Math 396. Integral curves

1. Motivation

Let M be a smooth manifold, and let $\vec{v}$ be a smooth vector field on M. We choose a point $m_0 \in M$. Imagine a particle placed at m_0, with $\vec{v}$ denoting a sort of "unchanging wind" on M. Does there exist a smooth map $c : I \to M$ with an open interval $I \subseteq \mathbf{R}$ around 0 such that $c(0) = m_0$ and at each time t the velocity vector $c'(t) \in \mathrm{T}_{c(t)}(M)$ is equal to the vector $\vec{v}(c(t))$ in the vector field at the point $c(t)$? In effect, we are asking for the existence of a parametric curve whose speed of trajectory through M arranges it to have velocity at each point exactly equal to the vector from $\vec{v}$ at that point. Note that it is natural to focus on the parameterization c and not just the image $c(I)$ because we are imposing conditions on the velocity vector $c'(t)$ at $c(t)$ and not merely on the "direction" of motion at each point. Since we are specifying both the initial position $c(0) = m_0$ and the initial velocity $c'(0) = \vec{v}(c(0)) = \vec{v}(m_0)$, physical intuition suggests that such a c should be unique on I *if it exists*.

We call such a c an *integral curve* in M for $\vec{v}$ through m_0. Note that this really is a mapping to M and is not to be confused with its image $c(I) \subseteq M$. (However, we will see in Remark 5.2 that knowledge of $c(I) \subseteq M$ and $\vec{v}$ suffices to determine c and I uniquely up to additive translation in time.) The reason for the terminology is that the problem of finding integral curves amounts to solving the equation $c'(t) = \vec{v}(c(t))$ that, in local coordinates, is a vector-valued non-linear first-order ODE with the initial condition $c(0) = m_0$. The process of solving such an ODE is called "integrating" the ODE, so classically it is said that the problem is to "integrate" the vector field to find the curve. In the homework you will work out the relationship between integral curves and the classical theory of first-order ODE's in more detail, as well as some basic examples.

One should consider the language of integral curves as the natural geometric and coordinate-free framework in which to think about first-order ODE's of "classical type" $u'(t) = \phi(t, u(t))$, but the local equations in the case of integral curves are of the special form $u'(t) = \phi(u(t))$; we will see later that such apparently special forms are no less general (by means of a change in how we define ϕ). The hard work is this: prove that in the classical theory of initial-value problems

$$u'(t) = \phi(t, u(t)), \ \ u(t_0) = v_0,$$

the solution has reasonable dependence on v_0 when we vary it, and in case ϕ depends on some auxiliary parameters the solution u has good dependence on variation of such parameters (at least as good as the dependence of ϕ).

Our aim in this handout is twofold: to develop the necessary technical enhancements in the local theory of first-order ODE's in order to prove the basic existence and uniqueness results for integral curves for smooth vector fields on open sets in vector spaces (in §2–§4), and to then apply these results (in §5) to study flow along vector fields on manifolds. In fact after reading §1 the reader is urged to skip ahead to §5 to see how such applications work out.

Example 1.1. Lest it seem "intuitively obvious" that solutions to differential equations should have nice dependence on initial conditions and auxiliary parameters, we now explicitly work out an elementary example to demonstrate why one cannot expect a trivial proof of such results.

Consider the initial-value problem

$$u'(t) = 1 + zu^2, \ \ u(0) = v$$

on $\mathbf{R}$ with $(z, v) \in \mathbf{R} \times \mathbf{R}$; in this family of ODE's, the initial time is fixed at 0 but z and v vary. For each $(v, z) \in \mathbf{R}^2$, the solution and its maximal open subinterval of existence $J_{v,z}$ around 0 in $\mathbf{R}$ may be worked out explicitly by the methods of elementary calculus, and there is a trichotomy of formulas for the solution, depending on whether z is positive, negative, or zero. The general solution $u_{v,z} : J_{v,z} \to \mathbf{R}$ is given as follows. We let $\tan^{-1} : \mathbf{R} \to (-\pi/2, \pi/2)$ be the usual arc-tangent function, and we let $\delta_z = \sqrt{|z|}$. The maximal interval $J_{v,z}$ is given by

$$J_{v,z} = \begin{cases} \{|t| < \min(\pi/2\delta_z, \tan^{-1}(1/\delta_z|v|))\}, & z > 0, \\ \mathbf{R}, & z \leq 0, \end{cases}$$

with the understanding that for $v = 0$ we ignore the $\tan^{-1}$ term, and for $t \in J_{v,z}$ we have

$$u_{v,z}(t) = \begin{cases} t + v, & z = 0, \\ (\delta_z^{-1} \cdot \tan(\delta_z v) + v)/(1 - \delta_z v \tan(\delta_z t)), & z > 0, \\ \delta_z^{-1} \cdot ((1 + \delta_z v)e^{2\delta_z t} - (1 - \delta_z v))/((1 + \delta_z v)e^{2\delta_z t} + (1 - \delta_z v)), & z < 0. \end{cases}$$

It is not too difficult to check (draw a picture) that the union $\mathscr{D}$ of the slices

$$J_{v,z} \times \{(v, z)\} \subseteq \mathbf{R} \times \{(v, z)\} \subseteq \mathbf{R}^3$$

is open in $\mathbf{R}^3$; explicitly, $\mathscr{D}$ is the set of triples (t, v, z) such that $u_{v,z}$ propogates to time t. A bit of examination of the trichotomous formula shows that $u : (t, v, z) \mapsto u_{v,z}(t)$ is a continuous mapping from $\mathscr{D}$ to $\mathbf{R}$. What is not at all obvious by inspection (due to the trichotomy of formulas and the intervention of $\sqrt{|z|}$) is that $u : \mathscr{D} \to \mathbf{R}$ is actually smooth (the difficulties are all concentrated along $z = 0$)! This makes it clear that it will not be a triviality to prove theorems asserting that solutions to certain ODE's have nice dependence on parameters and initial conditions.

The preceding example shows that (aside from very trivial examples) the study of differentiable dependence on parameters and initial values is a nightmare when carried out via explicit formulas. We should also point out another aspect of the situation: the *long-term* behavior of the solution can be very sensitive to initial conditions. For example, in Example 1.1 this is seen via the trichotomous nature of the formula for $u_{v,z}$. Since the C^p property is manifestly *local*, the drastically different long-term ("global") behavior of $u_{v,0}$ and $u_{v,z}$ as $|t| \to \infty$ for $0 < |z| \ll 1$ in Example 1.1 is *not* inconsistent with the assertion u is a smooth mapping. Another example of this sort of phenomenon was shown in Example 3.6 in the handout on ODE.

2. Local continuity results

Our attack on the problem of C^p dependence of solutions on parameters and initial conditions will use induction on p. We first treat a special case for the C^0-aspect of the problem, working only with t near the initial time t_0. There are two continuity problems: in the non-linear case and the linear case. The theory of existence for solutions to first-order initial-value problems is better in the linear case (a solution always exists "globally", on the entire interval of definition for the ODE, as we proved in Theorem 3.1 in the ODE handout). Correspondingly, we will have a global continuity result in the linear case. We begin with the general case, where the conclusions are local. (Global results in the general case will be proved in §3.)

Let V and V' be finite-dimensional vector spaces, $U \subseteq V$ and $U' \subseteq V'$ open sets. We view U' as a "space of parameters". Let $\phi : I \times U \times U' \to V$ be a C^p mapping with $p \geq 1$. For each $(t_0, v, z) \in I \times U \times U'$, consider the initial-value problem

$$u'(t) = \phi(t, u(t), z), \;\; u(t_0) = v \tag{2.1}$$

for $u : I \to U$. We regard z as an auxiliary parameter; since z is a point in an open set U' in vector space V', upon choosing a basis for V' we may say that z encodes the data of finitely many auxiliary *numerical* parameters. The power of the vector-valued approach is to reduce gigantic systems of $\mathbf{R}$-valued ODE's with initial conditions and parameters to a *single* ODE, a *single* initial condition, and a *single* parameter (all vector-valued).

Fix the choice of $t_0 \in I$. The local existence theorem (Theorem 2.1 in the ODE handout) ensures that for each $z \in U'$ and $v \in U$ there exists a (unique) solution $u_{v,z}$ to (2.1) on an interval around t_0 that may depend on z and v, and that there is a unique maximal connected open subset $J_{v,z} \subseteq I$ around t_0 on which this solution exists. Write $u(t, v, z) = u_{v,z}(t)$ for $t \in J_{v,z}$. Since the partials of ϕ are continuous, and continuous functions on compacts are uniformly continuous, an inspection of the *proof* of the local existence/uniqueness theorem (Theorem 2.1 in the ODE handout) shows that for each $v_0 \in U$ and $z_0 \in U'$ there is a connected open subset $I_0 \subseteq I$ around t_0 and small opens $U'_0 \subseteq U'$ around z_0 and $U_0 \subseteq U$ around v_0 such that $u_{v,z}$ exists on I_0 for all $(v, z) \in U_0 \times U'_0$. (The sizes of I_0, U_0, and U'_0 depend on the magnitude of the partials of ϕ at points near (t_0, v_0, z_0), but such magnitudes are bounded by uniform constants provided we do not move too far from (t_0, v_0, z_0).) We are interested in studying the mapping

$$u : I_0 \times U_0 \times U'_0 \to U$$

given by $(t, v, z) \mapsto u_{v,z}(t)$; properties of this map near (t_0, v_0, z_0) reflect dependence of the solution on initial conditions and auxiliary parameters if we do not flow too far in time from t_0.

Theorem 2.1. *For a small connected open $I_0 \subseteq I$ around t_0 and small opens $U'_0 \subseteq U'$ around z_0 and $U_0 \subseteq U$ around v_0, the unique mapping $u : I_0 \times U_0 \times U'_0 \to U$ that is differentiable in I_0 and satisfies*

$$(\partial_t u)(t, v, z) = \phi(t, u(t, v, z), z), \;\; u(t_0, v, z) = v$$

is a continuous mapping.

This local continuity condition will later be strengthened to a global continuity (and even C^p) property, once we work out how variation of (v, z) influences the position of the maximal connected open subset $J_{v,z} \subseteq I$ around t_0 on which the solution $u_{v,z}$ exists.

Proof. Fix a norm on V. The problem is local near $(t_0, v_0, z_0) \in I \times U \times U'$. In particular, we may assume that the interval I is compact. By inspecting the iteration method (contraction mapping) used to construct local solutions near t_0 for the initial-value problem

$$\widetilde{u}'(t) = \phi(t, \widetilde{u}(t), z), \;\; \widetilde{u}(t_0) = v$$

with $(v, z) \in U \times U'$ it is clear (from the C^1 property of ϕ) that the constants that show up in the construction may be chosen "uniformly" for all (v, z) near (v_0, z_0). That is, we may find a small $a > 0$ so that for all z (resp. v) in a small compact neighborhood $K'_0 \subseteq U'$ (resp. $K_0 \subseteq U$) around z_0 (resp. v_0), the integral operator

$$T_z(f) : t \mapsto f(t_0) + \int_{t_0}^{t} \phi(y, f(y), z)\mathrm{d}y$$

is a self-map of the complete metric space

$$X = \{f \in C(I \cap [t_0 - a, t_0 + a], V) \mid f(t_0) \in K_0, \mathrm{image}(f) \subseteq \overline{B}_{2r}(f(t_0))\}$$

(endowed with the sup norm) for a suitable small $r \in (0, 1)$ with $\overline{B}_{2r}(K_0) \subseteq U$.

Note that T_z preserves each closed "slice" $X_v = \{f \in X \mid f(t_0) = v\}$ for $v \in K_0$. By taking $a > 0$ sufficiently small and K_0 and K'_0 sufficiently small around v_0 and z_0, T_z is a contraction mapping on X_v with a contraction constant in $(0, 1)$ that is independent of $(v, z) \in K_0 \times K'_0$. Hence, for each $(v, z) \in K_0 \times K'_0$, on $I \cap [t_0 - a, t_0 + a]$ there exists a unique solution $u_{v,z}$ to the initial value problem

$$\widetilde{u}'(t) = \phi(t, \widetilde{u}(t), z), \;\; \widetilde{u}(t_0) = v,$$

and $u_{v,z}$ is the unique fixed point of $T_z : X_v \to X_v$.

Let $I_0 = I \cap (t_0 - a, t_0 + a)$, $U_0 = \mathrm{int}_V(K_0)$, $U_0' = \mathrm{int}_{V'}(K_0')$. We claim that $(t, v, z) \mapsto u_{v,z}(t)$ is continuous on $I_0 \times U_0 \times U_0'$. By the construction of fixed points in the proof of the contraction mapping theorem, the contraction constant controls the rate of convergence. Starting with the constant mapping $\underline{v} : t \mapsto v$ in X_v, $u_{v,z}(t) \in \overline{B}_{2r}(v)$ is the limit of the points $(T_z^n(\underline{v}))(t) \in \overline{B}_{2r}(v)$, and so it is enough to prove that $(t, v, z) \mapsto (T_z^n(\underline{v}))(t) \in V$ is continuous on $I_0 \times U_0 \times U_0'$ and that these continuous maps uniformly converge to the mapping $(t, v, z) \mapsto u_{v,z}(t)$. In fact, we shall prove such results on the slightly larger (compact!) domain $(I \cap [t_0 - a, t_0 + a]) \times K_0 \times K_0'$.

A bit more generally, for the continuity results it suffices to show that if $g \in X$ then

$$(t, v, z) \mapsto (T_z(g))(t) := g(t_0) + \int_{t_0}^{t} \phi(y, g(y), z)\mathrm{d}y \in V$$

is continuous on $(I \cap [t_0 - a, t_0 + a]) \times K_0 \times K_0'$. This continuity is immediate from uniform continuity of continuous functions on compact sets (check!). As for the uniformity, we want $T_z^n(\underline{v})$ to converge uniformly to $u_{v,z}$ in X (using the sup norm) with rate of convergence that is uniform in $(v, z) \in K_0 \times K_0'$. But the rate of convergence is controlled by the contraction constant for T_z on X_v, and we have noted above that this small constant may be taken to be the same for all $(v, z) \in K_0 \times K_0'$. ■

There is a stronger result in the linear case, and this will be used in §4:

Theorem 2.2. *With notation as above, suppose* $U = V$ *and* $\phi(t, v, z) = (A(t, z))(v) + f(t, z)$ *for continuous maps* $A : I \times U' \to \mathrm{Hom}(V, V)$ *and* $f : I \times U' \to V$*. For* $(v, z) \in U \times U'$*, let* $u_{v,z} : I \to V$ *be the unique solution to the linear initial-value problem*

$$\widetilde{u}'(t) = \phi(t, \widetilde{u}(t), z) = (A(t, z))(\widetilde{u}(t)) + f(t, z), \;\; \widetilde{u}(t_0) = v.$$

The map $u : (t, v, z) \mapsto u_{v,z}(t)$ *is continuous on* $I \times U \times U'$.

We make some preliminary comments before proving Theorem 2.2. Recall from the earlier handout on ODE's that since we are in the linear case the solution $u_{v,z}$ exists across the *entire* interval I (as a C^1 function of t) even though ϕ is now merely continuous in (t, v, z). This is why in the setup in Theorem 2.2 we really do have u defined on $I \times U \times U'$ (whereas in the general non-linear case the maximal connected open domain of $u_{v,z}$ around t_0 in I may depend on (v, z)). In Theorem 2.2 we only assume continuity of A and f, not even differentiability; such generality will be critical in the application of this theorem in §4. (The method of proof of Theorem 3.6 gives a direct proof of Theorem 2.2, so we could have opted to postpone the statement of Theorem 2.2 until later, deducing it from Theorem 3.6; however, it seems better to give a quick direct proof here.)

It should also be noted that although Theorem 2.2 is local in (v, z) near any particular (v_0, z_0), it is *not* local in t because the initial condition is at a fixed time t_0. Thus, the theorem is not a formal consequence of the local result in the general non-linear case in Theorem 2.1, as that result only gives continuity results for (t, z) near (t_0, z_0), with t_0 the fixed initial time. In the formulation of Theorem 2.2 we are not free to move the initial time, and so we need to essentially revisit our method of proof that the solution extends to all of I (*beyond* the range of applicability of the contraction method) for linear ODE's.

