

MATH 396. HOW TO COMPUTE INTEGRALS

In the homework, you developed the theory of absolute integrability for top-degree differential forms on smooth manifolds M with constant positive dimension $n > 0$, in both the oriented and non-oriented cases (though the integration operator $\int_M |\omega|$ in the non-oriented case is not linear in ω). You also proved that in the oriented case, if ω is such an absolutely integrable form then $|\int_{M,\mu} \omega| \leq \int_M |\omega|$. These definitions are well-suited to theoretical considerations but not to actual computation, since nobody can (or wants to) write down partitions of unity. In this handout, we wish to take up a few refinements to the theory, essentially to make the task of actually *computing* such integrals be much like the case of integration for functions on Euclidean space (and even reduce to such calculations when we understand the geometry of our domain sufficiently well).

For example, we certainly want to say that for $n > 0$ and an n -form ω on the sphere S^n (with a chosen orientation), $\int_{S^n} \omega = \int_{H^+} \omega|_{H^+} + \int_{H^-} \omega|_{H^-}$ where H^\pm are a pair of “complementary” hemispheres viewed as closed smooth submanifolds with boundary in S^n (sharing a common manifold-boundary). Ignoring the equator, these hemispheres are parameterized by open unit n -balls, so our integration problems should shift to old-fashioned function integrals on such n -balls.

Also, it should surely be the case that for any top-degree form $\omega \in \Omega_M^n(M)$ on a manifold with corners M , if $\omega' = \omega|_{M-\partial M}$ denotes the restriction of ω over the open submanifold complementary to the singular locus then absolute integrability for ω over M and for ω' over $M - \partial M$ are *equivalent*, and moreover that the resulting integrals agree (in both the non-oriented and oriented senses). Lest one dismiss this as “obvious”, it does require a bit of thought because (in view of how we *defined* integration of differential forms) if $\{\phi_i\}$ is a C^∞ partition of unity with compact supports (contained in coordinate domains) on M then $\{\phi_i|_{M-\partial M}\}$ is a C^∞ partition of unity on $M - \partial M$ with a locally finite collection of supports but *these supports are generally non-compact!* Hence, strictly speaking, this latter collection of functions is not the sort used in the *definition* of integration over $M - \partial M$.

Briefly put, to handle these and other related matters in a straightforward manner we need to revisit how partitions of unity are used in the calculation of integrals. More specifically, since we at least now have a general concept of integration of differential forms on M (in addition to the theory for functions on Euclidean space), we are now in position to try to use a wider class of partitions of unity than were permitted in the initial definition of such integrals. Once we develop some more efficient computational techniques, we will be able to prove everything that we expect to hold for any reasonable theory of integration.

1. PARTITIONS OF UNITY

Let M be a smooth manifold with corners and constant dimension $n > 0$. Let $\{\phi_i\}$ be a collection of non-negative smooth functions on M whose supports are locally finite and such that $\sum \phi_i = 1$. We do not assume that the ϕ_i 's have compact support and we do not assume that their supports lie in coordinate domains. It is reasonable to expect that for any n -form ω on M , ω is absolutely integrable if and only if two conditions hold: (i) the $\phi_i\omega$'s are absolutely integrable and (ii) $\sum_i \int_M |\phi_i\omega|$ is finite. In such cases, we expect $\int_M |\omega| = \sum_i \int_M |\phi_i\omega|$. If moreover M is oriented, the sum $\sum_i \int_M \phi_i\omega$ is absolutely convergent (as it is termwise bounded above in absolute value by the terms of the convergent sum $\sum_i \int_M |\phi_i\omega|$) and we expect that this sum should equal $\int_M \omega$. (We shall generally suppress explicit mention of the choice of orientation in our integration “without absolute values”.)

Note that the preceding desired properties *do not* just repeat the definition of integration of differential forms, since we are specifically avoiding two key assumptions on the ϕ_i 's that were used in the definition of such integrals (via partitions of unity), namely we allow that ϕ_i 's may have

non-compact support and we allow that these supports may fail to lie in coordinate domains. In the introductory discussion we saw why it was desired to allow for non-compact supports.

Theorem 1.1. *Let $\{\phi_i\}$ be a collection of non-negative smooth functions on M whose closed supports form a locally finite collection of closed sets in M , and assume $\sum \phi_i = 1$. For any top-degree differential form ω on M , ω is absolutely integrable if and only if all $\phi_i\omega$'s are absolutely integrable and $\sum_i \int_M |\phi_i\omega|$ is finite, in which case this sum is equal to $\int_M |\omega|$.*

If M is oriented and ω is absolutely integrable then $\sum_i \int_M \phi_i\omega$ is absolutely convergent and equal to $\int_M \omega$.

This theorem says that the “recipe” used to initially define integration of differential forms is *a posteriori* applicable for the widest class of partitions of unity $\{\phi_i\}$ that one could hope for. We couldn't state a result such as this prior to the initial definition of global integration of differential forms because to make sense of $\int_M \phi_i\omega$ for ϕ_i 's as general as in this theorem requires defining a concept of absolute integrability for general differential forms in the first place! (The point being that $\phi_i\omega$ is possibly not supported inside of a coordinate domain.)

