Math 396. Worked example of integral manifolds and PDE

1. Motivation

Let $E \subseteq T(\mathbf{R}^3)$ be the subbundle spanned by the pointwise independent vector fields

$$X = \partial_x + f\partial_z, \;\; Y = \partial_y + g\partial_z$$

with $f, g \in C^\infty(\mathbf{R}^3)$. Clearly

$$[X, Y] = ((\partial_y f + g\partial_z f) - (\partial_x g + f\partial_z g))\partial_z,$$

so E is stable under the bracket operation on the tangent bundle $T(\mathbf{R}^3)$ (i.e., it is an integrable subbundle) if and only if the non-linear PDE

$$\partial_y f + g\partial_z f = \partial_x g + f\partial_z g$$

is satisfied. By the local Frobenius integrability theorem, this PDE is necessary and sufficient for the existence of an integral manifold to E through every point of $\mathbf{R}^3$. The purpose of this handout is to give a more "bare hands" approach to deriving the necessity of this PDE as a condition for the local existence of integral manifolds to E through any point of $\mathbf{R}^3$.

The key moral to the story of these "integrability conditions" is that whereas for rather general first-order ODE's there is always a nice local existence and uniqueness theorem for any initial condition, for PDE's (and systems of PDE's) there are often genuine non-trivial constraints that must be satisfied in order for there to exist a solution with any initial condition. This is what makes the local theory of PDE's much harder than the local theory of ODE's, since it often requires a mixture of geometric and analytic skill to identify the right necessary conditions for local existence of solutions, as well as to prove their sufficiency. Moreover, as we shall see below, the inverse function theorem lets us always translate the problem of finding local integral manifolds through a point into the problem of solving an initial-value problem for a system of (non-linear) PDE's, with the integrability condition becoming a necessary condition for the PDE to admit a local solution without restriction on an initial value. So working with integral manifolds is really just a geometric language for working with certain systems of PDE's; this is just a higher-dimensional version of the old observation that the theory of integral curves for vector fields is a convenient geometric language for discussing the problem of solving certain kinds of first-order (typically non-linear) vector-valued ODE's.

In this handout we focus on how one would be led by rather elementary considerations (without bracket operations on vector fields) to the above PDE as a necessary condition for the existence of integral manifolds to E through any point in $\mathbf{R}^3$, and how this problem of integral manifolds is equivalent to the solvability of a certain PDE. Our approach is along the lines of what 19th century mathematicians did to arrive at necessary conditions for solving certain kinds of PDE's before the advent of modern geometric machinery gave another way to think about the problem. (The other classical method to discovering necessary conditions is to work with multivariable Taylor series expansions of unknown solutions to a PDE and to find necessary relations among the unknown Taylor coefficients that can be translated into the language of partials of the known functions entering into the given PDE problem. The point is that the elementary "method of undetermined coefficients" that is used to mechanically solve for the Taylor coefficients of a solution to an initial-value ODE does not work so easily for PDE's because one encounters the new phenomenon of non-trivial "integrability conditions" on the PDE.)

2. Application of the inverse function theorem

Let $m_0 = (x_0, y_0, z_0) \in \mathbf{R}^3$ be any point, and suppose that there exists an integral manifold N to E through m_0. In particular, $\mathrm{T}_{m_0}(N) \subseteq \mathrm{T}_{m_0}(\mathbf{R}^3)$ is the plane $E(m_0)$ spanned by $X(m_0)$ and $Y(m_0)$. Consider the projection $N \to \mathbf{R}^2$ to the xy-plane. By applying the Chain Rule to the factorization of this map as the inclusion $N \to \mathbf{R}^3$ followed by the linear projection $\mathbf{R}^3 \to \mathbf{R}^2$, the tangent map $\mathrm{T}_{m_0}(N) \to \mathrm{T}_{(x_0,y_0)}(\mathbf{R}^2)$ is given by $X(m_0) \mapsto \partial_x|_{(x_0,y_0)}$ and $Y(m_0) \mapsto \partial_y|_{(x_0,y_0)}$) (why?). Hence, this is a linear isomorphism, so by the inverse function theorem (!) if we shrink N around m_0 then we can arrange that this projection $N \to \mathbf{R}^2$ is a C^∞-isomorphism onto an open subset $U \subseteq \mathbf{R}^2$ around (x_0, y_0). Consider the C^∞ inverse map followed by inclusion:

$$U \simeq N \hookrightarrow \mathbf{R}^3.$$

This composite map is smooth, and being made as inverse to the projection it has the form $(x, y) \mapsto (x, y, h(x, y))$ for a smooth function h on U; clearly we must have $h(x_0, y_0) = z_0$ because $m_0 \in N$.

