Math 396. Map making

1. Motivation

When a cartographer makes a map of part of the surface of the earth, this amounts to choosing a C^∞ isomorphism from a 2-submanifold with corners S in the sphere $S^2(r)$ of radius r onto a 2-submanifold with corners R in the plane $\mathbf{R}^2$ (e.g., a square region, or a disc). To have an accurate map, there are two basic geometric features one must take into account: accuracy in angles and accuracy in area. For example, if two curves that cross transversally on the surface of the earth are translated onto the map, do they cross with the same angle on the map? For navigation by sea, the most basic manifestation of this question is: do the lines of latitude and longitude on the map cross at right angles (as they do on the surface of the earth)? Rigorously, the question is whether *tangent vectors* to the curves at the crossing points on both S and R meet with the same angle. In other words, does the C^∞ isomorphism $f : S \simeq R$ have the property that the induced mapping on tangent spaces $\mathrm{d}f(s) : \mathrm{T}_s(S) \simeq \mathrm{T}_{f(s)}(R) \simeq \mathbf{R}^2$ is angle-preserving with respect to the standard metric tensors on $S^2(r)$ and $\mathbf{R}^2$? The problem of area is this: do regions in R have areas (computed via the metric tensor of the plane) that are in the same proportion as the areas of the corresponding regions on S (computed via the metric tensor of the sphere)? Note that by considering *ratios* of areas, the specific radius of the earth and the choice of scale in the map are irrelevant.

Cartographers have long known as a matter of experience that it is impossible to create maps that are both angle-preserving and area-preserving (in the senses considered above). We will give a rigorous mathematical proof of this fact below. As a consequence of this state of affairs, one is led to seek maps that satisfy at least one of the two conditions: angle preservation with area distortion or area preservation with angle distortion. For the purposes of navigation by sea and air, angle preservation is extremely important because much local navigation is done via angles relative to lines of sight from the current position (e.g., planes flying along a loxodrome in order to get to a specific point when the wind conditions are generally known, or sea captains who need to travel along specific meridians). Most concretely, one wants to see the lines of latitude and longitude be meeting at local right angles. The *Mercator projection* achieves such angle preservation, but at a well-known cost in area distortion near the poles (just ask anyone who lives in Greenland or Australia).

In this handout, we use metric tensors to examine these problems in map making and we present the mathematical form of Mercator's projection as well as some others. In particular, we will calculate the distortion effect on area under Mercator's projection.

2. Conformal mappings

A C^∞ isomorphism between Riemannian manifolds with corners is an *isometry* when the tangent mappings respect the inner products (i.e., the metric tensor on the target pulls back to the metric tensor on the source). In other words, the tangent maps are orthogonal maps between inner product spaces. There is a weaker condition than orthogonality that one can ask to hold, namely preservation of angles. This is just slightly weaker than the isometry condition, but when applied over the family of tangents spaces to a manifold it incorporates a much wider range of possibilities. To understand this situation, we first need to clarify the distinction between orthogonality and angle preservation. We begin with some considerations in linear algebra.

Let $(V, \langle \cdot, \cdot \rangle)$ and $(V', \langle \cdot, \cdot \rangle')$ be inner product spaces with the same positive dimension n. For any two independent $v_1, v_2 \in V$, we define

$$\angle(v_1, v_2) \stackrel{\text{def}}{=} \cos^{-1}\left(\frac{|\langle v_1, v_2 \rangle|}{\|v_1\|\|v_2\|}\right) \in (0, \pi)$$

to be the *angle* between v_1 and v_2. Note that $\angle(v_1, v_2) = \angle(v_2, v_1)$. Clearly $\angle(v_1, v_2)$ is unaffected by nonzero scaling of v_1 and v_2. If $T : V \simeq V'$ is a linear isomorphism, we say T is *angle-preserving* if

$$\angle(T(v_1), T(v_2)) = \angle(v_1, v_2)$$

for any two independent $v_1, v_2 \in V$ (so $T(v_1), T(v_2)$ in V' are independent). Since 1-dimensional spaces do not admit a pair of independent vectors, we simply declare any linear isomorphism T to be angle-preserving in the 1-dimensional case (or it follows by logic from the empty condition, if you prefer). Obviously an orthogonal linear map T (i.e., one that respects the inner products) is angle-preserving. Also, if T is angle-preserving then clearly so is cT for any $c \neq 0$. This gives all possibilities:

Lemma 2.1. *A linear isomorphism $T : V \simeq V'$ is angle-preserving if and only if there exists $c > 0$ such that cT is orthogonal, in which case c is unique.*

This lemma makes angle-preservation an interesting property for a C^∞ isomorphism between Riemannian manifolds with corners (as a condition on the tangent mappings) because the positive constant as in the lemma (applied to tangent mappings) may *vary* from point to point. It is precisely the possibility of such variation that makes this condition a substantial weakening of the isometry condition.