Proof. Let $\{\psi_j\}$ be a smooth partition of unity with locally finite and compact supports contained in coordinate domains. First we assume that ω is absolutely integrable and we seek to show that the $\phi_i\omega$'s are absolutely integrable and $\sum_i \int_M |\phi_i\omega| = \int_M |\omega|$ (in particular, this summation is finite). Since ω is absolutely integrable, $\int_M |\omega| = \sum_j \int_M |\psi_j\omega|$ by definition. Since each ψ_j is compactly supported with support contained in a coordinate domain, $\psi_j\phi_i\omega = 0$ for all but finitely many i (depending on j) and $\int_M |\psi_j\omega| = \sum_i \int_M |\psi_j\phi_i\omega|$ due to finite additivity for integration of functions on opens in sectors in Euclidean space (here we also use that $\sum \phi_i = 1$ and all $\phi_i \geq 0$). Hence, the double sum $\sum_j \sum_i \int_M |\psi_j\phi_i\omega|$ is convergent and equal to $\int_M |\omega|$. We may rearrange it to get $\sum_i \sum_j \int_M |\psi_j\phi_i\omega| = \int_M |\omega|$. In particular, for each i the inner summation over j is convergent, but (due to the choice of the ψ_j 's) this is exactly the definition of $\int_M |\phi_i\omega|$. Hence, we conclude that each $\phi_i\omega$ is absolutely integrable and that $\sum_i \int_M |\phi_i\omega|$ is convergent and equal to $\int_M |\omega|$.

Now suppose that each $\phi_i\omega$ is absolutely integrable with $\sum_i \int_M |\phi_i\omega|$ convergent. We wish to deduce that ω is absolutely integrable. By definition, $\int_M |\phi_i\omega| = \sum_j \int_M |\psi_j\phi_i\omega|$ in the sense that the right side is convergent (due to the hypothesis on $\phi_i\omega$) and equal to the left side (by definition, given the convergence). Hence, the sum $\sum_i \sum_j \int_M |\psi_j\phi_i\omega|$ is convergent. Rearranging, $\sum_j \sum_i \int_M |\psi_j\phi_i\omega|$ is convergent. We want to prove that ω is absolutely integrable, which is to say that $\sum_j \int_M |\psi_j\omega|$ is convergent. Hence, it suffices to show that the inner sum $\sum_i \int_M |\psi_j\phi_i\omega|$ is equal to $\int_M |\psi_j\omega|$ for all j . Since each $\psi_j\omega$ is compactly supported inside of a coordinate domain, the problem reduces to the identity $\sum_i \int_U f_i h = \int_U h$ for a compactly supported smooth function h on an open set U in a sector in \mathbf{R}^n and a collection $\{f_i\}$ of non-negative smooth functions on U whose supports are locally finite in U and which satisfy $\sum f_i = 1$. Since h is compactly supported, $f_i h = 0$ for all but finitely many h and $\sum f_i h = h$ as a *finite* sum. Thus, finite additivity for integration of compactly supported continuous functions in a sector in \mathbf{R}^n does the job.

Finally, assume that M is oriented and ω is absolutely integrable. By the above, it follows that each $\phi_i\omega$ is absolutely integrable and that $\sum_i \int_M \phi_i\omega$ is absolutely convergent (as it is bounded above termwise in absolute value by the sum $\sum_i \int_M |\phi_i\omega|$ that we know to be finite and in fact equal to $\int_M |\omega|$). We want to prove that $\sum_i \int_M \phi_i\omega$ is equal to $\int_M \omega$. By definition, $\int_M \phi_i\omega = \sum_j \int_M \psi_j\phi_i\omega$ for all i , with this sum absolutely convergent (even bounded termwise in absolute value by $\sum_j \int_M |\psi_j\phi_i\omega|$). Hence, $\sum_i \int_M \phi_i\omega = \sum_i \sum_j \int_M \psi_j\phi_i\omega$ and this double sum is absolutely convergent because $\sum_i \sum_j \int_M |\psi_j\phi_i\omega|$ is finite (and equal to $\int_M |\omega|$) by the assumption that ω

is absolutely integrable. We can therefore rearrange to get that this double sum without absolute values is equal to $\sum_j \sum_i \int_M \psi_j \phi_i \omega$. But $\int_M \omega = \sum_j \int_M \psi_j \omega$, so we are reduced to proving $\sum_i \int_M \psi_j \phi_i \omega = \int_M \psi_j \omega$ for each j .