Proof. Fix a norm on V. Our goal is to prove continuity at each point $(t, v_0, z_0) \in I \times U \times U'$, so we may choose $(v_0, z_0) \in U \times U'$ and focus our attention on $(v, z) \in U \times U'$ near (v_0, z_0). Since I is a rising union of compact interval neighborhoods I_n around t_0, with each $t \in I$ admitting I_n as a neighborhood in I for large n (depending on t), it suffices to treat the I_n's separately. That is, we can assume I is a compact interval. Hence, since $u_{v_0,z_0} : I \to V$ is continuous and I is compact, there is a constant $M > 0$ such that $\|u_{v_0,z_0}(t)\| \le M$ for all $t \in I$. We wish to prove continuity of u at each point $(t, v_0, z_0) \in I \times U \times U'$. Choose a compact neighborhood $K' \subseteq U'$ around z_0, so the continuous $A : I \times K' \to \mathrm{Hom}(V, V)$ and $f : I \times K' \to V$ satisfy $\|A(t, z)\| \le N$ (operator norm!) and $\|f(t, z)\| \le \nu$ for all $(t, z) \in I \times K'$ and suitable $N, \nu > 0$.

By uniform continuity of A and f on the compact $I \times K'$, upon choosing $\varepsilon > 0$ we may find a sufficiently small open U'_ε around z_0 in $\mathrm{int}_{V'}(K')$ so that

$$\|A(t, z) - A(t, z_0)\| < \varepsilon, \ \ \|f(t, z) - f(t, z_0)\| < \varepsilon$$

for all $(t, z) \in I \times U'_\varepsilon$. Let $U_\varepsilon \subseteq U$ be an open around v_0 contained in the open ball of radius ε. Using the differential equations satisfied by $u_{v,z}$ and u_{v_0,z_0} on I, we conclude that for all $(t, v, z) \in I \times U_\varepsilon \times U'_\varepsilon$,

$$u'_{v,z}(t) - u'_{v_0,z_0}(t) = (A(t, z))(u_{v,z}(t) - u_{v_0,z_0}(t)) + (A(t, z) - A(t, z_0))(u_{v_0,z_0}(t)) + (f(t, z) - f(t, z_0)).$$

Thus, for all $(t, z) \in I \times U'_\varepsilon$ we have

$$\|u'_{v,z}(t) - u'_{v_0,z_0}(t)\| \le N\|u_{v,z}(t) - u_{v_0,z_0}(t)\| + \varepsilon(M + 1). \tag{2.2}$$

We now fix $z \in U'_\varepsilon$ and study the behavior of the restriction of $u_{v,z} - u_{v_0,z_0}$ to the closed subinterval $I_t \subseteq I$ with endpoints t and t_0 (and length $|t - t_0|$). By the Fundamental Theorem of Calculus applied to the C^1 mapping $g = u_{v,z} - u_{v_0,z_0} : I \to V$ whose value at t_0 is $v - v_0$ (initial conditions!), for any $t \in I$ we have $g(t) = (v - v_0) + \int_{t_0}^t g'$. Thus, the upper bound (2.2) on the pointwise norm of the integrand $g' = u'_{v,z} - u'_{v_0,z_0}$ therefore yields

$$\|g(t)\| \le \|v - v_0\| + \int_{I_t} (N\|g(y)\| + \varepsilon(M + 1))\mathrm{d}y = \varepsilon(1 + (M + 1)|t - t_0|) + N \int_{I_t} \|g(y)\|\mathrm{d}y.$$

Since I is compact, there is an $R > 0$ such that $|t - t_0| \le R$ for all $t \in I$. Hence,

$$\|g(t)\| \le \varepsilon(1 + (M + 1)R) + \int_{I_t} \|g(y)\| \cdot N\mathrm{d}y.$$

By Lemma 3.4 from the handout on ODE's, applied to (a translate of) the interval I_t (with h there taken to be the the continuous function $y \mapsto \|g(y)\|$ and α, β respectively taken to be the constant functions $\varepsilon(1 + (M + 1)R)$, and N), we get

$$\|g(t)\| \le \varepsilon(1 + (M + 1)R)(1 + \int_{I_t} Ne^{N(t-y)}\mathrm{d}y) = \varepsilon(1 + (M + 1)R)e^{N(t-t_0)} \le \varepsilon(M + 1)Re^{NR}$$

for all $t \in I$.

To summarize, for all $(t, v, z) \in I \times U_\varepsilon \times U'_\varepsilon$,

$$\|u(t, v, z) - u(t, v_0, z_0)\| \le \varepsilon Q$$

for a uniform constant $Q = (1+(M+1)R)e^{NR} > 0$ independent of ε. Thus, for $(t', v, z) \in I \times U_\varepsilon \times U'_\varepsilon$ with t' near t, $\|u(t', v, z) - u(t, v_0, z_0)\|$ is bounded above by

$$\|u(t', v, z) - u(t', v_0, z_0)\| + \|u(t', v_0, z_0) - u(t, v_0, z_0)\| \le \varepsilon Q + \|u_{v_0,z_0}(t') - u_{v_0,z_0}(t)\|. \tag{2.3}$$

Since u_{v_0,z_0} is continuous at $t \in I$, it follows from (2.3) (by taking ε to be sufficiently small) that $u(t', v, z)$ can be made as close as we please to $u(t, v_0, z_0)$ for (t', v, z) near enough to (t, v_0, z_0). In other words, $u : (t, v, z) \mapsto u(t, v, z) \in V$ on $I \times U \times U'$ is continuous at each point lying in a slice $I \times \{(v_0, z_0)\}$ with $(v_0, z_0) \in U \times U'$ arbitrary. Hence, u is continuous. ■

3. Domain of flow, main theorem on C^p-dependence, and reduction steps

Let $I \subseteq \mathbf{R}$ be a non-trivial interval, $U \subseteq V$ and $U' \subseteq V'$ open subsets of finite-dimensional vector spaces, and $\phi : I \times U \times U' \to V$ a C^p mapping with $p \geq 1$. For each $(t_0, v_0, z_0) \in I \times U \times U'$, we let $J_{t_0,v_0,z_0} \subseteq I$ be the maximal connected open subset around t_0 on which the initial-value problem

$$\widetilde{u}'(t) = \phi(t, \widetilde{u}(t), z_0), \;\; \widetilde{u}(t_0) = v_0 \tag{3.1}$$

has a solution; this solution will be denoted $u_{t_0,v_0,z_0} : J_{t_0,v_0,z_0} \to U$. For example, if the mapping $\phi(t_0, \cdot, z_0) : U \to V$ is the restriction of an affine-linear self-map $x \mapsto (A(t_0, z_0))(x) + f(t_0, z_0)$ of V for each $(t_0, z_0) \in I \times U'$ then $J_{t_0,v_0,z_0} = I$ for all (t_0, v_0, z_0) because linear ODE's on I (with an initial condition) have a (unique) solution on all of I.

We refer to equations of the form (3.1) as *time-dependent flow with parameters* in the sense that for each $(t, z) \in I \times U'$ the C^p-vector field $v \mapsto \phi(t, v, z) \in U \subseteq V \simeq \mathrm{T}_v(U)$ depends on both the time t and the auxiliary parameter z. That is, the visual picture for the equation (3.1) is that it describes the motion $t \mapsto \widetilde{u}(t)$ of a particle such that the velocity $\widetilde{u}'(t) = \phi(t, \widetilde{u}(t), z_0) \in V \simeq \mathrm{T}_{\widetilde{u}(t)}(U)$ at any time depends on not just the position $\widetilde{u}(t)$ and the fixed value of z_0 but also on the time t (via the "first variable" of ϕ). We wish to now state the most general result on C^p-dependence of such solutions as we vary the auxiliary parameter z, the initial time $t_0 \in I$, and the initial position $v_0 \in U$ at time t_0. In order to give a clean statement, we first need to introduce a new concept:

Definition 3.1. The *domain of flow* is

$$\mathscr{D}(\phi) = \{(t, \tau, v, z) \in I \times I \times U \times U' \,|\, t \in J_{\tau,v,z}\}.$$

In words, for each possible initial position $v_0 \in U$ and initial time $t_0 \in I$ and auxiliary parameter $z_0 \in U'$, we get a maximal connected open subset $J_{t_0,v_0,z_0} \subseteq I$ on which (3.1) has a solution, and this is where $\mathscr{D}(\phi)$ meets $I \times \{(t_0, v_0, z_0)\}$. For example, if $\phi(t_0, \cdot, z_0) : U \to V$ is the restriction of an affine-linear self-map of V for all $(t_0, z_0) \in I \times U'$ then $\mathscr{D}(\phi) = I \times I \times U \times U'$.

There is a natural set-theoretic mapping

$$u : \mathscr{D}(\phi) \to V$$

given by $(t, \tau, v, z) \mapsto u_{\tau,v,z}(t)$; this is called the *universal solution* to the given family (3.1) of time-dependent parametric ODE's with varying initial positions and initial times. On each "slice" $J_{t_0,v_0,z_0} = \mathscr{D}(\phi) \cap (I \times \{(t_0, v_0, z_0)\})$ this mapping is the unique solution to (3.1) on its maximal connected open subset around t_0 in I. Studying this mapping and its differentiability properties is tantamount to the most general study of how time-dependent flow with parameters depends on the initial position, initial time, and auxiliary parameters. Our goal is to prove that $\mathscr{D}(\phi)$ is *open* in $I \times I \times U \times U'$ and that if ϕ is C^p then so is $u : \mathscr{D}(\phi) \to V$. Such a C^p property for u on $\mathscr{D}(\phi)$ is the precise formulation of the idea that solutions to ODE's should depend "nicely" on initial conditions and auxiliary parameter.

In Example 1.1 we saw that even for rather simple ϕ's the nature of $\mathscr{D}(\phi)$ and the good dependence on parameters and initial conditions can look rather complicated when written out in explicit formulas. Before we address the openness and C^p problems, we verify an elementary topological property of the domain of flow.

Lemma 3.2. *If U and U' are connected then $\mathscr{D}(\phi) \subseteq I \times I \times U \times U'$ is connected.*

Proof. Pick $(t, t_0, v_0, z_0) \in \mathscr{D}(\phi)$. This lies in the connected subset $J_{t_0,v_0,z_0} \times \{(t_0, v_0, z_0)\}$ in $\mathscr{D}(\phi)$, so moving along this segment brings us to the point (t_0, t_0, v_0, z_0). But $\mathscr{D}(\phi)$ meets the subset of points $(t_0, t_0, v, z) \in I \times I \times U \times U'$ in exactly $\{(t_0, t_0)\} \times U \times U'$ because for *any* initial position $v \in U$ and auxiliary parameter $z \in U'$ the initial-value problem for the parameter z and initial condition $\widetilde{u}(t_0) = v$ does have a solution for $t \in I$ near t_0 (with nearness perhaps depending on

(v, z)). Hence, the problem is reduced to connectivity of $U \times U'$, which follows from the assumption that U and U' are connected. ■

The main theorem in this handout is:

Theorem 3.3. *The domain of flow $\mathscr{D}(\phi) \subseteq I \times I \times U \times U'$ is open and the mapping*

$$u : \mathscr{D}(\phi) \to V$$

given by $(t, \tau, v, z) \mapsto u_{\tau,v,z}(t)$ is C^p.

What does such openness really mean? The point is this: if we begin at some time t_0 with an initial position v_0 and parameter-value z_0, and if the resulting solution exists out to a time t (i.e., $(t, t_0, v_0, z_0) \in \mathscr{D}(\phi)$), then by slightly changing all three of these starting values we can still flow the solution to all times near t (in particular to time t). This fact is *not* obvious, though it is intuitively reasonable. Of course, as $|t| \to \infty$ we expect to have less and less room in which to slightly change t_0, v_0, and z_0 if we wish to retain the property of the solution flowing out to time t.

The proof of Theorem 3.3 will require two entirely different kinds of reduction steps. For the openness result it will be convenient to reduce the problem to the case when there are no auxiliary parameters ($U' = V' = \{0\}$), the interval I is open, the initial time is always 0, and the flow is not time-dependent (i.e., ϕ has domain U); in other words, the only varying quantity is the initial position. For the C^p aspect of the problem, it will be convenient (for purposes of induction on p) to reduce to the case when I is open, the initial time and position are fixed, and there is both an auxiliary parameter *and* time-dependent flow.

Let us first eliminate endpoints on I by reducing Theorem 3.3 to the case when $I \subseteq \mathbf{R}$ is open. Observe that to prove Theorem 3.3, we may pick $(v_0, z_0) \in U \times U'$ and study the problem on $I \times I \times U_0 \times U'_0$ for open subsets $U_0 \subseteq U$ and $U'_0 \subseteq U'$ around v_0 and z_0 with $U_0 \subseteq K_0$ and $U'_0 \subseteq K'_0$ for compact subsets $K_0 \subseteq U$ and $K'_0 \subseteq U'$. Hence, to reduce to the case of an open interval I we just need to prove:

Lemma 3.4. *Let $K_0 \subseteq U$ and $K'_0 \subseteq U'$ be compact neighborhoods of points $v_0 \in U$ and $z_0 \in U'$. Let $U_0 = \mathrm{int}_V(K_0)$ and $U'_0 = \mathrm{int}_{V'}(K'_0)$. There exists an open interval $J \subseteq \mathbf{R}$ containing I and a C^p mapping*

$$\widetilde{\phi} : J \times U_0 \times U'_0 \to V$$

restricting to ϕ on $I \times U \times U'$.

The point is that once we have such a $\widetilde{\phi}$, it is clear that $\mathscr{D}(\widetilde{\phi}) \subseteq J \times J \times U_0 \times U'_0$ satisfies

$$\mathscr{D}(\widetilde{\phi}) \cap (I \times I \times U_0 \times U'_0) = \mathscr{D}(\phi) \cap (I \times I \times U_0 \times U'_0)$$

and the "universal solution" $\mathscr{D}(\widetilde{\phi}) \to V$ agrees with u on the common subset $\mathscr{D}(\phi) \cap (I \times I \times U_0 \times U'_0)$. Hence, proving Theorem 3.3 for $\widetilde{\phi}$ will imply it for ϕ.

Proof. We may assume I is not open, and it suffices to treat the endpoints separately (if there are two of them). Thus, we fix an endpoint t_0 of $\overline{I}$ and we work locally on $\mathbf{R}$ near t_0. That is, it suffices to make J around t_0. For each point $(v, z) \in U \times U'$, by the Whitney extension theorem (or a cheap definition of the notion of "C^p mapping" on a sector) there is an open neighborhood $W_{v,z}$ of (v, z) in $U \times U'$ and an open interval $I_{v,z} \subseteq \mathbf{R}$ around t_0 such the mapping $\phi|_{(I \cap I_{v,z}) \times W_{v,z}}$ extends to a C^p mapping $\phi_{t,v,z} : I_{v,z} \times W_{v,z} \to V$. Finitely many $W_{v,z}$'s cover the compact $K \times K'$, say W_{v_n,z_n} for $1 \leq n \leq N$. Let J be the intersection of the *finitely many* I_{v_n,z_n}'s for these W_{v_n,z_n}'s that cover $K \times K'$, so J is an open interval around t_0 in $\mathbf{R}$ such that there are C^p mappings $\phi_n : J \times W_{v_n,z_n} \to V$ extending $\phi|_{(I \cap J) \times W_{v_n,z_n}}$ for each n.

Let X be the union of the W_{v_n,z_n}'s in $U \times U'$. Let $\{\alpha_i\}$ be a C^∞ partition of unity subordinate to the collection of opens $J \times W_{v_n,z_n}$ that covers $J \times X$, with α_i compactly supported in $J \times W_{v_{n(i)},z_{n(i)}}$. Thus, $\alpha_i \phi_{n(i)}$ is C^p and compactly supported in the open $J \times W_{v_{n(i)},z_{n(i)}} \subseteq J \times X$. It therefore "extends by zero" to a C^p mapping $\widetilde{\phi}_i : J \times X \to V$. Let $\widetilde{\phi} = \sum_i \widetilde{\phi}_i : J \times X \to V$; this is a locally finite sum since the supports of the α_i's are a locally finite collection. We claim that on $(J \cap I) \times K \times K'$ (and hence on $(J \cap I) \times U_0 \times U_0')$ the map $\widetilde{\phi}$ is equal to ϕ. By construction ϕ_n agrees with ϕ on $(J \cap I) \times W_{v_n,z_n}$ for all n, and hence $\widetilde{\phi}_i$ agrees with $\alpha_i \phi$ on $(J \cap I) \times W_{v_{n(i)},z_{n(i)}}$. Hence, $\widetilde{\phi}_i|_{(J\cap I)\times K \times K'}$ vanishes outside of the support of α_i and on this support it equals $\alpha_i \phi$. Thus, for $(t, v, z) \in (J \cap I) \times K \times K'$ we have $\widetilde{\phi}_i(t, v, z) = \alpha_i(t, v, z)\phi(t, v, z)$ for all i. Adding this up over all i (a finite sum), we get $\widetilde{\phi}(t, v, z) = \phi(t, v, z)$. ■

In view of this lemma, we may and do *assume I is open in* $\mathbf{R}$. We now exploit such openness to show how Theorem 3.3 may be reduced to each of two kinds of special cases.

