ultimately reducing to the fact that in the very definition of a premanifold we require at the level of local C^p-charts that the distinguished functions on an open set correspond to the usual C^p functions on the image open set in the target vector space for the local C^p-chart. Also, once we settle the case $n = 1$, the general case follows by the universal property of C^p products and the preceding example. In the case $n = 1$, the content is this: for a C^p premanifold X, a set-theoretic function $f : X \to \mathbf{R}$ lies in $\mathscr{O}(X)$ if and only if f is a C^p map when $\mathbf{R}$ is itself viewed as a C^p premanifold in the usual manner.

To verify this "obvious" claim, we may work locally on X (as both sides of the implication can be checked locally on X; e.g., $f \in \mathscr{O}(X)$ if and only if $f|_{X_i} \in \mathscr{O}(X_i)$ for opens X_i that cover X), and so we may assume that there is a global C^p-chart $\phi : X \simeq \phi(X)$ onto an open subset of a finite-dimensional $\mathbf{R}$-vector space V. Hence, by definition of charts on a C^p premanifold, $f \in \mathscr{O}(X)$ if and only if $f \circ \phi : U \to \mathbf{R}$ is a C^p map in the traditional sense. However, since ϕ is a C^p isomorphism when the open subset $\phi(X) \subseteq V$ is given the induced C^p-structure from V (again, this follows from the very definition of a C^p-chart), and since $f = (f \circ \phi) \circ \phi^{-1}$, it follows that f is a C^p map if and only if $f \circ \phi$ is a C^p map. Hence, we may replace f with $f \circ \phi$ so as to reduce to the case when X is an open subset of V with the induced C^p-structure. But the C^p-structure on V is *exactly* to declare the distinguished functions on an open subset are the C^p-functions in the usual sense, so we are done. ∎