Math 396. An application of Gram-Schmidt to prove connectedness

1. Motivation and background

Let V be an n-dimensional vector space over $\mathbf{R}$, and define $\mathrm{GL}(V)$ to be the set of invertible linear maps $V \simeq V$ (the notation stands for General Linear). In other words, this is the *open* locus in $\mathrm{Hom}_{\mathbf{R}}(V, V)$ where the continuous (multi-variate) "polynomial" function $\det : \mathrm{Hom}_{\mathbf{R}}(V, V) \to \mathbf{R}$ is non-vanishing. When $V = \mathbf{R}^n$, this is the set of invertible n by n matrices in $\mathrm{Mat}_{n\times n}(\mathbf{R})$, and it is usually called $\mathrm{GL}_n(\mathbf{R})$ rather than $\mathrm{GL}(\mathbf{R}^n)$.

For example, when $n = 2$ and we imagine the 4-dimensional space $\mathrm{Mat}_{2\times 2}(\mathbf{R})$ as coordinatized by matrix entries a, b, c, d, then $\mathrm{GL}_2(\mathbf{R})$ is the *complement* of the hypersurface in $\mathbf{R}^4$ cut out by the condition $ad - bc = 0$ in a 4-dimensional space. It's quite "big".

We make $\mathrm{GL}(V)$ into a topological space by viewing it as an open in the finite-dimensional $\mathbf{R}$-vector space $\mathrm{Hom}_{\mathbf{R}}(V, V)$. The concepts of open set, closed set, limit, etc. in $\mathrm{GL}(V)$ can be expressed in terms of any choice of linear coordinates on V used to identify the situation with $\mathrm{GL}_n(\mathbf{R})$ in which two matrices are "close" when the corresponding matrix entries (ij in each) are close in $\mathbf{R}$.

Consider the determinant map

$$\det : \mathrm{GL}(V) \to \mathbf{R} - \{0\}.$$

Being a polynomial function in matrix entries relative to any choice of basis of V, this is visibly continuous and trivially surjective (think of diagonal matrices). But the target is disconnected, so the source cannot be connected. More specifically,

$$U_+ = \{T \in \mathrm{GL}(V) \,|\, \det T > 0\}, \quad U_- = \{T \in \mathrm{GL}(V)) \,|\, \det T < 0\}$$

is a non-trivial separation of $\mathrm{GL}(V)$. But is this the only obstruction to connectedness? More specifically, if we define

$$\mathrm{GL}^+(V) = \{T \in \mathrm{GL}(V) \,|\, \det T > 0\},$$

then is this connected? In fact, we will even prove it is path-connected. This is hard to "see" right away, but the proof will exhibit an explicit geometrically constructed "path of matrices" joining up the identity map to any chosen T with positive determinant. The method will essentially amount to a vivid geometric perspective on the Gram-Schmidt process.

A related connectedness question concerns the orthogonal matrices. Suppose we fix a choice of an inner product $\langle \cdot, \cdot \rangle$ on V. We define

$$\mathrm{O}(V) = \mathrm{O}(V, \langle \cdot, \cdot \rangle) = \{T \in \mathrm{Hom}_{\mathbf{R}}(V, V) \,|\, \langle T(v), T(v') \rangle = \langle v, v' \rangle\},$$

called the *orthogonal group* for the data $(V, \langle \cdot, \cdot \rangle)$, though we usually suppress mention of $\langle \cdot, \cdot \rangle$ in the notation. In other words, if T^* is the adjoint map then the condition is $TT^* = 1$ (which forces $T^*T = 1$). In concrete terms, if we choose an *orthonormal* basis to identify V with $\mathbf{R}^n$ in such a way that our inner product goes over to the standard one, then $\mathrm{O}(V)$ becomes the "explicit"

$$\mathrm{O}_n(\mathbf{R}) = \{M \in \mathrm{GL}_n(\mathbf{R}) \,|\, MM^t = 1\}.$$

This is a closed subset of $\mathrm{GL}_n(\mathbf{R})$ since the condition $MM^t = 1$ amounts to a system of n^2 polynomial conditions on the matrix entries of M. For example, when $n = 2$ with

$$M = \begin{pmatrix} a & b \\ c & c \end{pmatrix}$$

we get the conditions

$$a^2 + b^2 = 1, \;\; c^2 + d^2 = 1, \;\; ac + bd = 0.$$

The elements of $O_n(\mathbf{R})$ are the linear maps from $\mathbf{R}^n$ to $\mathbf{R}^n$ that preserve the standard inner product on $\mathbf{R}^n$. We know that all eigenvalues of such a matrix over $\mathbf{C}$ (where it is unitary) have to have absolute value 1.