Proof. To see the uniqueness of c, if cT and $c'T$ are both orthogonal then for any nonzero $v \in V$ we have

$$\|v\| = \|cT(v)\|' = \|(c/c')(c'T(v))\|' = |c/c'| \cdot \|c'T(v)\|' = |c/c'|\|v\|,$$

so $|c/c'| = 1$. Since $c, c' > 0$, this forces $c = c'$. As for existence, we induct on the common dimension n. The case $n = 1$ is trivial (and uninteresting) for existence, so we now assume $n \geq 2$.

Consider the case $n = 2$. Let $\{e_1, e_2\}$ be an orthonormal basis of V, so $\{T(e_1), T(e_2)\}$ is an orthogonal pair of independent vectors in the plane V'. Composing T with multiplication by $1/\|T(e_1)\|' > 0$, we can assume $T(e_1)$ is a unit vector in V'. We want to prove that $T(e_2)$ is a unit vector, as then T carries an orthonormal basis to an orthonormal basis and so is orthogonal. By angle-preservation, $\angle(e_1, e_1 + e_2) = \angle(T(e_1), T(e_1 + e_2)) = \angle(T(e_1), T(e_1) + T(e_2))$. Since e_1 and $T(e_1)$ are unit vectors, these respective angles are the $\cos^{-1}$'s of $1/\sqrt{2}$ and $1/\sqrt{1 + \|T(e_2)\|'^2}$. This forces $\|T(e_2)\|' = 1$ as desired.

Now assume $n > 2$ and that the result is known in all lower dimensions. We pick a 2-dimensional subspace $P \subseteq V$ and let $P' \subseteq V'$ be its 2-dimensional image under T. By angle-preservation, T carries $P^\perp$ isomorphically onto $P'^\perp$, and these orthogonal complements have positive dimension $n - 2 < n$. Hence, by induction and the settled 2-dimensional case, there exist unique positive α, β such that αT is an orthogonal map from P onto P' and βT is an orthogonal map from $P^\perp$ to $P'^\perp$. In particular, on each line in P the mapping T distorts length by the factor $1/\alpha$ and on each line in $P^\perp$ it distorts length by the factor $1/\beta$. Thus, for a plane Q spanned by a line in P and a line in $P^\perp$ the restriction of T to Q distorts lengths by a factor of $1/\alpha$ on one line and $1/\beta$ on another line. However, by the 2-dimensional case we know that T must distort length by the same factor on all lines of the plane Q. Hence, $1/\alpha = 1/\beta$, so $\alpha = \beta$. If we let c denote this common value, then cT has orthogonal restriction as isomorphisms $P \simeq P'$ and $P^\perp \simeq P'^\perp$. It follows easily that $cT : V \simeq V'$ must therefore also be orthogonal. ■

We deduce the very important:

Theorem 2.2. *The linear map $T : V \simeq V'$ respects inner products if and only if it is angle-preserving and $\wedge^n(T^\vee) : \wedge^n(V'^\vee) \simeq \wedge^n(V^\vee)$ respects the volume forms.*

Recall that there are two volume forms, namely the wedge products of duals to an orthonormal basis (giving two possibilities up to sign). The condition on volume forms means that $\wedge^n(T^\vee)$ carries volume forms to volume forms.