Example 3.5. We first reduce the general case to that of time-independent flow without parameters and with a fixed initial time $t = 0$. Define the open subset

$$Y = \{(t, \tau) \in I \times \mathbf{R} \,|\, t + \tau \in I\} \subseteq \mathbf{R} \times \mathbf{R}$$

and let $W = \mathbf{R}^2 \oplus V \oplus V'$, so $U'' = Y \times U \times U'$ is an open subset of W. Define $\psi : U'' \to W$ to be the C^p mapping

$$(t, \tau, v, z) \mapsto (1, 0, \phi(t + \tau, v, z), 0).$$

Consider the initial-value problem

$$\widetilde{u}'(t) = \psi(\widetilde{u}'(t)), \;\; \widetilde{u}(0) = (t_0, 0, v_0, z_0) \in W \tag{3.2}$$

as a W-valued mapping on an unspecified open interval J around the origin in $\mathbf{R}$. A solution to this initial-value problem on J has the form $\widetilde{u} = (u_0, u_1, u_2, u_3)$ where $u_0, u_1 : J \rightrightarrows \mathbf{R}$, $u_2 : J \to U$, and $u_3 : J \to U'$ satisfy

$$(u_0'(t), u_1'(t), u_2'(t), u_3'(t)) = (1, 0, \phi(u_0(t) + u_1(t), u_2(t), u_3(t)), 0)$$

and

$$(u_0(0), u_1(0), u_2(0), u_3(0)) = (t_0, 0, v_0, z_0),$$

so $u_0(t) = t + t_0$, $u_1(t) = 0$, $u_3(t) = z_0$, and

$$u_2'(t) = \phi(t + t_0, u_2(t), z_0), \;\; u_2'(0) = v_0.$$

In other words, $u_2(t - t_0)$ is a solution to (3.1).

We define the *domain of flow* $\mathscr{D}(\psi) \subseteq \mathbf{R} \times U'' \subseteq \mathbf{R} \times W$ much like in Definition 3.1, except that we now consider initial-value problems

$$\widetilde{u}'(t) = \psi(\widetilde{u}(t)), \;\; \widetilde{u}(0) = (t_0, \tau_0, v_0, z_0) \in U'' \tag{3.3}$$

for which the initial time is fixed at 0. That is, $\mathscr{D}(\psi)$ is the set of points $(t_0, w_0) \in \mathbf{R} \times U''$ such that t_0 lies in the maximal open interval $J_{w_0} \subseteq \mathbf{R}$ on which the initial-value problem

$$\widetilde{u}'(t) = \psi(\widetilde{u}(t)), \;\; \widetilde{u}(0) = w_0$$

has a solution.

The above calculations show that the C^∞ isomorphism

$$U'' \simeq I \times I \times U \times U'$$

given by $(t, \tau, v, z) \mapsto (t + \tau, \tau, v, z)$ carries $\mathscr{D}(\psi) \cap (\mathbf{R} \times \mathbf{R} \times \{0\} \times U \times U')$ over to $\mathscr{D}(\phi)$ and carries the restriction of the "universal solution" to (3.3) on $\mathscr{D}(\psi)$ over to the universal solution u

on $\mathscr{D}(\phi)$. Hence, if we can prove Theorem 3.3 for ψ then it follows for ϕ. In this way, by studying ψ rather than ϕ we see that to prove Theorem 3.3 in general it suffices to consider time-independent parameter-free flow with initial time 0. Note also that the study of $\mathscr{D}(\psi)$ uses the time interval $I = \mathbf{R}$ since ψ is "time-independent".

The appeal of the preceding reduction step is that time-independent parameter-free flow with a varying initial position but fixed initial time is *exactly* the setup that is relevant for the local theory of integral curves to smooth vector fields on manifolds (with a varying initial point)! Thus, this apparently "special" case of Theorem 3.3 is in fact no less general (provided we allow ourselves to consider *all* cases at once). Unfortunately, in this special case it seems difficult to push through the C^p aspects of the argument. Hence, we will take care of the openness and continuity aspects of the problem in this special case (thereby giving such results in the general case), and then we will use an entirely different reduction step to dispose of the C^p property of the universal solution on the domain of flow.

We now restate the situation to which we have reduced ourselves, upon renaming ψ as ϕ. Let $I = \mathbf{R}$, and let U be an open subset in a finite-dimensional vector space V. Let $\phi : U \to V$ be a C^p mapping ($p \geq 1$) and consider the family of initial-value problems

$$u'(t) = \phi(u(t)), \;\; u(0) = v_0 \in U \tag{3.4}$$

with varying v_0. We define $J_{v_0} \subseteq \mathbf{R}$ to be the maximal open interval around the origin on which the unique solution u_{v_0} exists, and we define the *domain of flow*

$$\mathscr{D}(\phi) = \{(t, v) \in \mathbf{R} \times U \,|\, t \in J_v\} \subseteq \mathbf{R} \times U.$$

The openness and continuity aspects of Theorem 3.3 are a consequence of:

Theorem 3.6. *In the special situation just described,* $\mathscr{D}(\phi)$ *is an open subset of* $\mathbf{R} \times U$ *and* u *is continuous.*

We hold off on the proof of Theorem 3.6, because it requires knowing some continuity results of a local nature near the initial time. The continuity input we need was essentially proved in Theorem 2.1, except for the glitch that Theorem 2.1 has auxiliary parameters and a fixed initial position whereas Theorem 3.6 has no auxiliary parameters but a varying initial position! Thus, we will now explain how to reduce the general problems in Theorem 3.3 and Theorem 3.6 to another kind of special case.

Example 3.7. We already know it is sufficient to prove Theorem 3.3 in the special setup considered in Theorem 3.6: time-independent parameter-free flows with varying initial position but fixed initial time. Let us now show how the problem of proving Theorem 3.6 or even its strengthening with continuity replaced by the C^p property (and hence Theorem 3.3 in general) can be reduced to the case of a family of ODE's with time-dependent flow and auxiliary parameters but a fixed initial time and initial position. The idea is this: we rewrite the (3.4) so that the varying initial position becomes an auxiliary parameter! Using notation as in (3.4), let $\widetilde{V} = V \oplus V$ and define the open subset

$$\widetilde{U} = \{\widetilde{v} = (v_1, v_2) \in \widetilde{V} \,|\, v_1 + v_2 \in U, v_2 \in U\}.$$

Also define the "parameter space" $\widetilde{U}' = U$ as an open subset of $\widetilde{V}' = V$. Finally, define the C^p mapping $\widetilde{\phi} : \widetilde{U} \times \widetilde{U}' \to \widetilde{V} = V \oplus V$ by

$$\widetilde{\phi}((v_1, v_2), z) = (\phi(v_1 + z), 0).$$

Consider the parametric family of time-independent flows

$$\widetilde{u}'(t) = \widetilde{\phi}(\widetilde{u}(t), v), \;\; \widetilde{u}(0) = 0 \tag{3.5}$$

with varying $v \in \widetilde{U}' = U$ (now serving as an auxiliary parameter!). The unique solution (on a maximal open subinterval of $\mathbf{R}$ around 0) is seen to be $\widetilde{u}_v(t) = (u_v(t), 0)$ with $u_v(t) + v$ the unique solution (on a maximal open subinterval of $\mathbf{R}$ around 0) to

$$u'(t) = \phi(u(t)), \ \ u(0) = v.$$

Thus, the domain of flow $\mathscr{D}(\phi) \subseteq \mathbf{R} \times U$ for this latter family is a "slice" of the domain of flow $\mathscr{D}(\widetilde{\phi}) \subseteq \mathbf{R} \times \widetilde{U} \times \widetilde{U}' = \mathbf{R} \times \widetilde{U} \times U$, namely the subset of points of the form $(t, (0, v), v)$. The universal solution to (3.4) on $\mathscr{D}(\phi)$ is easily computed (by simple affine-linear formulas) in terms of the restriction of the universal solution to (3.5) on $\mathscr{D}(\widetilde{\phi})$.

It follows that the continuity property for the universal solution on the domain of flow for (3.5) with v as an auxiliary parameter implies the same for (3.4) with v as an initial position. The same implication works for the openness property of the domain of flow, as well as for the C^p property of the universal solution on the domain of flow. This reduces Theorem 3.3 (and even Theorem 3.6) to the special case of time-independent flows on $I = \mathbf{R}$ with an auxiliary parameter and fixed initial conditions. It makes the problem more general to permit the flow ϕ to even be time-dependent (again, keeping $I = \mathbf{R}$) and for the proof of Theorem 3.3 we will want such generality because it will be forced upon us by the method of proof that is going to be used for the C^p aspects of the problem.

We emphasize that although the preceding example proposes returning to the study of time-dependent flows with parameters (and $I = \mathbf{R}$), in so doing we have gained something on the generality in Theorem 3.3: the initial time and initial position are *fixed*. As has just been explained, the study of this special situation (with the definition of domain of flow adapted accordingly, namely as a subset of $\mathbf{R} \times U' = I \times U'$ rather than as a subset of $I \times I \times U \times U'$) is sufficient to solve the most general form of the problem as stated in Theorem 3.3. This new special situation is exactly the framework considered in §2.

Remark 3.8. For applications to manifolds, it is the case of time-independent flow with varying initial position but fixed initial time and no auxiliary parameters that is the relevant one. That is, as we have noted earlier, the setup in Theorem 3.6 is the one that describes the local situation for the theory of integral curves for smooth vector fields on a smooth manifold. However, such a setup is inadequate for the *proof* of the smooth-dependence properties on the varying initial conditions. This is why it was crucial for us to formulate Theorem 3.3 in the level of generality that we did: if we had stated it only in the context for which it would be applied on manifolds, then the inductive aspects of the proof would take us out of that context (and thereby create confusion as to what exactly we are aiming to prove), as we shall see in Remark 4.3.

We conclude this preliminary discussion by proving Theorem 3.6:

Proof. Obviously each point $(0, v_0) \in \mathbf{R} \times U$ lies in $\mathscr{D}(\phi)$, and we first make the local claim that there is a neighborhood of $(0, v_0)$ in $\mathbf{R} \times U$ contained in $\mathscr{D}(\phi)$ and on which u is continuous. That is, there exists $\varepsilon > 0$ such that for $v \in U$ sufficiently near v_0, the unique solution u_v to

$$u'(t) = \phi(u(t)), \ \ u(0) = v$$

exists on $(-\varepsilon, \varepsilon)$ and the map $(t, v) \mapsto u_v(t) \in V$ is continuous for $|t| < \varepsilon$ and $v \in U$ near v_0. If we ignore the continuity aspect, then the existence of ε follows from the method of proof of the local existence theorem for ODE's; this sort of argument was already used in the build-up to Theorem 2.1. Hence, $\mathscr{D}(\phi)$ contains an open set around $\{0\} \times U \subseteq \mathbf{R} \times U$, so the problem is to show that we acquire continuity for u on $\mathscr{D}(\phi)$ near $(0, v_0)$. The reduction technique in Example 3.7 reduces the continuity problem to the case when the initial position is fixed (as is the initial time at 0) but v is an auxiliary parameter. This is *exactly* the problem that was solved in Theorem 2.1!

We now fix a point $v_0 \in U$ and aim to prove that for *all* $t \in J_{v_0} \subseteq \mathbf{R}$ the point $(t, v_0) \in \mathscr{D}(\phi)$ is an interior point (relative to $\mathbf{R} \times U$) and u is continuous on an open around (t, v_0) in $\mathscr{D}(\phi)$. Let T_{v_0} be the set of $t \in J_{v_0}$ for which such openness and local continuity properties hold at (t', v_0) for $0 \leq t' < t$ when $t > 0$ and for $t < t' \leq 0$ when $t < 0$. It is a triviality that T_{v_0} is an open connected subset of J_{v_0}, but one has to do work to show it is non-empty! Fortunately, in the preceding paragraph we proved that $0 \in T_{v_0}$. Our goal is to prove $T_{v_0} = J_{v_0}$. Once this is shown, then since $v_0 \in U$ was arbitrary it will follow from the definition of $\mathscr{D}(\phi)$ that this domain of flow is a neighborhood of all of its points relative to the ambient space $\mathbf{R} \times U$ (hence it is open in $\mathbf{R} \times U$) and that $u : \mathscr{D}(\phi) \to V$ is continuous near each point of $\mathscr{D}(\phi)$ and hence is continuous. In other words, we would be done.

To prove that the open subinterval T_{v_0} in the open interval $J_{v_0} \subseteq \mathbf{R}$ around 0 satisfies $T_{v_0} = J_{v_0}$, we try to go as far as possible in both directions. Since $0 \in T_{v_0}$, we may separately treat the cases of moving to the right and moving to the left. We consider going to the right, and leave it to the reader to check that the same method applies in the left direction. If T_{v_0} does not exhaust all positive numbers in J_{v_0} then since T_{v_0} contains 0 it follows that the supremum of T_{v_0} is a finite positive number $t_0 \in J_{v_0}$ and we seek a contradiction by studying flow near (t_0, v_0). More specifically, since $t_0 \in J_{v_0}$ and J_{v_0} is open, the solution u_{v_0} does propogate past t_0. Define $v_1 = u_{v_0}(t_0) = u(t_0, v_0) \in U$.

The local openness and continuity results that we established at the beginning of the proof are applicable to time-independent parameter-free flows with varying initial positions and any fixed initial time in $\mathbf{R}$ (there is nothing sacred about the origin). Hence, there is some positive $\varepsilon > 0$ such that for $v \in B_\varepsilon(v_1) \subseteq U$ and $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ the point (t, v) is in the domain of flow for the family of ODE's (with varying v)

$$\widetilde{u}'(y) = \phi(\widetilde{u}(y)), \;\; \widetilde{u}(t_0) = v,$$

and the universal solution $\widetilde{u} : (t, v) \mapsto \widetilde{u}_v(t)$ continuous on the subset $(t_0 - \varepsilon, t_0 + \varepsilon) \times B_\varepsilon(v_1)$ in its domain of flow.

By continuity of the differentiable (even C^1) mapping $u_{v_0} : J_{v_0} \to V$, there is a $\delta > 0$ so that $u_{v_0}(t) \in B_{\varepsilon/4}(v_1)$ for $t_0 - \delta < t < t_0$. We may assume $\delta < \varepsilon/4$. Since t_0 is the supremum of the open interval T_{v_0}, we may choose δ sufficiently small so that $(t_0 - \delta, t_0) \subseteq T_{v_0}$. Pick any $t_1 \in (t_0 - \delta, t_0) \subseteq T_{v_0}$, so by definition of T_{v_0} there is an open interval J_1 around t_1 and an open U_1 around v_0 in U such that $J_1 \times U_1 \subseteq \mathscr{D}(\phi)$ and u is continuous on $J_1 \times U_1$. Since $u(t_1, v_0) = u_{v_0}(t_1) \subseteq B_{\varepsilon/4}(v_1)$ (as $t_1 \in (t_0 - \delta, t_0)$) and u is *continuous* (!) at $(t_1, u_0) \in J_1 \times U_1$ (by definition of T_{v_0}), we may shrink U_1 around v_0 and J_1 around t_1 so that

$$J_1 \subseteq (t_0 - \delta, t_0) \subseteq (t_0 - \varepsilon, t_0 + \varepsilon)$$

and $u(J_1 \times U_1) \subseteq B_{\varepsilon/2}(v_1)$. But $B_{\varepsilon/2}(v_1) \subseteq B_\varepsilon(v_1)$, so for all $v \in U_1$ the mapping

$$t \mapsto \widetilde{u}_{u(t_1, v_1)}(t + (t_0 - t_1))$$

extends u_v near t_1 out to time $t_0 + \varepsilon$ as a solution to the original initial-value problem

$$u'(t) = \phi(u(t)), \;\; u(0) = v.$$

We have shown $(0, t_0 + \varepsilon) \subseteq J_v$ for all $v \in U_1$ and that if $(t, v) \in (t_0 - \varepsilon, t_0 + \varepsilon) \times U_1$ then

$$u(t, v) = \widetilde{u}(t + (t_0 - t_1), u(t_1, v)) \tag{3.6}$$

(with $|t_0 - t_1| < \delta < \varepsilon/4$). But u is continuous on $J_1 \times U_1$, $\widetilde{u}$ is continuous on $(t_0 - \varepsilon, t_0 + \varepsilon) \times B_\varepsilon(v_1)$, and $u(J_1 \times U_1) \subseteq B_\varepsilon(v_1)$, so by inspection of continuity properties of the ingredients in the right side of (3.6) we conclude that

$$(t_0 - \varepsilon/4, t_0 + \varepsilon/4) \times U_1 \subseteq \mathscr{D}(\phi)$$

and that u is continuous on this domain. In particular, T_{v_0} contains $(t_0 - \varepsilon/4, t_0 + \varepsilon/4)$, and this contradicts the definition $t_0 = \sup T_{v_0}$. ■

Note that the proof of Theorem 3.6 would not have worked if we had not simultaneously proved continuity on the domain of flow. This continuity will be recovered in much stronger form below (namely, the C^p property), but the proof of the stronger properties rests on Theorem 3.6.