For any $M \in \mathrm{Mat}_{n\times n}(\mathbf{R})$ we know that the real number $\mathrm{Det}(M)$ has absolute value equal to $|\prod \lambda_i|$ where $\{\lambda_i\}$ is the set of eigenvalues of M in $\mathbf{C}$ (counting multiplicities in terms of roots of the characteristic polynomial). Since $|\lambda_i| = 1$ for all i in the orthogonal (or rather, unitary) case, we see that $|\prod \lambda_i| = 1$ for such matrices, so the determinant function on $O_n(\mathbf{R})$ has values in $\{\pm 1\}$. As with $\mathrm{GL}(V)$, the sign of the (continuous) determinant gives an evident non-trivial separation. Let's restrict our attention to

$$\mathrm{SO}(V) = \mathrm{SO}(V, \langle\cdot,\cdot\rangle) = \{T \in \mathrm{O}(V) \mid \det T = 1\} = \mathrm{O}(V) \cap \mathrm{GL}^+(V).$$

Here, S stands for "special", which is the usual terminology for when one imposes a "det = 1" condition (e.g., $\mathrm{SL}(V)$ denotes the subgroup of elements in $\mathrm{GL}(V)$ with determinant 1, called the *special linear* group of V; for $V = \mathbf{R}^n$ it is usually denoted $\mathrm{SL}_n(\mathbf{R}) \subseteq \mathrm{GL}_n(\mathbf{R})$). Is $\mathrm{SO}(V)$ connected? In fact, we'll prove it is path-connected.

Actually, the method of proof of the two connectedness results will be to first prove path connectedness of $\mathrm{SO}(V)$, and to then use the choice of an inner product and the Gram-Schmidt algorithm to deduce from this that $\mathrm{GL}^+(V)$ is path-connected. In order to motivate things with less clutter, we will first reduce the case of $\mathrm{GL}^+(V)$ to that of $\mathrm{SO}(V)$, and then we'll handle the latter case.

2. Path-connectedness of $\mathrm{GL}^+(V)$

Let $T \in \mathrm{GL}^+(V)$ be an element. We seek to find a continuous path in $\mathrm{GL}^+(V)$ which links up T to the identity map. We now fix a choice of inner product on V, which can certainly be done (in lots of ways), so we get a corresponding orthogonal group $\mathrm{O}(V)$. What we'll actually do is use the Gram-Schmidt algorithm to find a path in $\mathrm{GL}(V)$ joining up T to an element in $\mathrm{SO}(V)$. Then the path-connectedness of the latter (which we'll prove in the next section) will finish the job. Here is the basic idea. Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of V. Let $v_j = T(e_j)$ be the image of the jth basis vector under the linear map T. Let $\{v'_1, \ldots, v'_n\}$ be the orthonormal basis which results from applying the Gram-Schmidt process to the v_j's. Let $T' : V \to V$ be the linear map which sends e_j to v'_j (so T' is an isomorphism). We will "continuously deform" the ordered set $\{v_1, \ldots, v_n\}$ into $\{v'_1, \ldots, v'_n\}$ using the Gram-Schmidt formulas, and this will lead to a path joining up T to T' inside of $\mathrm{GL}^+(V)$. We'll then show that $T' \in \mathrm{SO}(V)$, so we'll be done (or rather, will be reduced to path-connectedness of $\mathrm{SO}(V)$).

More explicitly, consider the formulas which define the Gram-Schmidt algorithm. We first run through without normalizing:

$$\begin{aligned} w'_1 &= v_1, \\ w'_j &= v_j - \sum_{i=1}^{j-1} \frac{\langle v_j, w'_i\rangle}{\langle w'_i, w'_i\rangle} w'_i \end{aligned}$$

for $2 \le j \le n$. Thus, $v'_j = w'_j/\|w'_j\|$ for $1 \le j \le n$. We now define visibly continuous functions

$$w_i : [0,1] \to V$$

as follows:

$$w_1(t) = v_1$$
$$w_j(t) = v_j - t\sum_{i=1}^{j-1}\frac{\langle v_j, w'_i\rangle}{\langle w'_i, w'_i\rangle}w'_i$$

Note that for every t and $1 \leq i \leq n$ we have

$$\text{span}(w_1(t), \dots, w_i(t)) = \text{span}(v_1, \dots, v_i),$$

so $\{w_1(t), \dots, w_n(t)\}$ is a basis of V for all t. Also, for $t = 0$ this is the original basis $\{v_1, \dots\}$ and for $t = 1$ it is the non-normalized basis $\{w'_1, \dots\}$.