Proof. If T preserves the inner products then it obviously is angle-preserving and respects the volume forms. Conversely, if T is angle-preserving and respects volume forms then from angle-preservation we know (by the preceding lemma) that cT is orthogonal for a unique $c > 0$ and we must prove $c = 1$. Since cT is orthogonal it does respect volume forms. But $\wedge^n((cT)^\vee) = c^n \wedge^n (T^\vee)$, so since volume forms are unique up to sign we conclude $c^n = \pm 1$. Since $c > 0$, this forces $c = 1$. ■

We now wish to transport these notions to Riemannian geometry. Let $(M, \mathrm{d}s^2)$ and $(M', \mathrm{d}s'^2)$ be two Riemannian manifolds with corners, and assume they have constant dimension $n > 0$. Let $f : M \simeq M'$ be a C^∞ isomorphism. We say f is *conformal* if the linear isomorphism

$$\mathrm{d}f(m) : \mathrm{T}_m(M) \simeq \mathrm{T}_{f(m)}(M')$$

is angle-preserving for all $m \in M$. Then $(M, \mathrm{d}s^2)$ and $(M', \mathrm{d}s'^2)$ are said to be *conformally equivalent*. Since angle-preservation is uninteresting in dimension 1, we will now restrict attention to $n \geq 2$. (Riemannian geometry is also rather boring in dimension 1, since everything is flat; in fact, one can even classify connected Riemannian 1-manifolds with corners up to isometry: nontrivial intervals in $\mathbf{R}$ and circles in the plane with their standard metric.) The theory of conformal mapppings in the simplest case $n = 2$ already exhibits an astounding amount of depth and difficulty.

A pair of Riemannian manifolds with corners is *locally conformal* if each point on one admits an open neighborhood that is conformally equivalent to an open subset of the other. The theory of the curvature tensor will provide a method to prove that the plane is *nowhere* locally isometric to the 2-sphere (with their standard metrics). However, as we will explain later, the Mercator projection gives a conformal equivalence between a rather large open subset of a sphere and a rectangular strip in the plane. This demonstrates the huge gulf between conformal maps and isometries in Riemannian geometry.

The property of being conformal can be made more visual via velocity vectors:

Lemma 2.3. *A C^∞ isomorphism $f : M \simeq M'$ is conformal if and only if for any non-boundary point $m \in M$ and any two C^∞ embeddings $\sigma_1, \sigma_2 : (-\varepsilon, \varepsilon) \rightrightarrows M$ that cross transversally at $\sigma_1(0) = \sigma_2(0) = m$, the paths $f \circ \sigma_1$ and $f \circ \sigma_2$ in M' cross transversally at $f(m) \in M'$ at $t = 0$ with velocity vectors at the same angles:*

$$\angle(\sigma_1'(0), \sigma_2'(0)) = \angle((f \circ \sigma_1)'(0), (f \circ \sigma_2)'(0)).$$

Proof. Any nonzero tangent vector at $m \in M - \partial M$ can be expressed as the velocity vector to a curve through m across some open interval of time centered at 0, so by the Chain Rule the given condition does express conformality away from the boundary points. We shall deduce angle-preservation at the boundary points by continuity considerations. Pick $m \in \partial M$, and let $\vec{v}$ be a non-vanishing vector field over an open set U around m, so $\vec{v}' = (\mathrm{d}f)(\vec{v})$ is a non-vanishing vector field over the open set $U' = f(U)$ around $f(m)$. Let $\lambda = (\|\vec{v}_1'\|' \circ f)/\|\vec{v}_1\| \in C^\infty(U)$, so this is a positive smooth function on U. By the conformality over $U - \partial U$, the two Riemannian metric tensors $f^*(\mathrm{d}s'^2)$ and $\lambda^2 \cdot \mathrm{d}s^2$ concide over $U - \partial U$. Since $U - \partial U$ is dense in U, by continuity these tensors agree over U. Passing to fibers, this gives conformality by the criterion in Lemma 2.1. (Explicitly, for all $u \in U$, $\mathrm{d}f(u)$ carries $\langle \cdot, \cdot \rangle_{u'}$ to $\lambda(u)^2$ times $\langle \cdot, \cdot \rangle_u$.) ∎

It follows from the inverse function theorem that if a C^∞ mapping $f : M \to M'$ that is not assumed to be an isomorphism is an angle-preserving linear isomorphism on the level of tangent spaces then it is locally conformal. (Namely, on small opens where it is a C^∞ isomorphism it is conformal.) By the same method as used near the end of the proof of the preceding lemma, if $f : M \simeq M'$ is a conformal equivalence, there is a unique positive smooth function λ on M such that $f^*(\mathrm{d}s'^2) = \lambda^2 \cdot (\mathrm{d}s^2)$. For each $m \in M$, $\lambda(m)$ is the factor by which $\mathrm{d}f(m)$ distorts length. We call λ the *conformal multiplier*.