4. C^p dependence

Since Theorem 3.6 is now proved, in the setup of Theorem 3.3 the domain of flow $\mathscr{D}(\phi)$ is open and the universal solution $u : \mathscr{D}(\phi) \to V$ on it is continuous. To wrap up Theorem 3.3, we have to show that this universal solution is C^p. In view of the reduction step in Example 3.7, it suffices to solve the following analogous problem. Let $U \subseteq V$ and $U' \subseteq V'$ be open subsets of finite-dimensional vector spaces with $0 \in U$, and let $\phi : \mathbf{R} \times U \times U' \to V$ be a C^p mapping. Consider the family of ODE's

$$\widetilde{u}'(t) = \phi(t, \widetilde{u}(t), z), \quad \widetilde{u}(0) = 0 \tag{4.1}$$

for varying $z \in U'$. Let $u_z : J_z \to \mathbf{R}$ be the unique solution on its maximal open interval of definition around the origin in $\mathbf{R}$, and let $\mathscr{D}(\phi) \subseteq \mathbf{R} \times U'$ be the domain of flow: the set of points $(t, z) \in \mathbf{R} \times U'$ such that $t \in J_z$. This is an open subset of $\mathbf{R} \times U'$ since the openness aspect of Theorem 3.3 has been proved. We define the universal solution

$$u : \mathscr{D}(\phi) \to V$$

by $u(t, z) = u_z(t)$, so this is known to be continuous. Our goal is to prove that u is C^p.

Consider the problem of proving that u is C^p near a particular point (t_0, z_0). (**Warning**. The initial time in (4.1) is fixed at 0. Thus, t_0 does not denote an "initial time".) Obviously the parameter space U' only matters near z_0 for the purposes of the C^p property of u near (t_0, z_0). Let J be the compact interval in $\mathbf{R}$ with endpoints 0 and t_0. Since $\mathscr{D}(\phi)$ is *open* in $\mathbf{R} \times U'$ and contains the compact $J \times \{z_0\}$, we may shrink U' around z_0 and find $\varepsilon > 0$ such that $J_\varepsilon \times U' \subseteq \mathscr{D}(\phi)$ with $J_\varepsilon \subseteq \mathbf{R}$ the open interval obtained from J by appending open intervals of length ε at both ends. In other words, we may assume $I \times U' \subseteq \mathscr{D}(\phi)$ for an *open* interval $I \subseteq \mathbf{R}$ containing 0 and t_0.

To get the induction on p off the ground we claim that u is C^1 on $I \times U'$ (and so in particular at the arbitrarily chosen $(t_0, z_0) \in \mathscr{D}(\phi)$). This follows from a stronger result in the C^1 case that will be essential for the induction on p:

Theorem 4.1. *Let $I \subseteq \mathbf{R}$ be an open interval containing 0, and assume that the initial-value problem*

$$u'(t) = \phi(t, u(t), z), \quad u(0) = v_0$$

with C^1 mapping $\phi : \mathbf{R} \times U \times U' \to V$ has a solution $u_z : I \to U$ for all $z \in U'$. (That is, the domain of flow $\mathscr{D}(\phi) \subseteq \mathbf{R} \times U'$ contains $I \times U'$). Choose $z_0 \in U'$.

1. *For any connected open neighborhood $I_0 \subseteq I$ around 0 with compact closure in I, there is an open $U_0' \subseteq U'$ around z_0 and an open interval $I_0' \subseteq I$ containing $\overline{I}_0$ such that $u : (t, z) \mapsto u_z(t)$ is C^1 on $I_0' \times U_0'$.*

(2) *For any such* U_0' *and* I_0'*, the map* $I_0' \to \mathrm{Hom}(V', V)$ *given by the total* U'*-derivative* $t \mapsto (D_2 u)(t, z)$ *of the mapping* $u(t, \cdot) : U_0' \to V$ *at* $z \in U'$ *is the solution to the* $\mathrm{Hom}(V', V)$*-valued linear initial-value problem*

$$Y'(t) = A(t, z) \circ Y(t) + F(t, z), \ \ Y(0) = 0 \tag{4.2}$$

with $A(t, z) = (D_2\phi)(t, u_z(t), z) \in \mathrm{Hom}(V, V)$ *and* $F(t, z) = (D_3\phi)(t, u_z(t), z) \in \mathrm{Hom}(V', V)$ *continuous in* $(t, z) \in I_0' \times U_0'$*.*

The continuity of A and F follows from the C^1 property of ϕ and the continuity of $u_z(t)$ in (t, z) (which is ensured by the partial results we have obtained so far toward Theorem 3.3, especially Theorem 3.6). Our proof of Theorem 4.1 requires a lemma on the growth of "approximate solutions" to an ODE:

Lemma 4.2. *Let* $J \subseteq \mathbf{R}$ *be a non-empty open interval and let* $\phi : J \times U \to V$ *be a* C^1 *mapping, with* U *a convex open set in a finite-dimensional vector space* V*. Fix a norm on* V*, and assume that for all* $(t, v) \in J \times U$ *the linear map* $(D_2\phi)(t, v) \in \mathrm{Hom}(V, V)$ *has operator norm satisfying* $\|(D_2\phi)(t, v)\| \leq M$ *for some* $M > 0$*.*

Pick $\varepsilon_1, \varepsilon_2 \geq 0$ *and assume that* $u_1, u_2 : J \rightrightarrows U$ *are respectively* ε_1*-approximate and* ε_2*-approximate solutions to* $y'(t) = \phi(t, y(t))$ *in the sense that*

$$\|u_1'(t) - \phi(t, u_1(t))\| \leq \varepsilon_1, \ \ \|u_2'(t) - \phi(t, u_2(t))\| \leq \varepsilon_2$$

for all $t \in J$*. For any* $t_0 \in J$*,*

$$\|u_1(t) - u_2(t)\| \leq \|u_1(t_0) - u_2(t_0)\| e^{M|t - t_0|} + (\varepsilon_1 + \varepsilon_2)(e^{M|t - t_0|} - 1)/M. \tag{4.3}$$

In the special case $\varepsilon_1 = \varepsilon_2 = 0$ and $u_1(t_0) = u_2(t_0)$, the upper bound is 0 and hence we recover the global uniqueness theorem for a given initial condition. Thus, this lemma is to be understand as an analogue of the general uniqueness theorem when we move the initial condition (allow $u_1(t_0) \neq u_2(t_0)$) and allow approximate solutions to the ODE.

Proof. Using a translation allows us to assume $t_0 = 0$, and by negating if necessary it suffices to treat the case $t \geq t_0 = 0$. Since $u_j(t) = u_j(0) + \int_0^t u_j'(x)\mathrm{d}x$, the ε_j-approximation condition gives

$$\|u_j(t) - u_j(0) - \int_0^t \phi(x, u_j(x))\mathrm{d}x\| = \|\int_0^t (u_j'(x) - \phi(x, u_j(x)))\mathrm{d}x\| \leq \int_0^t \varepsilon_j \mathrm{d}x = \varepsilon_j t.$$

Thus, using the triangle inequality we get

$$\|u_1(t) - u_2(t)\| \leq \|u_1(0) - u_2(0)\| + \int_0^t \|\phi(x, u_1(x)) - \phi(x, u_2(x))\|\mathrm{d}x + (\varepsilon_1 + \varepsilon_2)t.$$

Consider the C^1 restriction $g(z) = \phi(x, zu_1(x) + (1 - z)u_2(z))$ of $\phi(x, \cdot)$ on the line segment in V joining the points $u_1(x), u_2(x) \in U$ (a segment lying entirely in U, since U is assumed to be convex). By the Fundamental Theorem of Calculus and the Chain Rule, $\phi(x, u_1(x)) - \phi(x, u_2(x))$ is equal to

$$g(1) - g(0) = \int_0^1 g'(z)\mathrm{d}z = \int_0^1 ((D_2\phi)(x, zu_1(x) + (1 - z)u_2(z)))(u_1(x) - u_2(x))\mathrm{d}z.$$

Thus, the assumed bound of M on the operator norm of $(D_2\phi)(t, v)$ for all $(t, v) \in J \times U$ gives

$$\|\phi(x, u_1(x)) - \phi(x, u_2(x))\| \leq M \cdot \int_0^1 \|u_1(x) - u_2(x)\|\mathrm{d}z = M\|u_1(x) - u_2(x)\|$$

for all $x \in J$. Hence, for $h(t) = \|u_1(t) - u_2(t)\|$ we have

$$h(t) \leq h(0) + (\varepsilon_1 + \varepsilon_2)t + \int_0^t Mh(x)\mathrm{d}x$$

for all $t \in J$ satisfying $t \geq 0$. By Lemma 3.4 in the handout on linear ODE, we thereby conclude that for all such t there is the bound

$$h(t) \leq h(0) + (\varepsilon_1 + \varepsilon_2)t + \int_0^t (h(0) + (\varepsilon_1 + \varepsilon_2)x)Me^{M(t-x)}\mathrm{d}x,$$

and by direct calculation this upper bound is exactly the one given in (4.3). ∎

Now we prove Theorem 4.1:

Proof. Fix norms on V and V'. Since A and F are continuous, by Theorem 2.2 there is a *continuous* mapping $y : I \times U' \to \mathrm{Hom}(V', V)$ such that $y(\cdot, z) : I \to \mathrm{Hom}(V', V)$ is the solution to (4.2) for all $z \in U'$. We need to prove (among other things) that if $z \in U'$ is near z_0 and $t \in I$ is near $\overline{I}_0$ then $y(t, z) \in \mathrm{Hom}(V', V)$ serves as a total U'-derivative for $u : I \times U' \to V$ at (t, z). This rests on getting estimates for the norm of $u(t, z + h) - u(t, z) - (y(t, z))(h)$ for $h \in V'$ near 0, at least with (t, z) near $\overline{I}_0 \times \{z_0\}$. Our estimate on this difference will be obtained via an application of Lemma 4.2. We first require some preliminary considerations to find the right I_0' and U_0' in which t and z should respectively live. The continuity of y on $I \times U'$ will not be used until near the end of the proof.

Since I_0 has compact closure in I, by shrinking I around $\overline{I}_0$, U around v_0, and U' around z_0 we may arrange that the operators norms of $(D_2\phi)(t, v, z) \in \mathrm{Hom}(V, V)$ and $(D_3\phi)(t, v, z) \in \mathrm{Hom}(V', V)$ are bounded above by some positive constants M and N for all $(t, v, z) \in I \times U \times U'$. We may also assume that U and U' are open balls centered at v_0 and z_0, so each is convex. For any points $(v_1, z_1), (v_2, z_2) \in U \times U'$ and $t \in I$, if we let $h(x) = \phi(t, xv_1 + (1 - x)v_2, z_1)$ and $g(x) = \phi(t, v_2, xz_1 + (1 - x)z_2)$ then

$$\phi(t, v_1, z_1) - \phi(t, v_2, z_2) = (h(1) - h(0)) + (g(1) - g(0)) = \int_0^1 h'(x)\mathrm{d}x + \int_0^1 g'(x)\mathrm{d}x$$

with

$$h'(x) = ((D_2\phi)(t, xv_1 + (1 - x)v_2, z))(v_1 - v_2), \;\; g'(x) = ((D_2\phi)(t, v_2, xz_1 + (1 - x)z_2))(z_1 - z_2)$$

by the Chain Rule. Hence, the operator-norm bounds give

$$\|\phi(t, v_1, z_1) - \phi(t, v_2, z_2)\| \leq M\|v_1 - v_2\| + N\|z_1 - z_2\|.$$

Setting $v_1 = v_2 = u_{z_1}(t)$ and using the equation $u_{z_1}'(t) = f(t, u_{z_1}(t), z_1)$ we get

$$\|u_{z_1}'(t) - f(t, u_{z_1}(t), z_2)\| \leq N\|z_1 - z_2\|$$

for all $t \in I$.

For $c = N\|z_1 - z_2\|$ we have shown that $u_{z_1} : I \to U$ is a c-approximate solution to the ODE $f'(t) = \phi(t, f(t), z_2)$ on I and its value at $t_0 = 0$ coincides with that of the 0-approximate (i.e., exact) solution u_{z_2} to the same ODE on I. Shrink I around $\overline{I}_0$ so that it has finite length, say bounded above by R. Hence, by Lemma 4.2 (U is convex!), for all $t \in I$ we have

$$\|u_{z_1}(t) - u_{z_2}(t)\| \leq c \cdot (e^{M|t|} - 1)/M \leq Q\|z_1 - z_2\|$$

with $Q = N(e^{MR} - 1)/M$.

Choose $\varepsilon > 0$. By working near the compact set $\overline{I}_0 \times \{(v_0, z_0)\}$, for h sufficiently near 0 the difference $u(t, z+h) - u(t, z)$ is as uniformly small as we please for all (t, z) near $\overline{I}_0 \times \{z_0\}$ because u

is *continuous* on $I \times U \times U'$ (and hence uniformly continuous around compacts). Hence, by taking h sufficiently small (depending on ε!) we may form a first-order Taylor approximation to

$$\phi(t, u(t, z+h), z+h) = \phi(t, u(t, z) + (u(t, z+h) - u(t, z)), z+h)$$

with error bounded in norm by $\varepsilon\|h\|$ for h near enough to 0 such that $u(t, z+h) - u(t, z)$ is uniformly small for (t, z) near $\overline{I}_0 \times \{z_0\}$. That is, for a suitable open ball $U'_0 \subseteq U'$ around z_0 and an open interval $I'_0 \subseteq I$ around $\overline{I}_0$ we have that for h sufficiently near 0 there is an estimate

$$\|\phi(t, u(t, z+h), z+h) - \phi(t, u(t, z), z) - (A(t, z))(u(t, z+h) - u(t, z)) - (F(t, z))(h)\| \leq \varepsilon\|h\|$$

for all $(t, v) \in I'_0 \times U'_0$. Here, we are of course using the *definitions* of A and F in terms of partials of ϕ. In view of the ODE's satisfied by u_z and u_{z+h}, we therefore get

$$\|u'_{z+h}(t) - u'_z(t) - (A(t, z))(u_{z+h}(t) - u_z(t)) - (F(t, z))(h)\| \leq \varepsilon\|h\| \tag{4.4}$$

for all $(t, v) \in I'_0 \times U'_0$ and h sufficiently near 0 (independent of (t, z)).