Renaming $\psi_j \omega$ as η , we have to prove that if η is compactly supported inside of a coordinate domain then $\sum_i \int_M \phi_i \eta$ is absolutely convergent and equal to $\int_M \eta$. Note that $\phi_i \eta = 0$ for all but finitely many i because the collection of $\text{supp}(\phi_i)$'s is locally finite and η is compactly supported. Passing into the coordinate domain containing the support of η , our problem is again reduced to the general identity " $\int_U h = \sum_i \int_U f_i h$ " on opens in sectors in Euclidean space as shown earlier in this proof. \blacksquare

Corollary 1.2. *Let $M^0 = M - \partial M$ as an open smooth submanifold of M . Let ω be an n -form on M and let $\omega^0 = \omega|_{M^0}$. We have that ω is absolutely integrable on M if and only if ω^0 is absolutely integrable on M^0 , in which case $\int_M |\omega| = \int_{M^0} |\omega^0|$. If M is oriented and we give M^0 the induced orientation, then for such ω we also have $\int_M \omega = \int_{M^0} \omega^0$.*

Proof. Let $\{\phi_i\}$ be a locally finite C^∞ partition of unity on M with compact supports contained in coordinate domains. Let $\phi_i^0 = \phi_i|_{M^0}$, so $\{\phi_i^0\}$ is a locally finite C^∞ partition of unity on M^0 whose supports are contained in coordinate domains but are generally non-compact.

By definition, ω is absolutely integrable if and only if $\sum_i \int_M |\phi_i \omega|$ is finite, in which case this sum is equal to $\int_M |\omega|$ and the sum $\sum_i \int_M \phi_i \omega$ is absolutely convergent and equal to $\int_M \omega$. By the preceding theorem, we can make the same assertion over M^0 using ω^0 and the ϕ_i^0 's (even though these ϕ_i^0 's generally have non-compact support in M^0). Hence, if we can prove that each $\phi_i^0 \omega^0$ is absolutely integrable on M^0 with $\int_{M^0} |\phi_i^0 \omega^0| = \int_M |\phi_i \omega|$ and (in the oriented case) $\int_{M^0} \phi_i^0 \omega^0 = \int_M \phi_i \omega$ then we will be done.

We may now pick i and rename $\phi_i \omega$ as ω , so we are reduced to the special case when ω is compactly supported on M with support contained in an open coordinate domain U in M . In this case, we wish to prove that ω^0 is absolutely integrable over M^0 with $\int_M |\omega| = \int_{M^0} |\omega^0|$ and (if M is oriented in a manner compatible with the coordinates on U) $\int_M \omega = \int_{M^0} \omega^0$. Let K be the compact support of ω , so $\{U, M - K\}$ is an open cover of M . Let $\{\psi_j\}$ be a smooth partition of unity on M subordinate to this cover, with ψ_j 's having locally finite and compact supports. Since each ψ_j is either supported in U or $M - K$, and ω vanishes over $M - K$, clearly $\psi_j \omega = 0$ except for those j such that ψ_j is supported in U . Any coordinate chart for M meets $M^0 = M - \partial M$ and U in coordinate charts for M^0 and U respectively, so it follows from the definition of integration of differential forms that:

- (1) The problems of computing $\int_M |\omega|$ and $\int_M \omega$ using $\{\psi_j\}$ are *the same* as the problems of computing $\int_U |\omega|$ and $\int_U \omega$ using $\{\psi_j|_U\}$ (dropping those $\psi_j|_U$'s that equal 0). More specifically, $\int_U |\omega| = \int_M |\omega|$ and (in the oriented case) $\int_U \omega = \int_M \omega$ in the sense that ω is absolutely integrable over M if and only if it is so over U , in which case both equalities hold.
- (2) The same goes over M^0 using ω^0 and $U \cap M^0$.

It follows that we can replace M with U , and so we are reduced to the case when M has global coordinates and is oriented by these coordinates. Thus, M^0 also has global coordinates and may be oriented by these coordinates. By the compatibility at the end of Exercise 4(iv) in Homework 9, our problem may therefore be translated into the classical setting of integration for compactly supported smooth functions on opens in sectors in Euclidean space. That is, if Σ is a sector in \mathbf{R}^n , $U \subseteq \Sigma$ is an open subset, and $h : U \rightarrow \mathbf{R}$ is a compactly supported smooth function, then we seek to prove that h is absolutely integrable on $U^0 = U - \partial \Sigma$ with $\int_{U^0} h = \int_U h$. We can use Change

of variables to reduce to the case when Σ is a “standard” sector $[0, \infty)^r \times \mathbf{R}^{n-r}$, and since h is compactly supported in U we can cover the support of h by a union Q of finitely many rectangles in U with sides parallel to the coordinate axes. By the theory of integration on Euclidean space, our integrals may be computed over Q and since Q has boundary with measure 0 it follows immediately that h is absolutely integrable over $Q^0 = Q - (Q \cap \partial\Sigma)$ with integral equal to $\int_Q h$. ■

We can now give a very important application that is easy to overlook if one does not think carefully about how everything has been defined.

Let M be an n -manifold with corners that is embedded in \mathbf{R}^n ; an example is the closed unit ball in \mathbf{R}^n (considered as a manifold with boundary). The inclusion $i : M \rightarrow \mathbf{R}^n$ induces a pullback isomorphism $TM \simeq i^*(T(\mathbf{R}^n))$. Passing to top exterior powers of duals, we get that $dt_1 \wedge \cdots \wedge dt_n$ on \mathbf{R}^n pulls back under i to a global generator for the line bundle of top-degree differential forms on M . Thus, any $\omega \in \Omega_M^n(M)$ may be uniquely written $\omega = f \cdot i^*(dt_1 \wedge \cdots \wedge dt_n)$, or more classically $\omega = f \cdot (dt_1 \wedge \cdots \wedge dt_n)|_M$, for some $f \in C^\infty(M)$.