Making one final modification, if we define functions $u_j : [0, 1] \to V$ by the rule

$$u_j(t) = \frac{w_j(t)}{\|w_j(t)\|^t}$$

then each u_j is continuous (why?) with $\{u_1(t), \dots, u_n(t)\}$ a basis of V for all t; this yields the original basis $\{v_1, \dots\}$ for $t = 0$ and the Gram-Schmidt output $\{v'_1, \dots\}$ for $t = 1$. We conclude that

$$[0, 1] \to V \times \cdots \times V = V^n$$

defined by

$$t \mapsto (u_1(t), \dots, u_n(t))$$

is a "continuous system of bases" which moves from $\{v_1, \dots, v_n\}$ to $\{v'_1, \dots\}$. Geometrically, we visualize a collection of n arrows sticking out of the origin, with this collection of arrows moving continuously from $\{v_i\}$ to $\{v'_i\}$. Such a visualization is sometimes called a *moving frame*.

Now recall we began with a linear map $T : V \simeq V$ determined by the condition $T(e_j) = v_j$ and we also defined a linear map $T' : V \to V$ by the property $T'(e_j) = v'_j$. Note that T' carries an orthonormal basis to an orthonormal basis. This at least makes T' orthogonal, thanks to:

Lemma 2.1. *Let $T' : (V, \langle\cdot,\cdot\rangle) \to (V', \langle\cdot,\cdot\rangle')$ be a map between finite-dimensional inner product spaces, with $\langle T'(e_i), T'(e_j)\rangle' = \langle e_i, e_j\rangle$ for a basis $\{e_1, \dots, e_n\}$ of V. Then T' respects the inner products. That is,*

$$\langle T'(v_1), T'(v_2)\rangle' = \langle v_1, v_2\rangle'$$

for all $v_1, v_2 \in V$.

Proof. The pairings

$$(v_1, v_2) \mapsto \langle T'(v_1), T'(v_2)\rangle', \quad (v_1, v_2) \mapsto \langle v_1, v_2\rangle$$

are bilinear forms on V which, by hypothesis, coincide on pairs from a basis. But by bilinearity, a bilinear form is uniquely determined by its values on pairs from a basis. Thus, these two bilinear forms coincide, and that's what we needed to prove. ■

Although this lemma shows that T' is orthogonal, it isn't immediately clear that $\det T' = 1$ (as opposed to $\det T' = -1$). The fact that $T' \in \text{SO}(V)$, which is to say $\det T' > 0$, will follow from our next observation: there is a continuous path in $\text{GL}(V)$ which begins at our initial T and ends at T'. Indeed, define $T_t : V \to V$ to be the linear map determined by the requirement

$$T_t(e_j) = u_j(t).$$

Note that $T_0 = T$ and $T_1 = T'$. Moreover, since $\{u_j(t)\}$ is a basis for all t, it follows that $T_t : V \to V$ is invertible for all t, which is to say $T_t \in \text{GL}(V)$.

We now show that the map $[0,1] \to \mathrm{GL}(V)$ defined by $t \mapsto T_t$ is actually *continuous.* To see the continuity, we impose coordinates via the orthonormal basis $\mathbf{e} = \{e_1, \ldots, e_n\}$. In such terms, T_t is the matrix whose jth column is the list of $\mathbf{e}$-coordinates of $T_t(e_j) = u_j(t)$. But recall that $t \mapsto u_j(t)$ is a *continuous* function $[0,1] \to V$, and a map to a finite-dimensional $\mathbf{R}$-vector space is continuous if and only if the resulting component functions relative to some (and then any) basis are continuous as maps to $\mathbf{R}$. That is, the "$\mathbf{e}$-coordinate functions" of the $u_j(t)$'s are continuous maps $[0,1] \to \mathbf{R}$. In more explicit terms, if we write