Having deduced that N near m_0 admits this xy-parameterization with some smooth function h, let us now compute the tangent planes to N in terms of h to deduce conditions on h (in terms of PDE's involving f and g) that are necessary *and* sufficient for such a parameteric surface to be integral for the subbundle E. We obviously can consider (x, y) as giving global coordinates on the submanifold N in $\mathbf{R}^3$ via the C^∞-isomorphism $N \simeq U$, and so for a point $(x_1, y_1) \in U$ the point $m_1 = (x_1, y_1, h(z_1, y_1)) \in N$ has its tangent plane $\mathrm{T}_{m_1}(N)$ spanned by the velocity vectors to the parameteric coordinate lines through this point. (These velocity vectors are exactly $\partial_{x'}|_{m_1}$ and $\partial_{y'}|_{m_1}$ where $\{x', y'\}$ is the coordinate system $N \simeq U \subseteq \mathbf{R}^2$ defined by the standard functions x and y on $U \subseteq \mathbf{R}^2$.) These parametric coordinate lines in N are the parametric curves

$$t \mapsto (t, y_1, h(t, y_1)), \;\; t \mapsto (x_1, t, h(x_1, t))$$

in $\mathbf{R}^3$, so the corresponding velocity vectors are

$$\partial_x|_{m_1} + (\partial_x h)(x_1, y_1)\partial_z|_{m_1}, \partial_y|_{m_1} + (\partial_y h)(x_1, y_1)\partial_z|_{m_1} \in \mathrm{T}_{m_1}(\mathbf{R}^3).$$

Hence, $\mathrm{T}_{m_1}(N) \subseteq \mathrm{T}_{m_1}(\mathbf{R}^3)$ is the plane spanned by these two visibly linearly independent vectors. But the integral manifold condition is $\mathrm{T}_{m_1}(N) = E(m_1)$ for all $m_1 \in N$ near m, and inspecting the trivializing frame $\{X, Y\}$ in the definition of E leaves as the only possibility

$$\partial_x|_{m_1} + (\partial_x h)(x_1, y_1)\partial_z|_{m_1} = X(m_1), \;\; \partial_y|_{m_1} + (\partial_y h)(x_1, y_1)\partial_z|_{m_1} = Y(m_1),$$

or in other words (upon comparing coefficients at $m_1 = (x_1, y_1, h(x_1, y_1))$) that for all $(x_1, y_1) \in \mathbf{R}^2$ near (x_0, y_0) we have

$$f(x_1, y_1, h(x_1, y_1)) = (\partial_x h)(x_1, y_1), \;\; g(x_1, y_1, h(x_1, y_1)) = (\partial_y h)(x_1, y_1).$$

To summarize our findings, in order that E admits an integral manifold through an arbitrary point in $\mathbf{R}^3$, it is necessary and sufficient that for any point $(x_0, y_0) \in \mathbf{R}^2$ there exists a smooth function h defined on an open U in $\mathbf{R}^2$ around (x_0, y_0) such that

$$f(x, y, h(x, y)) = \partial_x h, \;\; g(x, y, h(x, y)) = \partial_y h \tag{1}$$

and $h(x_0, y_0) \in \mathbf{R}$ may be assigned to be an arbitrary value $z_0 \in \mathbf{R}$. Our task is to translate this very explicit and concrete necessary and sufficient local existence criterion for integral manifolds into the initial PDE *not* involving h that we derived at the outset from the definition of integrability for the subbundle $E \subseteq T(\mathbf{R}^3)$ (the necessary and sufficient nature of which as a condition for the local existence of integral manifolds through any point of $\mathbf{R}^3$ is a special case of the local Frobenius integrability theorem).