For $h \in V'$ near 0 and $(t, z) \in I'_0 \times U'_0$, let

$$\delta(t, z, h) = u(t, z+h) - u(t, z) - (y(t, z))(h)$$

where $y_z = y(\cdot, z)$ is the solution to (4.2) on I. Using the ODE satisfied by y_z we get

$$(\partial_t\delta)(t, z, h) = u'_{z+h}(t) - u'_z(t) - (y'_z(t))(h) = u'_{z+h}(t) - u'_z(t) - (A(t, z))((y_z(t))(h)) - (F(t, z))(h).$$

Hence, (4.4) says

$$\|(\partial_t\delta)(t, z, h) - (A(t, z))(\delta(t, z, h))\| \leq \varepsilon\|h\|$$

for all $(t, z) \in I'_0 \times U'_0$ and h sufficiently near 0 (where "sufficiently near" is independent of (t, z)). This says that for $z \in U'_0$ and h sufficiently near 0, $\delta(\cdot, z, h)$ is an $\varepsilon\|h\|$-approximate solution to the V-valued ODE

$$X'(t) = (A(t, z))(X(t))$$

on I'_0 with initial value $\delta(0, z, h) = u_{z+w}(0) - u_z(0) - (y_z(0))(h) = v_0 - v_0 - 0 = 0$ at $t = 0$. The exact solution with this initial value is $X = 0$, and so Lemma 4.2 gives

$$\|\delta(t, z, h)\| \leq q\varepsilon\|h\|$$

for all $(t, z) \in I'_0 \times U'_0$ and sufficiently small h, with $q = (e^{MR} - 1)/M$ for an upper bound R on the length of I. The "sufficient smallness" of h depends on ε, but neither q nor $I'_0 \times U'_0$ have dependence on ε. Thus, we have proved that for $(t, z) \in I'_0 \times U'_0$

$$u(t, z+h) - u(t, z) - (y(t, z))(h) = \delta(t, z, h) = o(\|h\|)$$

in V as $h \to 0$ in V'. Hence, $(D_2u)(t, z)$ exists for all $(t, z) \in I'_0 \times U'_0$ and it is equal to $y(t, z) \in \mathrm{Hom}(V', V)$. But y depends *continuously* on (t, z), so $D_2u : I'_0 \times U'_0 \to \mathrm{Hom}(V', V)$ is continuous. Meanwhile, the ODE for u_z gives

$$(D_1u)(t, z) = u'_z(t) = \phi(t, u(t, z), z)$$

in $\mathrm{Hom}(\mathbf{R}, V) = V$, so by continuity of ϕ and of u in (t, z) it follows that $D_1u : I'_0 \times U'_0 \to V$ exists and is continuous.

We have shown that at each point of $I'_0 \times U'_0$ the mapping $u : I'_0 \times U'_0 \to V$ admits partials in the I'_0 and U'_0 directions with D_1u and D_2u both continuous on $I'_0 \times U'_0$. Thus, u is C^1. The preceding argument also yields that $(D_2u)(\cdot, z)$ is the solution to (4.2) on I'_0 for all $z \in U'_0$. ■

It has now been proved that, in the setup of Theorem 3.3, on the open domain of flow $\mathscr{D}(\phi)$ the universal solution u is always C^1. We shall use induction on p and the description of D_2u in Theorem 4.1 to prove that u is C^p when ϕ is C^p, with $1 \leq p \leq \infty$.

Remark 4.3. The inductive hypothesis will be applied to the ODE's (4.2) that are *time-dependent* and depend on parameters even if the initial ODE for the u_z's is time-independent. It is exactly for this aspect of induction that we have to permit time-dependent flow: without incorporating time-dependent flow into the inductive hypothesis, the argument would run into problems when we try to apply the inductive hypothesis to (4.2). (Strictly speaking, we could have kept time-dependence out of the inductive hypothesis by making repeated use of the reduction steps of the sort that preceded Theorem 3.6; however, it seems simplest to cut down on the use of such reduction steps when they're not needed.)

Since the domain of flow $\mathscr{D}(\phi)$ for a C^p mapping ϕ with $1 \leq p \leq \infty$ is "independent of p" (in the sense that it remains the domain of flow even if ϕ is viewed as being C^r with $1 \leq r < p$, as we must do in inductive arguments), it suffices to treat the case of *finite* $p \geq 1$. Thus, we now fix $p > 1$ and assume that the problem has been solved in the C^{p-1} case in general. As we have already seen, it suffices to treat the C^p case in the same setup considered in Theorem 4.1, which is to say time-dependent flow with an auxiliary parameter but fixed initial conditions. Moreover, since the domain $\mathscr{D}(\phi)$ is an *open* set in $\mathbf{R} \times U'$, the C^p problem near any particular point $(t_0, z_0) \in \mathscr{D}(\phi)$ is local around the compact product $I_{t_0} \times \{z_0\}$ in $\mathbf{R} \times U'$ where I_{t_0} is the compact interval in $\mathbf{R}$ with endpoints 0 and t_0. In particular, it suffices to prove:

Corollary 4.4. *Keep notation as in Theorem* 4.1*, and assume ϕ is C^p with $1 \leq p < \infty$. For sufficiently small open $U_0' \subseteq U_0$ around z_0 and an open subinterval $I_0' \subseteq I$ around $\overline{I}_0$, $(t, z) \mapsto u_z(t)$ is C^p as a mapping from $I_0' \times U_0'$ to V.*

As we will see in the proof, each time we use induction on p we will have to *shrink* U_0' and I_0' further. Hence, the method of proof does not directly give a result for $p = \infty$ across a neighborhood of $\overline{I}_0 \times \{z_0\}$ in $I \times U'$ because a shrinking family of opens (in $I \times U'$) around $\overline{I}_0 \times \{z_0\}$ need not have its intersection contain an open (in $I \times U'$) around $\overline{I}_0 \times \{z_0\}$. The reason we get a result in the C^∞ case is because we did the hard work to prove that the global domain of flow $\mathscr{D}(\phi)$ has good topological structure (i.e., it is an open set in $\mathbf{R} \times U'$); in the discussion preceding the corollary we saw how this openness enabled us to reduce the C^∞ case to the C^p case for *finite* $p \geq 1$. If we had not introduced the concept of domain of flow that is "independent of p" and proved its openness *a priori*, then we would run into a brick wall in the C^∞ case (the case we need in differential geometry!).

Proof. We proceed by induction, the case $p = 1$ being Theorem 4.1. Thus, we may and do assume $p > 1$. We emphasize (for purposes of the inductive step later) that our induction is really to be understood to be simultaneously applied to *all* time-dependent flows with an auxiliary parameter and a fixed initial condition.

By the inductive hypothesis, we can find open U_0' around z_0 in U' and an open interval $I_0' \subseteq I$ around $\overline{I}_0$ so that $u : (t, z) \mapsto u(t, z)$ is C^{p-1} on $I_0' \times U_0'$. Since u is C^{p-1} with $p - 1 \geq 1$, to prove that it is C^p on $I_0'' \times U_0''$ for some open $U_0'' \subseteq U_0'$ around z_0 and some open subinterval $I_0'' \subseteq I_0'$ around $\overline{I}_0$ it is equivalent to check that (as a V-valued mapping) for suitable such I_0'' and U_0'' the partials of u along the directions of I_0 and U' (via a basis of V', say) are all C^{p-1} at each point $(t, z) \in I_0'' \times U_0''$. By construction, $(D_1 u)(t, z) \in \mathrm{Hom}(\mathbf{R}, V) \simeq V$ is $\phi(t, u(t, z), z)$, and this has C^{p-1}-dependence on (t, z) because ϕ is C^p on $I \times U \times U'$ and $u : I_0' \times U_0' \to U$ is C^{p-1}.

To show that $(D_2 u)(t, z) \in \mathrm{Hom}(V', V)$ has C^{p-1}-dependence on $(t, z) \in I_0'' \times U_0''$ for suitable I_0'' and U_0'', first recall from Theorem 4.1 (viewing ϕ as a C^1 mapping) that on $I_0' \times U_0'$ the map $(t, z) \mapsto (D_2 u)(t, z)$ is the solution to the $\mathrm{Hom}(V', V)$-valued initial-value problem

$$Y'(t) = A(t, z) \circ Y(t) + F(t, z), \quad Y(0) = 0 \tag{4.5}$$

with $A(t,z) = (D_2\phi)(t,u_z(t),z) \in \mathrm{Hom}(V,V)$ and $F(t,z) = (D_3\phi)(t,u_z(t),z) \in \mathrm{Hom}(V',V)$ depending continuously on $(t,z) \in I_0' \times U_0'$. Since $u_z(t)$ has C^{p-1}-dependence on $(t,z) \in I_0' \times U_0'$ and ϕ is C^p, both A and F have C^{p-1}-dependence on $(t,z) \in I_0 \times U_0'$. But $p-1 \geq 1$ and the compact $\overline{I}_0$ is contained in I_0', so we may invoke the inductive hypothesis on $I_0' \times U_0'$ for the *time-dependent* flow (4.5) with a *varying parameter* but a *fixed* initial condition. More precisely, we have a "universal solution" D_2u to this latter family of ODE's across $I_0' \times U_0'$ and so by induction there exists an open $U_0'' \subseteq U_0'$ around z_0 and an open subinterval $I_0'' \subseteq I_0'$ around $\overline{I}_0$ such that the restriction to $I_0'' \times U_0''$ of the family of solutions $(D_2u)(\cdot,z)$ to (4.5) for $z \in U_0''$ has C^{p-1}-dependence on $(t,z) \in I_0'' \times U_0''$.

We have proved that for the C^1 map $u : I_0'' \times U_0'' \to U$ the maps $D_1u : I_0'' \times U_0'' \to V$ and $D_2u : I_0'' \times U_0'' \to \mathrm{Hom}(V',V)$ are C^{p-1}. Hence, u is C^p. ∎

5. Smooth flow on manifolds

Up to now we have proved some rather general results on the structure of solutions to ODE's (in both the ODE handout and in §2–§4 above). We now intend to use these results to study integral curves for smooth vector fields on smooth manifolds. The diligent reader will see that (with some modifications to statements of results) in what follows we can relax smoothness to C^p with $2 \leq p < \infty$, but such cases with finite p lead to extra complications (due to the fact that vector fields cannot be better than class C^{p-1}). Thus, we shall now restrict our development to the smooth case – all of the real ideas are seen here anyway, and it is by far the most important case in geometric applications. Let M be a smooth manifold, and let $\vec{v}$ be a smooth vector field on M. The first main theorem is the existence and uniqueness of a maximal integral curve to $\vec{v}$ through a specified point at time 0.

The following theorem is a manifold analogue of the existence and uniqueness theorem on maximal intervals around the initial time in the "classical" theory of ODE's (see §2 in the ODE handout). The extra novelty is that in the manifold setting we cannot expect the integral curve to lie in a single coordinate chart and so to prove the existence/uniqueness theorem (for the maximal integral curve) on manifolds we need to artfully reduce the problem to one in a single chart where we can exploit the established theory in open subsets of vector spaces.

Theorem 5.1. *Let $m_0 \in M$ be a point. There exists a unique maximal integral curve for $\vec{v}$ through m_0. That is, there exists an open interval $J_{m_0} \subseteq \mathbf{R}$ around 0 and a smooth mapping $c_{m_0} : J_{m_0} \to M$ satisfying*

$$c_{m_0}'(t) = \vec{v}(c_{m_0}(t)), \;\; c_{m_0}(0) = m_0$$

such that if $I \subseteq \mathbf{R}$ is any open interval around 0 and $c : I \to M$ is an integral curve for $\vec{v}$ with $c(0) = m_0$ then $I \subseteq J_{m_0}$ and $c_{m_0}|_I = c$.

Moreover, the map $c_{m_0} : J_{m_0} \to M$ is an immersion except if $\vec{v}(m_0) = 0$, in which case it is the constant map $c_{m_0}(t) = m_0$ for all $t \in J_{m_0} = \mathbf{R}$.

Beware that in general J_{m_0} may not equal $\mathbf{R}$. (It could be bounded, or perhaps bounded on one side.) For the special case $M = \mathbf{R}^n$, this just reflects the fact (as in Example 1.1) that solutions to non-linear initial-value problems $u'(t) = \phi(u(t))$ can fail to propogate for all time. Also, Example 5.4 below shows that c_{m_0} may fail to be injective. The immersion condition says that the image $c_{m_0}(J_{m_0})$ does not have "corners". In Example 5.7 we show that $c_{m_0}(J_{m_0})$ cannot "cross itself". However, c_{m_0} can fail to be an embedding: this is shown on the homework in the case of certain integral curves on a doughnut (for which the image is a densely-wrapped line).

Proof. We first construct a maximal integral curve, and then address the immersion aspect. Upon choosing local C^∞ coordinates around m_0, the homework exercise on integral curves shows that the problem of the existence of an integral curve for $\vec{v}$ through m_0 on a small open time interval around 0 is "the same" as the problem of solving (for small $|t|$) an ODE of the form

$$u'(t) = \phi(u(t)),\ \ u(0) = v_0$$

for $\phi : U \to V$ a C^∞ mapping on an open set U in a finite-dimensional vector space V (with $v_0 \in U$). Hence, the classical local existence/uniqueness theorem for ODE's (Theorem 2.1 in the ODE handout) ensures that for some $\varepsilon > 0$ there is an integral curve $c : (-\varepsilon, \varepsilon) \to M$ to $\vec{v}$ through m_0 (at time $t = 0$) and that any two such integral curves to $\vec{v}$ through m_0 (at time $t = 0$) coincide for t near 0.

For the existence of the maximal integral curve that recovers all others, all we have to show is that if $I_1, I_2 \subseteq \mathbf{R}$ are open intervals around 0 and $c_j : I_j \to M$ are integral curves for $\vec{v}$ through m_0 at time 0 then $c_1|_{I_1 \cap I_2} = c_2|_{I_1 \cap I_2}$. (Indeed, once this is proved then we can "glue" c_1 and c_2 to get an integral curve on the open interval $I_1 \cup I_2$, and more generally we can "glue" *all* such integral curves; on the union of their open interval domains we obviously get the unique maximal integral curve of the desired sort.) By replacing c_1 and c_2 with their restrictions to $I_1 \cap I_2$, we may rephrase the problem as a uniqueness problem: $I \subseteq \mathbf{R}$ is an open interval around 0 and $c_1, c_2 : I \rightrightarrows M$ are both solutions to the same "initial-value problem"

$$c'(t) = \vec{v}(c(t)),\ \ c(0) = m_0$$

with values in the manifold M. We wish to prove $c_1 = c_2$ on I. As we saw at the beginning of the present proof, by working in a local coordinate system near m_0 we may use the classical local uniqueness theorem to infer that $c_1(t) = c_2(t)$ for $|t|$ near 0.

To get equality on all of I we will treat the case $t > 0$ (the case $t < 0$ goes similarly). If $c_1(t) \neq c_2(t)$ for some $t > 0$ then the set $S \subseteq I$ of such t has an infimum $t_0 \in I$. Since c_1 and c_2 agree near the origin, necessarily $t_0 > 0$. Thus, c_1 and c_2 coincide on $[0, t_0)$, whence they agree on $[0, t_0]$. Let $x_0 \in M$ be the common point $c_1(t_0) = c_2(t_0)$. We can view c_1 and c_2 as integral curves for $\vec{v}$ through x_0 at time t_0. The local uniqueness for integral curves through a specified point at a specified time (the time $t = 0$ is obviously not sacred; $t = t_0$ works the same) implies that c_1 and c_2 must coincide for t near t_0. Hence, we get an ε-interval around t_0 in I on which c_1 and c_2 agree, whence they agree on $[0, t_0 + \varepsilon)$. Thus, t_0 cannot be the infimum of S after all. This contradiction completes the construction of maximal integral curves.

If $\vec{v}(m_0) = 0$ then the constant map $c(t) = m_0$ for all $t \in \mathbf{R}$ satisfies the conditions that uniquely characterize an integral curve for $\vec{v}$ through m_0. Thus, it remains to prove that if $\vec{v}(m_0) \neq 0$ then c is an immersion. By definition, $c'(t_0) = \mathrm{d}c(t_0)(\partial_t|_{t_0})$ for any $t_0 \in I$, with $\partial_t|_{t_0} \in \mathrm{T}_{t_0}(I)$ a basis vector. Thus, by the immersion theorem, c is an immersion around t_0 if and only if the tangent map $\mathrm{d}c(t_0) : \mathrm{T}_{t_0}(I) \to \mathrm{T}_{c(t_0)}(M)$ is injective, which is to say that the velocity $c'(t_0)$ is nonzero. In other words, we want to prove that if $\vec{v}(m_0) \neq 0$ then $c'(t_0) \neq 0$ for all $t_0 \in I$.

