Math 396. The C^∞ hairy ball theorem

1. Introduction

Consider the unit sphere $S^n \subseteq \mathbf{R}^{n+1}$ with $n > 0$. If n is odd then there exists a nowhere-vanishing smooth vector field on S^n. Indeed, if $n = 2k+1$ then consider the vector field $\vec{v}$ on $\mathbf{R}^{n+1} = \mathbf{R}^{2k+2}$ given by

$$\vec{v} = (-x_2\partial_{x_1} + x_1\partial_{x_2}) + \cdots + (-x_{2k+2}\partial_{x_{2k+1}} + x_{2k+1}\partial_{x_{2k+2}}) = \sum_{j=0}^{k}(-x_{2j+2}\partial_{x_{2j+1}} + x_{2j+1}\partial_{x_{2j+2}}).$$

For any point $p \in S^n$ it is easy to see that $\vec{v}(p) \in T_p(\mathbf{R}^{2k+2})$ is perpendicular to the line spanned by $\sum_i x_i(p)\partial_{x_i}|_p$, so it lies in the hyperplane $T_p(S^n)$ orthogonal to this line. In other words, the smooth section $\vec{v}|_{S^n}$ of the pullback bundle $(T(\mathbf{R}^{n+1}))|_{S^n}$ over S^n takes values in the subbundle $T(S^n)$, which is to say that $\vec{v}|_{S^n}$ is a smooth vector field on the manifold S^n. This is a visibly nowhere-vanishing vector field. For example, in the case $n = 1$ this vector field along the circle $S^1 \subseteq \mathbf{R}^2$ is the S^1-restriction of the angular vector field $-y\partial_x + x\partial_y = -\partial_\theta$.

The above construction does not work if n is even (think about it!), so there arises the question of whether there exists a nowhere-vanishing smooth vector field on S^n for even n. The answer is negative, and is called the *hairy ball* theorem (since it "explains" why one cannot continuously comb the hair on a ball without a bald spot):

Theorem 1.1. *A smooth vector field on S^n must vanish somewhere if n is even.*

In fact, a much stronger theorem is true: a *continuous* vector field on S^n must vanish somewhere when n is even. Our proof of the hairy ball theorem in the smooth case will use the smoothness in the context of deRham cohomology and its smooth-homotopy invariance with respect to pullback maps; at one key step we also use Stokes' theorem. A similar cohomology theory is constructed in algebraic topology (called *singular cohomology*) without the appeal to differential forms (and with the topological theory of orientation replacing the appeal to Stokes' theorem). This stronger tool enables our proof in the smooth case to be adapted to work in the continuous case. Remark 2.1 says more about this.

2. Proof of hairy ball theorem

Let $\vec{v}$ be a smooth vector field on S^n, and assume that it is nowhere-vanishing. (We do not use that n is even until the very end of the proof.) For each $p \in S^n$, let $\gamma_p : [0, \pi/\|\vec{v}(p)\|] \to S^n$ be the smooth parameteric great circle (with constant speed) going from p to $-p$ with velocity vector $\gamma_p'(0) = \vec{v}(p) \neq 0$ at $t = 0$. (This would not make sense if $\vec{v}(p) = 0$.) Working in the plane spanned by $p \in \mathbf{R}^{n+1}$ and $\vec{v}(p) \in T_p(\mathbf{R}^{n+1}) \simeq \mathbf{R}^{n+1}$ in $\mathbf{R}^{n+1}$, we get the formula

$$\gamma_p(t) = \cos(t\|\vec{v}(p)\|)p + \sin(t\|\vec{v}(p)\|)\frac{\vec{v}(p)}{\|\vec{v}(p)\|} \in S^n \subseteq \mathbf{R}^{n+1}.$$