Theorem 2.4. *A C^∞ isomorphism $f : M \simeq M'$ between Riemannian manifolds with corners is an isometry if and only if it is conformal and volume-preserving in the sense that* $\mathrm{Vol}(U) = \mathrm{Vol}(f(U))$ *for all open subsets $U \subseteq M$ (both volumes are required to be finite at the same time).*

See §3 in the handout on computing integrals for a discussion of volume on Riemannian manifolds with corners in the absence of orientation.

Proof. If f is an isometry then it pulls the metric tensor back to the metric tensor and so it is obviously conformal. The preservation of volume is easily deduced as well. (In the orientable case it follows from the fact that f pulls back a volume form to a volume form when M and M' are suitably oriented, and in the general case the localized nature of the definition of volume permits us to push through a variant on this argument.)

For the converse, suppose f is conformal and volume-preserving. Let $\lambda : M \to (0, \infty)$ be the smooth positive function that encodes the distortion effect of f on lengths in tangent spaces. We want to prove $\lambda = 1$, since $f^*(\mathrm{d}s'^2) = \lambda^2 \cdot \mathrm{d}s^2$. The hypotheses on f are inherited under restriction to open subsets, and the formation of λ is local. Hence, our problem is a local one. We may therefore assume M admits global coordinates, or more specifically that M is orientable. We pick an orientation, and use the unique orientation on M' so that f is orientation preserving. Let ω and ω' be the associated volume forms.

We first claim $f^*(\omega') = \lambda^n \cdot \omega$. For $m \in M$, this says that the linear orientation-preserving isomorphism $\mathrm{d}f(m) : \mathrm{T}_m(M) \simeq \mathrm{T}_{f(m)}(M')$ induces $\wedge^n(\mathrm{d}f(m)^\vee)$ that carries the volume form back to $\lambda(m)^n$ times the volume form. If $\{e_i\}$ is an oriented orthonormal basis of $\mathrm{T}_m(M)$ then the vectors $v_i' = (\mathrm{d}f(m))(e_i)$ give an oriented basis of $\mathrm{T}_{f(m)}(M')$ with v_i' having length $\lambda(m)$ for all i. Hence, $v_j' = \lambda(m) e_j'$ for a unit vector e_j' for each j, and $\{e_i'\}$ has the same orientation class as $\{v_i'\}$ since $\lambda(m) > 0$. It follows that $\{e_i'\}$ is an oriented orthonormal basis of $\mathrm{T}_{f(m)}(M')$, so $\omega'(m)$ is the ordered wedge product of the dual basis to the e_i''s. Since

$$\wedge^n(\mathrm{d}f(m))(e_1 \wedge \cdots \wedge e_n) = v_1' \wedge \cdots \wedge v_n' = \lambda(m)^n e_1' \wedge \cdots \wedge e_n',$$

passing to the dual mapping gives $(f^*(\omega'))(m) = \lambda(m)^n \omega(m)$, as desired.

Having proved $f^*(\omega') = \lambda^n \cdot \omega$, since f is a C^∞ isomorphism it follows that for any open set $U \subseteq M$ we have

$$\mathrm{Vol}(f(U)) = \int_{f(U)} \omega' = \int_U f^*(\omega') = \int_U \lambda^n \omega,$$

so the volume-preservation condition says $\int_U (\lambda^n - 1)\omega = 0$ for all open subsets $U \subseteq M$ with finite volume. Working in a small open neighborhood of a hypothetical point where $\lambda \neq 1$ (and so $\lambda^n - 1$

is bounded away from 0), we immediately get a contradiction. Hence, $\lambda = 1$ and so $f^*(\mathrm{d}s'^2) = \mathrm{d}s^2$. That is, f is an isometry. ■

Corollary 2.5. *If $f : M \simeq M'$ is a conformal equivalence such that* $\mathrm{Vol}(f(U_1))/\mathrm{Vol}(f(U_2)) = \mathrm{Vol}(U_1)/\mathrm{Vol}(U_2)$ *for all non-empty open subsets $U_1, U_2 \subseteq M$ with compact closure then the conformal multiplier $\lambda : M \to (0, \infty)$ is locally constant, and hence constant on connected components.*

Note that non-empty open subsets with compact closure in M necessarily have positive *finite* volume (check!), whence the ratios considered in the corollary make sense.