Assuming $c'(t_0) = 0$ for some $t_0 \in I$, in local coordinates near $c(t_0)$ the "integral curve" condition expresses c near t_0 as a solution to an initial-value problem of the form

$$u'(t) = \phi(u(t)),\ \ u(t_0) = v_0$$

with $c'(t_0) = 0$. Since $c'(t_0) = \vec{v}(c(t_0))$, in the initial-value problem we get the extra property $\phi(v_0) = \phi(u(t_0)) = 0$. The constant mapping $\xi : t \mapsto v_0$ for all $t \in \mathbf{R}$ therefore satisfies the initial-value problem (as $\phi(\xi(t)) = \phi(v_0) = 0$ and $\xi'(t) = 0$ for all t). Hence, by uniqueness it follows that c is constant for t near t_0, and so in particular c has vanishing velocity vectors for t near t_0. Since t_0 was an arbitrary point at which c has velocity zero, this shows that the subset $Z \subseteq I$ of t such that $c'(t) = 0$ is an *open* subset of I. However, by the local nature of closedness we may work on open parts of I carried by c into coordinate domains to see that Z is also a closed subset of I, and so since (by hypothesis) Z is non-empty we conclude from connectivity of I that $Z = I$. In particular, $0 \in Z$. This contradicts the assumption that $c'(0) = \vec{v}(m_0)$ is nonzero. Hence, c has to be an immersion when $\vec{v}(m_0) \neq 0$. ■

Remark 5.2. From a geometric point of view, it is unnatural to specify a "base point" on integral curves. Dropping reference to a specified "base point" at time 0, we can redefine the concept of integral curve for $\vec{v}$: a smooth map $c : I \to M$ on a non-empty open interval $I \subseteq \mathbf{R}$ (possibly not containing 0) such that $c'(t) = \vec{v}(c(t))$ for all $t \in I$. We have simply omitted the requirement that a particular number (such as 0) lies in I and that c has a specific image at that time. It makes sense to speak of maximal integral curves $c : I \to M$ for $\vec{v}$, namely integral curves that cannot be extended as such on a strictly larger open interval in $\mathbf{R}$. It is obvious (via Theorem 5.1) that any integral curve in this new sense uniquely extends to a maximal integral curve, and the only novelty is that the analogue of the uniqueness aspect of Theorem 5.1 requires a mild reformulation: if two maximal integral curves $c_1 : I_1 \to M$ and $c_2 : I_2 \to M$ for $\vec{v}$ have images that meet at a point, then there exists a unique $t_0 \in \mathbf{R}$ (usually nonzero) such that two conditions hold: $t_0 + I_1 = I_2$ (this determines t_0 if $I_1, I_2 \neq \mathbf{R}$) and $c_2(t_0 + t) = c_1(t)$ for all $t \in I_1$. This verification is left as a simple exercise via Theorem 5.1. (Hint: If $c_1(t_1) = c_2(t_2)$ for some $t_j \in I_j$, consider $t \mapsto c_1(t + t_1)$ and $t \mapsto c_2(t + t_2)$ on the open intervals $-t_1 + I_1$ and $-t_2 + I_2$ around 0.) In this new sense of integral curve, with no fixed base point, we consider the "interval of definition" in $\mathbf{R}$ to be well-defined up to additive translation. (That is, we tend to "identify" two integral curves that are related through additive translation in time.) Note that it is absolutely essential throughout the discussion that we are specifying velocities (via $\vec{v}$), as otherwise we cannot expect the subset $c(I) \subseteq M$ to determine its "time parameterization" uniquely up to additive translation in time.

Example 5.3. Consider the "inward" unit radial vector field

$$\vec{v} = -\partial_r = -\frac{x}{\sqrt{x^2 + y^2}}\partial_x - \frac{y}{\sqrt{x^2 + y^2}}\partial_y$$

on $M = \mathbf{R}^2 - \{(0,0)\}$. The integral curves are straight-line trajectories toward the origin at unit speed. Explicitly, an integral curve $c(t) = (c_1(t), c_2(t))$ satisfies an initial condition $c(0) = (x_0, y_0) \in M$ and an evolution equation

$$c'(t) = -\partial_r|_{c(t)} = -\frac{c_1(t)}{\sqrt{c_1(t)^2 + c_2(t)^2}}\partial_x|_{c(t)} - \frac{c_2(t)}{\sqrt{c_1(t)^2 + c_2(t)^2}}\partial_y|_{c(t)},$$

so since (by the Chain Rule) for any t_0 we must have

$$c'(t_0) \stackrel{\text{def}}{=} \mathrm{d}c(t_0)(\partial_t|_{t_0}) = c_1'(t_0)\partial_x|_{c(t_0)} + c_2'(t_0)\partial_y|_{c(t_0)}$$

the differential equation says

$$c_1' = -\frac{c_1}{\sqrt{c_1^2 + c_2^2}},\;\; c_2' = -\frac{c_2}{\sqrt{c_1^2 + c_2^2}},\;\; (c_1(0), c_2(0)) = (x_0, y_0).$$

These differential equations become a lot more transparent in terms of local polar coordinates (r, θ), with θ ranging through less than a full "rotation": $r'(t) = -1$ and $\theta'(t) = 0$. (Strictly speaking, whenever one computes an integral curve in local coordinates one must never forget the possibility that the integral curve might "escape the coordinate domain" in finite time, and so if the flow ceases at some time with the path approaching the boundary then the flow may well propogate within the manifold beyond the time for which it persists in the chosen coordinate chart. In the present case we get "lucky": the flow stops for global reasons unrelated to the chosen coordinate domain in which we compute.) It follows that in the coordinate domain the path must have $r(t) = r_0 - t$ with $r_0 = \sqrt{x_0^2 + y_0^2}$ and θ is constant (on the coordinate domain under consideration). In other words, the path is a half-line with motion towards the origin with a linear parameterization in time. Explicitly, if we let

$$(u_0, u_1) = (x_0/\sqrt{x_0^2 + y_0^2}, y_0/\sqrt{x_0^2 + y_0^2})$$

be the "unit vector" pointing in the same direction as $(x_0, y_0) \neq (0, 0)$ (i.e., the unique scaling of (x_0, y_0) by a positive number to have length 1) then in the chosen sector for polar coordinates we have

$$c_{(x_0,y_0)}(t) = (x_0 - u_0 t, y_0 - u_1 t) = (1 - t/(x_0^2 + y_0^2)^{1/2})x_0, (1 - t/(x_0^2 + y_0^2)^{1/2})y_0)$$

on the interval $J_{(x_0,y_0)} = (-\infty, r(x_0, y_0)) = (-\infty, \sqrt{x_0^2 + y_0^2})$ is the maximal integral curve through (x_0, y_0) at time $t = 0$. The failure of the solution to persist in the manifold to time $\sqrt{x_0^2 + y_0^2}$ is obviously not due to working in a coordinate sector, but rather because the flow viewed in $\mathbf{R}^2$ (containing M as an open subset) is approaching the point $(0, 0)$ not in the manifold (and so there cannot even be a continuous extension of the flow to time $\sqrt{x_0^2 + y_0^2}$ in the manifold M). Hence, the global integral curve really does cease to exist in M at this time. Of course, from the viewpoint of a person whose universe is M (and so cannot "see" the point $(0, 0) \in \mathbf{R}^2$), as they flow along this integral curve they will have a hard time understanding why spaceships moving along this curve encounter difficulties at this time.

As predicted by Theorem 5.1 and Remark 5.2, since the vector field $\vec{v}$ is everywhere non-vanishing there are no constant integral curves and all of them are immersions, with any two having images that are either disjoint or equal in M, and for those that are equal we see that varying the position at time $t = 0$ only has the effect of changing the parameterization mapping by an additive translation in time.

If we consider a vector field $\vec{v} = h(r)\partial_r$ for a non-vanishing smooth function h on $(0, \infty)$, then we naturally expect the integral curves to again be given by these rays, except that the direction of motion will depend on the constant sign of h (positive or negative) and the speed along the ray will depend on h. Indeed, by the same method as above it is clear that the integral curve $t \mapsto c_{(x_0,y_0)}(t)$ passing through (x_0, y_0) at time $t = 0$ is $c_{(x_0,y_0)}(t) = (x_0 + u_0 H(t), y_0 + u_1 H(t))$ where (u_0, u_1) is the unit-vector obtained through positive scaling of (x_0, y_0) and $H(t) = \int_{r_0}^{t+r_0} h$ for $r_0 = r(x_0, y_0) = \sqrt{x_0^2 + y_0^2}$.

Example 5.4. Let $M = \mathbf{R}^2$ and consider the "circular" (non-unit!) vector field

$$\vec{v} = \partial_\theta = -y\partial_x + x\partial_y$$

that is smooth on the entire plane (including the origin). The integral curve through the origin is the constant map to the origin, and the integral curve through any $(x_0, y_0) \neq (0, 0)$ at time 0 is the circular path

$$c_{(x_0,y_0)}(t) = (r_0 \cos(t + \theta_0), r_0 \sin(t + \theta_0)) = (x_0 \cos t - y_0 \sin t, x_0 \sin t + y_0 \cos t)$$

for $t \in \mathbf{R}$ with constant speed of motion $r_0 = r(x_0, y_0) = \sqrt{x_0^2 + y_0^2}$ (θ_0 is the "angle" parameter, only well-defined up to adding an integral multiple of 2π).

In Example 5.3 the obstruction to maximal integral curves being defined for all time is related to the hole at the origin. Quite pleasantly, on compact manifolds such a difficulty never occurs:

Theorem 5.5. *If M is compact, then maximal integral curves have interval of definition* $\mathbf{R}$.

Proof. We may assume M has constant dimension n. For each $m \in M$ we may choose a local coordinate chart $(\{x_1, \ldots, x_n\}, U_m)$ with parameterization by an open set $B_m \subseteq \mathbf{R}^n$ (i.e., B_m is the open image of U_m under the coordinate system). In such coordinates, the ODE for an integral curve takes the form $u'(t) = \phi(u(t))$ for a smooth mapping $\phi : B_m \to \mathbf{R}^n$. Give $\mathbf{R}^n$ its standard norm. By shrinking the coordinate domain U_m around m, we can arrange that $\phi(B_m)$ is bounded, say contained in a ball of radius R_m around the origin in $\mathbf{R}^n$, and that the total derivative for ϕ at each point of B_m has operator-norm bounded by some constant $L_m > 0$. Finally, choose $r_m \in (0,1)$ and an open U'_m around m in U_m so that B_m contains the set of points in $\mathbf{R}^n$ with distance at most $2r_m$ from the image of U'_m in B_m. Let $a_m = \min(1/2L_m, r_m/R_m) > 0$. By the *proof* of the local existence theorem for ODE's (Theorem 2.1 in the handout on linear ODE's), it follows that the equation $c'(t) = \vec{v}(c(t))$ with an initial condition $c(t_0) \in U'_m$ (for $c : I \to M$ on an unspecified open interval I around $0 \in \mathbf{R}$) can always be solved on $(t_0 - a_m, t_0 + a_m)$ for c with values in U_m. That is, if an integral curve has a point in U'_m at a time t_0 then it persists in U_m for a_m units of time in *both* directions.

The opens $\{U'_m\}_{m \in M}$ cover M, so by compactness of M there is a finite subcover $U'_{m_1}, \dots, U'_{m_N}$. Let $a = \min(a_{m_1}, \dots, a_{m_N}) > 0$. Let $c : I \to M$ be a *maximal* integral curve for $\vec{v}$. For any $t \in I$ we have $c(t) \in U'_{m_i}$ for some i, and so by the preceding argument (and maximality of c!)

$$(t - a_{m_i}, t + a_{m_i}) \subseteq I$$

with c having image inside of U_{m_i} on this interval. Hence, $(t - a, t + a) \subseteq I$. Since a is a positive constant independent of $t \in I$, this shows that for *all* $t \in I$ the interval $(t - a, t + a)$ is contained in I. Obviously (argue with supremums and infimums, or use the Archimedean property of $\mathbf{R}$) the only non-empty open interval (or even subset!) in $\mathbf{R}$ with such a property is $\mathbf{R}$. ■

Definition 5.6. A smooth vector field $\vec{v}$ on a smooth manifold M is *complete* if all of its maximal integral curves are defined on $\mathbf{R}$.

Theorem 5.5 says that on a compact C^∞ manifold all smooth vector fields are complete. Example 5.3 shows that some non-compact C^∞ manifolds can have non-complete smooth vector fields. In Riemannian geometry, the notion of completeness for (certain) vector fields is closely related to the notion of completeness in the sense of metric spaces (hence the terminology!).

Example 5.7. We now work out the interesting geometry when a maximal integral curve $c : I \to M$ is not injective. That is, $c(t_1) = c(t_2)$ for some distinct t_1 and t_2 in I. As one might guess, the picture will be that of a circle wound around infinitely many times (forward and backwards in time). Let us now prove that this is exactly what must happen, as an application of our earlier work with quotients by group actions.

Denote the point $c(t_1) = c(t_2)$ by x, so on $(t_2 - t_1) + I$ the map $\widetilde{c} : t \mapsto c(t + t_1 - t_2)$ is readily checked to be an integral curve whose value at t_2 is $c(t_1) = c(t_2)$. We can likewise run the process in reverse to recover c on I from $\widetilde{c}$, so the integral curve $\widetilde{c}$ is also maximal. The maximal integral curves c and $\widetilde{c}$ agree at t_2, so they must coincide: same interval domain and same map. In particular, $(t_2 - t_1) + I = I$ in $\mathbf{R}$ and $c = \widetilde{c}$ on this open interval. Since $t_2 - t_1 \neq 0$, the invariance of I under additive translation by $t_2 - t_1$ forces $I = \mathbf{R}$. We conclude that c is defined on $\mathbf{R}$ and (since $c = \widetilde{c}$) the map c is periodic with respect to additive time translation by $\tau = t_2 - t_1$. To keep matters interesting we assume c is not a constant map, and so (by Theorem 5.1) c is an immersion. In particular, for any $t \in I$ we have that c is injective on a neighborhood of t in I. Hence, there must be a *minimal* period $\tau > 0$ for the map c. (Indeed, if $t \in \mathbf{R}$ is a nonzero period for c then $c(t) = c(0)$ and hence t cannot get too close to zero. Since a limit of periods is a period, the infimum of the set of periods is both positive and a period, hence the least positive period.)

Any integral multiple of τ is clearly a period for c (by induction). Conversely, if τ' is any other period for c, it has to be an integral multiple of τ. Indeed, pick $n \in \mathbf{Z}$ so that $0 \leq \tau' - n\tau < \tau$ (visualize!), so we want $\tau' - n\tau$ to vanish. Any $\mathbf{Z}$-linear combination of periods for c is a period for

c (why?), so $\tau' - n\tau$ is a non-negative period less than the least positive period. Hence, it vanishes. In view of the preceding proof that if $c(t_1) = c(t_2)$ with $t_1 \neq t_2$ then the difference $t_2 - t_1$ is a nonzero period, it follows that $c(t_1) = c(t_2)$ if and only if t_1 and t_2 have the same image in $\mathbf{R}/\mathbf{Z}\tau$. Since $c : \mathbf{R} \to M$ is a smooth map that is invariant under the additive translation by τ, it factors uniquely through the projection $\mathbf{R} \to \mathbf{R}/\mathbf{Z}\tau$ via a *smooth* mapping

$$\overline{c} : \mathbf{R}/\mathbf{Z}\tau \to M$$

that we have just seen is injective. The injective map $\overline{c}$ is an immersion because c is an immersion and $\mathbf{R} \to \mathbf{R}/\mathbf{Z}\tau$ is a local C^∞ isomorphism, and since $\mathbf{R}/\mathbf{Z}\tau$ is compact (it's a circle!) any injective continuous map from $\mathbf{R}/\mathbf{Z}\tau$ to a Hausdorff space is automatically a homeomorphism onto its (compact) image. In other words, $\overline{c}$ is an embedded smooth submanifold.

To summarize, we have proved that any maximal integral curve that "meets itself" in the sense that the trajectory eventually returns to the same point twice (i.e., $c(t_1) = c(t_2)$ for some $t_1 \neq t_2$) must have a very simple form: it is a smoothly embedded circle parameterized by modified notion of angle (as $\tau > 0$ might not equal 2π). This does *not* say that the velocity vectors along the curve look "constant" if M is given as a submanifold of some $\mathbf{R}^n$, but rather than the time parameter induces a C^∞-embedding $\mathbf{R}/\mathbf{Z}\tau \hookrightarrow M$ for the minimal positive period τ.

The reader will observe that we have not yet used any input from ODE beyond the existence/uniqueness results from the ODE handout. That is, none of the work in §2–§4 has played any role in the present considerations. This shall now change: the geometry becomes very interesting when we allow m_0 to *vary*. This is the global analogue of varying the initial condition. Before we give the global results for manifolds, we make a definition:

Definition 5.8. The *domain of flow* is the subset $\mathscr{D}(\vec{v}) \subseteq \mathbf{R} \times M$ consisting of pairs (t, m) such that the integral curve to $\vec{v}$ through m (at time 0) flows out to time t. That is, $(t, m) \in \mathscr{D}(\vec{v})$ if and only if the maximal integral curve $c_m : I_m \to M$ for $\vec{v}$ with $c_m(0) = m$ has t contained in its open interval of definition I_m. For each nonzero $t \in \mathbf{R}$, $\mathscr{D}(\vec{v})_t \subseteq M$ denotes the set of $m \in M$ such that $t \in I_m$.