(These algebraic formulas would not make sense if $\vec{v}$ vanishes somewhere on S^n.) Consider the "flow" mapping

$$F : S^n \times [0,1] \to S^n$$

defined by $(p,t) \mapsto \gamma_p(\pi t/\|\vec{v}(p)\|)$. The formula for $\gamma_p(t)$ makes it clear that F is a smooth map (and is continuous if $\vec{v}$ is merely continuous and nowhere-vanishing). Now obviously $F(p,0) = p$ for all $p \in S^n$ and $F(p,1) = -p$ for all $p \in S^n$. Hence, F defines a smooth homotopy from the identity map on S^n to the antipodal map $p \mapsto -p$ on S^n (and it is a continuous homotopy if $\vec{v}$ is merely continuous and nowhere-vanishing). Thus, to prove the hairy ball theorem we just have to prove that if n is even then the identity and antipodal maps $S^n \rightrightarrows S^n$ are not smoothly homotopic to each other; likewise to get the continuous version we just need to prove that there is no continuous homotopy deforming one of these maps into the other.

To prove the *non-existence* of such a homotopy, we shall use the (smooth) homotopy invariance of deRham cohomology. Indeed, by this homotopy-invariance we get that under the existence of such a $\vec{v}$ the antipodal map $A : S^n \to S^n$ induces the identity map $A^* : \mathrm{H}^k_{\mathrm{dR}}(S^n) \to \mathrm{H}^k_{\mathrm{dR}}(S^n)$ on the kth deRham cohomology of S^n for all $k \geq 0$. Let us focus on the case $k = n$. To get a contradiction, we just have to prove that if n is even then A^* as a self-map of $\mathrm{H}^n_{\mathrm{dR}}(S^n)$ is *not* the identity map. (On HW 12, Exercise 4, you will prove that $\mathrm{H}^k_{\mathrm{dR}}(S^n) = 0$ for $0 < k < n$. This is why we do not try to study the action on these intermediate cohomologies of S^n. Likewise, we do not consider $k = 0$ since the action of A^* on $\mathrm{H}^0_{\mathrm{dR}}(S^n) = \mathbf{R}$ is clearly the identity map and so this is not useful either for getting a contradiction.)

Consider the n-form on $\mathbf{R}^{n+1}$ defined by

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x_i \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_{n+1}.$$

Clearly $\mathrm{d}\omega = (n+1)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1}$, so for the unit ball $B^{n+1} \subseteq \mathbf{R}^{n+1}$ with its standard orientation we have $\int_{B^{n+1}} \mathrm{d}\omega = (n+1)\mathrm{vol}(B^{n+1}) \neq 0$. (See the handout on "how to compute integrals" for the justification of this latter calculation; the point requiring a moment of thought is the fact that $\{x_1, \dots, x_{n+1}\}$ is *not* a coordinate system on the manifold with boundary B^{n+1}.) By Stokes' theorem for B^{n+1}, if we let $\eta = \omega|_{S^n}$ and we give $S^n = \partial B^{n+1}$ the induced boundary orientation then $\int_{S^n} \eta = \int_{B^{n+1}} \mathrm{d}\omega \neq 0$. Hence, by Stokes' theorem for the boundaryless smooth compact oriented manifold S^n (!) we conclude that the top-degree (hence closed!) differential form η on S^n is not exact. That is, its deRham cohomology class $[\eta] \in \mathrm{H}^n_{\mathrm{dR}}(S^n)$ is *non-zero*. (Note that ω is not closed as an n-form on $\mathbf{R}^{n+1}$, but its pullback η on S^n is necessarily closed on S^n purely for elementary reasons, as S^n is n-dimensional. There is no need to verify such closedness on S^n by bare-hands calculation in local coordinates on S^n.)