By working locally on M in \mathbf{R}^n and using that M is topologically embedded in \mathbf{R}^n , M is closed in an open subset $U \subseteq \mathbf{R}^n$ and it is not hard to check that the manifold boundary ∂M is the topological boundary of M in U . In particular, by the simple nature of manifold boundaries, the topological boundary of M in U has measure zero. Thus, the old theory of integration for functions on Euclidean space attaches a meaning to $\int_M f$ in the sense of absolute integrability (extending f by zero to U). Thus, it is tempting to think that essentially “by definition” ω is absolutely integrable on M in the sense of manifold integration if and only if f is absolutely integrable on $M \subseteq U$ in the sense of function integration on the open subset U in Euclidean space, in which case $\int_M |\omega| = \int_M |f|$ and that in such cases (using the induced orientation on M from \mathbf{R}^n) we also have $\int_M \omega = \int_M f$.

Of course such equality of theories of integration for functions and forms is true, but this is not a tautology! The point is that $\{t_1|_M, \dots, t_n|_M\}$ is *not* a coordinate system along the boundary points of M . That is, M does *not* generally have global coordinates in the sense of manifolds with corners; recall that “global coordinates” really refers to opens in sectors in a Euclidean space, and in practice M is not of this type inside of \mathbf{R}^n . Even the most basic example of restricting $\{x, y\}$ on \mathbf{R}^2 to the closed unit disc D illustrates this. The definitions of function integration and n -form integration are really quite different, as the latter uses partitions of unity subordinate to an atlas on the manifold and the former uses extension by zero to all of \mathbf{R}^n (or partitions of unity when working on an *open* subset of \mathbf{R}^n). If you think about it, you’ll see that there really is something to be checked in order to assert that $\int_M \omega = \int_M f$.

In practice we have to know that we can pass between the theory of integration for functions and n -forms as above, and this “obvious” fact is used all the time in calculations (since it is certainly impossible to write down partitions of unity!). I do not know of a single textbook that addresses the fact that this equality of integrals is not a rehash of the definitions of such integrals. The preceding results fortunately make it now a very easy matter to verify the desired equalities. By Corollary 1.2, for the n -form integration (in the absolute and oriented senses) we may replace M with $M - \partial M$. By the theory of function integration, since M has rectifiable boundary in U we can replace M with $M - \partial M$ for the purposes of function integration on M (using the theory of integration on U). But $M - \partial M$ is an open submanifold of \mathbf{R}^n and is an open subset of U with rectifiable boundary in U , so on both sides of the desired equalities we may replace M with $M - \partial M$ (and take this as U). Now $\{t_i|_M\}$ is a global coordinate system on the open set M in \mathbf{R}^n , so the *definition* of absolutely convergent integration of functions on open sets in \mathbf{R}^n (via partitions of unity!) concludes the argument.

Example 1.3. Let $\omega = (\sum t_j^2)^{-1/2} \cdot \sum_{i=1}^{n+1} (-1)^{i-1} t_i dt_1 \wedge \cdots \wedge \widehat{dt}_i \wedge \cdots \wedge dt_{n+1}$ in $\Omega_{\mathbf{R}^{n+1}}^n(\mathbf{R}^{n+1})$. For $n = 2$, this is $d\theta$ (and this is the reason for the ugly square root in front). For $n > 2$, it is an analogue of the differential of “solid angle”. Let $S^n(r)$ be the sphere of radius $r > 0$ centered at the origin; it is the boundary of the solid ball $B^{n+1}(r)$ of radius r , and as such we give it the outward normal orientation. This is the induced boundary orientation when the ball oriented via the standard orientation on \mathbf{R}^{n+1} . We wish to compute $\int_{S^n(r)} \omega$, by which we really mean $\int_{S^n(r)} i^*(\omega)$ with $i : S^n(r) \rightarrow \mathbf{R}^{n+1}$ the standard inclusion. The pullback $i^*(\omega)$ is a top-degree differential form on the sphere and the sphere is compact, so this integral makes sense. Note also that $i^*(\omega) = r^{-1} i^*(\eta)$ with η defined like ω but without the square-root factor in front.