Clearly $\mathbf{R} \times \{m\} \subseteq \mathbf{R} \times M$ meets $\mathscr{D}(\vec{v})$ in the domain of definition $I_m \subseteq \mathbf{R}$ for the maximal integral curve of $\vec{v}$ through m (at time 0). In particular, if M is connected then $\mathscr{D}(\vec{v})$ is connected (as in Lemma 3.2).

Example 5.9. Let us work out $\mathscr{D}(\vec{v})$ and $\mathscr{D}(\vec{v})_t$ for Example 5.3. Let $M = \mathbf{R}^2 - \{(0,0)\}$. In this case, $\mathscr{D}(\vec{v}) \subseteq \mathbf{R} \times M$ is the subset of pairs $(t, (x, y))$ with $t < \sqrt{x^2 + y^2}$. This is obviously an open subset. For each $t \in \mathbf{R}$, $\mathscr{D}(\vec{v})_t \subseteq M$ is the subset of points $(x, y) \in M$ such that $\sqrt{x^2 + y^2} > t$; hence, it is equal to M precisely for $t \leq 0$ and it is a proper open subset of M otherwise (exhausting M as $t \to 0^+$).

Example 5.10. For $t > 0$ we have $(-\varepsilon, t + \varepsilon) \times \{m\} \subseteq \mathscr{D}(\vec{v})$ for some $\varepsilon > 0$ if and only if $m \in \mathscr{D}(\vec{v})_t$, and similarly for $t < 0$ using $(t - \varepsilon, \varepsilon)$. Hence, if $t > 0$ then $\mathscr{D}(\vec{v})_t$ is the image under $\mathbf{R} \times M \to M$ of the union of the overlaps $\mathscr{D}(\vec{v}) \cap ((-\varepsilon, t + \varepsilon) \times M)$ over all $\varepsilon > 0$, and similarly for $t < 0$ using intervals $(t - \varepsilon, \varepsilon)$ with $\varepsilon > 0$.

The subsets $\mathscr{D}(\vec{v})_t \subseteq M$ grow as $t \to 0^+$, and the union of these loci is all of M: this just says that for each $m \in M$ there exists $\varepsilon_m > 0$ such that the maximal integral curve $c_m : J_m \to M$ has domain J_m that contains $(-\varepsilon_m, \varepsilon_m)$ (so $m \in \mathscr{D}(\vec{v})_t$ for $0 < |t| < \varepsilon_m$). Obviously $\mathscr{D}(\vec{v})_0 = M$.

Example 5.11. If M is compact, then by Theorem 5.5 the domain of flow $\mathscr{D}(\vec{v})$ is equal to $\mathbf{R} \times M$. Such equality is the notion of "completeness" for a smooth vector field on a smooth manifold, as in Definition 5.6.

In the case of opens in vector spaces, the above notion of domain of flow recovers the notion of domain of flow (as in §3) for time-independent parameter-free vector fields with fixed initial time (at 0) but varying initial position (a point on M at time 0). One naturally expects an analogue of Theorem 3.3; the proof is a mixture of methods and results from §3 (on opens in vector spaces):

Theorem 5.12. *The domain of flow $\mathscr{D}(\vec{v})$ is open in $\mathbf{R} \times M$, and the locus $\mathscr{D}(\vec{v})_t \subseteq M$ is open for all $t \in \mathbf{R}$. Moreover, if we give $\mathscr{D}(\vec{v})$ its natural structure of open C^∞ submanifold of $\mathbf{R} \times M$ then the set-theoretic mapping*

$$X_{\vec{v}} : \mathscr{D}(\vec{v}) \to M$$

defined by $(t, m) \mapsto c_m(t)$ is a smooth mapping (here; $c_m : I_m \to M$ is the maximal integral curve for $\vec{v}$ through m at time 0).

The mapping $X_{\vec{v}}$ is "vector flow along integral curves of $\vec{v}$"; it is the manifold analogue of the universal solution to a family of ODE's over the domain of flow in the classical case in §3. The openness in the theorem has a very natural intepretation: the ability to flow the solution to a given time is unaffected by small perturbations in the initial position. The smoothness of the mapping $X_{\vec{v}}$ is a manifold analogue of the C^∞-dependence on initial conditions in the classical case (as in Theorem 3.3, restricted to $\mathbf{R} \times \{0\} \times U \times \{0\}$ with $U' = V' = \{0\}$).

Proof. Since the map $\mathbf{R} \times M \to M$ is open and in Example 5.10 we have described $\mathscr{D}(\vec{v})_t$ as the image of a union of overlaps of $\mathscr{D}(\vec{v})$ with open subsets of $\mathbf{R} \times M$, the openness result for $\mathscr{D}(\vec{v})_t$ will follow from that for $\mathscr{D}(\vec{v})$ is open. Our problem is therefore to show that each $(t_0, m_0) \in \mathscr{D}(\vec{v})$ is an interior point (with respect to $\mathbf{R} \times M$) and that the set-theoretic mapping $(t, m) \mapsto c_m(t)$ is smooth near (t_0, m_0).

We first handle the situation for $t_0 = 0$. Pick a point $(0, m_0) \in \mathscr{D}(\vec{v})$, and choose a coordinate chart (φ, U) around m_0. In such coordinates the condition to be an integral curve with position at a point $m \in U$ at time 0 becomes a family of initial-value problems $u'(t) = \phi(u(t))$ with initial condition $u(0) = v_0 \in \varphi(U)$ for varying v_0 and $u : I \to \varphi(U)$ a C^∞ map on an unspecified open interval in $\mathbf{R}$ around 0. By using Theorem 3.3 (with $V' = \{0\}$) and the restriction to the slice $\mathbf{R} \times \{0\} \times \varphi(U) \times \{0\}$) it follows that for some $\varepsilon > 0$ and some $U_0 \subseteq U$ around m_0 the integral curve through any $m \in U_0$ at time 0 is defined on $(-\varepsilon, \varepsilon)$ and moreover the flow mapping $(t, m) \mapsto c_m(t)$ is smooth on $(-\varepsilon, \varepsilon) \times U_0$. Hence, the problem near points of the form $(0, m_0)$ is settled.

We now explain how to handle points $(t_0, m_0) \in \mathscr{D}(\vec{v})$ with $t_0 > 0$; the case $t_0 < 0$ will go in exactly the same way. Let $T_{m_0} \subseteq I_{m_0}$ be the subset of positive $\tau \in I_{m_0}$ such that $[0, \tau] \times \{m_0\}$ is interior to $\mathscr{D}(\vec{v})$ in $\mathbf{R} \times M$ and $(t, m) \mapsto c_m(t)$ is smooth at around (t', m_0) for all $t' \in [0, \tau)$. For example, the argument of the preceding paragraph shows $(0, \varepsilon) \subseteq T_{m_0}$ for some $\varepsilon > 0$ (depending on m_0). We want T_{m_0} to exhaust the set of positive elements of I_{m_0}, so we assume to the contrary and let τ_0 be the infimum of the set of positive numbers in $I_{m_0} - T_{m_0}$. Hence, $\tau_0 > 0$. Since $\tau_0 \in I_{m_0}$, the maximal integral curve c_{m_0} does propogate past τ_0. In particular, we get a well-defined point $m_1 = c_{m_0}(\tau_0) \in M$. Around m_1 we may choose a C^∞ coordinate chart (φ, U) with domain given by some open $U \subseteq M$ around m_1.

The integral curves for $\vec{v}|_U$ are described by time-independent parameter-free flow with a varying initial condition in the C^∞ coordinates on U. Thus, the argument from the final two paragraphs in the proof of Theorem 3.6 may now be carried over essentially *verbatim*. The only modification is that at all steps where the earlier argument said "continuous" (which was simultaneously being proved) we may use the word "smooth" (since Theorem 3.3 is now available to us on the open subset $\varphi(U)$ in a vector space). One also has to choose opens in $\varphi(U)$ so as to not wander outside of $\varphi(U)$ during the construction (as leaving this open set loses touch with the manifold M). We leave it to the reader to check that indeed the method of proof carries over with no substantive changes. ■

Definition 5.13. On the open subset $\mathscr{D}(\vec{v})_t \subseteq M$, the *flow to time t* is the mapping

$$X_{\vec{v},t} : \mathscr{D}(\vec{v})_t \to M$$

defined by $m \mapsto c_m(t)$.

In words, the points of $\mathscr{D}(\vec{v})_t$ are exactly those $m \in M$ such that the maximal integral curve for $\vec{v}$ through m at time 0 does propogate to time t (possibly $t < 0$), and $X_{\vec{v},t}(m)$ is the point $c_m(t)$ that this curve reaches after flowing for t units of time along $\vec{v}$ starting at m. For example, obviously $X_{\vec{v},0}$ is the identity map (since $c_m(0) = m$ for all $m \in M$). The perspective of integral curves is to focus on variation in time, but the perspective the mapping $X_{\vec{v},t}$ is to fix the time at t and to focus on variation in initial positions (at least for those initial positions for which the associated maximal integral curve persists to time t).

Corollary 5.14. *The mapping $X_{\vec{v},t}$ is a C^∞ isomorphism onto $\mathscr{D}(\vec{v})_{-t}$ with inverse given by $X_{\vec{v},-t}$.*

The meaning of the smoothness in this corollary is that the position the flow reaches at time t (if it lasts that long!) has smooth dependence on the initial position.

Proof. Set-theoretically, let us first check that the image of $X_{\vec{v},t}$ is $\mathscr{D}(\vec{v})_{-t}$ and that $X_{\vec{v},-t}$ is an inverse. For any $m \in \mathscr{D}(\vec{v})_t$ the point $X_{\vec{v},t}(m) = c_m(t) \in M$ sits on the maximal integral curve $c_m : I_m \to M$, and so (see Remark 5.2) the additive translate $c_m(t + (\cdot)) : (-t + I_m) \to M$ is the maximal integral curve for $\vec{v}$ through $m' = c_m(t)$ at time 0. Thus, $I_{m'} = -t + I_m$ and $c_{m'}(t') = c_m(t + t')$. Clearly $-t \in -t + I_m = I_{m'}$, so $m' \in \mathscr{D}(\vec{v})_{-t}$. Flowing by $-t$ units of time along this integral curve brings us from $m' = c_m(t)$ to $c_{m'}(-t) = c_m(t + (-t)) = c_m(0) = m$ as it should. Thus, $m = X_{\vec{v},-t}(m') = X_{\vec{v},-t}(c_t(m))$, as desired. We can run through the same argument with $-t$ in the role of t, and so in this way we see that $X_{\vec{v},t}$ and $X_{\vec{v},-t}$ are indeed inverse bijections between the open subsets $\mathscr{D}(\vec{v})_t$ and $\mathscr{D}(\vec{v})_{-t}$ in M. Hence, once we prove that $X_{\vec{v},t}$ and $X_{\vec{v},-t}$ are smooth maps (say when considered with target as M) then they are C^∞ isomorphisms between these open domains in M.

It remains to prove that $X_{\vec{v},t} : \mathscr{D}(\vec{v})_t \to M$ is smooth. Pick a point $m \in \mathscr{D}(\vec{v})_t$, so $(t, m) \in \mathscr{D}(\vec{v})$. By openness of $\mathscr{D}(\vec{v})$ in $\mathbf{R} \times M$ (Theorem 5.12), there exists $\varepsilon > 0$ and an open $U \subseteq M$ around m such that

$$(t - \varepsilon, t + \varepsilon) \times U \subseteq \mathscr{D}(\vec{v})$$

inside of $\mathbf{R} \times M$. Thus, $U \subseteq \mathscr{D}(\vec{v})_t$. On U, the mapping X_t is the composite of the C^∞ inclusion

$$U \to (t - \varepsilon, t + \varepsilon) \times U$$

given by $u \mapsto (t, u)$ and the restriction to this open target of the vector flow mapping $X_{\vec{v}} : \mathscr{D}(\vec{v}) \to M$ that has been proved to be C^∞ in Theorem 5.12. ■

Example 5.15. Suppose $\vec{v}$ is complete, so $\mathscr{D}(\vec{v}) = \mathbf{R} \times M$; i.e., $\mathscr{D}(\vec{v})_t = M$ for all $t \in \mathbf{R}$. (By Theorem 5.5, this is the case when M is compact.) For all $t \in \mathbf{R}$ we get a C^∞ automorphism $X_{\vec{v},t} : M \to M$ that flows each $m \in M$ to the point $c_m(t) \in M$ that is t units in time further away on the maximal integral curve of $\vec{v}$ through m. Explicitly, the vector flow mapping has the form $X_{\vec{v}} : \mathbf{R} \times M \to M$ and restricting it to the "slice" $\{t\} \times M$ in the source (or rather, composing X with the smooth inclusion $M \to \mathbf{R} \times M$ given by $m \mapsto (t, m)$) gives $X_{\vec{v},t}$.

This family of automorphisms $\{X_{\vec{v},t}\}_{t \in \mathbf{R}}$ is the 1-*parameter group generated by* $\vec{v}$. It is called a group because under composition it interacts well with the additive group structure on $\mathbf{R}$. More specifically, we have noted that $X_{\vec{v},0}$ is the identity and that $X_{\vec{v},t}$ is inverse to $X_{\vec{v},-t}$. We claim that $X_{\vec{v},t'} \circ X_{\vec{v},t} = X_{\vec{v},t'+t}$ for all $t, t' \in \mathbf{R}$ (so $X_{\vec{v},t'} \circ X_{\vec{v},t} = X_{\vec{v},t} \circ X_{\vec{v},t'}$ for all $t, t' \in \mathbf{R}$). In view of Remark 5.2, this says that if $c : \mathbf{R} \to M$ is (up to additive translation) the unique maximal integral curve for $\vec{v}$ with image containing $m \in M$, say $m = c(t_0)$, then $c(t' + (t + t_0)) = c((t' + t) + t_0)$; but this is (even physically) obvious!

Remark 5.16. In the case of non-complete $\vec{v}$ one can partially recover the group-like aspects of the $X_{\vec{v},t}$'s as in Example 5.15, except that one has to pay careful attention to domains of definition.

We conclude our tour of the elementary geometry of flow along integral curves by mentioning a marvelous application of the mappings $X_{\vec{v},t}$. Suppose that $f : M' \to M$ is a surjective submersion between smooth manifolds. Since each fiber $f^{-1}(m)$ is a smooth closed submanifold of M' (submersion theorem!) and these fibers cover M' without overlaps as $m \in M$ varies, we visualize the map f as a "smoothly varying family of manifolds" $\{f^{-1}(m)\}_{m \in M}$ indexed by the points of M. Such maps show up quite a lot in practice. Here is a basic example:

Example 5.17. For $A, B \in \mathbf{R}$ consider homogeneous polynomials

$$g_{A,B}(x, y, z) = y^2 z - x^3 - Axz^2 - Bz^3.$$

In an earlier homework we saw that for any homogeneous polynomial $h \in \mathbf{R}[x_0, \dots, x_n]$ it makes sense to define a "zero locus" $Z(h) \subseteq \mathbf{P}^n(\mathbf{R})$ as a closed subset. When is $Z(g_{A,B})$ a submanifold? This is a local problem, so we work in the three standard open charts that cover $\mathbf{P}^2(\mathbf{R})$. There is only one point on $Z(g_{A,B})$ outside of the chart $z \neq 0$, namely the point $\xi = [0, 1, 0]$. This point lies in the chart $y \neq 0$ on which we have coordinates $u = x/z$ and $v = y/z$, and in these coordinates the point ξ on $Z(g_{A,B})$ is the origin on the plane curve $v - u^3 - Auv^2 - Bv^3 = 0$; this is trivially seen to be a smooth point. To handle the rest of $Z(g_{A,B})$ we may work in the chart $z \neq 0$ (with coordinates $X = x/z$ and $Y = y/z$), so the problem is to check the Jacobian criterion for the plane curve $Y^2 = X^3 + AX + B$. One checks that the condition fails precisely at points $(X, 0)$ for which the polynomial $f(T) = T^3 + AT + B$ and its derivative vanish at T; that is, the double roots of f (if any) in $\mathbf{R}$. Such a double root can only exist if the discriminant $\Delta(A, B) = 4A^3 - 27B^2$ vanishes, so if we define $M \subseteq \mathbf{R}^2$ to be the open subset of pairs (A, B) with $\Delta(A, B) \neq 0$ then

$$M' = \{(A, B; [x, y, z]) \in M \times \mathbf{P}^2(\mathbf{R}) \,|\, g_{A,B}(x, y, z) = 0\}$$

is seen to be a smooth closed submanifold in $M \times \mathbf{P}^2(\mathbf{R})$. Of course, M is disconnected by the sign of $\Delta(A, B)$, and it can be shown that if $\Delta(A, B) > 0$ then $Z(g_{A,B})$ is connected and if $\Delta(A, B) < 0$ then $Z(g_{A,B})$ has exactly two connected components.