Proof. The problem is local, and so if we fix some U_2 with compact closure and only consider $U_1 \subseteq U_2$ then we are reduced to the case when M and M' have finite volume and $\mathrm{Vol}(f(U)) = c\mathrm{Vol}(U)$ for all open $U \subseteq M$, where $c = \mathrm{Vol}(M')/\mathrm{Vol}(M) > 0$. The same method as in the preceding proof shows that $\lambda^n = c$, so by positivity of λ and c we get $\lambda = c^{1/n}$. ■

By the theory of the curvature tensor to be studied later, the reason that the sphere is nowhere locally isometric to the plane is that the curvature tensor on the former is nowhere zero whereas on the latter it is everywhere zero. It will turn out that making a *constant* scaling on the metric tensor does not affect whether or not the curvature tensor vanishes at a point, and so it follows from the preceding corollary that the sphere and the plane cannot be locally conformally equivalent in a manner that respects ratios of volumes. Thus, from the viewpoint of cartography, no 2-submanifold with corners in a sphere admits a map (in the non-technical sense) on a piece of paper such that the map is both area-preserving and angle-preserving in the sense discussed at the beginning of this handout. Thus, we have rigorously proved the impossibility that is well-known to map-makers as a matter of experience.

The reasonable solutions to the "mapmaker's paradox" are to make maps that are conformal but distort area-ratios, or that preserve area-ratios but are not conformal. We shall make examples of both types of maps.

3. Non-conformal area-preserving maps

Consider a smooth embedded curve C in the upper half-plane $y > 0$ in the xy-plane such that C maps isomorphically onto its image I in the x-axis; that is, C is the graph of a smooth positive function over a non-trivial interval such that the function has non-vanishing derivative. It may be tempting to parameterize the manifold C by x, but we prefer to permit a more general parameterization of the form $t \mapsto (f(t), g(t))$ because in the case of a semi-circle we want to permit the parameterization $t \mapsto (\cos(t), \sin(t))$ with $0 < t < \pi$. The curve C generates an embedded smooth surface of revolution S in $\mathbf{R}^3$ that is the image of the embedding map $I \times S^1 \to \mathbf{R}^3$ given by

$$(t, \theta) \mapsto (f(t)\cos\theta, f(t)\sin\theta, g(t)).$$

Example 3.1. In the case $I = (0, \pi)$ and $f = \sin$, $g = \cos$ we get a unit sphere with the poles removed. Taking $\theta \in (0, 2\pi)$ (removing half of a great circle from the surface) thereby identifies our parameterization of the sphere with the spherical coordinate system.

In terms of this parameterization, the pullback of the metric tensor $\mathrm{d}x^{\otimes 2} + \mathrm{d}y^{\otimes 2} + \mathrm{d}z^{\otimes 2}$ is $f^2\mathrm{d}\theta^{\otimes 2} + (f'^2 + g'^2)\mathrm{d}t^{\otimes 2}$ and the area form is $\pm f\sqrt{f'^2 + g'^2}\mathrm{d}\theta \wedge \mathrm{d}t$ with the sign determined by the choice of orientation (which does not affect area calculations).

We now introduce a rather general mapping to the plane that will turn out to be area preserving but non-conformal (and hence angle distorting). In the special case of the sphere, it will have a simple geometric interpretation. Consider the mapping $p : S \to \mathbf{R}^2$ to the plane given by

$$p(t,\theta) = (\theta, \int_{t_0}^{t} f\sqrt{f'^2 + g'^2})$$

for a fixed $t_0 \in I$. This is a C^∞ mapping whose Jacobian matrix is diagonal with determinant $f\sqrt{f'^2+g'^2}$ that is non-vanishing at all points, so it is a local C^∞ isomorphism. Taking $\{u, v\}$ as the standard coordinate system on $\mathbf{R}^2$, the metric tensor in the plane is $\mathrm{d}u^{\otimes 2} + \mathrm{d}v^{\otimes 2}$ and its pullback under p is $\mathrm{d}\theta^{\otimes 2} + f\sqrt{f'^2+g'^2}\mathrm{d}t^{\otimes 2}$. This is not the metric tensor induced on S from $\mathbf{R}^3$, so p is certainly not an isometry, nor is it locally conformal (as the pullback $p^*(\mathrm{d}u^{\otimes 2} + \mathrm{d}v^{\otimes 2})$ is also not a multiple of the metric tensor on S by a (positive) smooth multiplier function). However,

$$p^*(\mathrm{d}u \wedge \mathrm{d}v) = f\sqrt{f'^2+g'^2}\mathrm{d}\theta \wedge \mathrm{d}t,$$

so the area forms *do* agree under pullback (up to sign, depending on the choice of orientation on S). Hence, p is area preserving.