By Stokes’ theorem, $\int_{S^n(r)} \omega = r^{-1} \int_{S^n(r)} \eta = r^{-1} \int_{B^{n+1}(r)} d\eta$; the abuse of notation by suppressing explicit mention of pullbacks from \mathbf{R}^{n+1} is harmless because d and pullback commute (such as for d on $B^{n+1}(r)$ as is relevant for Stokes’ theorem, and d on \mathbf{R}^{n+1} as is relevant to actually compute!). On \mathbf{R}^{n+1} , $d\eta = (n+1)dt_1 \wedge \cdots \wedge dt_n$. Thus, by the equality we have established between integration theory for functions and forms on N -submanifolds with corners in \mathbf{R}^N , the integral $\int_{B^{n+1}(r)} d\eta$ in the sense of manifolds (which is what comes out of Stokes’ theorem!) is the same as the old-fashioned integral $\int_{B^{n+1}(r)} (n+1)/r$ for a constant function over the rectifiable solid ball of radius r in \mathbf{R}^{n+1} . Hence, our initial fancy manifold integral is equal to $(n+1)\text{Vol}(B^{n+1}(r))/r > 0$, and this latter volume may be computed recursively via Fubini’s theorem.

Note in particular that since $S^n(r)$ is a boundaryless orientable manifold, exact forms of top degree on this sphere must have vanishing integral by Stokes’ theorem. We have exhibited an explicit top-degree (hence closed) form on this sphere whose integral we explicitly computed to be nonzero. Thus, $H_{\text{dR}}^n(S^n(r)) \neq 0$. In the case $n = 1$, this recovers the fact that $d\theta$ is a non-exact 1-form on the circle and hence represents a nonzero de Rham cohomology class.

2. CHOPPING UP SPACES

Recall one of our initial questions: can we compute an integral over the sphere as a sum of integrals over “complementary” hemispheres? We can now give an affirmative answer, with much more generality. We first require a lemma.

Lemma 2.1. *Let $f : M' \simeq M$ be a C^∞ isomorphism between smooth manifolds with corners that have constant positive dimension n . For $\omega \in \Omega_M^n(M)$ and $\omega' = f^*\omega$, ω is absolutely integrable over M if and only if ω' is absolutely integrable over M' , in which case $\int_M |\omega| = \int_{M'} |\omega'|$. Moreover, if M and M' are oriented such that f is orientation-preserving at all points, then $\int_M \omega = \int_{M'} \omega'$ in the absolutely integrable case.*

This lemma is used all the time to shift integration problems from one manifold to an isomorphic manifold.

Proof. Let $\{(U_j, \varphi_j)\}$ be a C^∞ atlas for M consisting of a locally finite collection of open subsets U_j , and let $U'_j = f^{-1}(U_j)$, $\varphi'_j = \varphi_j \circ f|_{U'_j}$. Let $\{\phi_i\}$ be a smooth partition of unity on M with compact supports subordinate to the U_j ’s, say ϕ_i supported inside of $U_{j(i)}$ for each i . Let $\phi'_i = \phi_i \circ f$, so $\{\phi'_i\}$ is a smooth partition of unity on M' with compact supports subordinate to the U'_j ’s, with ϕ'_i supported inside of $U'_{j(i)}$ for each i . The definitions of integration (in the absolute and oriented senses) on M and M' are defined in terms of integrals on sectors in Euclidean spaces as dictated by the atlases with their subordinate partition of unity. Hence, our problem is shifted to the case of compactly supported smooth functions on opens in sectors in \mathbf{R}^n and an *orientation-preserving* C^∞ isomorphism between such opens (using the standard orientation on \mathbf{R}^n). This orientation

condition ensures that the Jacobian determinant for the C^∞ isomorphism is everywhere positive, and hence is equal to its own absolute value. Thus, the Change of Variables formula gives what we need. \blacksquare

Theorem 2.2. *Let M_1, \dots, M_r be finitely many manifolds with corners that have constant dimension $n > 0$. Let $f_i : M_i \rightarrow M$ be smooth injective immersions that are homeomorphisms onto their images. Assume moreover that for the interiors $M_i^0 = M_i - \partial M_i$, the $f_i(M_i^0)$'s lie in M^0 and are pairwise disjoint. Finally, assume that the pairwise disjoint open submanifolds M_i^0 in M^0 have union whose closed complement is measure zero in M .*

For any n -form ω on M , ω is absolutely integrable if and only if each $\omega_i = f_i^\omega$ is absolutely integrable on M_i , in which case $\int_M |\omega| = \sum_i \int_{M_i} |f_i^*\omega|$. If moreover, M is oriented and each M_i is given the pullback orientation via the bundle isomorphism $TM_i \simeq f_i^*(TM)$ over M_i for all i then $\int_M \omega = \sum_i \int_{M_i} f_i^*\omega$.*

The meaning of this theorem is that if we can cover the n -manifold with corners M by finitely many n -submanifolds with corners that only meet along their boundaries, then integration problems over M can be reduced to ones over the M_i 's. The covering of a sphere by a pair of complementary hemispheres is a special case. Recall also that a subset of a manifold with corners M is said to be *of measure zero* if it is so under each of a collection of charts that cover M (in which case the same holds for where the subset meets any coordinate chart on M). This allows us to perform calculations by ignoring closed regions such as the “edge” of the spherical coordinate domain on the sphere.