The C^∞ surjective mapping $f : M' \to M$ induced by the standard projection $M \times \mathbf{P}^2(\mathbf{R}) \to M$ is *proper* since M' is closed in $M \times \mathbf{P}^2(\mathbf{R})$ and $\mathbf{P}^2(\mathbf{R})$ is compact. Moreover, it can be shown by direct calculation that f is a submersion. For $m = (A, B) \in M$, the fiber $f^{-1}(m) \subseteq \mathbf{P}^2(\mathbf{R})$ is exactly the smooth (possibly disconnected) submanifold $Z(g_{A,B})$ in $\mathbf{P}^2(\mathbf{R})$.

In general, if a surjective C^∞ submersion $f : M' \to M$ is (as in the preceding example) *proper* then we consider f to be a "smoothly varying family of compact manifolds"; the point is that not only is each fiber compact, but the properness of the total mapping f gives an extra "relative compactness" yielding very pleasant consequences (as we shall see soon).

For any C^∞ submersion $f : M' \to M$, the submersion theorem says that if we work locally on both M *and* M' then (up to C^∞ isomorphism) the mapping f looks like projection to a factor space. However, this description requires us to work locally on the source and hence it loses touch with the *global* geometry of the smooth fibers $f^{-1}(m)$ for $m \in M$. It is a remarkable fact that for *proper* surjective submersions $f : M' \to M$, the local (on M' and M) description of f as projection to the factor of a product can be achieved by shrinking *only* on M. The result is this:

Theorem 5.18. *If $f : M' \to M$ is a proper surjective submersion, then M is covered by opens U_i such that for each i there is a C^∞ isomorphism $f^{-1}(U_i) \simeq U_i \times X_i$ for a compact manifold X_i with this isomorphism carrying the C^∞ map f on $f^{-1}(U_i)$ over to the standard projection $U_i \times X_i \to U_i$.*

We make some general remarks before proving the theorem. One consequence is that $f^{-1}(m_i)$ is C^∞-isomorphic to X_i for all $m_i \in U_i$, and hence the C^∞-isomorphism class of a fiber $f^{-1}(m)$ is "locally constant" in M. By using path-connectivity of connected components, it follows that if $f : M' \to M$ is a proper surjective C^∞ submersion to a connected smooth base M then *all* fibers $f^{-1}(m)$ are C^∞-isomorphic to each other!

The property of f as given in the conclusion of the theorem is usually summarized by saying that f is a "C^∞-fibration with compact fibers". Such a fibration result is an incredibly powerful topological tool, especially in the study of families of compact manifolds, and the technique of its proof (vector flow) is a basic ingredient in getting Morse theory off the ground.

Proof. (There is one step we shall have to skim over, as it requires some notions to be developed later.) We may work locally over M, so without loss of generality M is the open unit ball in $\mathbf{R}^n$ with coordinates $x_1, \dots, x_n$, and that we work around the origin m in this ball. We first need to construct smooth vector fields $\vec{v}_1, \dots, \vec{v}_n$ on M' such that $\mathrm{d}f(m') : \mathrm{T}_{m'}(M) \to \mathrm{T}_{f(m')}(M)$ sends $\vec{v}_i(m')$ to $\partial_{x_i}|_{f(m')}$ for all $m' \in M'$. Using the submersion property of f, such vector fields can be constructed. (In language of pullback bundles to be discussed later, the submersion condition implies that the map of vector bundles $\mathrm{d}f : \mathrm{T}M' \to f^*(\mathrm{T}M)$ over M' is fiberwise surjective. The induced map on smooth M'-sections is consequently surjective, a general fact to be proved later via Riemannian metrics. One takes $\vec{v}_i \in (\mathrm{T}M')(M') = \mathrm{Vec}_{M'}(M')$ to lift the section $f^*(\partial_{t_i})$ of $f^*(\mathrm{T}M)$.)

The open set $\mathscr{D}(\vec{v}_i) \subseteq \mathbf{R} \times M'$ contains $\{0\} \times M'$, and so it contains the subset $\{0\} \times f^{-1}(m)$ that is *compact* (since f is proper). Hence, it contains $(-\varepsilon_i, \varepsilon_i) \times U_i'$ for some open set $U_i' \subseteq M'$ around $f^{-1}(m)$. But since $f : M' \to M$ is *proper*, an open set around a fiber $f^{-1}(m)$ must contain an open of the form $f^{-1}(U_i)$ for an open $U_i \subseteq M$ around m. Thus, we conclude that $\mathscr{D}(\vec{v}_i)$ contains $(-\varepsilon_i, \varepsilon_i) \times f^{-1}(U_i)$ for some $\varepsilon_i > 0$ and some open U_i around m. Let $\varepsilon = \min_i \varepsilon_i > 0$ and $U = \cap_i U_i$, so for all i the domain of flow $\mathscr{D}(\vec{v}_i)$ contains $(-\varepsilon, \varepsilon) \times f^{-1}(U)$, with $\varepsilon > 0$ and $U \subseteq M$ an open around m. Hence, there is a flow mapping

$$X_{\vec{v}_i} : (-\varepsilon, \varepsilon) \times \pi^{-1}(U) \to M'$$

for $1 \le i \le n$.

Fix $1 \le i \le n$, and consider the composite mapping $h_i : M' \to M \stackrel{p_i}{\to} (-1, 1)$ where p_i is projection to the ith coordinate on the open unit ball $M \subseteq \mathbf{R}^n$. Since $\mathrm{d}f(m')(\vec{v}_i(m')) = \partial_{x_i}|_{f(m')}$ for all $m' \in M'$, integral curves for $\vec{v}_i$ in M' map to integral curves for ∂_{x_i} in M, and these are straight lines in the open unit ball M. For any $m' \in f^{-1}(U) \cap h_i^{-1}(t_0)$ with $|t_0| < \varepsilon$, for $|t| < \varepsilon - |t_0|$ the integral curve for $\vec{v}_i$ passing through m' therefore flows out to time t with $X_{\vec{v}_i, t}(m') \in h_i^{-1}(t)$. Provided that we begin at m' sufficiently close to the compact $f^{-1}(m)$, this endpoint $X_{\vec{v}_i, t}(m')$ will be in the open set $\pi^{-1}(U)$ around $f^{-1}(m)$.

Arguing in this way and again using properness of f (to know that opens around $f^{-1}(m)$ contain f-preimages of opens around m), we can find $\varepsilon_0 \in (0, \varepsilon)$ and an open $U_0 \subseteq U$ around m such that flow along $\vec{v}_1$ over time $(-\varepsilon_0, \varepsilon_0)$ with initial point in $\pi^{-1}(U_0)$ ends at a point in $\pi^{-1}(U)$. We repeat this procedure for $\vec{v}_2$ with U_0 in the role of U, and so on, to eventually arrive (after n iterations of this argument) at a very small $\eta \in (0, \varepsilon)$ and open sets $U_n \subseteq U_{n-1} \subseteq \cdots \subseteq U_0$ such that $X_{\vec{v}_i, t}(\pi^{-1}(U_i)) \subseteq \pi_i^{-1}(U_{i-1})$ for $|t| < \eta$. Hence, we arrive at an "iterated flow" mapping

$$(-\eta, \eta)^n \times \pi^{-1}(U_n) \to \pi^{-1}(U)$$

defined by

$$(t_1, \ldots, t_n, m') \mapsto (X_{\vec{v}_1, t_1} \circ X_{\vec{v}_2, t} \circ \cdots \circ X_{\vec{v}_n, t_n})(m')$$

that is certainly C^∞. We restrict this by replacing $\pi^{-1}(U_n)$ with the closed smooth submanifold $f^{-1}(m) \subseteq \pi^{-1}(U_n)$ to arrive at a smooth mapping

$$(-\eta, \eta)^n \times f^{-1}(m) \to \pi^{-1}(U)$$

given by the same iterated flow formula.

Geometrically, the map we have just constructed is a flow away from the fiber $f^{-1}(m)$ by flowing for time t_i in the ith coordinate direction over the base, done in the order "first x_1-direction, then x_2-direction, and so on." The image is contained in $\cap h_i^{-1}(-\eta, \eta) \subseteq f^{-1}((-\varepsilon, \varepsilon)^n)$. More specifically, by recalling that flow along $\vec{v}_i$ on M' lies over straight-line flow in the ith coordinate direction in the ball M (with the same time parameter!), we have built a smooth mapping

$$(-\eta, \eta)^n \times f^{-1}(m) \to f^{-1}((\varepsilon, \varepsilon)^n)$$

that lies over the inclusion

$$(-\eta, \eta)^n \hookrightarrow (-\varepsilon, \varepsilon)^n.$$

Hence, this map has image contained in $f^{-1}((-\eta, \eta)^n)$, and so for the open set $U = (-\eta, \eta)^n$ around $m \in M$ we have a smooth map

$$\psi : U \times f^{-1}(m) \to f^{-1}(U)$$

compatible with the projections from each side onto U (using $f : f^{-1}(U) \to U$).

We want to prove that after shrinking U around m the map ψ becomes a smooth isomorphism. By the *definition* of ψ (and of the iterated flow!), the restriction of ψ to the fiber over the origin m is the *identity map* on $f^{-1}(m)$. Since the $\vec{v}_i$'s were constructed to "lift" the ∂_{x_i}'s, it follows that for any $m' \in f^{-1}(m)$ the tangent mapping

$$\mathrm{d}\psi((0, \ldots, 0), m') : \mathbf{R}^n \times \mathrm{T}_{m'}(f^{-1}(m)) \to \mathrm{T}_{m'}(M)$$

is an isomorphism: it carries $\mathrm{T}_{m'}(f^{-1}(m)) = \ker(\mathrm{d}f(m'))$ to itself by the identity, and carries the standard basis of $\mathbf{R}^n$ to the vectors $\vec{v}_i(m')$ in $\mathrm{T}_{m'}(M')$ whose images in $\mathrm{T}_{m'}(M')/\mathrm{T}_{m'}(f^{-1}(m)) \simeq \mathrm{T}_m(M)$ are the $\partial_{x_i}|_m$'s that are a basis of this quotient! Thus, by the inverse function theorem we conclude that ψ is a local C^∞ isomorphism near all points over $m \in M$. Since the projection $U \times f^{-1}(m) \to U$ and the map $f : f^{-1}(U) \to U$ are proper C^∞ submersions, we may conclude the result by Theorem 5.19 below. ■

The following interesting general theorem was used in the preceding proof:

Theorem 5.19. *Let Z be a smooth manifold and let $\pi' : X' \to Z$ and $\pi : X \to Z$ be proper C^∞ submersions of smooth manifolds. Let*

$$h : X' \to X$$

be a mapping "over Z" (in the sense that $\pi \circ h = \pi'$). If $z_0 \in Z$ is a point such that h restricts to a C^∞ isomorphism ${\pi'}^{-1}(z_0) \simeq \pi^{-1}(z_0)$ and h is a local C^∞ isomorphism around points of ${\pi'}^{-1}(z_0)$, then there exists an open subset $U \subseteq Z$ around z_0 such that the mapping ${\pi'}^{-1}(U) \to \pi^{-1}(U)$ induced by the Z-map h is a C^∞-isomorphism.

The principle of this result is that for maps between proper objects over a base space, whatever happens on the fibers over a single point of the base space also happens over an open around the point. This is not literally a true statement in such generality, but in many contexts it can be given a precise meaning.

Proof. There is an open set in X' around ${\pi'}^{-1}(z_0)$ on which h is a local smooth isomorphism. Such an open set contains the π'-preimage of an open around z_0, due to properness of π'. Hence, by replacing Z with this open subset around z_0 and X and X' with the preimages of this open, we may assume that $h : X' \to X$ is a local C^∞-isomorphism. Since X and X' are proper over Z and all spaces under consideration are Hausdorff, it is not difficult to check that h must also be proper! Hence, h is a proper local isomorphism. The fibers of h are compact (by properness) and discrete (by the local isomorphism condition), whence they are *finite*.

Consider the function $s : X \to \mathbf{Z}$ that sends x to the size of $h^{-1}(x)$. This function is equal to 1 on $\pi^{-1}(z_0)$ by the hypothesis on h. I claim that it is a locally constant function. Grant this for a moment, so $s^{-1}(1)$ is an open set in X around $\pi^{-1}(z_0)$. By properness of π we can find an open set $U \subseteq Z$ around z_0 such that $\pi^{-1}(U) \subseteq s^{-1}(1)$. Replacing Z, X, and X' with U, $\pi^{-1}(U)$, and ${\pi'}^{-1}(U) = h^{-1}(\pi^{-1}(U))$ we get to the case when $h : X' \to X$ is a local C^∞-isomorphism whose fibers all have size 1. Such an h is bijective, and hence a C^∞ isomorphism (as desired).

How are we to show that s is locally constant? Rather generally, if $h : X' \to X$ is any proper local C^∞ isomorphism between smooth manifolds (so h has finite fibers, by the same argument used above), then we claim that the size of the fibers of h is locally constant on X. We can assume that X is connected, and in this case we claim that all fibers have the same size. Suppose $x_1, x_2 \in X$ are two points over which the fibers of h have different sizes. Make a continuous path $\sigma : [0,1] \to X$ with $\sigma(0) = x_1$ and $\sigma(1) = x_2$. For each $t \in [0,1]$ we can count the size of $h^{-1}(\sigma(t))$, and this has distinct values at $t = 0, 1$. Thus, the subset of $t \in [0,1]$ such that $\#h^{-1}(\sigma(t)) \neq \#h^{-1}(\sigma(0))$ is non-empty. We let t_0 be its infimum, and $x_0 = \sigma(t_0)$. Hence, there exist points t arbitrarily close to t_0 such that $\#h^{-1}(\sigma(t)) = \#h^{-1}(\sigma(0))$ and there exist other points t arbitrarily close to t_0 such that $\#h^{-1}(\sigma(t)) \neq \#h^{-1}(\sigma(0))$. (Depending on whether or not $h^{-1}(\sigma(t_0))$ has the same size as $h^{-1}(\sigma(0))$, we can always take $t = t_0$ for one of these two cases.)

It follows that the size of $h^{-1}(x_0)$ is distinct from that of $h^{-1}(\xi_n)$ for a sequence $\xi_n \to x_0$ in X. Since h is a local C^∞ isomorphism, if there are exactly r points in $h^{-1}(x_0)$ (perhaps $r = 0$) and we enumerate them as $x'_1, \dots, x'_r$ then by the Hausdorff and local C^∞-isomorphism conditions we may choose pairwise disjoint small opens U'_i around x'_i mapping isomorphically onto a common open U around x_0. Thus, for all $x \in U$ there are at least r points in $h^{-1}(x)$. It follows that for large n (so $\xi_n \in U$) the fiber $h^{-1}(\xi_n)$ has size at least r and hence has size strictly larger than r. In particular, $h^{-1}(\xi_n)$ contains a point ξ'_n *not* equal to any of the r points where $h^{-1}(\xi_n)$ meets $\coprod U'_i$ ($h : U'_i \to U$ is bijective for all i). In particular, for each n we have that $\xi'_n \notin U'_i$ for all i.

Let K be a compact neighborhood of x_0 in X, so $K' = h^{-1}(K)$ is a compact subset of X' by properness. Taking large n so that $\xi_n \in K$, the sequence ξ'_n lies in the compact K'. Passing to a subequence, we may suppose $\{\xi'_n\}$ has a limit $\xi' \in K'$. But $h(\xi'_n) = \xi_n \to x_0$, so $h(\xi') = x_0$. In other words, $\xi' = x'_i$ for some i. This implies that $\xi'_n \in U'_i$ for large n, a contradiction! ■