By the existence of the smooth homotopy between A and the identity map (due to the assumed existence of the nowhere-vanishing smooth vector field $\vec{v}$ on S^n), it follows that A^* on $\mathrm{H}^n_{\mathrm{dR}}(S^n)$ is the identity map, so $[A^*(\eta)] = A^*([\eta])$ is equal to $[\eta]$. That is, the top-degree differential forms $A^*(\eta)$ and η on S^n differ by an exact form. But the antipodal map $A : S^n \to S^n$ is induced by the negation map $N : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$, and by inspection of the definition of $\omega \in \Omega^n_{\mathbf{R}^{n+1}}(\mathbf{R}^{n+1})$ we have $N^*(\omega) = (-1)^{n+1}\omega$. Hence, pulling back this equality to the sphere gives $A^*(\eta) = (-1)^{n+1} \cdot \eta$ in $\Omega^n_{S^n}(S^n)$. Thus, in $\mathrm{H}^n_{\mathrm{dR}}(S^n)$ we have

$$[\eta] = A^*([\eta]) = [A^*(\eta)] = [(-1)^{n+1}\eta] = (-1)^{n+1}[\eta].$$

If n is even we therefore have $[\eta] = -[\eta]$, so $[\eta] = 0$. But we have already seen via Stokes' theorem for the boundaryless manifold S^n and for the manifold with boundary B^{n+1} that $[\eta]$ is nonzero. This completes the proof.

Remark 2.1. In HW 12, Exercise 4, it is proved that $\mathrm{H}^n_{\mathrm{dR}}(S^n)$ is 1-dimensional (and $\mathrm{H}^k_{\mathrm{dR}}(S^n) = 0$ for $0 < k < n$) via induction on n and a Mayer-Vietoris calculation. In the topological theory of singular cohomology there is an analogous Mayer-Vietoris machine, and by the same method it can thereby be proved that the top-degree singular cohomology $\mathrm{H}^n_{\mathrm{sing}}(S^n)$ is also 1-dimensional over $\mathbf{R}$. (This also follows from deRham's theorem that compares the two cohomology theories on smooth manifolds, but it is much easier to compute in each cohomology theory separately in this particular case.) Thus, for the antipodal map $A : S^n \to S^n$ it follows by pure thought that the pullback map $A^* : \mathrm{H}^n_{\mathrm{dR}}(S^n) \to \mathrm{H}^n_{\mathrm{dR}}(S^n)$ that is a linear self-map of a 1-dimensional vector space must be multiplication by some scalar $c_n \in \mathbf{R}$. But $A \circ A = \mathrm{id}_{S^n}$, so by functoriality of cohomology with respect to pullback we get $c_n^2 = 1$, or in other words $c_n = \pm 1$ (perhaps depending on n). The same goes in singular cohomology. If for some n we can calculate $c_n = -1$, it then follows that A^* is *not* the identity on the 1-dimensional degree-n cohomology of S^n, so A cannot be smoothly homotopic to the identity map on S^n for such n; since singular cohomology has homotopy-invariance as well (with respect to continuous homotopy), it likewise follows that if this sign for the action of A on degree-n singular cohomology of S^n can be computed to be -1 for some n then A cannot even be continuously homotopic to the identity map on S^n.

To summarize, the key point of the proof of the hairy ball theorem (in either the smooth or continuous cases) is to compute that $(-1)^{n+1}$ is the sign that gives the action of antipodal-pullback on top-degree cohomology (deRham or singular) of S^n for any $n > 0$. So what has to be done is that some *nonzero* cohomology class ξ has to be found for which its antipodal pullback can be determined (as either ξ or $-\xi$). In the proof above we found such a class in deRham cohomology, namely the one represented by the closed form $\omega|_{S^n}$ (for which we obtained $A^*(\omega|_{S^n}) = (-1)^{n+1}\omega|_{S^n}$ in $\Omega^n_{S^n}(S^n)$ and not even just modulo exact forms). In the topological setting one has to use what is called the *fundamental class* of the manifold S^n; this is a certain degree-n singular cohomology class that is closely tied up with the *topological* theory of orientation (which can be given in purely topological terms for any topological manifold, without any mention of a tangent bundle or oriented atlases which do not make sense without some differentiability anyway). From this viewpoint, the meaning of the sign $(-1)^{n+1}$ is the rather concrete fact that the antipodal map $A : S^n \to S^n$ is orientation-preserving for odd n and orientation-reserving for even n (something that is very obvious in suitable coordinates for the C^∞-manifold S^n).