Proof. By Corollary 1.2, we may remove ∂M without affecting any of the integrals under consideration (in the sense of absolute convergence or value). The same goes for removing the ∂M_i 's. Hence, we may assume that the M_i 's are pairwise disjoint open submanifolds of M whose union has closed complement of measure zero. By Lemma 2.1, $\omega|_{f_i(M_i)}$ is absolutely integrable over the open submanifold $f_i(M_i)$ in M if and only if $f_i^*\omega$ is absolutely integrable over M_i , in which case $\int_{M_i} |f_i^*\omega| = \int_{f_i(M_i)} |\omega|$ and moreover when M is oriented (and each M_i is compatibly oriented) $\int_{M_i} f_i^*\omega = \int_{f_i(M_i)} \omega$.

Thus, for our purposes we may replace M_i with $f_i(M_i)$ and so we reduce ourselves to the following question on the smooth manifold M : the M_i 's are pairwise disjoint open submanifolds, and we seek to prove that ω is absolutely integrable over M if and only if it is absolutely integrable over each open subset M_i , in which case $\int_M |\omega| = \sum_i \int_{M_i} |\omega|$ and moreover if M is oriented then $\int_M \omega = \sum_i \int_{M_i} \omega$ (using the induced orientation on the open subsets M_i).

Let $\{\psi_j\}$ be a smooth partition of unity with compact supports subordinate to an atlas on M , so each $\{\psi_j|_{M_i}\}$ has the same property on M_i except that the supports on M_i are generally non-compact. Since we can study integration on M and the M_i 's via this partition of unity, it is clearly enough (check!) to solve the problem for each $\psi_j\omega$. Hence, we can assume that ω has compact support K contained in an open coordinate domain U in M . Letting $U_i = U \cap M_i$, the compactness of K lets us argue as in the proof of Corollary 1.2 to show that replacing M and the M_i 's with U and the U_i 's does not affect any of the integrals under consideration. Hence, we may assume M has global coordinates. Our problem is now shifted to the setting of function integrals on opens in sectors in Euclidean space (due to the compatibility at the end of Exercise 4(iv) in Homework 9). That is, U is open in a sector in \mathbf{R}^n , h is a continuous compactly supported function on U , and U_1, \dots, U_r are disjoint opens in U whose union has closed complement (in U) with measure zero. Under these conditions, we need to prove that h is absolutely integrable on each U_i and that

$\int_U h = \sum \int_{U_i} h$. Since $U - (\cup U_i)$ is closed with measure zero in U , this equality (including the absolute integrability over U_i) follows from the theory of integration on Euclidean space. ■

Example 2.3. We can compute an integral over a sphere as a sum of integrals over a pair of “complementary” hemispheres, or over the open locus on which spherical coordinates are defined. Likewise, for any manifold endowed with a coordinate chart complementary to a closed set of measure zero, all integration problems on the manifold can be shifted to the coordinate chart (i.e., we can ignore the region off of the chart). This is used all the time when we compute integrals on the plane using polar coordinates, or integrals on a torus using the “angle coordinatization”, etc.

Also, on Grassmannians any of the standard open subsets U_I has complement that is closed with measure zero (essentially contained in a union of zero loci of nonzero polynomial functions on a vector space), and so any integration on a Grassmannian may be computed by working in a single U_I .

3. NON-ORIENTED VOLUME

In the homework we have seen how to define volume on oriented Riemannian manifolds with corners (and without open points): we integrate the volume form. This was seen to be independent of the choice of orientation. But surely we do not need orientability to define volume. After all, it is obvious that a Möbius strip embedded in \mathbf{R}^3 (and given the induced metric tensor) *must* have an area (just make a 1-dimensional slit to restore orientability and to undo the twist). In the absence of orientability there is no global volume form, since the definition of the volume form required specifying an orientation in the fibers (to tell which of the two unit vectors in each $\det(T_m(M)^\vee)$ to pick). But this cannot be a serious obstacle, since *locally* we can always find orientations and the choice has no effect on the local volume. By some vague principle of additivity for volume, this convinces us that there has to be a way around the problem.

Here is what to do. Let (M, ds^2) be the given Riemannian manifold with corners (and without open points). Let $\{\phi_i\}$ be a C^∞ partition of unity with locally finite collection of compact supports such that each ϕ_i is supported inside of an orientable open subset $U_i \subseteq M$ (such as a coordinate domain). Pick an orientation on U_i , and let ω_i be the resulting volume form; if we change the orientation on U_i then ω_i will change by a sign on some connected components of U_i . The form $\phi_i \omega_i$ on U_i is smooth and compactly supported on U_i , so $\int_{U_i} |\phi_i \omega_i|$ makes sense and is clearly independent of the choice of orientation on U_i . If we let ω'_i denote the “extension by zero” of $\phi_i \omega_i$, then $\int_{U_i} |\phi_i \omega_i| = \int_M |\omega'_i|$ and so in this way we see that it only depends on ϕ_i and not the choice of U_i . Roughly speaking, $\int_M |\omega'_i|$ is a “dampened volume” for the support of ϕ_i , with dampening due to the decrease of ϕ_i toward 0 near the boundary of its support.