$$u_j(t) = \sum_i a_{ij}(t) e_i$$

then $a_{ij} : [0,1] \to \mathbf{R}$ is continuous. Thus, if we stare at the matrix

$$T_t = (a_{ij}(t))$$

in the $\mathbf{e}$-coordinates, then every matrix entry is a continuous $\mathbf{R}$-valued function of t. Since continuity for a matrix-valued function is equivalent to continuity of the matrix entry functions, it follows that

$$[0,1] \to \mathrm{Hom}_{\mathbf{R}}(V,V) \simeq \mathrm{Mat}_{n\times n}(\mathbf{R})$$

defined by $t \mapsto T_t$ really is continuous. The topology on $\mathrm{GL}(V)$ is induced by $\mathrm{Hom}_{\mathbf{R}}(V,V)$, which is to say that continuity of $t \mapsto T_t$ as a $\mathrm{GL}(V)$-valued map is a consequence of its continuity as a $\mathrm{Hom}_{\mathbf{R}}(V,V)$-valued map.

Summarizing what we have done so far, given a linear isomorphism $T \in \mathrm{GL}(V)$, we have constructed a continuous path inside of $\mathrm{GL}(V)$ which begins at T and ends at $T' \in \mathrm{O}(V)$ (where we chose an inner product on V). Crucial to this was the explicit nature of the Gram-Schmidt algorithm.

This basic construction never actually needed that $\det T > 0$. But now we use the condition $\det T > 0$ to prove $\det T' > 0$ (and hence $T' \in \mathrm{SO}(V)$, as T' is *orthogonal*). The point is simply that the map

$$\det : \mathrm{GL}(V) \to \mathbf{R} - \{0\}$$

is continuous and hence the map $[0,1] \to \mathbf{R} - \{0\}$ defined by $t \mapsto \det(T_t)$ is continuous (being a composite of continuous maps). Since a continuous map $\varphi : [0,1] \to \mathbf{R} - \{0\}$ must have connected (and hence interval) image, the sign of $\varphi(t)$ must be the same throughout (Intermediate Value Theorem!). In our situation, it follows that the function $t \mapsto \det(T_t)$ has constant sign. Since the sign is positive at $t = 0$, it must then be positive at $t = 1$. We conclude that not only is $T' \in \mathrm{SO}(V)$ but in fact we have constructed a continuous path from T to T' entirely inside of $\mathrm{GL}^+(V)$. Now we just need to prove the path-connectedness of $\mathrm{SO}(V)$ to find a path in here linking up T' to the identity. This is done in the next section.

3. Path-connectedness of SO(V)

Choose any $T \in \mathrm{SO}(V)$. We will find a continuous path in $\mathrm{SO}(V)$ which begins at T and ends at the identity map. This will yield the desired path connectedness. Choose an orthonormal basis $\{e_j\}$ of V, and let $v_j = T(e_j)$, so by orthogonality of T we know that $\{v_j\}$ is an orthonormal basis of V as well. We will define a continuous function $u : [0,1] \to V \times \cdots \times V = V^n$ described by

$$t \mapsto (u_1(t), \ldots, u_n(t))$$

such that $u(0) = \{e_1, \ldots, e_n\}$, $u(1) = \{v_1, \ldots, v_n\}$, and $u(t) = \{u_1(t), \ldots, u_n(t)\}$ is an orthonormal basis of V for all $t \in [0,1]$. Suppose for a moment that we have such a continuous system of orthonormal bases. Define the linear maps $T_t : V \to V$ by the condition $T_t(e_j) = u_j(t)$. The map T_t is *orthogonal* since it takes an orthonormal basis to an orthonormal basis. Note that $T_0 = \mathrm{id}_V$

and $T_1 = T$. By the same method as in the previous section, the continuity of u implies that $t \mapsto T_t$ is a continuous map from $[0,1]$ to $\mathrm{GL}(V)$, and even into $\mathrm{O}(V)$.

In particular, the function $\det(T_t)$ is a *continuous* non-vanishing function on $[0,1]$ with values in $\{\pm 1\}$ since orthogonal maps from V to V have determinant ± 1, whence this determinant is constant. The value at $t = 0$ is $\det(T_0) = \det(\mathrm{id}_V) = 1$, so $t \mapsto T_t$ is a continuous path in $\mathrm{SO}(V)$ connecting the identity map to T, thereby finishing the proof of path-connectedness once we have constructed the above continuous system u of orthonormal bases moving from $\{e_i\}$ to $\{v_i\}$. The construction of such a continuous u *must* somewhere use that the orthogonal map $T : V \to V$ sending e_j to v_j has determinant 1 rather than -1 (as otherwise no such T can exist!).