Consider the special case when S is a sphere, so $f = \sin$ and $g = \cos$. In this case, $p(t,\theta) = (\theta, -\cos(t))$ with $t \in (0,\pi)$, so p carries the domain of spherical coordinates onto the rectangle $(0,\pi) \times (-1,1)$. What is the meaning of this p? If we wrap a cylinder around the unit sphere with axis of symmetry given by the z-axis, then there is a natural projection from the sphere (minus the poles) onto the surface of the cylinder: we draw the half-line parallel to the xy-plane linking the z-axis to a point on the sphere, and we mark where that ray meets the surrounding cylinder. This point has height $\cos t$ above the xy-plane, so its cylindrical coordinates are (θ, t). Hence, up to a reflection through the xy-plane, our mapping p is this: we project the sphere onto the circumscribed cylinder with axis of symmetry along the z-axis, and then we make a slit in the cylinder along $\theta = 0$. Unrolling the cylinder to a flat piece of paper then gives the image in the $t\theta$-plane (up to reflection through the θ-axis).

Area preserving maps such as p above are mathematically interesting, but they tend to be less useful for navigational purposes. (However, there are some area preserving maps in use, particular by governmental organizations that are trying not to offend other countries by distorting their true size and shape.)

4. Conformal area-distorting maps

There are two conformal mappings from the sphere to the plane that have been popularly used by cartographers, the Mercator projection and the stereographic projection. The latter corresponds to those old maps in which the two hemispheres are presented as separate discs with longitude lines given by radial lines heading to the pole at the origin of each disc and lattitude lines given by concentric circles centered at the origin (with the outside edge on each disc equal to the equator). In this setup, we use stereographic projection with a plane that passes through the equator (and not through a pole on the other side of the world): projection from the pole in the opposite hemisphere is a C^∞ isomorphism of the chosen closed hemisphere onto a closed disc.

Algebraically, if we consider a unit sphere (for simplicity) and we perform stereographic projection from the north pole $(0,0,1)$ onto the xy-plane, then for the point P on the sphere given in spherical coordinates by

$$P = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$

some elementary geometry (check!) shows that the point where the line linking P to the north pole meets the xy-plane is

$$h(\theta,\phi)=\left(\frac{\sin\phi\cos\theta}{1-\cos\phi},\frac{\sin\phi\sin\theta}{1-\cos\phi}\right).$$

We know that stereographic projection sets up a C^∞ isomorphism between $S^2-\{(0,0,1)\}$ and the plane (allowing the plane of projection to pass through the sphere does not affect this property; the algebraic formula for the inverse map is not hard to find, and is easier to sort out if one avoids spherical coordinates).

We compute the pullback of the metric tensor from the plane:

$$h^*(\mathrm{d}x^{\otimes 2}+\mathrm{d}y^{\otimes 2})=(1-\cos\phi)^{-2}(\sin^2(\phi)\mathrm{d}\theta^{\otimes 2}+\mathrm{d}\phi^{\otimes 2}).$$

The induced metric tensor on S^2 from $\mathbf{R}^3$ is $\sin^2(\phi)\mathrm{d}\theta^{\otimes 2}+\mathrm{d}\phi^{\otimes 2}$, so we have a conformal mapping into the plane with conformal multiplier $\lambda(\phi,\theta)=(1-\cos\phi)^{-1}$. On area forms, $h^*(\mathrm{d}x\wedge\mathrm{d}y)$ is $(1-\cos\phi)^{-2}$ times the area form on the sphere. Thus, along the equator ($\phi=\pi/2$) the distortion factor on area is 1 and as long as we remain in the opposite (i.e., southern) hemisphere the distortion gets worse as we approach the south pole (i.e., the center of the disc) with distorion factor $1/2$ there. This is not so bad: by projecting a hemisphere onto a disc using stereographic projection from the opposite pole we get a conformal mapping whose area distortion is uniform across concentric circles around the origin and at worst distorts area by a factor of $1/2$. That is, if we have two regions of equal area in the southern hemisphere on the earth but one is near the south pole and the other is near the equator then the one near the south pole will be assigned only half as much area in the disc as the region near the equator.