If M is orientable and ω is the volume form associated to an orientation, then it is clear that $\int_M |\omega'_i| = \int_M |\phi_i \omega|$, so $\sum_i \int_M |\phi_i \omega| = \int_M |\omega| = \int_M \omega$ (using the orientation on M for the final integral), in the sense that the convergence of this sum is precisely the condition for ω to be absolutely integrable, in which case we see that it calculates the volume of M . Motivated by this, in general we define the *volume* of M to be $\sum_i \int_M |\omega'_i|$ if this sum converges. Of course, it has to be proved that the property of convergence for this sum (which *a priori* depends on the initial choice of partition of unity $\{\phi_i\}$ with locally finite and compact supports) is independent of the choice of $\{\phi_i\}$, in which case the value of the sum is also independent of the choice. This verification is left as an exercise; it follows the same paradigm of expanding a double sum in two different ways exactly as in the proof of well-definedness of integrals for “absolute values” of differential forms over manifolds with corners. Obviously volume is positive.

It follows by the same method as used to prove Theorem 2.2 that this notion of volume has reasonable geometric properties that we expect. For example, $\text{Vol}(M) = \text{Vol}(M - \partial M)$ (in the sense that one is finite if and only if the other is, in which case they are equal), and in the setup of Theorem 2.2 we have $\text{Vol}(M) = \sum_i \text{Vol}(M_i)$ (in the sense that M has finite volume if and only if the M_i 's all do, in which case the equality holds). Also, if $U \subseteq M$ is an open subset with compact closure, then $\text{Vol}(U)$ is finite (can you prove this?). Such manipulations are used all the time when computing the volume of specific Riemannian manifolds with corners. From the viewpoint of measure theory, it can be proved that the assignment of volume to open sets in M uniquely extends to a (reasonable) measure on M . In this sense, via the method of integration of differential forms we see that Riemannian manifolds always admit a canonical measure (even in the absence of an orientation). In the special case $M = \mathbf{R}^n$ with its standard flat metric, we recover Lebesgue measure.

4. DECIPHERING CLASSICAL NOTATION

Let S be an embedded oriented smooth surface with boundary in \mathbf{R}^3 and C an embedded oriented curve with boundary in \mathbf{R}^n . One often sees the following expressions:

$$(1) \quad \int_S (Pdx dy + Qdx dz + Rdz dy), \quad \int_C \sum f_j dx_j$$

for smooth functions P, Q, R on an open $U \subseteq \mathbf{R}^3$ containing S and smooth functions f_1, \dots, f_n on an open U' containing C . What do such notations mean?!? From the modern point of view, if we let $i : S \hookrightarrow U \subseteq \mathbf{R}^3$ and $\sigma : C \hookrightarrow U' \subseteq \mathbf{R}^n$ denote the embedding maps then such integrals should be understood to mean

$$\int_S i^*(Pdx \wedge dy + Qdx \wedge dz + Rdz \wedge dy), \quad \int_C \sigma^*(\sum f_j dx_j).$$

The suppression of the pullback notation fortunately does not create problems with applications of Stokes' theorem, because the d operator commutes with pullback (so whether we compute d on an ambient Euclidean space and then apply pullback, or compute d on the submanifold after pullback, it all comes to the same thing as far as the end result on the submanifold is concerned).

This is all fine and well, but it avoids the real issue: does our modern definition for the meaning of (1) *compute* what the old-timers worked with in the setting of embedded submanifolds (with corners) in Euclidean space? After all, in the 19th century nobody used partitions of unity. Likewise, in multivariable calculus books one does not see partitions of unity. What one *does* see in the old books and modern textbooks for physics students are parametric curves in \mathbf{R}^n and parametric surfaces in \mathbf{R}^3 . The classical method of computation of the integrals in (1) (by whatever method of "definition" is employed) consists of two steps. First, away from certain closed subsets of measure zero, the manifold is expressed as a disjoint union of finitely many subsets that are explicitly presented as the images under embeddings from open regions in a Euclidean space. (For example, on a sphere S^{n-1} we ignore the equator and express the remaining two hemispheres as images of the open unit ball in \mathbf{R}^{n-1} . On a surface of revolution we make a slit and parameterize by angle and x .) In other words, we cover the manifold (with corners) by coordinate charts that have minimal overlap. Once this is done, the problem classically is "reduced" to computing integrals (in whatever sense has been defined) over the specified parametric charts. We also have an analogue of this first step, namely Theorem 2.2! Hence, everyone agrees that in classical situations the problem of actually computing an integral is reduced to the case when the submanifold with corners admits global coordinates. By passing to connected components so that there are exactly two orientations, we can always choose the coordinate chart to be positive for the chosen orientation.