3. Mixed partials

Now we come to the real point, which is that simple considerations resting on equality of mixed partials basically lead to the integrability conditions in the setup of the Frobenius theorems. In our situation, it goes as follows. We have arrived at the expressions in (1) for $\partial_x h$ and $\partial_y h$ in terms of f and g, so the necessary equality $\partial_y(\partial_x h) = \partial_x(\partial_y h)$ forces

$$\partial_y(f(x,y,h(x,y))) = \partial_x(g(x,y,h(x,y)))$$

if such an h is to exist at all. Now we artfully eliminate the appearance of h! Using the Chain Rule to expand both sides of the above identity, we get the reformulated equality

$$(\partial_y f)(x,y,h) + (\partial_y h)(x,y)\cdot(\partial_z f)(x,y,h) = (\partial_x g)(x,y,h) + (\partial_x h)(x,y)\cdot(\partial_z g)(x,y,h)$$

that still involves h and its partials. By (1) we get another reformulation:

$$(\partial_y f)(x,y,h) + g(x,y,h)\cdot(\partial_z f)(x,y,h) = (\partial_x g)(x,y,h) + f(x,y,h)\cdot(\partial_z g)(x,y,h)$$

with h only appearing through its value $h(x,y)$ and not through its partials. This physically corresponds to rewriting the equations of a dynamical system purely in terms of positions without involving velocities, and as such as the hallmark feature of "integrable systems".

In particular, in order that we can find h satisfying (1) such that the value $h(x_0,y_0)$ may be arbitrarily assigned for any fixed $(x_0,y_0)\in\mathbf{R}^2$, by specializing at $x=x_0$ and $y=y_0$ with any desired value $h(x_0,y_0)=z_0$ we get the necessary condition

$$(\partial_y f)(x_0,y_0,z_0) + g(x_0,y_0,z_0)(\partial_z f)(x_0,y_0,z_0) = (\partial_x g)(x_0,y_0,z_0) + f(x_0,y_0,z_0)\cdot(\partial_z g)(x_0,y_0,z_0)$$

where $(x_0,y_0,z_0)\in\mathbf{R}^3$ is arbitrary. In other words, we have obtained the *necessary* condition

$$\partial_y f + g\partial_z f = \partial_x g + f\partial_z g \tag{2}$$

for the existence of integral manifolds to E through any point of $\mathbf{R}^3$. This is *exactly* the explicit description of the condition that E be an integrable subbundle of $T(\mathbf{R}^3)$, with h eliminated. So indeed by elementary considerations (with the inverse function theorem) we have deduced the necessity of the integrability conditions without needing to ever discuss vector fields or the bracket operation. Put another way, the PDE (1) for the unknown h can be solved *only* if (2) holds, and the local Frobenius theorem says that (2) is even sufficient for local (and unique!) solvability near any fixed point $(x_0,y_0)\in\mathbf{R}^2$ subject to an arbitrary choice of initial value $h(x_0,y_0)=z_0$.

Of course, to analyze general situations the geometric language of vector fields and flow along integral curves is a lot more convenient and efficient than explicit manipulations with PDE's. But also note that it is *not* evident that the preceding reasoning can be reversed to recover the local existence of h with any prescribed initial value $h(x_0,y_0)\in\mathbf{R}$ via elementary arguments with the final PDE constraint (2) (where we artfully eliminated the appearance of h). That is, by eliminating h we seem to have possibly lost the *sufficiency* aspect. The local Frobenius theorem says that sufficiency is actually not lost (and that moreover there is uniqueness upon specifying $h(x_0,y_0)$ arbitrarily and working over a small connected open neighborhood of (x_0,y_0) in $\mathbf{R}^2$). This should make you appreciate the content of the local Frobenius integrability theorem (which does provide exactly the local existence of such an h under the derived necessary condition (2), via a proof that rests on essentially geometric ideas related to flow along integral curves to vector fields).