Now we give the construction of u. If $\dim V = 1$, then the only orthogonal map on V with determinant 1 is the identity, so $\mathrm{SO}(V)$ consists of a single element and hence path-connectedness is trivial. We induct on $\dim V$, so we can assume $\dim V > 1$. Consider the two ordered orthonormal bases $\{e_i\}$ and $\{v_i\}$ related by the orthogonal map T with $\det T = 1$. If e_1 and v_1 are linearly independent, let W be the 2-dimensional span of e_1 and v_1. If we have linear dependence, let W be a 2-dimensional subspace containing the common line spanned by e_1 and v_1.

We have an orthogonal decomposition $V = W \oplus W^{\perp}$ (note $W^{\perp} = 0$ in case $\dim V = 2$). Choose an ordered orthonormal basis of W of the form $\{v_1, v_1'\}$. We have $e_1 = av_1 + a'v_1'$ with $a^2 + a'^2 = 1$. We can find $\theta \in [0, 2\pi)$ such that

$$(a, a') = (\cos(\theta), \sin(\theta)),$$

so if we let $r_t : W \to W$ be the rotation by angle $t\theta$ for $0 \le t \le 1$, then r_0 is the identity and r_1 is a rotation which sends v_1 to e_1.

Define the linear map $T_t : V \simeq V$ on $V = W \oplus W^{\perp}$ by the requirement that on $W^{\perp}$ it acts as the identity and on W it acts by r_t. It is clear from the construction on W and $W^{\perp}$ that T_t is an orthogonal map for all t, and even has determinant 1 for all t. The continuity of the trigonometric matrix function entries for r_t makes it clear that $t \mapsto T \circ T_t$ is a continuous map from $[0,1]$ to $\mathrm{SO}(V)$. Moreover, $T \circ T_0 = T$ and $T \circ T_1$ sends e_1 to e_1. Thus, by moving along the *continuous* path $t \mapsto T \circ T_t$ in $\mathrm{SO}(V)$ we link up our original map T to one which fixes e_1. If we can find a continuous path in $\mathrm{SO}(V)$ from T_1 to the identity map, we'll be done by simply moving along the concatentation of the two paths.

Since T_1 fixes e_1, if we let $V' = (\mathbf{R}e_1)^{\perp}$ then $V = \mathbf{R}e_1 \oplus V'$ is an orthogonal decomposition and the *orthogonal* T_1 must take V' back into V'. If we let $T' : V' \to V'$ denote the orthogonal map induced by T_1, then the action of T_1 on $V = \mathbf{R}e_1 \oplus V'$ is described by $\mathrm{id}_{\mathbf{R}e_1} \oplus T'$. Since $\dim V' < \dim V$ and

$$1 = \det T_1 = \det(\mathrm{id}_{\mathbf{R}e_1}) \det T' = \det T',$$

we have $T' \in \mathrm{SO}(V')$, so by induction there is a continuous path $[0,1] \to \mathrm{SO}(V')$ written as $t \mapsto T_t'$ which begins at T' and ends at $\mathrm{id}_{V'}$. Thus, the maps $\mathrm{id}_{\mathbf{R}e_1} \oplus T_t'$ form a continuous path in $\mathrm{SO}(V)$ beginning at T_1 and ending at the identity.

Remark 3.1. We conclude with a challenge question. Observe that $\mathbf{C}^{\times}$ is connected (in contrast with $\mathbf{R}^{\times}$). Hence, there is no determinant obstruction to connectivity of $\mathrm{GL}_n(\mathbf{C})$. Thus, one may be led to guess that if V is a nonzero finite-dimensional $\mathbf{C}$-vector space then the open subset $\mathrm{GL}(V)$ of $\mathbf{C}$-linear automorphisms in $\mathrm{Hom}_{\mathbf{C}}(V, V)$ is connected, and even path-connected. (Here we give any finite-dimensional $\mathbf{C}$-vector space, such as $\mathrm{Hom}_{\mathbf{C}}(V, V)$, its natural topology as a finite-dimensional $\mathbf{R}$-vector space.) Prove the correctness of this guess by using moving frames in the $\mathbf{C}$-vector space V.