Now comes the second step, assuming that there is an open domain D in a sector in \mathbf{R}^k and a smooth embedding $f : D \rightarrow \mathbf{R}^N$ such that $M = f(D)$ is the submanifold under consideration and f is orientation-preserving with respect to the given orientation on M and the orientation on D from \mathbf{R}^k . (In the case of oriented curves, this amounts to taking f so that the velocity vectors $f'(t)$ along the curve always lie in the positive half-line for the orientation on the curve.) If ω is the differential form being “integrated” (in whatever sense) over M , then the orientation compatibility for f allows us to use Lemma 2.1 to infer that $\int_M \omega = \int_D f^* \omega$ when D is oriented from \mathbf{R}^k . Since D is open in a sector in \mathbf{R}^k , it is a k -submanifold with corners in \mathbf{R}^k . Hence, if $\{y_1, \dots, y_k\}$ are the standard coordinates on \mathbf{R}^k , we can uniquely write $f^* \omega = h dy_1 \wedge \dots \wedge dy_k$ for $h \in C^\infty(D)$. By the discussion following the proof of Corollary 1.2, our fancy manifold integral $\int_D f^* \omega$ is equal to the old-fashioned function integral $\int_D h$.

Our task is to show that $\int_D h$ is *exactly* how the old-timers and modern-day physics students compute their “integral” (however it is defined) over the image M of the embedding $f : D \rightarrow \mathbf{R}^N$, at least for curves in space and surfaces in \mathbf{R}^3 (as these are the only cases used by people who don’t know the modern theory of differential forms). Let us return to the mysterious surface integral in (1), in the case when there is a mapping $f : D \simeq S \subseteq \mathbf{R}^3$ with $f(u, v) = (f_1(u, v), f_2(u, v), f_3(u, v))$ for smooth functions f_1, f_2, f_3 on D . The classical recipe to perform the calculation begins by replacing the given “surface element” in x, y, z with one in terms of u and v . What this means in practice is the following. First, formally replace $P(x, y, z)$ with $P(u, v)$ (which really means $P \circ f = f^*(P)$) and similarly for Q and R . Next, replace the differential dx with the “total differential” $\Delta f_1 = \partial_u f_1 du + \partial_v f_1 dv$ (which really means $df_1 = f^*(dx)$). There are also the formal rules for manipulation: $du du = 0 = dv dv$ and $du dv = -dv du$ (justified by some hocus-pocus with infinitesimals), which we recognize as the algebraic properties of wedge products. To summarize, once the algebra of substitution is done the classical integrand

$$P(x, y, z) dx dy + Q(x, y, z) dz dx + R(x, y, z) dz dy$$

(whatever it is supposed to mean) is formally replaced with

$$((P \circ f) \cdot (\partial_u f_1 \partial_v f_2 - \partial_v f_1 \partial_u f_2) + (Q \circ f) \cdot (\partial_u f_3 \partial_v f_1 - \partial_v f_3 \partial_u f_1) + (R \circ f) \cdot (\partial_u f_3 \partial_v f_2 - \partial_v f_3 \partial_u f_2)) du dv$$

considered as a “surface element” on the domain D in \mathbf{R}^2 . If this is written as $g du dv$, then what the old-timers compute is $\int_D g$. But a review of the meaning of the various substitutions shows that $f^*(P dx \wedge dy + Q dz \wedge dx + R dz \wedge dy) = g du \wedge dv$, so indeed we’re computing the same thing!

The case of parametric curves goes almost exactly the same way, except that the algebra is a lot easier: there are no wedge products and so no mystical “rules” on how to manipulate products of infinitesimals. If $\sigma(t) = (\sigma_1(t), \dots, \sigma_n(t))$ is the mapping that parameterizes our oriented curve in \mathbf{R}^n with non-vanishing velocity vectors such that $\sigma'(t) \in T_{\sigma(t)}(C)$ “points the right way” (i.e., motion in the direction specified by the orientation of the curve) then $\{t\}$ is an *oriented* coordinate chart for the submanifold $C \subseteq \mathbf{R}^n$. Hence, if t ranges through a nontrivial interval I in \mathbf{R} (given the standard orientation) then *our* calculation of the line integral in (1) proceeds as follows:

$$\int_I \sigma^* \left(\sum f_j dx_j \right) = \sum \int_I (f_j \circ \sigma) \sigma'_j dt = \sum \int_I (f_j \circ \sigma) \sigma'_j$$

where the final integrals are old calculus-style integrals of functions on an interval. The way the old-timers integrate the “line elements” $\sum f_j dx_j$ along the parameterized curve C is to replace f_j with $f_j(t)$ (i.e., $f_j \circ \sigma$) and to replace dx_j with $d\sigma_j(t) = \sigma'_j(t) dt$ (i.e., $\sigma'_j dt = \sigma_j^*(dx_j)$) to get the infinitesimal $\sum (f_j \circ \sigma) \sigma'_j dt$, and they integrate the coefficient function $\sum (f_j \circ \sigma) \sigma'_j$ (and also keep

the symbol dt that has some vague meaning to them). This is exactly the integral we wound up with in the end.

Hence, for line integrals in space and surface integrals in \mathbf{R}^3 , our modern definition gives the same answers as in earlier approaches to integration on such manifolds. Moreover, even the formal mechanics of the *method* of calculation is essentially the same. The main difference is that we have given vastly more general definitions via partitions of unity and so on, and we have given genuine meaning to every step of the calculation. In any concrete situation where one wants to compute a number, one must use the “chopping up the space” method to reduce to calculations in charts since it is impossible to actually do an explicit calculation with partitions of unity.