Although the distortion effects on area are bounded within a given hemisphere under this conformal mapping, the drawback is that the lines of latitude and longitude are not actually *straight lines* in the plane. The Mercator projection avoids this defect, but it has unbounded area distortion near the poles. We now turn to the mathematical preparation for the Mercator projection. Rather generally, given a surface of revolution S as above, the mapping $P:S\to\mathbf{R}^2$ defined by

$$P(t,\theta)=\left(\theta,\int_{t_0}^{t}\frac{\sqrt{f'^2+g'^2}}{f}\right)$$

for a fixed $t_0\in I$ is clearly a local C^∞ isomorphism, and we have $P^*(\mathrm{d}x)=\mathrm{d}\theta$ and $P^*(\mathrm{d}y)=(\sqrt{f'^2+g'^2}/f)\mathrm{d}t$, so the metric tensor $f^2\mathrm{d}\theta^{\otimes 2}+(f'^2+g'^2)\mathrm{d}t^{\otimes 2}$ on S from its embedding into $\mathbf{R}^3$ can be written as $f\cdot P^*(\mathrm{d}x^{\otimes 2}+\mathrm{d}y^{\otimes 2})$ with $f>0$. To express f in terms of x and y requires locally inverting P to express t in terms of x and y, which seems unpleasant. Regardless, we at least see explicitly that P is a locally conformal mapping since its failure to respect the metric tensors on S and on the plane is given by a smooth positive function multiplier. One can likewise compute the behavior on area forms to see the area distortion, but we prefer to work this out in the special case of the sphere.

Now consider the special case when S is the unit sphere, so we take $f=\sin$ and $g=\cos$ (and $0<t<\pi$) to get the sphere endowed with spherical coordinates (t playing the role of ϕ). In this case, $P(t,\theta)=(\theta,\int_{\pi/2}^{t}\csc)=(\theta,\log\tan(t/2))$. This is a C^∞ isomorphism from the domain of spherical coordinates (i.e., the complement of a longitude line) onto the infinite slab $(0,2\pi)\times\mathbf{R}$. This is the *Mercator projection*, though the version found in stores tends to stop at finite height. Note that $y=\log\tan(t/2)$ is equal to 0 at precisely the points with $t=\pi/2$. That is, the equatorial points go in the center of the slab. The latitude lines have fixed t and varying θ, so these go over to the horizontal lines in the image. The longitude lines have fixed θ and varying t, so these go over to the vertical lines in the image. The extreme cases $t \to 0, \pi$ correspond to $y \to \infty, -\infty$, so the poles are (in principle) infinitely far off to the top and bottom.

Let us work out the metric tensor in terms of the coordinates from the plane. We shall write ϕ instead of t now. Since $\log \tan(\phi/2) = y$, we have $\tan(\phi/2) = e^y$, so passing to differentials gives $\sec^2(\phi/2)\mathrm{d}(\phi/2) = e^y \mathrm{d}y$. But $\sec^2 = 1 + \tan^2$, so with a bit of algebra we find

$$\mathrm{d}\phi = \frac{2e^y}{e^{2y}+1}\mathrm{d}y = \frac{2}{e^y + e^{-y}}\mathrm{d}y = \frac{1}{\cosh(y)}\mathrm{d}y = \mathrm{sech}(y)\mathrm{d}y$$

(in terms of the hyperbolic trig functions). Since $\mathrm{d}\theta = \mathrm{d}x$ and

$$\mathrm{sech}(y) = \frac{2e^y}{e^{2y}+1} = \frac{2\tan(\phi/2)}{\sec^2(\phi/2)} = \sin\phi,$$

the metric tensor $\sin^2(\phi)\mathrm{d}\theta^{\otimes 2} + \mathrm{d}\phi^{\otimes 2}$ on the sphere is $(\mathrm{sech}(y))^2(\mathrm{d}x^{\otimes 2} + \mathrm{d}y^{\otimes 2})$ in terms of the xy-coordinates. We see the conformal nature of this mapping, with area distorted by the factor $\mathrm{sech}(y) = 2/(e^y + e^{-y})$. Hence, there is no area distortion along the equator ($y = 0$) but near the poles the distortion factor grows exponentially as a function of the vertical distance from the equator in the Mercator map. More specifically, for a "large" xy-box in the Mercator map far from the line $y = 0$ the corresponding region on the sphere has exponentially smaller area (as a function of y). This is exactly the gross distortion near the poles that everyone knows. But since the map is conformal, it is perfectly useful for navigational purposes.